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THE TWO-PHASE NAVIER-STOKES EQUATIONS WITH SURFACE TENSION IN CYLINDRICAL DOMAINS

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Abstract. This article is concerned with the well-posedness of a model for the dynamics of two immiscible and incompressible fluids in cylindrical domains, which are separated by a sharp interface, forming a contact angle with the solid wall of the container. We prove that the nonlinear system has a unique strong global solution in the L_p-sense, provided that the initial data is small. To this end, we show maximal L_p-regularity of the linearized problem and apply the contraction mapping principle in order to solve the nonlinear problem.

1. Introduction

In a wider sense, this article is concerned with the mathematical analysis of the dynamics of fluids. To be more precise, the behavior of two fluids inside a bounded container, separated by a sharp interface, is investigated.

Let $u = u(t, x)$ and $\pi = \pi(t, x)$ denote the velocity field and the pressure field of a single incompressible fluid in a domain Ω. By saying that the fluid is incompressible, we mean that its density $\rho > 0$ is constant. Then the dynamics of the fluid is described by the Navier-Stokes equations

$$\begin{aligned} \partial_t(\rho u) - \mu\Delta u + \rho(u \cdot \nabla)u + \nabla\pi = \rho f, \quad t > 0,\ x \in \Omega, \\ \operatorname{div} u = 0, \quad t > 0,\ x \in \Omega, \end{aligned} \tag{1.1}$$

where $\mu > 0$ represents the viscosity of the fluid and f is some external force (e.g. gravity). The first equation reflects balance of momentum, while the second equation states conservation of mass.

Let us consider a more comprehensive situation, where the domain Ω is occupied by two incompressible and immiscible fluids, *fluid 1* and *fluid 2*, which are separated by a sharp interface $\Gamma(t)$ for each $t \geq 0$. We denote by $\Omega_j(t)$ the subset of Ω which is filled with *fluid j*, $j \in \{1, 2\}$ with ρ_j,μ_j being the density and viscosity, respectively, of *fluid j*. If u^j and π^j are the velocity fields and the pressure fields of *fluid j*, respectively, then, for $t \geq 0$, one sets

$$u(t,x) := \begin{cases} u^1(t,x), & x \in \Omega_1(t), \\ u^2(t,x), & x \in \Omega_2(t), \end{cases} \qquad \pi(t,x) := \begin{cases} \pi^1(t,x), & x \in \Omega_1(t), \\ \pi^2(t,x), & x \in \Omega_2(t). \end{cases}$$

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Assuming that (u^j, π^j) satisfies the Navier-Stokes equations in each of the phases $\Omega_j(t)$, we may conclude that (u, π) satisfies (1.1) for all $t > 0$ and $x \in \Omega \backslash \Gamma(t)$, where ρ and μ are defined by

$$\rho(x) := \begin{cases} \rho_1, & x \in \Omega_1(t), \\ \rho_2, & x \in \Omega_2(t), \end{cases} \quad \mu(x) := \begin{cases} \mu_1, & x \in \Omega_1(t), \\ \mu_2, & x \in \Omega_2(t). \end{cases}$$

Clearly one expects that the two fluids should affect each other in their dynamics. Therefore, it is natural to ask for relations that describe the coupling of the two fluids across the interface $\Gamma(t)$. If one neglects effects of phase transitions between the phases $\Omega_1(t)$ and $\Omega_2(t)$ (e.g. the exchange of mass) then the motion of the moving boundary $\Gamma(t)$ is only caused by the velocity fields of the both fluids. Therefore it is natural to propose that $u^2|_{\Gamma(t)} = u^1|_{\Gamma(t)}$. Then the *normal velocity* V_Γ of $\Gamma(t)$ is given by

$$V_\Gamma = u \cdot \nu_\Gamma, \tag{1.2}$$

where ν_Γ denotes the unit normal field on $\Gamma(t)$ pointing from $\Omega_1(t)$ to $\Omega_2(t)$. We call the quantity $[\![u]\!] := u^2|_{\Gamma(t)} - u^1|_{\Gamma(t)}$ the *jump of* u *across* $\Gamma(t)$. Note that

$$[\![u]\!] = 0 \tag{1.3}$$

if and only if the velocity field u is continuous across the interface $\Gamma(t)$. Another condition on $\Gamma(t)$ reads

$$-[\![\mu(\nabla u + \nabla u^{\mathsf{T}})]\!]\nu_\Gamma + [\![\pi]\!]\nu_\Gamma = \sigma H_\Gamma \nu_\Gamma, \tag{1.4}$$

where $\sigma > 0$ denotes the (constant) *surface tension* of $\Gamma(t)$ and $H_\Gamma := -\operatorname{div}_\Gamma \nu_\Gamma$ is the *mean curvature* of $\Gamma(t)$ with $\operatorname{div}_\Gamma$ being the surface divergence on $\Gamma(t)$. Condition (1.4) describes the balance of forces on the interface. To be precise, there is no contribution to the rate of change of the momentum coming from the interface $\Gamma(t)$.

If the fixed boundary $\partial\Omega$ of Ω is not empty, then the system (1.1)-(1.4) with $[\![u]\!] = 0$ has to be equipped with appropriate boundary conditions on $\partial\Omega$ as well as some initial conditions on $u(0) = u_0$ and $\Gamma(0) = \Gamma_0$. There is a vast literature concerning the mathematical treatment of free boundary problems for the Navier-Stokes equations with or without surface tension. To this end we can only give a selection and refer the reader to [2, 5, 6, 7, 8, 9, 10, 11, 19, 23, 24, 25, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43]. For a derivation of (1.1)-(1.4) we refer to [18] or [27].

It is the main purpose of this article to extend the results on well-posedness obtained in [26] to the framework of bounded cylindrical domains. To be precise, we assume that $\Omega = G \times (H_1, H_2)$, where $G \subset \mathbb{R}^{n-1}$, $n \in \{2, 3\}$ is a bounded domain with smooth boundary and $H_1 < 0 < H_2$. Suppose furthermore that there is a family of hypersurfaces $\{\Gamma(t)\}_{t \geq 0}$ given as a graph of some height function h over G, i.e.

$$\Gamma(t) = \{(x', x_n) \in \Omega : x_n = h(t, x'),\ x' \in G\}, \quad t > 0,$$

such that for each $t \geq 0$ the interface $\Gamma(t)$ divides Ω into two subdomains $\Omega_1(t)$ and $\Omega_2(t)$ which are filled with two fluids, respectively. Let us use the convention that $\Omega_2(t)$ is the upper phase. Assuming that the equations (1.1)-(1.4) are be satisfied, we are led in a natural way to the problem of finding suitable boundary conditions on the vertical part $S_1 := \partial G \times (H_1, H_2)$ and the horizontal part $S_2 := (G \times \{H_1\}) \cup (G \times \{H_2\})$ of the boundary $\partial\Omega$ of Ω. This turns out to be a delicate question, since within the above setting we are on the one side concerned with two parts S_1 and S_2 of the boundary such that $\partial S_1 = \partial S_2$. Therefore the boundary conditions on S_1 and S_2 have to be chosen in such a way that they are compatible to each other. On the other side we have to deal with a *contact angle problem*, as $\partial\Gamma(t)$ is a moving contact line on S_1. At this point we want to emphasize that the choice of the periodic setting in [44] allows to circumvent the formation of a contact angle.

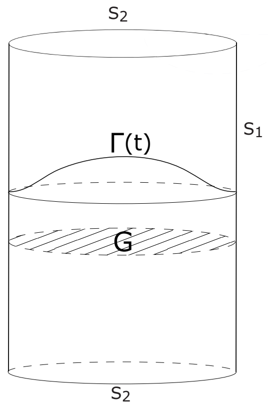

FIGURE 1. Cylindrical domain

The theory of contact angle problems, in particular with a dynamic contact angle which depends on t, is yet not well understood. In fact, there exist different points of view about how to model such a problem. Some researchers argue that the dynamic contact angle is determined by an additional equation, while others assume that the contact angle will be determined by the dynamic equations for the interface and the fluid, hence the equation for the contact angle should be redundant. We refer to [3] & [31] and to the references given therein.

Therefore, in order to avoid this lack of clarity, we assume throughout this article that the contact angle is constant and equal to 90 degrees. One can interpret this ansatz as an idealization. It is possible to translate the condition on the contact angle to a condition on the height function h from above. Indeed, if h is sufficiently smooth, then the unit normal on $\Gamma(t)$ with respect to $\Omega_1(t)$ is given by

$$\nu_\Gamma = \frac{1}{\sqrt{1 + |\nabla_{x'} h|^2}} \begin{pmatrix} -\nabla_{x'} h \\ 1 \end{pmatrix}.$$

Since the outer unit normal on S_1 is given by $\nu_{S_1} = (\nu_{\partial G}, 0)^\mathsf{T}$, the condition on the contact angle reads $\nu_\Gamma \cdot \nu_{S_1} = 0$ or equivalently $\partial_{\nu_{\partial G}} h = 0$ at the contact line. Concerning S_1 it is not possible to propose Dirichlet boundary conditions, the so-called *no-slip* boundary conditions, since this leads to a paradoxon for the moving contact line, see e.g. [28]. The next canonical choice are the so-called *Navier* boundary conditions or *partial-slip* boundary conditions

$$u \cdot \nu_{S_1} = 0, \quad P_{S_1}(\mu(\nabla u + \nabla u^\mathsf{T})\nu_{S_1}) + \alpha u = 0,$$

where $P_{S_1} := I - \nu_{S_1} \otimes \nu_{S_1}$ denotes the projection to the tangent space on S_1. The parameter $\alpha > 0$ has the physical meaning of a friction coefficient. However, it turns out that for $\alpha > 0$, these boundary conditions do not allow the interface to move along S_1 which is not very reasonable, as numerical simulations show. To see this defect, consider for simplicity the case $n = 2$. The equation (1.2) in terms of h then reads

$$\partial_t h = u_2 - u_1 \partial_1 h, \tag{1.5}$$

where $u = (u_1, u_2)$. Observe that for $n = 2$ the partial slip conditions read as follows

$$u_1 = 0, \quad \mu(\partial_1 u_2 + \partial_2 u_1) + \alpha u_2 = 0.$$

Therefore it holds that $\mu \partial_1 u_2 + \alpha u_2 = 0$, which is a Robin boundary condition for u_2 on S_1. Differentiating (1.5) with respect to x_1, and taking into account that $\partial_1 h = 0$ at S_1 (by the contact angle condition) we obtain $\partial_1 u_2 = 0$, hence $u_2 = 0$ if $\alpha > 0$. Consequently it holds that $\partial_t h = 0$ at S_1 and therefore $h(t)$ is constant with respect to t.

In order to circumvent this problem, we will consider the case $\alpha = 0$, the so-called *pure-slip* boundary conditions. From a physical point of view this means that there is no friction on the boundary S_1. Having fixed the boundary conditions on S_1 we may choose suitable boundary conditions on S_2, having in mind that these conditions have to match those on S_1. It turns out that homogeneous Dirichlet boundary conditions are a good choice, since they are compatible with the pure-slip boundary conditions on S_1. Note that the no-slip boundary conditions on S_2 do not cause any problems with the moving interface, since we will always have $\Gamma(t) \cap S_2 = \emptyset$ for all $t \geq 0$. We are thus led to the problem

$$\begin{aligned}
\partial_t(\rho u) - \mu \Delta u + \rho(u \cdot \nabla) u + \nabla \pi &= -\rho \gamma_a e_n, \quad \text{in } \Omega \backslash \Gamma(t),\\
\operatorname{div} u &= 0, \quad \text{in } \Omega \backslash \Gamma(t),\\
-[\![\mu(\nabla u + \nabla u^{\mathsf{T}})]\!] \nu_\Gamma + [\![\pi]\!] \nu_\Gamma &= \sigma H_\Gamma \nu_\Gamma, \quad \text{on } \Gamma(t),\\
[\![u]\!] &= 0, \quad \text{on } \Gamma(t),\\
V_\Gamma &= u \cdot \nu_\Gamma, \quad \text{on } \Gamma(t),\\
P_{S_1}\Big(\mu(\nabla u + \nabla u^{\mathsf{T}}) \nu_{S_1}\Big) &= 0, \quad \text{on } S_1 \backslash \partial\Gamma(t),\\
u \cdot \nu_{S_1} &= 0, \quad \text{on } S_1 \backslash \partial\Gamma(t),\\
u &= 0, \quad \text{on } S_2,\\
\nu_\Gamma \cdot \nu_{S_1} &= 0, \quad \text{on } \partial\Gamma(t),\\
u(0) &= u_0, \quad \text{in } \Omega \backslash \Gamma(0),\\
\Gamma(0) &= \Gamma_0,
\end{aligned} \tag{1.6}$$

where we denote by $\gamma_a > 0$ the acceleration constant due to gravity.

With this article, we present a rather complete analysis of (1.6) with respect to the existence and uniqueness of strong L_p-solutions. In Section 2 we will first transform (1.6) to a fixed domain which does not vary in time. This will be done by means of a height function h, assuming that $\Gamma(t)$ is given as the graph of h over the domain G. By means of local charts the transformed problem can be pulled back to certain model problems. As the analysis of two types of these model problems, namely the Stokes equations in quarter-spaces and the two-phase Stokes equations in half spaces is not known, we will provide a systematic treatment of these problems subsequently. At this point we want to emphasize that the analysis of the latter problems is more involved than the usual model problems in half spaces. This is due to the fact that one has to deal with mixed boundary conditions meeting at the contact line. However, our assumption on the contact angle enables us to use reflection techniques in order to pull back the quarter space to a half space with Dirichlet boundary conditions and the two-phase half space to a two-phase full space with a flat interface.

In Section 3 we use the results from Section 2 combined with a localization procedure to prove existence and uniqueness of a solution of the principal linearization having maximal regularity of type L_p. To be precise, if u and π denote the (transformed) velocity field and pressure field, respectively, we will show that $(u, \pi, [\![\pi]\!], h)$ enjoys the regularity

$$u \in H_p^1(J; L_p(\Omega)^n) \cap L_p(J; H_p^2(\Omega\backslash\Sigma)^n), \quad \pi \in L_p(J; \dot{H}_p^1(\Omega)),$$

$$[\![\pi]\!] \in W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma)).$$

and

$$h \in W_p^{2-1/2p}(J; L_p(\Sigma)) \cap H_p^1(J; W_p^{2-1/p}(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma)),$$

where $J = [0, T]$ is some nonempty bounded interval. This optimal regularity result in turn allows to apply the contraction mapping principle in Section 4 to obtain a unique solution of the nonlinear problem having optimal regularity as well. In particular, problem (1.6) generates a local semiflow in a natural phase space.

Finally, all technical results which are needed for the execution of the above program are collected in an appendix. Several results concerning extension operators, auxiliary elliptic and parabolic problems in quarter spaces and two-phase half spaces but also in bounded cylindrical domains are provided. In addition, we state the divergence theorem for bounded Lipschitz domains.

Notation: The symbols H_p^s, W_p^s, $s \geq 0$ refer to the Bessel potential spaces and Sobolev-Slobodeckii spaces, respectively. If $J = [0, T]$ is some interval and X a suitable Banach space, then ${}_0W_p^s(J; X)$ denotes the subspace of $W_p^s(J; X)$ consisting of all functions having a vanishing trace at $t = 0$, whenever it exists. We denote by $\dot{W}_p^k(\Omega) = \dot{H}_p^k(\Omega)$ the homogeneous Sobolev space of order $k \in \mathbb{N}$, where $\Omega \subset \mathbb{R}^n$ is some domain. The symbol $(\cdot|\cdot)$ denotes the standard inner product in $\mathbb{R}^n$ and we sometimes also make use of the notation $u \cdot v = (u|v)$ for $u, v \in \mathbb{R}^n$.

2. Preliminaries and model problems

For the sake of readability we will assume throughout this article that the space dimension n is equal to 3. This is the most important case from a viewpoint of applications. Furthermore we will assume from now on that $p > n + 2 = 5$. In Section 4 about the well-posedness of the nonlinear model, this condition on p is a result of some Sobolev embeddings which are needed for the proof.

It is convenient to introduce the *modified pressure* $\tilde{\pi} := \pi + \rho\gamma_a x_3$ in (1.6), where $x_3 = x \cdot e_3$, $x \in \mathbb{R}^3$ and $e_3 = (0, 0, 1)$. Then we obtain the following problem.

$$
\begin{aligned}
\partial_t(\rho u) - \mu\Delta u + \rho(u\cdot\nabla)u + \nabla\tilde{\pi} &= 0, \quad \text{in } \Omega\backslash\Gamma(t),\\
\operatorname{div} u &= 0, \quad \text{in } \Omega\backslash\Gamma(t),\\
-[\![\mu(\nabla u + \nabla u^{\mathsf{T}})]\!]\nu_\Gamma + [\![\tilde{\pi}]\!]\nu_\Gamma &= \sigma H_\Gamma \nu_\Gamma + [\![\rho]\!]\gamma_a x_3 \nu_\Gamma, \quad \text{on } \Gamma(t),\\
[\![u]\!] &= 0, \quad \text{on } \Gamma(t),\\
V_\Gamma &= u\cdot\nu_\Gamma, \quad \text{on } \Gamma(t),\\
P_{S_1}\Big(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}\Big) &= 0, \quad \text{on } S_1\backslash\partial\Gamma(t),\\
u\cdot\nu_{S_1} &= 0, \quad \text{on } S_1\backslash\partial\Gamma(t),\\
u &= 0, \quad \text{on } S_2,\\
\nu_\Gamma\cdot\nu_{S_1} &= 0, \quad \text{on } \partial\Gamma(t),\\
u(0) &= u_0, \quad \text{in } \Omega\backslash\Gamma(0),\\
\Gamma(0) &= \Gamma_0.
\end{aligned}
\tag{2.1}
$$

Here $\Omega = G \times (H_1, H_2)$, $H_1 < 0 < H_2$, is a cylindrical domain where $G \subset \mathbb{R}^2$ is an open bounded domain with a smooth boundary ∂G. The compact free boundary $\Gamma(t)$ divides Ω into two unbounded disjoint phases $\Omega_j(t)$, $j = 1, 2$, so that $\Omega = \Omega_1(t) \cup \Gamma(t) \cup \Omega_2(t)$. The convention is that $\Omega_2(t)$ is the upper phase while $\Omega_1(t)$ is the lower one with the unit normal ν_Γ at $x \in \Gamma(t)$ pointing from $\Omega_1(t)$ to $\Omega_2(t)$. We denote by ν_{S_1} the outer unit normal at the fixed boundary S_1. The operator $P_{S_1} := I - \nu_{S_1} \otimes \nu_{S_1}$ stands for the projection to the tangential space on S_1.

2.1. **Reduction to a flat interface.** In this subsection we transform the time-dependent domain $\Omega\backslash\Gamma(t)$ to a fixed domain. To this end, we assume that

$$\Gamma(t) = \{x = (x_1, x_2, x_3) \in G \times (H_1, H_2) : x_3 = h(t, x'),\ x' = (x_1, x_2) \in G\}$$

$t \geq 0$. Let $\varphi \in C^\infty(\mathbb{R}; [0, 1]))$ such that $\varphi(s) = 1$ if $|s| \leq \delta/2$ and $\varphi(s) = 0$ if $|s| \geq \delta$, where $\delta < \min\{-H_1, H_2\}/2$. Define a mapping

$$\Theta_h(t, \bar{x}) := \bar{x} + \varphi(\bar{x}_3)h(t, \bar{x}')e_3 =: \bar{x} + \theta_h(t, \bar{x}),$$

where $e_3 = (0, 0, 1)$, $\bar{x} := (\bar{x}', \bar{x}_3)$ and for fixed $t > 0$ set $x = \Theta_h(t, \bar{x})$. An easy computation shows

$$\theta_h'^{\mathsf{T}} = \begin{pmatrix} 0 & 0 & \partial_1 h\varphi \\ 0 & 0 & \partial_2 h\varphi \\ 0 & 0 & h\varphi' \end{pmatrix},$$

It follows that Θ_h' is invertible if $\|h\|_{\infty,\infty} < 1/(2|\varphi'|_\infty)$ and

$$(\Theta_h')^{-\mathsf{T}} = (I + \theta_h'^{\mathsf{T}})^{-1} = \frac{1}{1 + h\varphi'}\begin{pmatrix} 1 + h\varphi' & 0 & -\partial_1 h\varphi \\ 0 & 1 + h\varphi' & -\partial_2 h\varphi \\ 0 & 0 & 1 \end{pmatrix}.$$

Here $\|\cdot\|_{\infty,\infty}$ denotes the L_∞-norm in the time-space cylinder $(0, a)\times\Omega$, $a \in (0, \infty]$. In the sequel, let $\|h\|_{\infty,\infty} < \eta$ with $0 < \eta \leq 1/(2|\varphi'|_\infty)$ being sufficiently small. Then the inverse $\Theta_h^{-1} : \Omega \to \Omega$ is well defined and it transforms the free interface

$\Gamma(t)$ to the flat and fixed interface $\Sigma := G \times \{0\}$. Now we define the transformed quantities

$$\begin{aligned}\bar{u}(t,\bar{x}) &:= u(t,\Theta_h(t,\bar{x}))\\ \bar{\pi}(t,\bar{x}) &:= \tilde{\pi}(t,\Theta_h(t,\bar{x}))\end{aligned}$$

and compute $\nu_\Gamma = (-\nabla_{x'}h, 1)^{\mathsf{T}}/\sqrt{1+|\nabla_{x'}h|^2}$,

$$\begin{aligned}\nabla\tilde{\pi} &= \nabla\bar{\pi} - M_0(h)\nabla\bar{\pi}\\ \operatorname{div} u &= \operatorname{div}\bar{u} - (M_0(h)\nabla|\bar{u})\\ \Delta u &= \Delta\bar{u} - M_1(h) : \nabla^2\bar{u} - M_2(h)\nabla\bar{u}\\ \partial_t u &= \partial_t\bar{u} - \varphi\partial_t h(1+\varphi' h)^{-1}\partial_3\bar{u},\end{aligned}$$

where $M_0(h) := \theta_h'^{\mathsf{T}}(I+\theta_h'^{\mathsf{T}})^{-1}$,

$$M_1(h) : \nabla^2\bar{u} := \Big[2\operatorname{sym}(\theta_h'^{\mathsf{T}}[I+\theta_h']^{-\mathsf{T}}) - [I+\theta_h']^{-1}\theta_h'\theta_h'^{\mathsf{T}}[I+\theta_h']^{-\mathsf{T}}\Big] : \nabla^2\bar{u},$$

and

$$M_2(h)\nabla\bar{u} := \big([\Delta\Theta_h^{-1}]\circ\Theta_h|\nabla\big)\,\bar{u}.$$

Furthermore it holds that $V_\Gamma = (\partial_t\Theta_h|\nu_\Gamma) = \partial_t h(e_3|\nu_\Gamma) = \partial_t h/\sqrt{1+|\nabla_{x'}h|^2}$. This yields the following transformed problem for $\bar{u}$ and $\bar{\pi}$ (for convenience we drop the bars in the sequel).

$$\begin{aligned}\partial_t(\rho u) - \mu\Delta u + \nabla\pi &= F(u,\pi,h), \quad \text{in } \Omega\backslash\Sigma,\\ \operatorname{div} u &= F_d(u,h), \quad \text{in } \Omega\backslash\Sigma,\\ -[\![\mu\partial_3 v]\!] - [\![\mu\nabla_{x'}w]\!] &= G_v(u,h), \quad \text{on } \Sigma,\\ -2[\![\mu\partial_3 w]\!] + [\![\pi]\!] - \sigma\Delta_{x'}h - [\![\rho]\!]\gamma_a h &= G_w(u,h), \quad \text{on } \Sigma,\\ [\![u]\!] &= 0, \quad \text{on } \Sigma,\\ \partial_t h - w &= H_1(u,h), \quad \text{on } \Sigma,\\ P_{S_1}\Big(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}\Big) &= H_2(u,h), \quad \text{on } S_1\backslash\partial\Sigma,\\ u\cdot\nu_{S_1} &= 0, \quad \text{on } S_1\backslash\partial\Sigma,\\ u &= 0, \quad \text{on } S_2,\\ \partial_{\nu_{\partial G}}h &= 0, \quad \text{on } \partial\Sigma,\\ u(0) &= u_0, \quad \text{in } \Omega\backslash\Sigma\\ h(0) &= h_0, \quad \text{on } \Sigma.\end{aligned} \tag{2.2}$$

Here

$$\begin{aligned}F(u,p,h) &:= \rho\varphi\partial_t h(1+\varphi' h)^{-1}\partial_3 u - \mu(M_1(h):\nabla^2 u + M_2(h)\nabla u) + M_0(h)\nabla\pi\\ F_d(u,h) &:= (M_0(h)\nabla|u)\\ G_v(u,h) &:= -[\![\mu(\nabla v + \nabla v^{\mathsf{T}})]\!]\nabla h + |\nabla h|^2[\![\mu\partial_3 v]\!]\\ &\qquad + \big((1+|\nabla h|^2)[\![\mu\partial_3 w]\!] - (\nabla h|[\![\mu\nabla w]\!])\big)\nabla h\\ G_w(u,h) &:= -(\nabla h|[\![\mu\nabla w]\!]) - (\nabla h|[\![\mu\partial_3 v]\!]) + |\nabla h|^2[\![\mu\partial_3 w]\!] + \sigma G_\kappa(h)\end{aligned}$$

$$\begin{aligned}
G_\kappa(h) &:= \operatorname{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right) - \Delta h\\
H_1(u,h) &:= -(v|\nabla h)\\
H_2(u,h) &:= P_{S_1}(\mu(M_0(h)\nabla u + \nabla u^{\mathsf{T}} M_0(h)^{\mathsf{T}})\nu_{S_1}),
\end{aligned}$$

where we have set $v := (u_1, u_2)$, $w := u_3$ and $\nabla w = \nabla_{x'} w$, $\nabla v = \nabla_{x'} v$, $\nabla h = \nabla_{x'} h$ for the sake of readability.

2.2. **Linearization, regularity and compatibility conditions.** We consider first the principal linearization of (2.2), that is

$$\begin{aligned}
\partial_t(\rho u) - \mu\Delta u + \nabla\pi &= f, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u &= f_d, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu\partial_3 v]\!] - [\![\mu\nabla_{x'} w]\!] &= g_v, && \text{on } \Sigma,\\
-2[\![\mu\partial_3 w]\!] + [\![\pi]\!] - \sigma\Delta_{x'} h &= g_w, && \text{on } \Sigma,\\
[\![u]\!] &= u_\Sigma, && \text{on } \Sigma,\\
\partial_t h - m[w] &= g_h, && \text{on } \Sigma,\\
P_{S_1}\left(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}\right) &= P_{S_1} g_1, && \text{on } S_1\backslash\partial\Sigma,\\
u\cdot\nu_{S_1} &= g_2, && \text{on } S_1\backslash\partial\Sigma,\\
u &= g_3, && \text{on } S_2,\\
\partial_{\nu_{\partial G}} h &= g_4, && \text{on } \partial\Sigma,\\
u(0) &= u_0, && \text{in } \Omega\backslash\Sigma\\
h(0) &= h_0, && \text{on } \Sigma,
\end{aligned} \tag{2.3}$$

where $m[w] := (w_+ + w_-)/2$ is the arithmetic mean of the directional traces $w_\pm$ of w to Σ from Ω_2 and Ω_1. This arithmetic mean is introduced, since the jump of $w = u_3$ across Σ is not necessarily zero. However, note that $m[w] = w$ in case $[\![w]\!] = 0$. Note further, that we neglected the term $[\![\rho]\!]\gamma_a h$ in the jump of the stress tensor, as it is of lower order compared to $\Delta_{x'} h$.

Let $J = [0, T]$ with $T \in (0,\infty)$. We are looking for solutions (u,π) of the Stokes equation with

$$u \in H_p^1(J; L_p(\Omega)^3) \cap L_p(J; H_p^2(\Omega\backslash\Sigma)^3), \quad \pi \in L_p(J; \dot{H}_p^1(\Omega)),$$

and

$$[\![\pi]\!] \in W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma)).$$

Note that the latter regularity condition on $[\![\pi]\!]$ is determined by the regularity of the Neumann trace of u on Σ. For the height function h this yields

$$\Delta_{x'} h \in W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma))$$

and

$$\partial_t h \in W_p^{1-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma)),$$

hence the optimal regularity class for h is given by

$$h \in W_p^{2-1/2p}(J; L_p(\Sigma)) \cap H_p^1(J; W_p^{2-1/p}(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma)).$$

Let us discuss the necessary regularity and compatibility conditions on the data $(f, f_d, g_v, g_w, g_h, g_1, g_2, g_3, g_4, u_\Sigma, u_0, h_0)$. If $(u, \pi, [\![\pi]\!], h)$ is a solution of (2.3) in the regularity classes stated above, then it holds that $f \in L_p(J; L_p(\Omega)^3)$, $f_d \in L_p(J; H^1_p(\Omega\backslash\Sigma))$

$$\begin{gathered}
(g_v, g_w) \in W^{1/2-1/2p}_p(J; L_p(\Sigma)^3) \cap L_p(J; W^{1-1/p}_p(\Sigma)^3),\\
u_\Sigma \in W^{1-1/2p}_p(J; L_p(\Sigma)^3) \cap L_p(J; W^{2-1/p}_p(\Sigma)^3),\\
g_h \in W^{1-1/2p}_p(J; L_p(\Sigma)) \cap L_p(J; W^{2-1/p}_p(\Sigma)),\\
P_{S_1} g_1 \in W^{1/2-1/2p}_p(J; L_p(S_1)^3) \cap L_p(J; W^{1-1/p}_p(S_1\backslash\partial\Sigma)^3),\\
g_2 \in W^{1-1/2p}_p(J; L_p(S_1)) \cap L_p(J; W^{2-1/p}_p(S_1\backslash\partial\Sigma)),\\
g_3 \in W^{1-1/2p}_p(J; L_p(S_2)^3) \cap L_p(J; W^{2-1/p}_p(S_2)^3),\\
g_4 \in W^{3/2-1/p}_p(J; L_p(\Sigma)) \cap H^1_p(J; W^{1-2/p}_p(\Sigma)) \cap L_p(J; W^{2-2/p}_p(\Sigma)),\\
u_0 \in W^{2-2/p}_p(\Omega\backslash\Sigma)^3, \quad h_0 \in W^{3-2/p}_p(\Sigma).
\end{gathered}$$

Concerning compatibility conditions at $t = 0$ we have $\operatorname{div} u_0 = f_d|_{t=0}$,

$$g_v|_{t=0} = -[\![\mu\partial_3 v_0]\!] - [\![\mu\nabla_{x'} w_0]\!],$$

$[\![u_0]\!] = u_\Sigma|_{t=0}$, $u_0 \cdot \nu_{S_1} = g_2|_{t=0}$, $u_0 = g_3|_{t=0}$, $\partial_{\nu_{\partial G}} h_0 = g_4|_{t=0}$ and

$$P_{S_1}(\mu(\nabla u_0 + \nabla u_0^{\mathsf{T}})\nu_{S_1}) = P_{S_1} g_1|_{t=0}.$$

Since $\partial\Sigma \subset S_1 \neq \emptyset$ and $\partial S_1 \cap \partial S_2 \neq \emptyset$, there are additional compatibility conditions which have to be satisfied.

If $(u, \pi, [\![\pi]\!], h)$ is a solution of (2.3) with the above regularity, then the following compatibility conditions at $\partial\Sigma$ and ∂S_2 have necessarily to be satisfied.

- $[\![g_2]\!] = u_\Sigma \cdot \nu_{S_1}$, $[\![(g_1 \cdot e_3)/\mu - \partial_3 g_2]\!] = \partial_{\nu_{S_1}}(u_\Sigma \cdot e_3)$, at $\partial\Sigma$,
- $P_{\partial G}[(D' v_\Sigma)\nu'] = [\![P_{\partial G} g_1'/\mu]\!]$, $\partial_t g_4 - m[(g_1 \cdot e_3)/\mu - \partial_3 g_2] = \partial_{\nu_{\partial G}} g_h$, at $\partial\Sigma$,
- $(g_v|\nu_{\partial G}) = -[\![g_1 \cdot e_3]\!]$ at $\partial\Sigma$, $(g_3|\nu_{S_1}) = g_2$ at ∂S_2,
- $P_{\partial G}[\mu(D' g_3')\nu'] = (P_{\partial G} g_1')$, $\mu\partial_{\nu_{S_1}}(g_3 \cdot e_3) + \mu\partial_3 g_2 = g_1 \cdot e_3$ at ∂S_2.

Here we use the notation $\nu' = \nu_{\partial G}$, $P_{\partial G} := I - \nu' \otimes \nu'$, $D'v := \nabla_{x'} v + \nabla_{x'} v^{\mathsf{T}}$ and $g' := \sum_{k=1}^2 (g \cdot e_k)e_k$. These conditions follow easily by comparing the equations $(2.3)_3$ and $(2.3)_{5-10}$ with each other.

There is another compatibility and regularity condition hidden in the system, which stems from the divergence equation. Multiply $\operatorname{div} u = f_d$ by $\phi \in H^1_{p'}(\Omega)$, $p' = p/(p-1)$ and integrate by parts (see Proposition 5.11) to the result

$$\begin{aligned}
(2.4)\qquad \int_\Omega f_d \phi\, dx - \int_{S_1} g_2 \phi|_{S_1} dS_1 - \int_{S_2} (g_3 \cdot \nu_{S_1})\phi|_{S_2} dS_2 + \\
+ \int_\Sigma (u_\Sigma \cdot \nu_\Sigma)\phi|_\Sigma d\Sigma = -\int_\Omega u \cdot \nabla\phi\, dx,
\end{aligned}$$

where $\nu_{S_2}(x', H_2) = e_3$, $\nu_{S_2}(x', H_1) = -e_3$ for $x' \in G$ and $\nu_\Sigma = e_3$. It follows that the functional $[\phi \mapsto \langle (f_d, g_2, g_3, u_\Sigma), \phi\rangle]$ defined by the left side of (2.4) is continuous on $H^1_{p'}(\Omega)$ with respect to the semi-norm $\|\nabla \cdot\|_{L_{p'}(\Omega)}$. Since $H^1_{p'}(\Omega)$ is dense in the homogeneous Sobolev space $\dot{H}^1_{p'}(\Omega)$ (the constants are already factorized) with respect to $\|\nabla\cdot\|_{L_{p'}(\Omega)}$ for all domains Ω which are considered in this article, it follows that $(f_d, g_2, g_3, u_\Sigma)$ determines a functional on $\dot{H}^1_{p'}(\Omega)$, i.e. $(f_d, g_2, g_3, u_\Sigma) \in \hat{H}^{-1}_p(\Omega) := (\dot{H}^1_{p'}(\Omega))^*$. The norm of $(f_d, g_2, g_3, u_\Sigma)$ in $\hat{H}^{-1}_p(\Omega)$ is then given by

$$\|(f_d, g_2, g_3, u_\Sigma)\|_{\hat{H}^{-1}_p} = \sup\{\langle (f_d, g_2, g_3, u_\Sigma), \phi\rangle / \|\nabla\phi\|_{L_{p'}} : \phi \in H^1_{p'}(\Omega)\}.$$

Moreover, if $u \in H^1_p(J; L_p(\Omega)^3)$, then $\frac{d}{dt}(f_d, g_2, g_3, u_\Sigma)$ is well defined by the computation above, hence

$$(f_d, g_2, g_3, u_\Sigma) \in H^1_p(J; \hat{H}^{-1}_p(\Omega))$$

is another necessary compatibility and regularity condition on the data. In particular, if Ω is bounded, then we may choose $\phi = 1$ in (2.4) to obtain

$$\int_\Omega f_d dx - \int_{S_1} g_2 dS_1 - \int_{S_2} (g_3\cdot\nu_{S_1}) dS_2 + \int_\Sigma (u_\Sigma\cdot\nu_\Sigma) d\Sigma = 0.$$

2.3. **Model problems.** The proof of existence and uniqueness of a solution $(u, \pi, [\![\pi]\!], h)$ to (2.3) is based on a localization procedure. We will obtain six different types of charts, which yield six different types of model problems. These are

- the full space Stokes equations (without any boundary- or interface conditions)
- the two-phase Stokes equations with a flat interface and without any boundary condition
- the Stokes equations with pure slip boundary conditions in a half-space and no interface
- the Stokes equations with no-slip boundary conditions in a half-space and no interface
- the Stokes equations in a quarter space with pure slip boundary conditions on one part of the boundary and no-slip boundary conditions on the other part
- the two-phase Stokes equations with pure slip boundary conditions in a half-space, a flat interface and a contact angle of 90 degrees.

While the first four of these problems are well understood, there seem to be no results on well-posedness of the last two problems.

2.3.1. *The Stokes equations in quarter-spaces.* Consider the problem

$$\begin{aligned}
\partial_t(\rho u) - \mu\Delta u + \nabla\pi &= f, && x_1\in\mathbb{R},\ x_2>0,\ x_3>0,\\
\operatorname{div} u &= f_d, && x_1\in\mathbb{R},\ x_2>0,\ x_3>0,\\
\mu[\partial_2 u_1 + \partial_1 u_2, \partial_3 u_2 + \partial_2 u_3]^\mathsf{T} &= g_1, && x_1\in\mathbb{R},\ x_2=0,\ x_3>0,\\
u_2 &= g_2, && x_1\in\mathbb{R},\ x_2=0,\ x_3>0,\\
u &= g_3, && x_1\in\mathbb{R},\ x_2>0,\ x_3=0,\\
u(0) &= u_0, && x_1\in\mathbb{R},\ x_2>0,\ x_3>0.
\end{aligned} \tag{2.5}$$

For convenience, let $\Omega := \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+$, $S_1 := \mathbb{R}\times\{0\}\times\mathbb{R}_+$ and $S_2 := \mathbb{R}\times\mathbb{R}_+\times\{0\}$.

In a first step we extend $u_0 \in W_p^{2-2/p}(\Omega)^3$ with respect to x_2 via the reflection

$$\tilde{u}_0(x_1,x_2,x_3) = \begin{cases} u_0(x_1,x_2,x_3), & \text{if } x_2 > 0, \\ -u_0(x_1,-2x_2,x_3) + 2u_0(x_1,-x_2/2,x_3), & \text{if } x_2 < 0. \end{cases}$$

Then $\tilde{u}_0 \in W_p^{2-2/p}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+)^3$. Applying the same method to

$$g_3 \in W_p^{1-1/2p}(J; L_p(S_2)^3) \cap L_p(J; W_p^{2-1/p}(S_2)^3)$$

yields an extension

$$\tilde{g}_3 \in W_p^{1-1/2p}(J; L_p(\mathbb{R}\times\mathbb{R})^3) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}\times\mathbb{R})^3).$$

Furthermore it holds that $\tilde{g}_3|_{t=0} = \tilde{u}_0|_{x_3=0}$, since $g_3|_{t=0} = u_0|_{S_2}$. Then we solve the half-space problem

$$\begin{aligned} \partial_t\tilde{u} - \Delta\tilde{u} &= 0, \quad (x_1,x_2,x_3) \in \mathbb{R}^2\times\mathbb{R}_+, \\ \tilde{u}|_{x_3=0} &= \tilde{g}_3, \quad (x_1,x_2) \in \mathbb{R}^2,\ x_3 = 0, \\ \tilde{u}(0) &= \tilde{u}_0, \quad (x_1,x_2,x_3) \in \mathbb{R}^2\times\mathbb{R}_+, \end{aligned} \tag{2.6}$$

to obtain a unique solution

$$\tilde{u} \in H_p^1(J; L_p(\mathbb{R}^3_+)^3) \cap L_p(J; H_p^2(\mathbb{R}^3_+)^3).$$

If (u,π) is a solution of (2.5), then the (restricted) function $(u-\tilde{u},\pi)$ solves (2.5) with $u_0 = g_3 = 0$ and some modified data (f,g_1,g_2) (not to be relabeled) in the right regularity classes having a vanishing trace at $t=0$ whenever it exists.

In a next step we extend

$$g_1 \in {}_0W_p^{1/2-1/2p}(J; L_p(S_1)^2) \cap L_p(J; W_p^{1-1/p}(S_1)^2),$$

and

$$g_2 \in {}_0W_p^{1-1/2p}(J; L_p(S_1)) \cap L_p(J; W_p^{2-1/p}(S_1)),$$

w.r.t. x_3 to some functions

$$\tilde{g}_1 \in {}_0W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^2)^2) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^2)^2),$$

and

$$\tilde{g}_2 \in {}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^2)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^2)),$$

and solve the half-space problem

$$\begin{aligned} \partial_t\bar{u} - \Delta\bar{u} &= 0, \quad x_1,x_3 \in \mathbb{R},\ x_2 > 0, \\ \mu[\partial_2\bar{u}_1 + \partial_1\bar{u}_2, \partial_3\bar{u}_2 + \partial_2\bar{u}_3]^{\mathsf{T}} &= \tilde{g}_1, \quad x_1,x_3 \in \mathbb{R},\ x_2 = 0, \\ \bar{u}_2 &= \tilde{g}_2, \quad x_1,x_3 \in \mathbb{R},\ x_2 = 0, \\ \bar{u}(0) &= 0, \quad x_1,x_3 \in \mathbb{R},\ x_2 > 0, \end{aligned} \tag{2.7}$$

to obtain a unique solution

$$\bar{u} \in {}_0H_p^1(J; L_p(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3) \cap L_p(J; H_p^2(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3).$$

If (u,π) is a solution of (2.5) it follows that the (restricted) function $(u-\tilde{u}-\bar{u},\pi)$ solves the problem

$$\begin{aligned} \partial_t(\rho u) - \mu\Delta u + \nabla\pi &= f, \quad (x_1,x_2,x_3)\in\Omega,\\ \operatorname{div} u &= f_d, \quad (x_1,x_2,x_3)\in\Omega,\\ \mu[\partial_2 u_1 + \partial_1 u_2, \partial_3 u_2 + \partial_2 u_3]^\mathsf{T} &= 0, \quad (x_1,x_2,x_3)\in S_1,\\ u_2 &= 0, \quad (x_1,x_2,x_3)\in S_1,\\ u &= g_3, \quad (x_1,x_2,x_3)\in S_2,\\ u(0) &= 0, \quad (x_1,x_2,x_3)\in\Omega, \end{aligned} \tag{2.8}$$

with some modified data (f,f_d,g_3) in the right regularity classes having a vanishing trace at $t=0$ whenever it exists. Note that $g_3 := \bar{u}|_{x_3=0}$ and the compatibility conditions $(g_3)_2 = \partial_2(g_3)_{1,3} = 0$ hold if $x_1\in\mathbb{R}$, $x_2=0$ and $x_3=0$ (here we use the abbreviation $(g_3)_j := g_3\cdot e_j$, $j\in\{1,2,3\}$). We will now extend $(f_1,f_3,f_d,(g_3)_{1,3})$ by even reflection and $(f_2,(g_3)_2)$ by odd reflection to $\{x_2<0\}$. Then we consider the (reflected) half-space problem

$$\begin{aligned} \partial_t(\rho\hat{u}) - \mu\Delta\hat{u} + \nabla\hat{\pi} &= \tilde{f}, \quad x_1,x_2\in\mathbb{R},\ x_3>0,\\ \operatorname{div}\hat{u} &= \tilde{f}_d, \quad x_1,x_2\in\mathbb{R},\ x_3>0,\\ \hat{u} &= \tilde{g}_3, \quad x_1,x_2\in\mathbb{R},\ x_3=0,\\ \hat{u}(0) &= 0, \quad x_1,x_2\in\mathbb{R},\ x_3>0, \end{aligned} \tag{2.9}$$

which has a unique solution

$$\begin{aligned} \hat{u} &\in {}_0H_p^1(J;L_p(\mathbb{R}^3_+)^3)\cap L_p(J;H_p^2(\mathbb{R}^3_+)^3),\\ \hat{\pi} &\in L_p(J;\dot{H}_p^1(\mathbb{R}^3_+)), \end{aligned}$$

by [4, Theorem 6.1].

The (restricted) pair $(u,\pi) := (\tilde{u}+\bar{u}+\hat{u},\hat{\pi})$ is the desired unique solution to (2.5). We have thus proven the following

Theorem 2.1. *Let $n=3$, $p>5$, $T>0$, $\rho,\mu>0$ and $J=[0,T]$. Then there exists a unique solution*

$$\begin{aligned} u &\in H_p^1(J;L_p(\Omega)^3)\cap L_p(J;H_p^2(\Omega)^3)\\ \pi &\in L_p(J;\dot{H}_p^1(\Omega)) \end{aligned}$$

of (2.5) if and only if the data satisfy the following regularity and compatibility conditions.

(1) $f\in L_p(J;L_p(\Omega)^3)$;
(2) $f_d\in L_p(J;H_p^1(\Omega))$;
(3) $g_1\in W_p^{1/2-1/2p}(J;L_p(S_1)^2)\cap L_p(J;W_p^{1-1/p}(S_1)^2)$;
(4) $g_2\in W_p^{1-1/2p}(J;L_p(S_1))\cap L_p(J;W_p^{2-1/p}(S_1))$;
(5) $g_3\in W_p^{1-1/2p}(J;L_p(S_2)^3)\cap L_p(J;W_p^{2-1/p}(S_2)^3)$;
(6) $u_0\in W_p^{2-2/p}(\Omega)^3$;
(7) $\operatorname{div} u_0 = f_d|_{t=0}$, $\mu[\partial_2(u_0)_1+\partial_1(u_0)_2, \partial_3(u_0)_2+\partial_2(u_0)_3]|_{x_2=0}^\mathsf{T} = g_1|_{t=0}$;
(8) $(u_0)_2|_{x_2=0} = g_2|_{t=0}$, $u_0|_{x_3=0} = g_3|_{t=0}$;

(9) $(g_3)_2|_{x_2=0} = g_2|_{x_3=0}$, $\mu[\partial_2(g_3)_1+\partial_1(g_3)_2, \partial_3 g_2|_{x_3=0}+\partial_2(g_3)_3]|^{\mathsf{T}}_{x_2=0} = g_1|_{x_3=0}$;
(10) $(f_d, g_2, g_3) \in H^1_p(J; \hat{H}^{-1}_p(\Omega))$.

2.3.2. *The Stokes equations in bent quarter-spaces.* Let $\theta \in BC^3(\mathbb{R})$ such that

$$G_\theta := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > \theta(x_1)\} \quad \text{and} \quad \Omega_\theta = G_\theta \times \mathbb{R}_+.$$

We assume furthermore that $|\theta'|_\infty \le \eta$ and $|\theta^{(j)}|_\infty \le M$, $j \in \{2, 3\}$, where we may choose $\eta > 0$ as small as we wish. Let $S_{1,\theta} := \partial G_\theta \times \mathbb{R}_+$ and $S_{2,\theta} := G_\theta \times \{0\}$. Furthermore, let $\nu_{S_{1,\theta}} = (\nu_{G_\theta}, 0)^{\mathsf{T}}$ with $\nu_{G_\theta} := \frac{1}{\sqrt{1+\theta'(x_1)^2}}(\theta'(x_1), -1)^{\mathsf{T}}$ denote the outer unit normal to $S_{1,\theta}$ at $(x_1, \theta(x_1), x_3)$, $(x_1, x_3) \in \mathbb{R} \times \mathbb{R}_+$ and let $P_{S_{1,\theta}}$ be the tangential projection to $S_{1,\theta}$. Consider the problem

$$\begin{aligned} \partial_t(\rho u) - \mu\Delta u + \nabla\pi &= f, \quad x \in \Omega_\theta, \\ \operatorname{div} u &= f_d, \quad x \in \Omega_\theta, \\ P_{S_{1,\theta}}[\mu(Du)\nu_{S_{1,\theta}}] &= P_{S_1} g_1, \quad x \in S_{1,\theta}, \\ (u|\nu_{S_{1,\theta}}) &= g_2, \quad x \in S_{1,\theta}, \\ u &= g_3, \quad x \in S_{2,\theta} \\ u(0) &= u_0, \quad x \in \Omega_\theta. \end{aligned} \tag{2.10}$$

Here $\rho, \mu > 0$ are given constants. Note that since $\nu_{S_1} = (\nu_{\partial G}, 0)^{\mathsf{T}}$ it holds that

$$P_{S_{1,\theta}}[\mu(Du)\nu_{S_{1,\theta}}] = \begin{pmatrix} P_{\partial G_\theta}[\mu(D'v)\nu_{G_\theta}] \\ \mu\partial_3(v|\nu_{G_\theta}) + \mu\partial_{\nu_{G_\theta}} w, \end{pmatrix} \tag{2.11}$$

where $D' = D_{(x_1,x_2)}$ and $u = (v, w)$. Therefore, the given data $(f, f_d, g_1, g_2, g_3, u_0)$ is subject to the compatibility conditions $(g_3|\nu_{S_{1,\theta}}) = g_2|_{S_{2,\theta}}$,

$$P_{\partial G_\theta}[\mu(D' g_3')\nu_{S_{1,\theta}}] = P_{\partial G_\theta} g_1',$$

and $\mu(\partial_3 g_2 + \partial_{\nu_{G_\theta}}(g_3 \cdot e_3)) = g_1 \cdot e_3$ at the contact line $\{(x_1, \theta(x_1), 0) : x_1 \in \mathbb{R}\}$, where

$$g_j' := \sum_{k=1}^{2} (g_j \cdot e_k) e_k$$

for $j \in \{1, 3\}$. Furthermore, at $t = 0$ we have $\operatorname{div} u_0 = f_d|_{t=0}$, $u_0|_{S_{2,\theta}} = g_3|_{t=0}$, $(u_0|\nu_{S_{1,\theta}}) = g_2|_{t=0}$ and $P_{S_{1,\theta}}[\mu(Du_0)\nu_{S_{1,\theta}}] = P_{S_{1,\theta}} g_1|_{t=0}$. Lastly, $(f_d, g_2, g_3) \in H^1_p(J; \hat{H}^{-1}_p(\Omega_\theta))$.

For convenience we shall reduce (2.10) to the case $u_0 = f = g_3 = 0$. To this end we first extend u_0 and f to some $\tilde{u}_0 \in W^{2-2/p}_p(\mathbb{R}^3)^3$ and $f \in L_p(J; L_p(\mathbb{R}^3)^3)$ and solve the full-space problem

$$\begin{aligned} \partial_t(\rho\tilde{u}) - \mu\Delta\tilde{u} &= \tilde{f}, \quad \text{in } \mathbb{R}^3, \\ \tilde{u}(0) &= \tilde{u}_0, \quad \text{in } \mathbb{R}^3, \end{aligned}$$

to obtain a unique solution

$$\tilde{u} \in H^1_p(J; L_p(\mathbb{R}^3)^3) \cap L_p(J; H^1_p(\mathbb{R}^3)^3).$$

Let now $\tilde{g}_3 := g_3 - \tilde{u}|_{S_2}$. Then $\tilde{g}_3|_{t=0} = 0$ by construction and we may extend $\tilde{g}_3$ to some

$$\hat{g}_3 \in {}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^2)^3) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^2)^3).$$

With $\hat{g}_3$ at hand we solve the half-space problem

$$\begin{aligned} \partial_t(\rho\hat{u}) - \mu\Delta\hat{u} &= 0, \quad \text{in } \mathbb{R}^3_+, \\ \hat{u} &= \hat{g}_3, \quad \text{on } \mathbb{R}^2, \\ \hat{u}(0) &= 0, \quad \text{in } \mathbb{R}^3_+, \end{aligned}$$

to obtain a unique solution

$$\hat{u} \in H_p^1(J; L_p(\mathbb{R}^3_+)^3) \cap L_p(J; H_p^1(\mathbb{R}^3_+)^3).$$

If u is a solution of (2.10), it follows that the (restricted) function $\bar{u} := u - \tilde{u} - \hat{u}$ solves (2.10) with $f = u_0 = g_3 = 0$ and some modified functions $\bar{f}_d, \bar{g}_j$, $j \in \{1, 2\}$ in the correct regularity classes satisfying the compatibility conditions $\bar{g}_2|_{S_{2,\theta}} = 0$, $P_{\partial G_\theta}\bar{g}_1' = 0$ and $\bar{g}_1 \cdot e_3 = \partial_3\bar{g}_2$ at the contact line. Moreover, $(\bar{f}_d, \bar{g}_2, 0) \in {}_0H_p^1(J; \hat{H}_p^{-1}(\Omega_\theta))$.

Observe that by the identity $(P_{S_{1,\theta}}w|\nu_{S_{1,\theta}}) = 0$, $w \in \mathbb{R}^3$, the second component of $P_{S_{1,\theta}}w$ is redundant (it can always be calculated from the first one). Therefore we may replace the term $P_{S_{1,\theta}}[\mu(Du)\nu_{S_{1,\theta}}]$ by its first and last component, i.e. we consider the two equations

$$P_{S_{1,\theta}}[\mu(Du)\nu_{S_{1,\theta}}] \cdot e_j = P_{S_{1,\theta}}g_1 \cdot e_j$$

for $j \in \{1, 3\}$. Observe also that $P_{S_{1,\theta}}g_1 \cdot e_3 = g_1 \cdot e_3$, since the last component of $\nu_{S_{1,\theta}}$ is identically zero..

In what follows we will transform the domain G_θ to $G := \mathbb{R} \times \mathbb{R}_+$, the boundaries $S_{1,\theta}$ and $S_{2,\theta}$ to $S_1 := \partial G \times \mathbb{R}_+$ and $S_2 := G \times \{0\}$, respectively, and, hence, Ω_θ to $\Omega := G \times \mathbb{R}_+$. To this end we introduce the new variables $\bar{x}_1 = x_1$, $\bar{x}_2 = x_2 - \theta(x_1)$ and $\bar{x}_3 = x_3$ for $x \in \Omega_\theta = G_\theta \times \mathbb{R}_+$. Suppose that (u, π) is a solution of (2.10) and define the new functions

$$\bar{u}(\bar{x}) := u(\bar{x}_1, \bar{x}_2 + \theta(\bar{x}_1), \bar{x}_3)$$

and

$$\bar{\pi}(\bar{x}) := \pi(\bar{x}_1, \bar{x}_2 + \theta(\bar{x}_1), \bar{x}_3),$$

where $\bar{x} := (\bar{x}_1, \bar{x}_2, \bar{x}_3)$. In the same way we transform the data (f_d, g_1, g_2) to $(\bar{f}_d, \bar{g}_1, \bar{g}_2)$. It holds that $\partial_k^j\bar{u} = \partial_k^j u$ for $k \in \{2, 3\}$, $j \in \{1, 2\}$,

$$\partial_1 u = \partial_1\bar{u} - \theta'(\bar{x}_1)\partial_2\bar{u}$$

and

$$\partial_1^2 u = \partial_1^2\bar{u} - 2\theta'(\bar{x}_1)\partial_1\partial_2\bar{u} - \theta''(\bar{x}_1)\partial_2\bar{u} + \theta'(\bar{x}_1)^2\partial_2^2\bar{u}.$$

Therefore, the pair $(\bar{u}, \bar{\pi})$ solves the following problem

$$
\begin{aligned}
\partial_t(\rho\bar{u}) - \mu\Delta\bar{u} + \nabla\bar{\pi} &= M_1(\theta, \bar{u}, \bar{\pi}), \quad \bar{x} \in \Omega,\\
\operatorname{div} \bar{u} &= M_2(\theta, \bar{u}) + \bar{f}_d, \quad \bar{x} \in \Omega,\\
\mu(\partial_1\bar{u}_2 + \partial_2\bar{u}_1) &= M_3(\theta, \bar{u}) - \sqrt{1+\theta'^2}^3[P_{S_{1,\theta}}\bar{g}_1 \cdot e_1], \quad \bar{x} \in S_1,\\
\mu(\partial_2\bar{u}_3 + \partial_3\bar{u}_2) &= M_4(\theta, \bar{u}) - \sqrt{1+\theta'^2}[\bar{g}_1 \cdot e_3], \quad \bar{x} \in S_1,\\
\bar{u}_2 &= M_5(\theta, \bar{u}) - \sqrt{1+\theta'^2}\bar{g}_2, \quad \bar{x} \in S_1,\\
\bar{u} &= 0, \quad \bar{x} \in S_2,\\
\bar{u}(0) &= 0, \quad \bar{x} \in \Omega,
\end{aligned}
\tag{2.12}
$$

where the functions M_j are given by

$$M_1(\theta, \bar{u}, \bar{\pi}) := 2\theta'(\bar{x}_1)\partial_1\partial_2\bar{u} + \theta''(\bar{x}_1)\partial_2\bar{u} - \theta'(\bar{x}_1)^2\partial_2^2\bar{u} + \theta'(\bar{x}_1)\partial_2\bar{\pi}e_1,$$

$$M_2(\theta, \bar{u}) := \theta'(\bar{x}_1)\partial_2\bar{u}_1,$$

$$M_3(\theta, \bar{u}) := \mu\theta'(\bar{x}_1)[2\partial_1\bar{u}_1 + \theta'(\bar{x}_1)(\partial_1\bar{u}_2 - \partial_2\bar{u}_1) - (1 + \theta'(\bar{x}_1)^2)\partial_2\bar{u}_2],$$

$$M_4(\theta, \bar{u}) := \mu\theta'(\bar{x}_1)(\partial_1\bar{u}_3 - \theta'(\bar{x}_1)\partial_2\bar{u}_3 + \partial_3\bar{u}_1),$$

$$M_5(\theta, \bar{u}) := \theta'(\bar{x}_1)\bar{u}_1.$$

Now we want to go back from (2.12) to (2.10). To this end we consider the functions on the right hand side of (2.12) as given data in the right regularity classes. Our aim is the to interpret (2.12) as a perturbation of (2.5), provided $|\theta'|_\infty < \eta$ and $\eta > 0$ is sufficiently small. It is therefore reasonable to solve (2.12) by a Neumann series argument. To this end let

$${}_0\mathbb{E}_u(T) := \{u \in {}_0H_p^1(J; L_p(\Omega)^3) \cap L_p(J; H_p^2(\Omega)^3) : u|_{S_2} = 0\},$$

$$\mathbb{E}_\pi(T) := L_p(J; \dot{H}_p^1(\Omega)),$$

${}_0\mathbb{E}(T) := {}_0\mathbb{E}_u(T) \times \mathbb{E}_\pi(T)$,

$$\tilde{\mathbb{F}}(T) := \mathbb{F}_1(T) \times \mathbb{F}_2(T) \times_{j=3}^5 {}_0\mathbb{F}_j(T),$$

where

$$\mathbb{F}_1(T) := L_p(J; L_p(\Omega)^3),$$

$$\mathbb{F}_2(T) := L_p(J; H_p^1(\Omega)),$$

$${}_0\mathbb{F}_3(T) := {}_0W_p^{1/2-1/2p}(J; L_p(S_1)) \cap L_p(J; W_p^{1-1/p}(S_1)),$$

${}_0\mathbb{F}_4(T) := {}_0\mathbb{F}_3(T)$, and

$${}_0\mathbb{F}_5(T) := {}_0W_p^{1-1/2p}(J; L_p(S_1)) \cap L_p(J; W_p^{2-1/p}(S_1)).$$

Finally, we set

$${}_0\mathbb{F}(T) := \{(f_1, \ldots, f_5) \in \tilde{\mathbb{F}}(T) : (9)\ \&\ (10) \text{ in Theorem 2.1 are satisfied}\}.$$

Define an operator $L : {}_0\mathbb{E}(T) \to {}_0\mathbb{F}(T)$ by

$$L(\bar{u}, \bar{\pi}) := \begin{bmatrix} \partial_t(\rho\bar{u}) - \mu\Delta\bar{u} + \nabla\bar{\pi} \\ \operatorname{div}\bar{u} \\ \mu(\partial_2\bar{u}_1 + \partial_1\bar{u}_2)|_{S_1} \\ \mu(\partial_3\bar{u}_2 + \partial_2\bar{u}_3)|_{S_1} \\ \bar{u}_2|_{S_1} \end{bmatrix}$$

and note that $L : {}_0\mathbb{E}(T) \to {}_0\mathbb{F}(T)$ is an isomorphism by Theorem 2.1. Define

$$M(\theta, \bar{u}, \bar{\pi}) := (M_1(\theta, \bar{u}, \bar{\pi}), M_2(\theta, \bar{u}), M_3(\theta, \bar{u}), M_4(\theta, \bar{u}), M_5(\theta, \bar{u}))^{\mathsf{T}}$$

and

$$F := (0, f_2, f_3, f_4, f_5)^{\mathsf{T}},$$

with $f_2 := \bar{f}_d$,

$$f_3 := -\sqrt{1 + \theta'^2}^3 [P_{S_{1,\theta}}\bar{g}_1 \cdot e_1], \; f_4 := -\sqrt{1 + \theta'^2}[\bar{g}_1 \cdot e_3]$$

and $f_5 := -\sqrt{1 + \theta'^2}\bar{g}_2$. By the smoothness of θ, it follows that $F \in \tilde{\mathbb{F}}(T)$. So it remains to check that the compatibility conditions (9) & 10 in Theorem 2.1 are satisfied. Since $\bar{g}_2|_{S_2} = 0$, $P_{\partial G_\theta}\bar{g}_{1,v} = 0$ and $\bar{g}_{1,w} = \partial_3\bar{g}_2$ at the contact line, the compatibility conditions in Theorem 2.1 (9) are easily verified. To verify (10) in Theorem 2.1 we have to show that $(f_2, f_5, 0) \in {}_0H^1_p(J; \hat{H}^{-1}_p(\Omega))$. Note that for the reduced data from above we have $(f_d, g_2, 0) \in {}_0H^1_p(J; \hat{H}^{-1}_p(\Omega_\theta))$, hence for a.e. $t \in J$ the functional $\Psi(t) : H^1_{p'}(\Omega_\theta) \to \mathbb{R}$ defined by

$$\langle \Psi(t), \phi \rangle := \int_{\Omega_\theta} f_d(t)\phi \; dx - \int_{S_{1,\theta}} g_2(t)\phi|_{S_{1,\theta}} \; dS_\theta$$

as well as its derivative with respect to t are continuous with respect to the norm $\|\nabla \cdot\|_{L_{p'}(\Omega_\theta)}$. Transforming Ω_θ to the quarter space Ω and $S_{1,\theta}$ to S_1 via the above diffeomorphism $\Phi(x_1, x_2, x_3) = (x_1, x_2 - \theta(x_1), x_3)$ yields

$$\int_{\Omega_\theta} f_d(t)\phi \; dx - \int_{S_{1,\theta}} g_2(t)\phi|_{S_{1,\theta}} \; dS_\theta =$$
$$= \int_{\Omega} \bar{f}_d(t)\bar{\phi} \; d\bar{x} - \int_{S_1} \sqrt{1 + \theta'(\bar{x})^2}\bar{g}_2(t)\bar{\phi}|_{S_1} \; dS,$$

where $\bar{\phi}(\bar{x}_1, \bar{x}_2, \bar{x}_3) := \phi(x_1, x_2 - \theta(x_1), x_3)$. This shows that for a.e. $t \in J$ the functional $\bar{\Psi}(t) : H^1_{p'}(\Omega) \to \mathbb{R}$ given by

$$\langle \bar{\Psi}(t), \bar{\phi} \rangle := \int_{\Omega} \bar{f}_d(t)\bar{\phi} \; d\bar{x} - \int_{S_1} \sqrt{1 + \theta'(\bar{x})^2}\bar{g}_2(t)\bar{\phi}|_{S_1} \; dS$$

and its derivative with respect to t are continuous with respect to the norm $\|\nabla \cdot\|_{L_{p'}(\Omega)}$, hence $(f_2, f_5, 0) \in {}_0H^1_p(J; \hat{H}^{-1}_p(\Omega))$. This implies $F \in {}_0\mathbb{F}(T)$.

Concerning $M(\theta, \bar{u}, \bar{\pi})$, we observe that for $\bar{u} \in {}_0\mathbb{E}_u(T)$ we have $\bar{u} = 0$ as well as $\partial_j\bar{u} = 0$ at S_2 for $j \in \{1, 2\}$ and therefore also at the line

$$\partial S_1 = \partial S_2 = \overline{S_1} \cap \overline{S_2} = \mathbb{R} \times \{0\} \times \{0\},$$

by continuity of $\bar{u}$ and $\partial_j \bar{u}$ in $\overline{\Omega}$. Therefore $M_3(\theta,\bar{u}) = M_5(\theta,\bar{u}) = 0$ at $\overline{S_1} \cap \overline{S_2}$. Moreover,

$$M_4(\theta,\bar{u}) = \mu\theta'(\bar{x}_1)\partial_3\bar{u}_1$$

at $\overline{S_1} \cap \overline{S_2}$, hence $\mu\partial_3 M_5(\theta,\bar{u}) = M_4(\theta,\bar{u})$. It remains to verify the condition

$$(M_2(\theta,\bar{u}), -M_5(\theta,\bar{u}), 0) \in {}_0H_p^1(J;\hat{H}_p^{-1}(\Omega))$$

for $\bar{u} \in {}_0\mathbb{E}_u(T)$. We compute

$$\begin{aligned}\int_\Omega M_2(\theta,\bar{u})\phi\, d\bar{x} - &\int_{S_1} (-M_5(\theta,\bar{u}))\phi|_{S_1}\, dS\\ &= \int_\Omega \theta'(\bar{x}_1)(\partial_2\bar{u}_1)\phi\, d\bar{x} + \int_{S_1} \theta'(\bar{x}_1)\bar{u}_1\phi|_{S_1}\, dS\\ &= -\int_\Omega \theta'(\bar{x}_1)\bar{u}_1\partial_2\phi\, d\bar{x},\end{aligned}$$

for each $\phi \in H_{p'}^1(\Omega)$, where we integrated by parts with respect to the variable $\bar{x}_2$. This yields the claim.

It follows that $M(\theta,\bar{u}) \in {}_0\mathbb{F}(T)$ for each $(\bar{u},\bar{\pi}) \in {}_0\mathbb{E}(T)$ and therefore we may rewrite (2.12) shortly as $(\bar{u},\bar{\pi}) = L^{-1}M(\theta,\bar{u},\bar{\pi}) + L^{-1}F$ in ${}_0\mathbb{E}(T)$. We intend to show that for each $\varepsilon > 0$ there exist $T_0 > 0$ and $\eta_0 > 0$ such that

$$\|M(\theta,\bar{u},\bar{\pi})\|_{\mathbb{F}(T)} \leq \varepsilon\|(\bar{u},\bar{\pi})\|_{\mathbb{E}(T)}, \tag{2.13}$$

provided that $T \in (0,T_0)$ and $\eta \in (0,\eta_0)$.

The above computation for $(M_2, M_5, 0)$ readily yields that

$$\|(M_2(\theta,\bar{u}), M_5(\theta,\bar{u}), 0)\|_{H_p^1(J;\hat{H}_p^{-1}(\Omega))} \leq \|\theta'\|_\infty \|\bar{u}\|_{\mathbb{E}_u(T)}.$$

Moreover, it holds that

$$\begin{aligned}\|M_2(\theta,\bar{u})\|_{L_p(J;H_p^1(\Omega))} &\leq \|\theta'\|_\infty\|\bar{u}\|_{\mathbb{E}_u(T)} + \|\theta''\|_\infty\|\bar{u}\|_{L_p(J;H_p^1(\Omega))}\\ &\leq \|\theta'\|_\infty\|\bar{u}\|_{\mathbb{E}_u(T)} + T^{1/2p}\|\theta''\|_\infty\|\bar{u}\|_{L_{2p}(J;H_p^1(\Omega))}\\ &\leq (\|\theta'\|_\infty + T^{1/2p}C\|\theta''\|_\infty)\|\bar{u}\|_{\mathbb{E}_u(T)},\end{aligned}$$

where the constant $C > 0$ stems from the embeddings

$${}_0H_p^1(J;L_p(\Omega)) \cap L_p(J;H_p^2(\Omega)) \hookrightarrow {}_0H_p^{1/2}(J;H_p^1(\Omega)) \hookrightarrow L_{2p}(J;H_p^1(\Omega)),$$

valid for each $p > 1$. Note that $C > 0$ does not depend on $T > 0$, since $\bar{u}|_{t=0} = 0$. The estimate for M_1 is very easy. Indeed, by Hölder's inequality we obtain

$$\|M_1(\theta,\bar{u},\bar{\pi})\|_{L_p(J;L_p(\Omega))} \leq C\left[\|\theta'\|_\infty(1+\|\theta'\|_\infty) + T^{1/2p}\|\theta''\|_\infty\right]\|(\bar{u},\bar{\pi})\|_{\mathbb{E}(T)}.$$

Again, $C > 0$ does not depend on $T > 0$. The estimates for M_3, M_4 are nearly the same. So we just concentrate on M_4.

$$\begin{aligned}\|M_4(\theta,\bar{u})\|_{\mathbb{F}_4(T)} &\leq \|M_4(\theta,\bar{u})\|_{W_p^{1/2-1/2p}(J;L_p(S_1))} + \|M_4(\theta,\bar{u})\|_{L_p(J;W_p^{1-1/p}(S_1))}\\ &\leq C\left[\|\theta'\|_\infty\|\bar{u}\|_{\mathbb{E}_u(T)} + \|M_4(\theta,\bar{u})\|_{L_p(J;W_p^{1-1/p}(S_1))}\right].\end{aligned}$$

To estimate last term, it suffices to consider a term of the form $\theta'\partial_j\bar{u}$ in $L_p(J;W_p^{1-1/p}(S_1))$ for some $j\in\{1,2,3\}$. Making use of the embedding

$$L_p(J;H_p^1(\Omega))\hookrightarrow L_p(J;W_p^{1-1/p}(S_1))$$

we obtain

$$\begin{aligned}\|\theta'\partial_j\bar{u}\|_{L_p(J;W_p^{1-1/p}(S_1))}\le C\|\theta'\partial_j\bar{u}\|_{L_p(J;H_p^1(\Omega))}\le\\ \le C\left[\|\theta'\|_\infty+T^{1/2p}\|\theta''\|_\infty\right]\|\bar{u}\|_{\mathbb{E}_u(T)},\end{aligned}$$

with $C>0$ being independent of $T>0$. Finally, it remains to estimate M_5 in $\mathbb{F}_5(T)$. We employ the embedding

$$L_p(J;H_p^2(\Omega))\hookrightarrow L_p(J;W_p^{2-1/p}(S_1))$$

to the result

$$\begin{aligned}\|\theta'\bar{u}_1\|_{L_p(J;W_p^{2-1/p}(S_1))}&\le C\|\theta'\bar{u}_1\|_{L_p(J;H_p^2(\Omega))}\\ &\le C\left[\|\theta'\|_\infty+T^{1/2p}(\|\theta''\|_\infty+\|\theta'''\|_\infty)\right]\|\bar{u}\|_{\mathbb{E}_u(T)}.\end{aligned}$$

Collecting everything together, we have shown that

$$\|M(\theta,\bar{u},\bar{\pi})\|_{\mathbb{F}(T)}\lesssim\left[\|\theta'\|_\infty+T^{1/2p}(\|\theta''\|_\infty+\|\theta'''\|_\infty)\right]\|(\bar{u},\bar{\pi})\|_{\mathbb{E}(T)}.$$

Recall that $\|\theta'\|_\infty<\eta$. Therefore, choosing first $\eta>0$, then $T>0$ small enough, we obtain the desired estimate (2.13). A Neumann series argument in ${}_0\mathbb{E}(T)$ finally implies that there exists a unique solution $(\bar{u},\bar{\pi})\in{}_0\mathbb{E}(T)$ of the equation $L(\bar{u},\bar{\pi})=M(\theta,\bar{u},\bar{\pi})+F$ or equivalently a solution (u,π) of (2.10), provided that the data satisfy all relevant compatibility conditions at the contact line $\overline{S_1}\cap\overline{S_2}$.

This in turn yields a solution operator $S_{QS}:\mathbb{F}_{QS}\to\mathbb{E}_{QS}$ for (2.10), where $\mathbb{E}_{QS}$ and $\mathbb{F}_{QS}$ are the solution space and data space, respectively, for the bent quarter-space and the data in $\mathbb{F}_{QS}$ satisfy all relevant compatibility conditions at the contact line $\{(x_1,\theta(x_1),0):x_1\in\mathbb{R}\}$.

2.3.3. *The two-phase Stokes equations in half-spaces.* Consider the problem

$$\begin{aligned}\partial_t(\rho u)-\mu\Delta u+\nabla\pi&=f, &&x_1\in\mathbb{R},\ x_2>0,\ x_3\in\dot{\mathbb{R}},\\ \operatorname{div}u&=f_d, &&x_1\in\mathbb{R},\ x_2>0,\ x_3\in\dot{\mathbb{R}},\\ -[\![\mu\partial_3 v]\!]-[\![\mu\nabla_{x'}u_3]\!]&=g_v, &&x_1\in\mathbb{R},\ x_2>0,\ x_3=0,\\ -2[\![\mu\partial_3u_3]\!]+[\![\pi]\!]-\sigma\Delta_{x'}h&=g_w, &&x_1\in\mathbb{R},\ x_2>0,\ x_3=0,\\ [\![u]\!]&=u_\Sigma, &&x_1\in\mathbb{R},\ x_2>0,\ x_3=0,\\ \partial_t h-m[u_3]&=g_h, &&x_1\in\mathbb{R},\ x_2>0,\ x_3=0,\\ \mu[\partial_2u_1+\partial_1u_2,\partial_3u_2+\partial_2u_3]^\mathsf{T}&=g_1, &&x_1\in\mathbb{R},\ x_2=0,\ x_3\in\dot{\mathbb{R}},\\ u_2&=g_2, &&x_1\in\mathbb{R},\ x_2=0,\ x_3\in\dot{\mathbb{R}},\\ \partial_2 h&=g_3, &&x_1\in\mathbb{R},\ x_2=0,\ x_3=0,\\ u(0)&=u_0, &&x_1\in\mathbb{R},\ x_2>0,\ x_3\in\dot{\mathbb{R}},\\ h(0)&=h_0 &&x_1\in\mathbb{R},\ x_2>0,\ x_3=0.\end{aligned}\tag{2.14}$$

Here $m[w] := (w_+ + w_-)/2$, where $w_\pm$ denote the traces of w at $x_3 = 0$ from above and below, respectively. Note that $m[w] = w|_{x_3=0}$ if w is continuous at $x_3 = 0$, that is, if $[\![w]\!] = w_+ - w_- = 0$. Furthermore $x' := (x_1, x_2)$.

For convenience we set $\Omega := \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$, $S_1 := \mathbb{R} \times \{0\} \times \mathbb{R}$, $\Sigma := \mathbb{R} \times \mathbb{R}_+ \times \{0\}$ and $\partial\Sigma := \mathbb{R} \times \{0\} \times \{0\}$. We will prove the following existence and uniqueness result.

Theorem 2.2. *Let $n = 3$, $p > 5$, $T > 0$, $\rho_j, \mu_j > 0$, $j = 1, 2$, $J = [0, T]$. The problem* (2.14) *has a unique solution (u, π, h) with regularity*

$$u \in H^1_p(J; L_p(\Omega)^3) \cap L_p(J; H^2_p(\Omega\backslash\Sigma)^3), \quad \pi \in L_p(J; \dot{H}^1_p(\Omega\backslash\Sigma)),$$

$$[\![\pi]\!] \in W^{1/2-1/2p}_p(J; L_p(\Sigma)) \cap L_p(J; W^{1-1/p}_p(\Sigma)),$$

$$h \in W^{2-1/2p}_p(J; L_p(\Sigma)) \cap H^1_p(J; W^{2-1/p}_p(\Sigma)) \cap L_p(J; W^{3-1/p}_p(\Sigma)),$$

if and only if the data satisfy the following regularity and compatibility conditions.

(1) $f \in L_p(J; L_p(\Omega)^3)$,
(2) $f_d \in L_p(J; H^1_p(\Omega\backslash\Sigma))$,
(3) $g = (g_v, g_w) \in W^{1/2-1/2p}_p(J; L_p(\Sigma)^3) \cap L_p(J; W^{1-1/p}_p(\Sigma))^3$,
(4) $u_\Sigma \in W^{1-1/2p}_p(J; L_p(\Sigma)^3) \cap L_p(J; W^{2-1/p}_p(\Sigma)^3)$;
(5) $g_h \in W^{1-1/2p}_p(J; L_p(\Sigma)) \cap L_p(J; W^{2-1/p}_p(\Sigma))$,
(6) $g_1 \in W^{1/2-1/2p}_p(J; L_p(S_1))^2 \cap L_p(J; W^{1-1/p}_p(S_1\backslash\partial\Sigma))^2$,
(7) $g_2 \in W^{1-1/2p}_p(J; L_p(S_1)) \cap L_p(J; W^{2-1/p}_p(S_1\backslash\partial\Sigma))$,
(8) $g_3 \in W^{3/2-1/p}_p(J; L_p(\partial\Sigma)) \cap H^1_p(J; W^{1-2/p}_p(\partial\Sigma)) \cap L_p(J; W^{2-2/p}_p(\partial\Sigma))$;
(9) $u_0 = (v_0, w_0) \in W^{2-2/p}_p(\Omega)^3$, $h_0 \in W^{3-2/p}_p(\Sigma)$
(10) $\operatorname{div} u_0 = f_d|_{t=0}$, $[\![u_0]\!] = u_\Sigma|_{t=0}$,
(11) $\mu[\partial_2(u_0)_1 + \partial_1(u_0)_2, \partial_3(u_0)_2 + \partial_2(u_0)_3]|^{\mathsf{T}}_{x_2=0} = g_1|_{t=0}$,
(12) $(u_0)_2|_{x_2=0} = g_2|_{t=0}$, $\partial_2 h_0|_{x_2=0} = g_3|_{t=0}$, $-[\![\mu\partial_3 v_0]\!] - [\![\mu\nabla_{x'}(u_0)_3]\!] = g_v|_{t=0}$,
(13) $(g_v)_2 + [\![(g_1)_2]\!] = 0$, $[\![(g_1)_1/\mu]\!] = \partial_2(u_\Sigma)_1 + \partial_1(u_\Sigma)_2$ *at* $\partial\Sigma$;
(14) $[\![(g_1)_2/\mu - \partial_3 g_2]\!] = \partial_2(u_\Sigma)_3$, $[\![g_2]\!] = (u_\Sigma)_2$ *at* $\partial\Sigma$,
(15) $\partial_t g_3 - m[(g_1)_2/\mu - \partial_3 g_2] = \partial_2 g_h$ *at* $\partial\Sigma$,
(16) $(f_d, u_\Sigma \cdot e_3, g_2) \in H^1_p(J; \hat{H}^{-1}_p(\Omega))$.

Proof. In a first step we will show that without loss of generality we may assume $u_0 = 0$ and $h_0 = 0$. We start with h_0. For that purpose we extend h_0 and g_h with respect to x_2 to some functions $\tilde{h}_0 \in W^{3-2/p}_p(\mathbb{R}^2)$ and

$$\tilde{g}_h \in W^{1-1/2p}_p(J; L_p(\mathbb{R}^2)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^2)),$$

respectively. Furthermore, we extend u_0 with respect to x_2 to some function $\tilde{u}_0 \in W^{2-2/p}_p(\mathbb{R}^2 \times \dot{\mathbb{R}})^3$, where $\dot{\mathbb{R}} := \mathbb{R}\backslash\{0\}$. The extensions for u_0 and g_h can be achieved by applying a higher order reflection method as in Subsection 2.3.1. In general, for the extension of h_0, one cannot apply the reflection technique from Subsection 2.3.1, since for large p one has $W^{3-2/p}_p \hookrightarrow C^2$. However, the extension for $\tilde{h}_0$ exists due to the results in [45, 46]. Let now

$$\tilde{h}(t) = [2e^{-(I-\Delta_{x'})^{1/2}t} - e^{-2(I-\Delta_{x'})^{1/2}t}]\tilde{h}_0+$$

$$[e^{-(I-\Delta_{x'})t} - e^{-2(I-\Delta_{x'})t}](I-\Delta_{x'})^{-1}\{m[\tilde{u}_0\cdot e_3] + \tilde{g}_h|_{t=0}\}, \quad t\geq 0,$$

where $\Delta_{x'}$ denotes the Laplace operator with respect to the variables $x'=(x_1,x_2)\in\mathbb{R}^2$. Since $\tilde{h}_0\in W_p^{3-2/p}(\mathbb{R}^2)$ and $m[\tilde{u}_0\cdot e_3], \tilde{g}_h|_{t=0}\in W_p^{2-3/p}(\mathbb{R}^2)$, it follows from elementary semigroup theory that

$$\tilde{h}\in W_p^{2-1/2p}(J;L_p(\mathbb{R}^2))\cap H_p^1(J;W_p^{2-1/p}(\mathbb{R}^2))\cap L_p(J;W_p^{3-1/p}(\mathbb{R}^2))$$

with $\tilde{h}(0)=\tilde{h}_0$ and $\partial_t\tilde{h}(0)=m[\tilde{u}_0\cdot e_3]+\tilde{g}_h|_{t=0}$.

Let us turn to u_0. Consider the extension $\tilde{u}_0\in W_p^{2-2/p}(\mathbb{R}^2\times\dot{\mathbb{R}})^3$ from above and let $\tilde{u}_0^{\pm}:=\tilde{u}_0|_{x_3\gtrless 0}\in W_p^{2-2/p}(\mathbb{R}^2\times\mathbb{R}_{\pm})^3$. Extend $\tilde{u}_0^+$ with respect to the variable x_3 to $\hat{u}_0^+\in W_p^{2-2/p}(\mathbb{R}^3)^3$. Then we solve the full space problem

$$\begin{aligned}\partial_t\hat{u}^+ - \Delta\hat{u}^+ &= 0, \quad x\in\mathbb{R}^3,\\ \hat{u}^+(0) &= \hat{u}_0^+, \quad x\in\mathbb{R}^3,\end{aligned}$$

to obtain a unique solution

$$\hat{u}^+\in H_p^1(J;L_p(\mathbb{R}^3)^3)\cap L_p(J;H_p^2(\mathbb{R}^3)^3).$$

Extending $\tilde{u}_0^-$ with respect to x_3 to some $\hat{u}_0^-\in W_p^{2-2/p}(\mathbb{R}^3)^3$ and solving the latter full space problem with $\hat{u}_0^+$ being replaced by $\hat{u}_0^-$ yields a unique solution

$$\hat{u}^+\in H_p^1(J;L_p(\mathbb{R}^3)^3)\cap L_p(J;H_p^2(\mathbb{R}^3)^3).$$

Then we define

$$\hat{u}:=\begin{cases}\hat{u}^+|_\Omega, & x_3>0,\\ \hat{u}_-|_\Omega, & x_3<0.\end{cases}$$

Then $\hat{u}\in H_p^1(J;L_p(\Omega)^3)\cap L_p(J;H_p^2(\Omega\backslash\Sigma)^3)$ and $\hat{u}|_{t=0}=u_0$ in $\Omega\backslash\Sigma$. If $(u,\pi,[\![\pi]\!],h)$ is a solution of (2.14), then $(u-\hat{u},\pi,[\![\pi]\!],h-\tilde{h})$ solves (2.14) with $u_0=0$, $h_0=0$ and some modified data

$$(f,f_d,g_v,g_w,u_\Sigma,g_h,g_1,g_2,g_3)$$

(not to be relabeled) in the right regularity classes, having vanishing traces at $t=0$ and satisfying the compatibility conditions at $\partial\Sigma$ stated in Theorem 2.2. Note also that by construction $\partial_t(h-\tilde{h})|_{t=0}=0$.

By Proposition 5.2 we may also assume that $g_3=0$. Indeed, there exists

$$h_*\in {}_0W_p^{2-1/2p}(J;L_p(\Sigma))\cap {}_0H_p^1(J;W_p^{2-1/p}(\Sigma))\cap L_p(J;W_p^{3-1/p}(\Sigma))$$

such that $\partial_2 h_*|_{x_2=0}=g_3$. Replacing h by $h-h_*$ it follows that $\partial_2(h-h_*)|_{x_2=0}=0$. The functions g_h and g_w have to be replaced by $g_h-\partial_t h_*$ and $g_w+\sigma\Delta_{x'}h_*$, respectively.

Next we extend

$$g_1^+:=g_1|_{x_3>0}\in {}_0W_p^{1/2-1/2p}(J;L_p(\mathbb{R}^2_+)^2)\cap L_p(J;W_p^{1-1/p}(\mathbb{R}^2_+)^2)$$

by even reflection and

$$g_2^+:=g_2|_{x_3>0}\in {}_0W_p^{1-1/2p}(J;L_p(\mathbb{R}^2_+))\cap L_p(J;W_p^{2-1/p}(\mathbb{R}^2_+))$$

by means of the reflection

$$\tilde{g}_2^+(t,x_1,x_3) = \begin{cases} g_2^+(t,x_1,x_3), & \text{if } x_3 > 0, \\ -g_2^+(t,x_1,-2x_3) + 2g_2^+(t,x_1,-x_3/2), & \text{if } x_3 < 0. \end{cases}$$

to functions

$$\tilde{g}_1^+ \in {}_0W_p^{1/2-1/2p}(J;L_p(\mathbb{R}^2)^2) \cap L_p(J;W_p^{1-1/p}(\mathbb{R}^2)^2)$$

and

$$\tilde{g}_2^+ \in {}_0W_p^{1-1/2p}(J;L_p(\mathbb{R}^2)) \cap L_p(J;W_p^{2-1/p}(\mathbb{R}^2)).$$

Let $\mu^+ := \mu|_{x_3>0}$ and solve the parabolic system

(2.15)

$$\begin{aligned} \partial_t u_* - \Delta u_* &= 0, && (x_1,x_3) \in \mathbb{R}^2,\ x_2 > 0, \\ \mu^+[\partial_2(u_*)_1 + \partial_1(u_*)_2, \partial_3(u_*)_2 + \partial_2(u_*)_3]^{\mathsf{T}} &= \tilde{g}_1^+, && (x_1,x_3) \in \mathbb{R}^2,\ x_2 = 0, \\ (u_*)_2 &= \tilde{g}_2^+, && (x_1,x_3) \in \mathbb{R}^2,\ x_2 = 0, \\ u_*(0) &= 0, && (x_1,x_3) \in \mathbb{R}^2,\ x_2 > 0, \end{aligned}$$

by [13], to obtain a solution

$$u_* \in {}_0H_p^1(J;L_p(\mathbb{R}^2_+ \times \mathbb{R}))^3 \cap L_p(J;H_p^2(\mathbb{R}^2_+ \times \mathbb{R}))^3.$$

Then we repeat the same procedure for $g_j^- := g_j|_{x_3<0}$ to obtain a function

$$u_{**} \in {}_0H_p^1(J;L_p(\mathbb{R}^2_+ \times \mathbb{R}))^3 \cap L_p(J;H_p^2(\mathbb{R}^2_+ \times \mathbb{R}))^3$$

as a solution of (2.15) with $\tilde{g}_j^+$ being replaced by the extensions $\tilde{g}_j^-$ of g_j^- and μ^+ being replaced by $\mu^- := \mu|_{x_3<0}$.

Define

$$v := \begin{cases} u_*, & x_3 > 0, \\ u_{**}, & x_3 < 0. \end{cases}$$

It follows that the function $\bar{u} := u - v$ satisfies $\bar{u}|_{t=0} = 0$, $[\![\bar{u}]\!] = u_\Sigma - [\![v]\!] =: k$ and

$$\mu[\partial_2\bar{u}_1 + \partial_1\bar{u}_2, \partial_3\bar{u}_2 + \partial_2\bar{u}_3] = 0, \quad \bar{u}_2 = 0$$

at $S_1\backslash\overline{\Sigma}$. In order to remove the jump of $\bar{u}$, we note that by the compatibility conditions it holds that $k_2 = 0$ and $\partial_2 k_1 = \partial_2 k_3 = 0$ on $\partial\Sigma$. Therefore it is possible to extend

$$k \in {}_0W_p^{1-1/2p}(J;L_p(\mathbb{R}^2_+))^3 \cap L_p(J;W_p^{2-1/p}(\mathbb{R}^2_+))^3$$

to a function

$$\tilde{k} \in {}_0W_p^{1-1/2p}(J;L_p(\mathbb{R}^2))^3 \cap L_p(J;W_p^{2-1/p}(\mathbb{R}^2))^3$$

by even reflection of k_1, k_3 and odd reflection of k_2. Then we solve the Dirichlet problem

$$\begin{aligned} \partial_t w - \Delta w &= 0, && (x_1,x_2) \in \mathbb{R}^2,\ x_3 > 0, \\ \operatorname{tr}_{x_3=0} w &= \tilde{k}, && (x_1,x_2) \in \mathbb{R}^2,\ x_3 = 0, \\ w(0) &= 0, && (x_1,x_2) \in \mathbb{R}^2,\ x_3 > 0, \end{aligned} \tag{2.16}$$

to obtain a unique solution

$$w \in {}_0H_p^1(J;L_p(\mathbb{R}^3_+)) \cap L_p(J;H_p^2(\mathbb{R}^3_+)).$$

Note that by symmetry the function

$$\bar{w}(t,x) = \begin{bmatrix} w_1(t,x_1,-x_2,x_3), \\ -w_2(t,x_1,-x_2,x_3), \\ w_3(t,x_1,-x_2,x_3) \end{bmatrix}$$

is a solution of (2.16) too, hence $w = \bar{w}$ and therefore it holds that $w_2 = 0$ as well as $\partial_2 w_1 + \partial_1 w_2 = \partial_3 w_2 + \partial_2 w_3 = 0$ at $S_1\backslash\overline{\Sigma}$. Let $\bar{u}_\pm := \bar{u}|_{x_3 \gtrless 0}$ and define

$$u^* := \begin{cases} \bar{u}_+ - w, & \text{if } x_3 > 0, \\ \bar{u}_-, & \text{if } x_3 < 0. \end{cases}$$

Then $[\![u^*]\!] = 0$ and

$$\mu[\partial_2 u_1^* + \partial_1 u_2^*, \partial_3 u_2^* + \partial_2 u_3^*] = 0, \quad u_2^* = 0$$

on $S_1\backslash\overline{\Sigma}$. We arrive at the problem

$$\begin{aligned} \partial_t(\rho u) - \mu\Delta u + \nabla\pi &= f, & x &\in \Omega\backslash\Sigma, \\ \operatorname{div} u &= f_d, & x &\in \Omega\backslash\Sigma, \\ -[\![\mu\partial_3 v]\!] - [\![\mu\nabla_{x'} w]\!] &= g_v, & x &\in \Sigma, \\ -2[\![\mu\partial_3 u_3]\!] + [\![\pi]\!] - \sigma\Delta_{x'} h &= g_w, & x &\in \Sigma, \\ [\![u]\!] &= 0, & x &\in \Sigma, \\ \partial_t h - u_3 &= g_h, & x &\in \Sigma, \\ \mu[\partial_2 u_1 + \partial_1 u_2, \partial_3 u_2 + \partial_2 u_3]^{\mathsf{T}} &= 0, & x &\in S_1\backslash\partial\Sigma, \\ u_2 &= 0, & x &\in S_1\backslash\partial\Sigma, \\ \partial_2 h &= 0, & x &\in \partial\Sigma, \\ u(0) &= 0, & x &\in \Omega\backslash\Sigma, \\ h(0) &= 0 & x &\in \Sigma, \end{aligned} \tag{2.17}$$

with modified data $f \in L_p(J; L_p(\Omega))^3$,

$$f_d \in L_p(J; H_p^1(\Omega\backslash\Sigma)),$$

$$(g_v, g_w) \in {}_0W_p^{1/2-1/2p}(J; L_p(\Sigma)^3) \cap L_p(J; W_p^{1-1/p}(\Sigma)^3),$$

and

$$g_h \in {}_0W_p^{1-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma)),$$

satisfying the compatibility conditions $(g_v)_2 = \partial_2 g_h = 0$ at $\partial\Sigma$ and $(f_d, 0, 0) \in {}_0H_p^1(J; \hat{H}_p^{-1}(\Omega))$.

Therefore it is possible to extend $(f_1, f_3, f_d, (g_v)_1, g_w, g_h)$ by even reflection to $\{x_2 < 0\}$. On the other side we may extend $(f_2, (g_v)_2)$ by odd reflection to $\{x_2 < 0\}$.

In a next step we consider the (reflected) problem

$$
\begin{aligned}
\partial_t(\rho\tilde{u}) - \mu\Delta\tilde{u} + \nabla\tilde{\pi} &= \tilde{f}, \quad (x_1,x_2)\in\mathbb{R}^2,\ x_3\in\dot{\mathbb{R}},\\
\operatorname{div}\tilde{u} &= \tilde{f}_d, \quad (x_1,x_2)\in\mathbb{R}^2,\ x_3\in\dot{\mathbb{R}},\\
-[\![\mu\partial_3\tilde{v}]\!] - [\![\mu\nabla_{x'}\tilde{u}_3]\!] &= \tilde{g}_v, \quad (x_1,x_2)\in\mathbb{R}^2,\ x_3=0,\\
-2[\![\mu\partial_3\tilde{u}_3]\!] + [\![\tilde{\pi}]\!] - \sigma\Delta_{x'}\tilde{h} &= \tilde{g}_w, \quad (x_1,x_2)\in\mathbb{R}^2,\ x_3=0,\\
[\![\tilde{u}]\!] &= 0, \quad (x_1,x_2)\in\mathbb{R}^2,\ x_3=0,\\
\partial_t\tilde{h} - \tilde{u}_3 &= \tilde{g}_h, \quad (x_1,x_2)\in\mathbb{R}^2,\ x_3=0,\\
\tilde{u}(0) &= 0, \quad (x_1,x_2)\in\mathbb{R}^2,\ x_3\in\dot{\mathbb{R}},\\
\tilde{h}(0) &= 0, \quad (x_1,x_2)\in\mathbb{R}^2,\ x_3=0,
\end{aligned}
\tag{2.18}
$$

with given reflected data $\tilde{f}\in L_p(J;L_p(\mathbb{R}^2\times\mathbb{R}))^3$,

$$\tilde{f}_d\in L_p(J;H^1_p(\mathbb{R}^2\times\dot{\mathbb{R}})),$$

$$(\tilde{g}_v,\tilde{g}_w)\in {}_0W^{1/2-1/2p}_p(J;L_p(\mathbb{R}^2)^3)\cap L_p(J;W^{1-1/p}_p(\mathbb{R}^2)^3),$$

and

$$\tilde{g}_h\in {}_0W^{1-1/2p}_p(J;L_p(\mathbb{R}^2))\cap L_p(J;W^{2-1/p}_p(\mathbb{R}^2)),$$

where $(\tilde{f}_d,0)\in{}_0H^1_p(J;\hat{H}^{-1}_p(\mathbb{R}^2\times\mathbb{R}))$.

By [24, Theorem 5.1] there exists a unique solution $(\tilde{u},\tilde{\pi},[\![\tilde{\pi}]\!],\tilde{h})$ of (2.18) with regularity

$$\tilde{u}\in{}_0H^1_p(J;L_p(\mathbb{R}^3))^3\cap L_p(J;H^2_p(\dot{\mathbb{R}}^3))^3,$$

$$\tilde{\pi}\in L_p(J;\dot{H}^1_p(\dot{\mathbb{R}}^3)),$$

$$[\![\tilde{\pi}]\!]\in{}_0W^{1/2-1/2p}_p(J;L_p(\mathbb{R}^2))\cap L_p(J;W^{1-1/p}_p(\mathbb{R}^2)),$$

and

$$\tilde{h}\in{}_0W^{2-1/2p}_p(J;L_p(\mathbb{R}^2))\cap{}_0H^1_p(J;W^{2-1/p}_p(\mathbb{R}^2))\cap L_p(J;W^{3-1/p}_p(\mathbb{R}^2)).$$

Note that by symmetry the function $(\bar{u},\bar{\pi},\bar{h})$ with $\bar{u}_j(x):=\tilde{u}_j(x_1,-x_2,x_3)$, $j\in\{1,3\}$, $\bar{u}_2(x):=-\tilde{u}_2(x_1,-x_2,x_3)$, $\bar{\pi}(x):=\tilde{\pi}(x_1,-x_2,x_3)$ and $\bar{h}(x'):=\tilde{h}(x_1,-x_2)$ is a solution of (2.18) too. Therefore, by uniqueness, it follows that

$$\tilde{u}_j(x_1,-x_2,x_3)=\tilde{u}_j(x_1,x_2,x_3),\ j\in\{1,3\},$$

$$\tilde{u}_2(x_1,x_2,x_3)=-\tilde{u}_2(x_1,-x_2,x_3),\ \tilde{\pi}(x_1,x_2,x_3)=\tilde{\pi}(x_1,-x_2,x_3)$$

and $\tilde{h}(x_1,x_2)=\tilde{h}(x_1,-x_2)$. This necessarily yields

$$\tilde{u}_2=(\partial_2\tilde{u}_1+\partial_1\tilde{u}_2)=(\partial_3\tilde{u}_2+\partial_2\tilde{u}_3)=0,$$

as well as $\partial_2\tilde{h}=0$ at $S_1\backslash\overline{\Sigma}$. Hence the restriction $(\tilde{u},\tilde{\pi},[\![\tilde{\pi}]\!],\tilde{h})|_\Omega$ is the unique strong solution of (2.17). $\square$

2.3.4. *The two-phase Stokes equations in bent half-spaces.* Let $\theta \in BC^3(\mathbb{R})$ such that

$$G_\theta := \{(x_1,x_2) \in \mathbb{R}^2 : x_2 > \theta(x_1)\} \quad \text{and} \quad \Omega_\theta = G_\theta \times \mathbb{R}.$$

We assume furthermore that $|\theta'|_\infty \leq \eta$ and $|\partial_x^j \theta|_\infty \leq M$, $j \in \{2,3\}$, where we may choose $\eta > 0$ as small as we wish. Let $S_{1,\theta} := \partial G_\theta \times \mathbb{R}$. Furthermore, let $\nu_{S_{1,\theta}} = (\nu_{G_\theta}, 0)^\mathsf{T}$ with $\nu_{G_\theta} := \frac{1}{\sqrt{1+\theta'(x_1)^2}}(\theta'(x_1),-1)^\mathsf{T}$ denote the outer unit normal to $S_{1,\theta}$ at $(x_1,\theta(x_1),x_3)$, $(x_1,x_3) \in \mathbb{R}\times\mathbb{R}$ and let $P_{S_{1,\theta}}$ be the tangential projection to $S_{1,\theta}$. Furthermore, let $\Sigma_\theta := G_\theta \times \{0\}$ and $\partial\Sigma_\theta := \partial G_\theta \times \{0\}$.

Consider the problem

$$\begin{aligned}
\partial_t(\rho u) - \mu\Delta u + \nabla\pi &= f, \quad x \in \Omega_\theta\backslash\Sigma_\theta,\\
\operatorname{div} u &= f_d, \quad x \in \Omega_\theta\backslash\Sigma_\theta,\\
-[\![\mu\partial_3 v]\!] - [\![\mu\nabla_{x'} w]\!] &= g_v, \quad x \in \Sigma_\theta,\\
-2[\![\mu\partial_3 w]\!] + [\![\pi]\!] - \sigma\Delta_{x'} h &= g_w, \quad x \in \Sigma_\theta,\\
[\![u]\!] &= u_\Sigma, \quad x \in \Sigma_\theta,\\
\partial_t h - m[w] &= g_h, \quad x \in \Sigma_\theta,\\
P_{S_{1,\theta}}\left(\mu(\nabla u + \nabla u^\mathsf{T})\nu_{S_{1,\theta}}\right) &= P_{S_{1,\theta}} g_1, \quad x \in S_{1,\theta}\backslash\partial\Sigma_\theta,\\
u\cdot\nu_{S_{1,\theta}} &= g_2, \quad x \in S_{1,\theta}\backslash\partial\Sigma_\theta,\\
\partial_{\nu_{G_\theta}} h &= g_3, \quad x \in \partial\Sigma_\theta\\
u(0) &= u_0, \quad x \in \Omega_\theta\backslash\Sigma_\theta,\\
h(0) &= h_0, \quad x \in \Sigma_\theta,
\end{aligned} \tag{2.19}$$

where $u = (v,w)$ and $v = (u_1,u_2)$, $w = u_3$. Without loss of generality we may consider $u_0 = 0$ and $h_0 = 0$ in (2.19). Literally, this can be seen as in Subsection 2.3.3, we will not go into the details. The remaining modified data (not to be relabeled) belong to the right regularity classes and they have vanishing traces at $t = 0$.

Next, we will show that we may assume $u_\Sigma = 0$. For that purpose, extend u_Σ with respect to x_2 to some function

$$\tilde u_\Sigma \in {}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^2)^3) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^2)^3),$$

and solve the half space problem

$$\begin{aligned}
\partial_t u_* - \Delta u_* &= 0, \quad x \in \mathbb{R}^2\times\mathbb{R}_+,\\
u_* &= \tilde u_\Sigma, \quad x \in \mathbb{R}^2\times\{0\},\\
u_*(0) &= 0, \quad x \in \mathbb{R}^2\times\mathbb{R}_+,
\end{aligned}$$

by [13] to obtain a unique solution

$$u_* \in {}_0H_p^1(J; L_p(\mathbb{R}^2\times\mathbb{R}_+)^3) \cap L_p(J; H_p^2(\mathbb{R}^2\times\mathbb{R}_+)^3).$$

If $(u,\pi,[\![\pi]\!],h)$ is a solution of (2.19) with $u_0 = 0$ and $h_0 = 0$, and

$$u_{**} := \begin{cases} \bar u_+ - u_*, & \text{if } x_3 > 0,\\ \bar u_-, & \text{if } x_3 < 0.\end{cases}$$

where $u^\pm := u|_{x_3 \gtrless 0}$, then $[\![u_{**}]\!] = 0$. Again the remaining modified data have the correct regularity and vanishing traces at $t = 0$. Note that in this case $m[w_{**}] = w_{**}$, where $u_{**} = (v_{**}, w_{**})$.

Let us show that we may reduce (2.19) with $u_0 = 0$, $h_0 = 0$ and $u_\Sigma = 0$ to the case $g_v = 0$, $g_w = 0$ and $g_h = 0$. To this end we extend the data (g_v, g_w, g_h) with respect to x_2 to some functions

$$(\tilde{g}_v, \tilde{g}_w) \in {}_0W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^2)^3) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^2)^3),$$

and

$$\tilde{g}_h \in {}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^2)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^2)).$$

Then we consider the two-phase problem

$$\begin{aligned}
\partial_t \tilde{u} - \Delta \tilde{u} &= 0, \quad x \in \mathbb{R}^2 \times \dot{\mathbb{R}},\\
-[\![\mu \partial_3 \tilde{v}]\!] - [\![\mu \nabla_{x'} \tilde{w}]\!] &= \tilde{g}_v, \quad x \in \mathbb{R}^2 \times \{0\},\\
-2[\![\mu \partial_3 \tilde{w}]\!] - \sigma \Delta_{x'} \tilde{h} &= \tilde{g}_w, \quad x \in \mathbb{R}^2 \times \{0\},\\
[\![\tilde{u}]\!] &= 0, \quad x \in \mathbb{R}^2 \times \{0\},\\
\partial_t \tilde{h} - \tilde{w} &= \tilde{g}_h, \quad x \in \mathbb{R}^2 \times \{0\},\\
\tilde{u}(0) &= 0, \quad x \in \mathbb{R}^2 \times \dot{\mathbb{R}},\\
\tilde{h}(0) &= 0, \quad x \in \mathbb{R}^2 \times \{0\},
\end{aligned} \tag{2.20}$$

for the unknowns $(\tilde{u}, \tilde{h})$. Interestingly, the equations for $\tilde{v}$ and $\tilde{w}$ decouple. Therefore we study for the moment the problem

$$\begin{aligned}
\partial_t \tilde{w} - \Delta \tilde{w} &= 0, \quad x \in \mathbb{R}^2 \times \dot{\mathbb{R}},\\
-2[\![\mu \partial_3 \tilde{w}]\!] - \sigma \Delta_{x'} \tilde{h} &= \tilde{g}_w, \quad x \in \mathbb{R}^2 \times \{0\},\\
[\![\tilde{w}]\!] &= 0, \quad x \in \mathbb{R}^2 \times \{0\},\\
\partial_t \tilde{h} - \tilde{w} &= \tilde{g}_h, \quad x \in \mathbb{R}^2 \times \{0\},\\
\tilde{w}(0) &= 0, \quad x \in \mathbb{R}^2 \times \dot{\mathbb{R}},\\
\tilde{h}(0) &= 0, \quad x \in \mathbb{R}^2 \times \{0\},
\end{aligned} \tag{2.21}$$

for the unknowns $(\tilde{w}, \tilde{h})$. Assume that $(\tilde{w}, \tilde{h})$ are already known. Then, $\tilde{w}$ is explicitly given by

$$\tilde{w}(x_3) = \frac{1}{2(\mu_+ + \mu_-)} \begin{cases} e^{-Lx_3} L^{-1}(\sigma \Delta_{x'} \tilde{h} + \tilde{g}_w), & \text{if } x_3 > 0,\\ e^{-L(-x_3)} L^{-1}(\sigma \Delta_{x'} \tilde{h} + \tilde{g}_w), & \text{if } x_3 < 0, \end{cases} \tag{2.22}$$

where $L := (\partial_t - \Delta_{x'})^{1/2}$. Therefore,

$$\tilde{w}|_{x_3=0} = \frac{1}{2(\mu_+ + \mu_-)} L^{-1}(\sigma \Delta_{x'} \tilde{h} + \tilde{g}_w)$$

and it follows that we may reduce (2.21) to a single equation for $\tilde{h}$ which reads

$$\partial_t \tilde{h} - \frac{\sigma}{2(\mu_+ + \mu_-)} L^{-1} \Delta_{x'} \tilde{h} = \frac{1}{2(\mu_+ + \mu_-)} L^{-1} \tilde{g}_w + \tilde{g}_h, \tag{2.23}$$

and which is subject to the initial condition $\tilde{h}(0) = 0$. Making use of the $\mathcal{R}$-boundedness of the operator $\Delta_{x'}$ in $K_p^s(\mathbb{R}^2)$, $K \in \{W, H\}$, the operator-valued H^∞-calculus for ∂_t in ${}_0H_p^r(J; K_p^s(\mathbb{R}^2))$ and real interpolation one can show as in [24, Section 5] that the operator $\partial_t - \frac{\sigma}{2(\mu_+ + \mu_-)} L^{-1}\Delta_{x'}$ is invertible in ${}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^2)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^2))$ with domain

$$ {}_0W_p^{2-1/2p}(J; L_p(\mathbb{R}^2)) \cap {}_0H_p^1(J; W_p^{2-1/p}(\mathbb{R}^2)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^2)). $$

Hence there exists a unique solution

$$ \tilde{h} \in {}_0W_p^{2-1/2p}(J; L_p(\mathbb{R}^2)) \cap {}_0H_p^1(J; W_p^{2-1/p}(\mathbb{R}^2)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^2)) $$

of (2.23). Then $\tilde{w}$ is given by (2.22) and, finally, $\tilde{v}$ is the unique solution of the two-phase problem

$$ \begin{aligned} \partial_t\tilde{v} - \Delta\tilde{v} &= 0, \quad x \in \mathbb{R}^2 \times \dot{\mathbb{R}}, \\ -[\![\mu\partial_3\tilde{v}]\!] &= [\![\mu\nabla_{x'}\tilde{w}]\!] + \tilde{g}_v, \quad x \in \mathbb{R}^2 \times \{0\}, \\ [\![\tilde{v}]\!] &= 0, \quad x \in \mathbb{R}^2 \times \{0\}, \\ \tilde{v}(0) &= 0, \quad x \in \mathbb{R}^2 \times \dot{\mathbb{R}}. \end{aligned} $$

In summary, we have shown that we may reduce (2.19) to the problem

$$ \begin{aligned} \partial_t(\rho u) - \mu\Delta u + \nabla\pi &= f, \quad x \in \Omega_\theta\backslash\Sigma_\theta, \\ \operatorname{div} u &= f_d, \quad x \in \Omega_\theta\backslash\Sigma_\theta, \\ -[\![\mu\partial_3 v]\!] - [\![\mu\nabla_{x'}w]\!] &= 0, \quad x \in \Sigma_\theta, \\ -2[\![\mu\partial_3 w]\!] + [\![\pi]\!] - \sigma\Delta_{x'}h &= 0, \quad x \in \Sigma_\theta, \\ [\![u]\!] &= 0, \quad x \in \Sigma_\theta, \\ \partial_t h - w &= 0, \quad x \in \Sigma_\theta, \\ P_{S_{1,\theta}}\left(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_{1,\theta}}\right) &= P_{S_{1,\theta}}g_1, \quad x \in S_{1,\theta}\backslash\partial\Sigma_\theta, \\ u \cdot \nu_{S_{1,\theta}} &= g_2, \quad x \in S_{1,\theta}\backslash\partial\Sigma_\theta, \\ \partial_{\nu_{G_\theta}} h &= g_3, \quad x \in \partial\Sigma_\theta \\ u(0) &= 0, \quad x \in \Omega_\theta\backslash\Sigma_\theta, \\ h(0) &= 0, \quad x \in \Sigma_\theta, \end{aligned} \tag{2.24} $$

with given data (f, g_1, g_2, g_3) having vanishing traces at $t = 0$ and which satisfy the compatibility conditions

$$ [\![g_2]\!] = 0, \; [\![g_1 \cdot e_3]\!] = 0, \; [\![P_{S_{1,\theta}}g_1 \cdot e_1/\mu]\!] = 0, \; [\![\partial_3 g_2 - g_1 \cdot e_3/\mu]\!] = 0, $$

and

$$ \partial_t g_3 + \partial_3 g_2 - g_1 \cdot e_3/\mu = 0 $$

at the contact line $\{(x_1, \theta(x_1), 0) : x_1 \in \mathbb{R}\}$. To see this, one can apply the representation (2.11) from Subsection 2.3.2. Note also that the second component of $P_{S_{1,\theta}}w$ is redundant, as it can always be reproduced from the first component. Finally, it holds that $(f_d, 0, g_2) \in {}_0H_p^1(J; \hat{H}_p^{-1}(\Omega_\theta))$.

We will now transform Ω_θ to $\Omega := \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}$, $S_{1,\theta}$ to $S_1 := \mathbb{R}\times\{0\}\times\mathbb{R}$, Σ_θ to $\Sigma := \mathbb{R}\times\mathbb{R}_+\times\{0\}$ and $\partial\Sigma_\theta$ to $\partial\Sigma := \mathbb{R}_+\times\{0\}\times\{0\}$. To this end we introduce the new variables $\bar{x}_1 = x_1$, $\bar{x}_2 = x_2 - \theta(x_1)$ and $\bar{x}_3 = x_3$ for $x\in\Omega_\theta$. Suppose that (u,π,h) is a solution of (2.19) and define the new functions

$$\bar{u}(\bar{x}) := u(\bar{x}_1,\bar{x}_2+\theta(\bar{x}_1),\bar{x}_3)$$
$$\bar{\pi}(\bar{x}) := \pi(\bar{x}_1,\bar{x}_2+\theta(\bar{x}_1),\bar{x}_3)$$

and

$$\bar{h}(\bar{x}') := h(\bar{x}_1,\bar{x}_2+\theta(\bar{x}_1)),$$

where $\bar{x}' := (\bar{x}_1,\bar{x}_2)$. In the same way we transform all of the data. Then, as in Subsection 2.3.2, $(\bar{u},\bar{\pi},\bar{h})$ satisfies the problem

(2.25)
$$\begin{aligned}
\partial_t(\rho\bar{u}) - \mu\Delta\bar{u} + \nabla\bar{\pi} &= M_1(\theta,\bar{u},\bar{\pi}) + \bar{f}, \quad \bar{x}\in\Omega\backslash\Sigma,\\
\operatorname{div}\bar{u} &= M_2(\theta,\bar{u}) + \bar{f}_d, \quad \bar{x}\in\Omega\backslash\Sigma,\\
-[\![\mu\partial_3\bar{v}]\!] - [\![\mu\nabla_{\bar{x}'}\bar{w}]\!] &= M_3(\theta,\bar{u}), \quad \bar{x}\in\Sigma\\
-2[\![\mu\partial_3\bar{w}]\!] + [\![\bar{\pi}]\!] - \sigma\Delta_{\bar{x}'}\bar{h} &= M_4(\theta,\bar{h}), \quad \bar{x}\in\Sigma,\\
[\![\bar{u}]\!] &= 0, \quad \bar{x}\in\Sigma\\
\partial_t\bar{h} - \bar{w} &= 0, \quad \bar{x}\in\Sigma,\\
\mu(\partial_1\bar{u}_2 + \partial_2\bar{u}_1) &= M_5(\theta,\bar{u}) - \sqrt{1+\theta'^2}^3[P_{S_{1,\theta}}\bar{g}_1\cdot e_1], \quad \bar{x}\in S_1\backslash\partial\Sigma,\\
\mu(\partial_2\bar{u}_3 + \partial_3\bar{u}_2) &= M_6(\theta,\bar{u}) - \sqrt{1+\theta'^2}[\bar{g}_1\cdot e_3], \quad \bar{x}\in S_1\backslash\partial\Sigma,\\
\bar{u}_2 &= M_7(\theta,\bar{u}) - \sqrt{1+\theta'^2}\bar{g}_2, \quad \bar{x}\in S_1\backslash\partial\Sigma,\\
\partial_2\bar{h} &= M_8(\theta,\bar{h}) - \sqrt{1+\theta'^2}\bar{g}_3, \quad \bar{x}\in\partial\Sigma,\\
\bar{u}(0) &= 0, \quad \bar{x}\in\Omega\backslash\Sigma\\
\bar{h}(0) &= 0, \quad \bar{x}\in\Sigma,
\end{aligned}$$

where $\bar{u} = (\bar{v},\bar{v})$. The functions M_j are given by

$$\begin{aligned}
M_1(\theta,\bar{u},\bar{\pi}) &:= 2\theta'(\bar{x}_1)\partial_1\partial_2\bar{u} + \theta''(\bar{x}_1)\partial_2\bar{u} - \theta'(\bar{x}_1)^2\partial_2^2\bar{u} + \theta'(\bar{x}_1)\partial_2\bar{\pi}e_1,\\
M_2(\theta,\bar{u}) &:= \theta'(\bar{x}_1)\partial_2\bar{u}_1,\\
M_3(\theta,\bar{u}) &= [-\theta'(\bar{x}_1)[\![\mu\partial_2\bar{w}]\!],0]^{\mathsf{T}},\\
M_4(\theta,\bar{h}) &= \sigma\left(-2\theta'(\bar{x}_1)\partial_1\partial_2\bar{h} - \theta''(\bar{x}_1)\partial_2\bar{h} + \theta'(\bar{x}_1)^2\partial_2^2\bar{h}\right),\\
M_5(\theta,\bar{u}) &:= \mu\theta'(\bar{x}_1)[2\partial_1\bar{u}_1 + \theta'(\bar{x}_1)(\partial_1\bar{u}_2 - \partial_2\bar{u}_1) - (1+\theta'(\bar{x}_1)^2)\partial_2\bar{u}_2],\\
M_6(\theta,\bar{u}) &:= \mu\theta'(\bar{x}_1)(\partial_1\bar{u}_3 - \theta'(\bar{x}_1)\partial_2\bar{u}_3 + \partial_3\bar{u}_1),\\
M_7(\theta,\bar{u}) &:= \theta'(\bar{x}_1)\bar{u}_1.
\end{aligned}$$

and

$$M_8(\theta,\bar{h}) = \theta'(\bar{x}_1)\left(\partial_1\bar{h} - \theta'(\bar{x}_1)\partial_2\bar{h}\right).$$

Let us define the function spaces

$$\begin{aligned}
{}_0\mathbb{E}_u(T) &:= \{u\in{}_0H_p^1(J;L_p(\Omega)^3)\cap L_p(J;H_p^2(\Omega\backslash\Sigma)^3) : [\![u]\!] = 0, \text{ on } \Sigma\},\\
\mathbb{E}_\pi(T) &:= L_p(J;\dot{H}_p^1(\Omega\backslash\Sigma)),
\end{aligned}$$

$$
\begin{aligned}
{}_0\mathbb{E}_q(T) &:= {}_0W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma)),\\
{}_0\mathbb{E}_h(T) &:= {}_0W_p^{2-1/2p}(J; L_p(\Sigma)) \cap {}_0H_p^1(J; W_p^{2-1/p}(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma))
\end{aligned}
$$

$$
\begin{aligned}
{}_0\mathbb{E}(T) := \{(u,\pi,q,h) \in {}_0\mathbb{E}_u(T) \times \mathbb{E}_\pi(T) \times {}_0\mathbb{E}_q(T) \times {}_0\mathbb{E}_h(T) :&\\
q = [\![\pi]\!],\ \partial_t h - u\cdot e_3 = 0 \text{ on } \Sigma\},&
\end{aligned}
$$

$$
\tilde{\mathbb{F}}(T) := \mathbb{F}_1(T) \times \mathbb{F}_2(T) \times_{j=3}^{8} {}_0\mathbb{F}_j(T),
$$

where

$$
\begin{aligned}
\mathbb{F}_1(T) &:= L_p(J; L_p(\Omega)^3),\\
\mathbb{F}_2(T) &:= L_p(J; H_p^1(\Omega\backslash\Sigma)),\\
{}_0\mathbb{F}_3(T) &:= {}_0W_p^{1/2-1/2p}(J; L_p(\Sigma)^2) \cap L_p(J; W_p^{1-1/p}(\Sigma)^2),\\
{}_0\mathbb{F}_4(T) &:= {}_0W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma)),\\
{}_0\mathbb{F}_5(T) &:= {}_0W_p^{1/2-1/2p}(J; L_p(S_1)) \cap L_p(J; W_p^{1-1/p}(S_1\backslash\partial\Sigma)),
\end{aligned}
$$

${}_0\mathbb{F}_6(T) := {}_0\mathbb{F}_5(T)$,

$$
{}_0\mathbb{F}_7(T) := {}_0W_p^{1-1/2p}(J; L_p(S_1)) \cap L_p(J; W_p^{2-1/p}(S_1)),
$$

and

$$
{}_0\mathbb{F}_8(T) := {}_0W_p^{3/2-1/p}(J; L_p(\partial\Sigma)) \cap H_p^1(J; W_p^{1-2/p}(\partial\Sigma)) \cap L_p(J; W_p^{2-2/p}(\partial\Sigma)).
$$

Finally, we set

$$
{}_0\mathbb{F}(T) := \{(f_1,\dots,f_8) \in \tilde{\mathbb{F}}(T) : (13)\ \&\ (16) \text{ in Theorem 2.2 are satisfied}\}.
$$

Define an operator $L : {}_0\mathbb{E}(T) \to {}_0\mathbb{F}(T)$ by

$$
L(\bar{u},\bar{\pi},\bar{q},\bar{h}) := \begin{bmatrix}
\partial_t(\rho\bar{u}) - \mu\Delta\bar{u} + \nabla\bar{\pi}\\
\operatorname{div}\bar{u}\\
-[\![\mu\partial_3\bar{v}]\!] - [\![\mu\nabla_{\bar{x}'}\bar{w}]\!]\\
-2[\![\mu\partial_3\bar{w}]\!] + \bar{q} - \sigma\Delta_{\bar{x}'}\bar{h}\\
\mu(\partial_2\bar{u}_1 + \partial_1\bar{u}_2)|_{S_1}\\
\mu(\partial_3\bar{u}_2 + \partial_2\bar{u}_3)|_{S_1}\\
\bar{u}_2|_{S_1}\\
\partial_2\bar{h}|_{\partial\Sigma}
\end{bmatrix}
$$

and note that $L : {}_0\mathbb{E}(T) \to {}_0\mathbb{F}(T)$ is an isomorphism by Theorem 2.2. Define

$$
M(\theta,\bar{u},\bar{\pi},\bar{h}) := (M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8)^{\mathsf{T}}(\theta,\bar{u},\bar{\pi},\bar{h})
$$

and

$$
F := (f_1, f_2, 0, 0, f_5, f_6, f_7, f_8)^{\mathsf{T}},
$$

with $f_1 := \bar{f}$, $f_2 := \bar{f}_d$,

$$
f_5 := -\sqrt{1+\theta'^2}^{\,3}[P_{S_{1,\theta}}\bar{g}_1 \cdot e_1],\ \ f_6 := -\sqrt{1+\theta'^2}[\bar{g}_1 \cdot e_3],
$$

$f_7 := -\sqrt{1+\theta'^2}\bar{g}_2$ and $f_8 := -\sqrt{1+\theta'^2}\bar{g}_3$. It can be readily checked that the components of F satisfy the compatibility conditions (13)-(16) in Theorem 2.2. In fact, this can be seen as in Subsection 2.3.2. Since $\theta \in BC^3(\mathbb{R})$ this implies that $F \in {}_0\mathbb{F}(T)$. In the same way one can show that the components of $M(\theta,\bar{u},\bar{\pi},\bar{h})$ satisfy the compatibility conditions (14)-(16) as well as the second compatibility condition in (13) in Theorem 2.2. Unfortunately the first condition in Theorem 2.2 (13) for M_6, which reads

$$[\![M_6(\theta,\bar u)]\!] = 0 \text{ on } \Sigma,$$

is in general not satisfied. To circumvent this problem, we modify $M_3(\theta,\bar u)$ as follows

$$\bar M_3(\theta,\bar u) := \theta'(\bar x_1)\left[[\![\mu\partial_2\bar w]\!], -\mathrm{ext}_\Sigma\left([\![\mu(\partial_1\bar w - \theta'(\bar x_1)\partial_2\bar w + \partial_3\bar u_1)|_{S_1\setminus\partial\Sigma}]\!]\right)\right]^{\mathsf T}.$$

Here ext_Σ is a suitable bounded and linear extension operator from

$${}_0W_p^{1/2-1/p}(J;L_p(\partial\Sigma))\cap L_p(J;W_p^{1-2/p}(\partial\Sigma))$$

to

$${}_0W_p^{1/2-1/2p}(J;L_p(\Sigma))\cap L_p(J;W_p^{1-1/p}(\Sigma)),$$

such that $[\mathrm{ext}_\Sigma z]|_{\partial\Sigma} = z$ for all $z\in {}_0W_p^{1/2-1/p}(J;L_p(\partial\Sigma))\cap L_p(J;W_p^{1-2/p}(\partial\Sigma))$, which exists due to Proposition 5.1. Note that if we have a solution $(u,\pi,q,h)\in {}_0\mathbb E(T)$ of (2.25) with $M_3(\theta,\bar u)$ replaced by $\bar M_3(\theta,\bar u)$, then, by the first component of the third line in (2.25), we obtain that

$$[\![\mu(\partial_1\bar w - \theta'(\bar x_1)\partial_2\bar w + \partial_3\bar u_1)]\!] = 0$$

on Σ, hence $\bar M_3(\theta,\bar u) = M_3(\theta,\bar u)$ in this case.

Let us define

$$\bar M(\theta,\bar u,\bar\pi,\bar h) := (M_1,M_2,\bar M_3,M_4,M_5,M_6,M_7,M_8)^{\mathsf T}(\theta,\bar u,\bar\pi,\bar h).$$

Since the modification in M_3 does not affect the other compatibility conditions in Theorem 2.2, it follows readily that $\bar M(\theta,\bar u,\bar\pi,\bar h)\in {}_0\mathbb F(T)$ for each $(\bar u,\bar\pi,\bar q,\bar h)\in {}_0\mathbb E(T)$. Therefore, we may rewrite (2.25), with M_3 replaced by $\bar M_3$, in the more condensed form

$$(\bar u,\bar\pi,\bar q,\bar h) = L^{-1}\bar M(\theta,\bar u,\bar\pi,\bar h) + L^{-1}F \tag{2.26}$$

in the space ${}_0\mathbb E(T)$. As in Subsection 2.3.2 we will apply a Neumann series argument to show that (2.26) has a unique solution $(\bar u,\bar\pi,\bar q,\bar h)\in {}_0\mathbb E(T)$. For that purpose we need to show the following property for $\bar M$. For each $\varepsilon>0$ there exist $T_0>0$ and $\eta_0>0$ such that

$$\|\bar M(\theta,\bar u,\bar\pi,\bar h)\|_{\mathbb F(T)} \le \varepsilon\|(\bar u,\bar\pi,\bar q,\bar h)\|_{\mathbb E(T)},$$

provided that $T\in(0,T_0)$ and $\eta\in(0,\eta_0)$. Mimicking the estimates of Subsection 2.3.2 for the components of $\bar M$ and taking into account that the operator ext_Σ is linear and bounded, one obtains an estimate of the form

$$\|\bar M(\theta,\bar u,\bar\pi,\bar h)\|_{\mathbb F(T)} \le C\left[\|\theta'\|_\infty + T^{1/2p}(\|\theta''\|_\infty + \|\theta'''\|_\infty)\right]\|(\bar u,\bar\pi,\bar q,\bar h)\|_{\mathbb E(T)},$$

with a uniform constant $C>0$. Since $\|\theta'\|_\infty\le\eta$, we may first choose $\eta>0$ sufficiently small and then $T>0$ sufficiently small, to obtain the desired estimate for the function $\bar M$.

Then we may apply a Neumann series argument in ${}_0\mathbb E(T)$ to conclude that there exists a unique solution $(\bar u,\bar h,\bar\pi)\in {}_0\mathbb E(T)$ of the equation

$$L(\bar u,\bar\pi,\bar q,\bar h) = \bar M(\theta,\bar u,\bar\pi,\bar h) + F$$

or equivalently a unique solution (u,π,q,h) of (2.19) as explained above.

This in turn yields a solution operator $S_{HS} : \mathbb{F}_{HS} \to \mathbb{E}_{HS}$ for (2.10), where $\mathbb{E}_{HS}$ and $\mathbb{F}_{HS}$ are the solution space and data space, respectively, for the bent half-space and the data in $\mathbb{F}_{HS}$ satisfy all relevant compatibility conditions at the contact line $\partial\Sigma_\theta$.

3. General bounded cylindrical domains

Let $n = 3$ and $p > 5$. In this section we will prove that system (2.3) admits a unique solution. To this end we apply the method of localization. We want to emphasize that this localization procedure cannot be simply carried over from standard parabolic systems. This is due to the divergence equation and the presence of the pressure in (2.3). Let

$$\mathbb{E}_u(J) := H_p^1(J; L_p(\Omega)^3) \cap L_p(J; H_p^2(\Omega\backslash\Sigma)^3), \quad \mathbb{E}_\pi(J) := L_p(J; \dot{H}_p^1(\Omega)),$$
$$\mathbb{E}_q(J) := W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma)).$$
$$\mathbb{E}_h(J) := W_p^{2-1/2p}(J; L_p(\Sigma)) \cap H_p^1(J; W_p^{2-1/p}(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma)),$$

and

$$\mathbb{E}(J) := \{(u, \pi, q, h) \in \mathbb{E}_u(J) \times \mathbb{E}_\pi(J) \times \mathbb{E}_q(J) \times \mathbb{E}_h(J) : q = [\![\pi]\!]\}.$$

3.1. **Regularity of the pressure.** Let $(u)_\Omega := u - \frac{1}{|\Omega|}\int_\Omega u dx$ denote the part of $u \in L_1(\Omega)$ with mean value zero. We start with an auxiliary lemma which provides some additional regularity for the pressure, which is needed for the localization procedure.

Lemma 3.1. *Let $(u, \pi, [\![\pi]\!], h) \in \mathbb{E}(J)$ be a solution of* (2.3) *with*

$$(f_d, u_0, h_0, g_2, u_\Sigma \cdot \nu_\Sigma, g_3 \cdot \nu_{\partial\Omega}) = 0,$$

and $f \in {}_0W_p^\alpha(J; L_p(\Omega)^3)$ for some $\alpha \in (0, 1/2 - 1/2p)$. Then the following assertions hold.

(1) *If Ω is bounded, then $(\pi)_\Omega \in {}_0W_p^\alpha(J; L_p(\Omega))$ and the estimate*
$$\|(\pi)_\Omega\|_{W_p^\alpha(L_p)} \le C\left(\|u\|_{\mathbb{E}_u} + \|[\![\pi]\!]\|_{\mathbb{E}_q} + \|f\|_{W_p^\alpha(L_p)}\right)$$
is valid, where $C > 0$ does not depend on the length of the interval J.

(2) *If Ω is a full space, a (bent) quarter space or a (bent) half space, then $(\pi)_K \in {}_0W_p^\alpha(J; L_p(K))$ for each bounded set $K \subset \overline{\Omega}$. Furthermore there exists a constant $C_K > 0$ which does not depend on the length of the interval J such that the estimate*
$$\|(\pi)_K\|_{W_p^\alpha(L_p(K))} \le C_K\left(\|u\|_{\mathbb{E}_u} + \|[\![\pi]\!]\|_{\mathbb{E}_q} + \|f\|_{W_p^\alpha(L_p)}\right)$$
is valid.

Proof. 1. Let $g \in L_{p'}(\Omega)$ be given and solve the problem

$$\begin{aligned} \Delta\psi &= g - (g|\mathbb{1}) \quad \text{in } \Omega\backslash\Sigma, \\ [\![\rho\psi]\!] &= 0 \quad \text{on } \Sigma, \\ [\![\partial_{\nu_\Sigma}\psi]\!] &= 0 \quad \text{on } \Sigma, \\ \partial_{\nu_{\partial\Omega}}\psi &= 0 \quad \text{on } \partial\Omega\backslash\partial\Sigma = (S_1\backslash\partial\Sigma) \cup S_2, \end{aligned} \tag{3.1}$$

by Lemma 5.6 and define $\phi := \rho\psi$. Since $((\pi)_\Omega|\mathbb{1}) = (u|\nabla\phi) = 0$ we obtain by integration by parts

$$\begin{aligned}
((\pi)_\Omega|g) &= ((\pi)_\Omega|(g)_\Omega)\\
&= \Big(\frac{(\pi)_\Omega}{\rho}|\Delta\phi\Big) = -\int_\Sigma [\![\frac{(\pi)_\Omega}{\rho}\partial_{\nu_\Sigma}\phi]\!]d\Sigma - \Big(\frac{\nabla\pi}{\rho}|\nabla\phi\Big)\\
&= -\int_\Sigma [\![\pi]\!]\frac{\partial_{\nu_\Sigma}\phi}{\rho}d\Sigma - \Big(\frac{\mu}{\rho}\Delta u|\nabla\phi\Big) - (f|\nabla\phi)\\
&= \int_\Omega \frac{\mu}{\rho}\nabla u : \nabla^2\phi dx - \int_{\partial\Omega}\frac{\mu\partial_{\nu_{\partial\Omega}}u}{\rho}\nabla\phi d\sigma + \int_\Sigma\{[\![\frac{\mu\partial_{\nu_\Sigma}u}{\rho}\nabla\phi]\!] - [\![\pi]\!]\frac{\partial_{\nu_\Sigma}\phi}{\rho}\}d\Sigma\\
&\quad - (f|\nabla\phi).
\end{aligned}$$

Note that there exists a constant $C > 0$ such that $\|\phi\|_{W^2_{p'}} \le C\|g\|_{L_{p'}}$. Hence, taking the supremum of the left hand side over all functions $g \in L_{p'}(\Omega)$ with norm less or equal to one, we obtain

$$\begin{aligned}
\|(\pi)_\Omega(t)\|_{L_p(\Omega)} \le C\Big(&\|\nabla u(t)\|_{L_p(\Omega)} + \|\partial_{\nu_{\partial\Omega}}u(t)\|_{L_p(\partial\Omega)}\\
&+ \|\left(\partial_{\nu_\Sigma}u(t)\right)_\pm\|_{L_p(\Sigma)} + \|[\![\pi(t)]\!]\|_{L_p(\Sigma)} + \|f(t)\|_{L_p(\Omega)}\Big),
\end{aligned}$$

for almost all $t \in J$. The same strategy yields the estimate

$$\begin{aligned}
\|(\pi)_\Omega(t) - (\pi)_\Omega(s)\|_{L_p(\Omega)} &\le C\Big(\|\nabla(u(t)-u(s))\|_{L_p(\Omega)} + \|\partial_{\nu_{\partial\Omega}}(u(t)-u(s))\|_{L_p(\partial\Omega)}\\
&+ \|\left(\partial_{\nu_\Sigma}(u(t)-u(s))\right)_\pm\|_{L_p(\Sigma)} + \|[\![\pi(t)]\!] - [\![\pi(s)]\!]\|_{L_p(\Sigma)} + \|f(t)-f(s)\|_{L_p(\Omega)}\Big),
\end{aligned}$$

for almost all $s, t \in J$.

By the mixed derivative theorem and trace theory it holds that $\partial_k u_l \in {}_0H^{1/2}_p(J; L_p(\Omega))$,

$$(\partial_k u_l)_\pm|_\Sigma \in {}_0W^{1/2-1/2p}_p(J; L_p(\Sigma))$$

and

$$\partial_k u_l|_{\partial\Omega} \in {}_0W^{1/2-1/2p}_p(J; L_p(\partial\Omega)),$$

for $k, l \in \{1, 2, 3\}$. Moreover, $[\![\pi]\!] \in {}_0W^{1/2-1/2p}_p(J; L_p(\Sigma))$. Since $H^s_p \hookrightarrow W^{s-\varepsilon}_p$ for each $s > 0$, $\varepsilon \in (0, s)$, the claim follows.

2. The proof of the second assertion follows essentially the lines of the proof of the first assertion. We fix a bounded set $K \subset \overline{\Omega}$. Let $g \in L_p(K)$ and define $(g)_K := g - \frac{1}{|K|}(g|1)_K$, where $(u|v)_K := \int_K uvdx$. Extend $(g)_K$ by zero to $\tilde{g} \in L_p(\Omega)$. Then $\tilde{g} \in \hat{W}^{-1}_p(\Omega) \cap L_p(\Omega)$ and we may solve the elliptic problem (3.1) with $\tilde{g}$ as an inhomogeneity in the first equation by Lemma 5.6. This yields a solution $\psi \in \dot{H}^1_p(\Omega\backslash\Sigma) \cap \dot{H}^2_p(\Omega\backslash\Sigma)$ satisfying the estimate

$$\|\nabla\psi\|_{L_p(\Omega)} + \|\nabla^2\psi\|_{L_p(\Omega)} \le C\|\tilde{g}\|_{L_p(\Omega)} \le C_K\|g\|_{L_p(K)}.$$

We have $((\pi)_K|g)_K = ((\pi)_K|(g)_K)_K = ((\pi)_K|\tilde{g})_\Omega := \int_\Omega(\pi)_K\tilde{g}dx$. We are now in a position to imitate the steps in the proof of the first assertion. This yields the validity of the second assertion. □

3.2. **Reduction of the data.** It is convenient to reduce the data in (2.3) to the special case

$$f = f_d = u_0 = h_0 = g_2 = u_\Sigma \cdot \nu_\Sigma = g_3 \cdot \nu_{\partial\Omega} = 0.$$

Extend $h_0 \in W_p^{3-2/p}(\Sigma)$ and $g_h|_{t=0}, m[u_0 \cdot e_3] \in W_p^{2-3/p}(\Sigma)$ to some functions $\tilde{h}_0 \in W_p^{3-2/p}(\mathbb{R}^2)$ and $\tilde{g}_h^0, \tilde{m}_0 \in W_p^{2-3/p}(\mathbb{R}^2)$, respectively, and define

$$\tilde{h}_*(t) = [2e^{-(I-\Delta_{x'})^{1/2}t} - e^{-2(I-\Delta_{x'})^{1/2}t}]\tilde{h}_0 + [e^{-(I-\Delta_{x'})t} - e^{-2(I-\Delta_{x'})t}](I-\Delta_{x'})^{-1}\left(\tilde{m}_0 + \tilde{g}_h^0\right), \quad t \geq 0.$$

Then

$$\tilde{h}_* \in W_p^{2-1/2p}(J; L_p(\mathbb{R}^2)) \cap H_p^1(J; W_p^{2-1/p}(\mathbb{R}^2)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^2))$$

and it holds that $\tilde{h}_*(0) = \tilde{h}_0$ as well as $\partial_t \tilde{h}_*(0) = \tilde{m}_0 + \tilde{g}_h^0$. Defining $h_* := \tilde{h}_*|_\Sigma$ it follows that $h_*(0) = h_0$ and $\partial_t h_*(0) = m[u_0] + g_h|_{t=0}$. Setting $h_1 := h - h_*$ we have $h_1|_{t=0} = \partial_t h_1|_{t=0} = 0$.

Next, let $u_0 = (v_0, w_0)$ and $q_0 := 2[\![\mu\partial_3 w_0]\!] + \sigma\Delta_{x'}h_0 + g_w|_{t=0} \in W_p^{1-3/p}(\Sigma)$. Extend q_0 to some $\tilde{q}_0 \in W_p^{1-3/p}(\mathbb{R}^2)$ and define $\tilde{q}_*(t) := e^{\Delta_{x'}t}\tilde{q}_0$. Then

$$\tilde{q}_* \in W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^2)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^2)).$$

Setting $q_* := \tilde{q}_*|_\Sigma$ it follows that

$$q_* \in W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma))$$

and $q_*|_{t=0} = q_0$. Given q_*, we solve the weak elliptic transmission problem

$$\begin{aligned} (\nabla\pi_*|\nabla\phi) &= 0, \quad \phi \in H_{p'}^1(\Omega), \\ [\![\pi_*]\!] &= q_*, \quad \text{on } \Sigma \end{aligned}$$

to obtain a unique solution $\pi_* \in L_p(J; \dot{H}_p^1(\Omega\backslash\Sigma))$ by Lemma 5.7.

Next we solve the parabolic transmission problem

$$\begin{aligned} \partial_t(\rho u_*) - \mu\Delta u_* &= -\nabla\pi_* + \rho f, \quad \text{in } \Omega\backslash\Sigma, \\ -[\![\mu\partial_3 v_*]\!] - [\![\mu\nabla_{x'}w_*]\!] &= g_v, \quad \text{on } \Sigma, \\ -2[\![\mu\partial_3 w_*]\!] &= g_w - q_* + \sigma\Delta_{x'}h_*, \quad \text{on } \Sigma, \\ [\![u_*]\!] &= u_\Sigma, \quad \text{on } \Sigma, \\ P_{S_1}\left(\mu(\nabla u_* + \nabla u_*^\mathsf{T})\nu_{S_1}\right) &= P_{S_1}g_1, \quad \text{on } S_1\backslash\partial\Sigma, \\ u_* \cdot \nu_{S_1} &= g_2, \quad \text{on } S_1\backslash\partial\Sigma, \\ u_* &= g_3, \quad \text{on } S_2, \\ u_*(0) &= u_0, \quad \text{in } \Omega\backslash\Sigma. \end{aligned} \tag{3.2}$$

to obtain a solution $u_* \in H_p^1(J; L_p(\Omega)^3) \cap L_p(J; H_p^2(\Omega\backslash\Sigma)^3)$ by Lemma 5.10. Note that all relevant compatibility conditions of the data are satisfied by assumption.

Setting $u_1 = u - u_*$ and $\pi_1 = \pi - \pi_*$ we see that w.l.o.g. we may assume that $u_0 = h_0 = f = 0$. To remove f_d we solve the transmission problem

$$\begin{aligned} \Delta\psi &= f_d - \operatorname{div} u_* && \text{in} \quad \Omega\backslash\Sigma, \\ [\![\rho\psi]\!] &= 0 && \text{on} \quad \Sigma, \\ [\![\partial_{e_3}\psi]\!] &= 0 && \text{on} \quad \Sigma, \\ \partial_{\nu_{\partial\Omega}}\psi &= 0 && \text{on} \quad \partial\Omega\backslash\partial\Sigma = (S_1\backslash\partial\Sigma) \cup S_2, \end{aligned} \tag{3.3}$$

by Lemma 5.8. We remark that $\int_\Omega (f_d - \operatorname{div} u_*)dx = 0$ by the compatibility conditions on $(f_d, u_\Sigma, g_2, g_3)$ and

$$f_d - \operatorname{div} u_* \in {}_0H^1_p(J; \hat{H}^{-1}_p(\Omega)) \cap L_p(J; H^1_p(\Omega\backslash\Sigma)).$$

Therefore we obtain a solution $\nabla\psi \in {}_0\mathbb{E}_u(J)$. Setting $u_2 := u_1 - \nabla\psi$, $\pi_2 := \pi_1 + \rho\partial_t\psi - \mu\Delta\psi$ and $h_2 := h_1$ we see that we may assume that $f_d = g_2 = u_\Sigma \cdot e_3 = g_3 \cdot e_3 = 0$. The time trace of all the remaining data at $t = 0$ vanishes.

3.3. **Localization procedure.** Before we can state the main result of this subsection, we introduce some function spaces. Let

$$\mathbb{F}_1(J) := L_p(J; L_p(\Omega)^3), \quad \mathbb{F}_2(J) := L_p(J; H^1_p(\Omega\backslash\Sigma)).$$

$$\begin{aligned} \mathbb{F}_3(J) &:= W_p^{1/2-1/2p}(J; L_p(\Sigma)^2) \cap L_p(J; W_p^{1-1/p}(\Sigma)^2), \\ \mathbb{F}_4(J) &:= W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma)), \\ \mathbb{F}_5(J) &:= W_p^{1-1/2p}(J; L_p(\Sigma)^3) \cap L_p(J; W_p^{2-1/p}(\Sigma)^3), \\ \mathbb{F}_6(J) &:= W_p^{1-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma)), \\ \mathbb{F}_7(J) &:= W_p^{1/2-1/2p}(J; L_p(S_1)^3) \cap L_p(J; W_p^{1-1/p}(S_1\backslash\partial\Sigma)^3), \\ \mathbb{F}_8(J) &:= W_p^{1-1/2p}(J; L_p(S_1)) \cap L_p(J; W_p^{2-1/p}(S_1\backslash\partial\Sigma)), \\ \mathbb{F}_9(J) &:= W_p^{1-1/2p}(J; L_p(S_2)) \cap L_p(J; W_p^{2-1/p}(S_2)), \\ \mathbb{F}_{10}(J) &:= W_p^{3/2-1/p}(J; L_p(\partial\Sigma)) \cap H^1_p(J; W_p^{1-2/p}(\partial\Sigma)) \cap L_p(J; W_p^{2-2/p}(\partial\Sigma)), \end{aligned}$$

and $\tilde{\mathbb{F}}(J) := \times_{j=1}^{10}\mathbb{F}_j(J)$ as well as

$$\mathbb{F}(J) := \{(f_1, \dots, f_{10}) \in \tilde{\mathbb{F}}(J) : (f_2, f_5, f_8, f_9) \in H^1_p(J; \hat{H}^{-1}_p(\Omega))\}.$$

Furthermore, we set $X_\gamma := X_{\gamma,u} \times X_{\gamma,h}$, where $X_{\gamma,u} := W_p^{2-2/p}(\Omega\backslash\Sigma)^3$ and $X_{\gamma,h} := W_p^{3-2/p}(\Sigma)$.

The main result of this subsection reads as follows.

Theorem 3.2. *Let $\mu_j, \rho_j, H_j, \sigma > 0$, $n = 3$, $p > 5$ and let $G \in \mathbb{R}^{n-1}$ be open and bounded with $\partial G \in C^4$. Define $\Omega := G \times (H_1, H_2)$ and let $\Sigma := G \times \{0\}$. Let $S_1 := \partial G \times (H_1, H_2)$ and $S_2 := (G \times \{H_1\}) \cup (G \times \{H_2\})$ be the vertical and horizontal parts of the boundary of Ω, respectively. Then there exists a unique solution*

$$u \in H^1_p(J; L_p(\Omega)^3) \cap L_p(J; H^2_p(\Omega\backslash\Sigma)^3), \quad \pi \in L_p(J; \dot{H}^1_p(\Omega\backslash\Sigma)),$$

$$[\![\pi]\!] \in W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma))$$

$$h \in W_p^{2-1/2p}(J; L_p(\Sigma)) \cap H^1_p(J; W_p^{2-1/p}(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma)),$$

of (2.3) *if and only if the data are subject to the following regularity and compatibility conditions.*

(1) $(f, f_d, g_v, g_w, u_\Sigma, g_h, g_1, g_2, g_3, g_4) \in \mathbb{F}(J)$,
(2) $(u_0, h_0) \in X_\gamma$,
(3) $\operatorname{div} u_0 = f_d|_{t=0}$, $-[\![\mu\nabla_{x'}w_0]\!] - [\![\mu\partial_3 v_0]\!] = g_v|_{t=0}$, $[\![u_0]\!] = u_\Sigma|_{t=0}$,
(4) $P_{S_1}(\mu(\nabla u_0 + \nabla u_0^{\mathsf{T}})\nu_{S_1}) = P_{S_1} g_1|_{t=0}$, $u_0 \cdot \nu_{S_1} = g_2|_{t=0}$, $u_0 = g_3|_{t=0}$,
(5) $\partial_{\nu_{\partial G}} h_0 = g_4|_{t=0}$,
(6) $[\![g_2]\!] = u_\Sigma \cdot \nu_{S_1}$,
(7) $[\![(g_1 \cdot e_3)/\mu - \partial_3 g_2]\!] = \partial_{\nu_{S_1}}(u_\Sigma \cdot e_3)$,
(8) $P_{\partial G}[(D' v_\Sigma)\nu'] = [\![P_{\partial G} g_1'/\mu]\!]$,
(9) $\partial_t g_4 - m[(g_1 \cdot e_3)/\mu - \partial_3 g_2] = \partial_{\nu_{S_1}} g_h$,
(10) $(g_v|\nu_{S_1}) = -[\![g_1 \cdot e_3]\!]$, $(g_3|\nu_{S_1}) = g_2$,
(11) $P_{\partial G}[\mu(D' g_3')\nu'] = (P_{\partial G} g_1')$,
(12) $\mu\partial_{\nu_{S_1}}(g_3 \cdot e_3) + \mu\partial_3 g_2 = g_1 \cdot e_3$,

where $\nu' := \nu_{\partial G}$.

Proof. We will split the proof in two parts.

(I) Existence of a left inverse

Let $(u, \pi, [\![\pi]\!], h)$ be a solution of (2.3). By the results of the last subsection there exists $(\bar{u}, \bar{\pi}, [\![\bar{\pi}]\!], \bar{h})$ such that

$$(\tilde{u}, \tilde{\pi}, [\![\tilde{\pi}]\!], \tilde{h}) := (u, \pi, [\![\pi]\!], h) - (\bar{u}, \bar{\pi}, [\![\bar{\pi}]\!], \bar{h})$$

solves the problem

$$\begin{aligned}
\partial_t(\rho\tilde{u}) - \mu\Delta\tilde{u} + \nabla\tilde{\pi} &= 0, \quad \text{in } \Omega\backslash\Sigma,\\
\operatorname{div}\tilde{u} &= 0, \quad \text{in } \Omega\backslash\Sigma,\\
-[\![\mu\partial_3\tilde{v}]\!] - [\![\mu\nabla_{x'}\tilde{w}]\!] &= \tilde{g}_v, \quad \text{on } \Sigma,\\
-2[\![\mu\partial_3\tilde{w}]\!] + [\![\tilde{\pi}]\!] - \sigma\Delta_{x'}\tilde{h} &= \tilde{g}_w, \quad \text{on } \Sigma,\\
[\![\tilde{u}]\!] &= \tilde{u}_\Sigma, \quad \text{on } \Sigma,\\
\partial_t\tilde{h} - m[\tilde{w}] &= \tilde{g}_h, \quad \text{on } \Sigma,\\
P_{S_1}\left(\mu(\nabla\tilde{u} + \nabla\tilde{u}^{\mathsf{T}})\nu_{S_1}\right) &= P_{S_1}\tilde{g}_1, \quad \text{on } S_1\backslash\partial\Sigma,\\
\tilde{u}\cdot\nu_{S_1} &= 0, \quad \text{on } S_1\backslash\partial\Sigma,\\
\tilde{u} &= \tilde{g}_3, \quad \text{on } S_2,\\
\partial_{\nu_{\partial G}}\tilde{h} &= \tilde{g}_4, \quad \text{on } \partial\Sigma,\\
\tilde{u}(0) &= 0, \quad \text{in } \Omega\backslash\Sigma\\
\tilde{h}(0) &= 0, \quad \text{on } \Sigma,
\end{aligned} \tag{3.4}$$

and $(\tilde{g}_3|e_3) = (\tilde{u}_\Sigma|e_3) = 0$. Choose open sets $U_k = B_r(x_k)$ with

- $\partial\Sigma \subset \bigcup_{k=7}^{N_1} U_k$,
- $\partial S_2 \subset \bigcup_{k=N_1+1}^{N} U_k$,

and choose $r > 0$ sufficiently small such that the corresponding solution operators from Subsections 2.3.2 & 2.3.4 are well-defined. According to Proposition 5.3 there exist open and connected sets

- $U_0 \cap \Sigma \neq \emptyset$, $U_0 \cap \partial\Omega = \emptyset$;
- $U_k \subset \Omega_k$, $k = 1, 2$;
- $U_k \cap S_1 \neq \emptyset$, $U_k \cap (\Sigma \cup S_2) = \emptyset$, $k = 3, 4$;
- $U_k \cap S_2 \neq \emptyset$, $U_k \cap (\Sigma \cup S_1) = \emptyset$, $k = 5, 6$,

and a family of functions $\{\varphi\}_{k=0}^N \subset C_c^3(\mathbb{R}^3; [0,1])$ such that $\overline{\Omega} \subset \bigcup_{k=0}^N U_k$, $\operatorname{supp} \varphi_k \subset U_k$, $\sum_{k=0}^N \varphi_k = 1$ and $\partial_{\nu_{S_1}} \varphi_k(x) = \partial_{e_3} \varphi_k(x) = 0$ for $x \in U_k \cap (\partial\Sigma \cup \partial S_2)$, $k \geq 7$.

Multiplying each equation in (3.4) by φ_k we obtain the following local problems

$$
\begin{aligned}
\partial_t(\rho \tilde{u}_k) - \mu\Delta \tilde{u}_k + \nabla \tilde{\pi}_k &= F_k(\tilde{u}, \tilde{\pi}), \quad \text{in } \Omega^k \backslash \Sigma^k,\\
\operatorname{div} \tilde{u}_k &= F_{dk}(\tilde{u}), \quad \text{in } \Omega^k \backslash \Sigma^k,\\
-[\![\mu\partial_3 \tilde{v}_k]\!] - [\![\mu \nabla_{x'} \tilde{w}_k]\!] &= \tilde{g}_{vk} + G_{vk}(\tilde{u}), \quad \text{on } \Sigma^k,\\
-2[\![\mu\partial_3 \tilde{w}_k]\!] + [\![\tilde{\pi}_k]\!] - \sigma\Delta_{x'} \tilde{h}_k &= \tilde{g}_{wk} + G_{wk}(\tilde{u}, \tilde{h}), \quad \text{on } \Sigma^k,\\
[\![\tilde{u}_k]\!] &= \tilde{u}_{\Sigma k}, \quad \text{on } \Sigma^k,\\
\partial_t \tilde{h}_k - m[\tilde{w}_k] &= \tilde{g}_{hk}, \quad \text{on } \Sigma^k,\\
P_{S_1^k}\Big(\mu(\nabla \tilde{u}_k + \nabla \tilde{u}_k^{\mathsf{T}})\nu_k\Big) &= P_{S_1^k} \tilde{g}_{1k} + G_{1k}(\tilde{u}), \quad \text{on } S_1^k \backslash \partial\Sigma^k,\\
\tilde{u}_k \cdot \nu_k &= 0, \quad \text{on } S_1^k \backslash \partial\Sigma^k,\\
\tilde{u}_k &= \tilde{g}_{3k}, \quad \text{on } S_2^k,\\
\partial_{\nu_k} \tilde{h}_k &= \tilde{g}_{4k}, \quad \text{on } \partial\Sigma^k,\\
\tilde{u}_k(0) &= 0, \quad \text{in } \Omega^k \backslash \Sigma^k\\
\tilde{h}_k(0) &= 0, \quad \text{on } \Sigma^k,
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
F_k(\tilde{u}, \tilde{\pi}) &:= [\nabla, \varphi_k]\tilde{\pi} - \mu[\Delta, \varphi_k]\tilde{u},\\
F_{dk}(\tilde{u}) &:= \tilde{u} \cdot \nabla\varphi_k,\\
G_{vk}(\tilde{u}) &:= (I - e_3 \otimes e_3) G_k(\tilde{u}, \tilde{h}),\\
G_{wk}(\tilde{u}, \tilde{h}) &:= G_k(\tilde{u}, \tilde{h}) e_3,\\
G_k(\tilde{u}, \tilde{h}) &:= [\![-\mu(\nabla\varphi_k \otimes \tilde{u} + \tilde{u} \otimes \nabla\varphi_k)]\!] e_3 - \sigma[\Delta_\Sigma, \varphi_k]\tilde{h} e_3,
\end{aligned}
$$

and

$$G_{1k}(\tilde{u}) := (I - \nu_k \otimes \nu_k)(\mu(\nabla\varphi_k \otimes \tilde{u} + \tilde{u} \otimes \nabla\varphi_k))\nu_k.$$

Furthermore we have set $P_{S_1^k} := I - \nu_k \otimes \nu_k$.

For $k = 0$ we obtain a pure two-phase problem with a flat interface in $\mathbb{R}^n$. This case has been treated in [24]. If $k \in \{1, 2\}$ then we are lead to one-phase Stokes equations in $\mathbb{R}^n$. An analysis of these problems can be found in [4]. If $k \in \{7, \ldots, N_1\}$ and $k \in \{N_1 + 1, \ldots, N\}$ then we rotate the coordinate system (with respect to the x_3 axis) and translate it to obtain two-phase Stokes equations in bent half-spaces and one-phase Stokes equations in bent quarter-spaces, respectively. These problems have been treated in Subsections 2.3.2 and 2.3.4. Hence, the solution operators for the charts U_k, $k \geq 7$ are well defined by the results in 2.3.2 and 2.3.4. Finally, if $k \in \{3, 4\}$ then we obtain the Stokes equations in bent half-spaces with pure-slip conditions, while for $k \in \{5, 6\}$ we are lead to the Stokes equations in half-spaces with no-slip boundary condition, see e.g. [4] for the theory of the last two type of problems. We denote the corresponding solution operators for each chart U_k by $\mathcal{S}_k$.

Note that all functions F_j, G_j carry additional time regularity (take into account Lemma 3.1) with exception of F_{dk}. To circumvent this problem we will reduce (3.5) to the case $F_{dk} = 0$. For this purpose we apply Lemma 5.8 and solve the transmission problem

$$\begin{aligned} \Delta\phi_k &= F_{dk}(\tilde{u}) && \text{in} \quad \Omega^k\backslash\Sigma^k, \\ [\![\rho\phi_k]\!] &= 0 && \text{on} \quad \Sigma^k, \\ [\![\partial_{e_3}\phi_k]\!] &= 0 && \text{on} \quad \Sigma^k, \\ \partial_{\nu_k}\phi_k &= 0 && \text{on} \quad \partial\Omega^k\backslash\partial\Sigma^k. \end{aligned}$$

This yields a solution

$$\nabla\phi_k \in {}_0H_p^1(J; H_p^1(\Omega^k\backslash\Sigma^k)^3) \cap L_p(J; H_p^3(\Omega^k\backslash\Sigma^k)^3) =: {}_0Z(J)$$

satisfying the estimate

$$\|\nabla\phi_k\|_{Z(J)} \le C_N\|\tilde{u}\|_{\mathbb{E}_u(J)}. \tag{3.6}$$

The constant $C_N > 0$ depends on N but not on the length of J. We define $\hat{u}_k := \tilde{u}_k - \nabla\phi_k$ and $\hat{\pi}_k := \tilde{\pi}_k + \rho\partial_t\phi_k - \mu\Delta\phi_k$. With $\hat{h} = \tilde{h}$ we obtain the system

$$\begin{aligned} \partial_t(\rho\hat{u}_k) - \mu\Delta\hat{u}_k + \nabla\hat{\pi}_k &= F_k(\tilde{u},\tilde{\pi}), \quad \text{in } \Omega^k\backslash\Sigma^k, \\ \operatorname{div}\hat{u}_k &= 0, \quad \text{in } \Omega^k\backslash\Sigma^k, \\ -[\![\mu\partial_3\hat{v}_k]\!] - [\![\mu\nabla_{x'}\hat{w}_k]\!] &= \tilde{g}_{vk} + \hat{G}_{vk}(\tilde{u}), \quad \text{on } \Sigma^k, \\ -2[\![\mu\partial_3\hat{w}_k]\!] + [\![\hat{\pi}_k]\!] - \sigma\Delta_{x'}\hat{h}_k &= \tilde{g}_{wk} + \hat{G}_{wk}(\tilde{u},\tilde{h}), \quad \text{on } \Sigma^k, \\ [\![\hat{u}_k]\!] &= \tilde{u}_{\Sigma k} - [\![\nabla\phi_k]\!], \quad \text{on } \Sigma^k, \\ \partial_t\hat{h}_k - m[\hat{w}_k] &= \tilde{g}_{hk} + m[\partial_3\phi_k], \quad \text{on } \Sigma^k, \\ P_{S_1^k}\Big(\mu(\nabla\hat{u}_k + \nabla\hat{u}_k^{\mathsf{T}})\nu_k\Big) &= P_{S_1^k}\tilde{g}_{1k} + \hat{G}_{1k}(\tilde{u}), \quad \text{on } S_1^k\backslash\partial\Sigma^k, \\ \hat{u}_k\cdot\nu_k &= 0, \quad \text{on } S_1^k\backslash\partial\Sigma^k, \\ \hat{u}_k &= \tilde{g}_{3k} - \nabla\phi_k, \quad \text{on } S_2^k, \\ \partial_{\nu_k}\hat{h}_k &= \tilde{g}_{4k}, \quad \text{on } S_1^k\cap\Sigma^k, \\ \hat{u}_k(0) &= 0, \quad \text{in } \Omega^k\backslash\Sigma^k \\ \hat{h}_k(0) &= 0, \quad \text{on } \Sigma^k, \end{aligned} \tag{3.7}$$

where

$$\hat{G}_k(\tilde{u},\tilde{h}) := G_k(\tilde{u},\tilde{h}) + 2[\![\mu\nabla^2\phi_k]\!]e_3 - [\![\mu\Delta\phi_k]\!]e_3,$$

$\hat{G}_{kv}$, $\hat{G}_{kw}$ defined as above and

$$\hat{G}_{1k}(\tilde{u}) := G_{1k}(\tilde{u}) - 2\mu(I - \nu_k\otimes\nu_k)\nabla^2\phi_k\nu_k.$$

With the help of the solution operators $\mathcal{S}_k$, we may rewrite (3.7) as

$$(\hat{u}_k, \hat{\pi}_k, \hat{h}_k) = \mathcal{S}_k \left(\tilde{H}_k + H_k(\tilde{u}, \tilde{\pi}, \tilde{h}) \right), \tag{3.8}$$

where $\tilde{H}_k$ stands for the set of given data and $H_k(\tilde{u}, \tilde{\pi}, \tilde{h})$ denotes the remaining part on the right hand side of (3.7). Let $\{\theta_k\}_{k=0}^N \subset C_c^\infty(U_k)$ such that $\theta_k|_{\operatorname{supp}\varphi_k} = 1$ and multiply (3.8) by θ_k. By Lemma 3.1 it holds that $(\tilde{\pi}_k \nabla^j \theta_k), (\hat{\pi}_k \nabla^j \theta_k) \in {}_0W_p^\alpha(J; L_p(\Omega^k))$ for each $j \in \{0,1,2\}$ and $k \in \{0, \dots, N\}$, since $\operatorname{supp}\theta_k \subset U_k$ is bounded. In addition, the estimate

$$\begin{aligned} &\|\tilde{\pi}_k \nabla^j \theta_k\|_{W_p^\alpha(J;L_p(\Omega^k))} + \|\hat{\pi}_k \nabla^j \theta_k\|_{W_p^\alpha(J;L_p(\Omega^k))} \\ &\qquad\qquad \le C \left(\|\tilde{u}\|_{\mathbb{E}_u(J)} + \|\tilde{h}\|_{\mathbb{E}_u(J)} + \|\tilde{H}\|_{\mathbb{F}(J)} \right) \end{aligned}$$

is valid, where $C > 0$ does not depend on $T > 0$. This implies

$$\begin{aligned} \|(\nabla^j \theta_k)(\rho \partial_t \phi_k - \mu \Delta \phi_k)\|_{W_p^\alpha(J;L_p(\Omega^k))} &= \|(\nabla^j \theta_k)(\hat{\pi}_k - \tilde{\pi}_k)\|_{W_p^\alpha(J;L_p(\Omega^k))} \\ &\le C \left(\|\tilde{u}\|_{\mathbb{E}_u(J)} + \|\tilde{h}\|_{\mathbb{E}_u(J)} + \|\tilde{H}\|_{\mathbb{F}(J)} \right) \end{aligned}$$

and since $\Delta \phi_k = F_{dk}(\tilde{u}) \in {}_0\mathbb{E}_u(J)$, it follows that

$$\|(\nabla^j \theta_k) \partial_t \phi_k\|_{{}_0W_p^\alpha(J;L_p(\Omega^k))} \le C \left(\|\tilde{u}\|_{\mathbb{E}_u(J)} + \|\tilde{h}\|_{\mathbb{E}_u(J)} + \|\tilde{H}\|_{\mathbb{F}(J)} \right)$$

for each $j \in \{0,1,2\}$ and $k \in \{0, \dots, N\}$. Hence, by Hölder's inequality and Sobolev embedding

$$\|(\nabla^j \theta_k) \partial_t \phi_k\|_{L_p(J;L_p(\Omega^k))} \le T^{1/2p} \|(\nabla^j \theta_k) \partial_t \phi_k\|_{{}_0W_p^\alpha(J;L_p(\Omega^k))}.$$

Next, we apply Hölder's inequality, Sobolev embeddings and the mixed derivative theorem to obtain

$$\begin{aligned} \|\theta_k \partial_t \phi_k\|_{L_p(J;H_p^1(\Omega^k))} &\le T^{1/2p} \|\theta_k \partial_t \phi_k\|_{L_{2p}(J;H_p^1(\Omega^k))} \\ &\le C T^{1/2p} \|\theta_k \partial_t \phi_k\|_{W_p^{\alpha/2-\varepsilon}(J;H_p^1(\Omega^k))} \\ &\le C T^{1/2p} \|\theta_k \partial_t \phi_k\|_{H_p^{\alpha/2-\varepsilon/2}(J;H_p^1(\Omega^k))} \\ &\le C T^{1/2p} \|\theta_k \partial_t \phi_k\|_{H_p^{\alpha-\varepsilon}(J;L_p(\Omega^k)) \cap L_p(J;H_p^2(\Omega^k))} \\ &\le C T^{1/2p} \|\theta_k \partial_t \phi_k\|_{W_p^{\alpha}(J;L_p(\Omega^k)) \cap L_p(J;H_p^2(\Omega^k))} \end{aligned}$$

for some $\alpha \in (0, 1/2 - 1/2p)$ and a sufficiently small $\varepsilon > 0$. Note that

$$\|\nabla \partial_t \phi_k\|_{L_p(J;L_p(\Omega^k))} + \|\nabla^2 \partial_t \phi_k\|_{L_p(J;L_p(\Omega^k))} \le C \|\tilde{u}\|_{\mathbb{E}_u(J)},$$

by (3.6), hence

$$\|\theta_k \partial_t \phi_k\|_{L_p(J;H_p^1(\Omega^k))} \le C T^{1/2p} \left(\|\tilde{u}\|_{\mathbb{E}_u(J)} + \|\tilde{h}\|_{\mathbb{E}_u(J)} + \|\tilde{H}\|_{\mathbb{F}(J)} \right).$$

In particular, this implies

$$\begin{aligned} \|\theta_k \partial_t \nabla \phi_k\|_{L_p(J;L_p(\Omega^k))} &\le \|\theta_k \partial_t \phi_k\|_{L_p(J;H_p^1(\Omega^k))} + \|(\nabla \theta_k) \partial_t \phi_k\|_{L_p(J;L_p(\Omega^k))} \\ &\le C T^{1/2p} \left(\|\tilde{u}\|_{\mathbb{E}_u(J)} + \|\tilde{h}\|_{\mathbb{E}_u(J)} + \|\tilde{H}\|_{\mathbb{F}(J)} \right). \end{aligned}$$

Moreover, by Sobolev embedding and the mixed derivative theorem, we obtain

$$\|\theta_k \nabla\phi_k\|_{L_p(J;H_p^2(\Omega^k))} \le CT^{1/2p}\|\nabla\phi_k\|_{{}_0H_p^{1/2}(J;H_p^2(\Omega^k))} \le CT^{1/2p}\|\tilde u\|_{\mathbb{E}_u(J)}.$$

Since all terms in $H_k(\tilde u,\tilde\pi,\tilde h)$ carry additional time regularity, there exists some $\gamma>0$ such that

$$\|H_k(\tilde u,\tilde\pi,\tilde h)\|_{\mathbb{F}(J)} \le CT^{\gamma}\|(\tilde u,\tilde\pi,\tilde h)\|_{\mathbb{E}(J)}.$$

We may now replace $\theta_k\hat u_k$ by $\theta_k(\tilde u_k-\nabla\phi_k)$ and $\theta_k\hat\pi_k$ by $\theta_k(\tilde\pi_k+\rho\partial_t\phi_k-\mu\Delta\phi_k)$ in (3.8) to obtain the estimate

$$\|\theta_k(\tilde u_k,\tilde\pi_k,\tilde h_k)\|_{\mathbb{E}(J)} \le C\left(\|\theta_k\tilde H_k\|_{\mathbb{F}(J)}+T^{\tilde\gamma}\|(\tilde u,\tilde\pi,\tilde h)\|_{\mathbb{E}(J)}\right), \tag{3.9}$$

with a constant $C>0$ being independent of $T>0$. Here $\tilde\gamma:=\max\{1/2p,\gamma\}$. Since $\theta_k(\tilde u_k,\tilde\pi_k,\tilde h_k)=(\tilde u_k,\tilde\pi_k,\tilde h_k)$ we may take the sum over all charts to obtain

$$\|(\tilde u,\tilde\pi,\tilde h)\|_{\mathbb{E}(J)} \le C_N\left(\|\tilde H\|_{\mathbb{F}(J)}+T^{\tilde\gamma}\|(\tilde u,\tilde\pi,\tilde h)\|_{\mathbb{E}(J)}\right).$$

Therefore, choosing $T>0$ sufficiently small, we obtain the a priori estimate

$$\|(\tilde u,\tilde\pi,\tilde h)\|_{\mathbb{E}(J)} \le C_N\|\tilde H\|_{\mathbb{F}(J)}$$

for the solution of (3.4). A successive application of the above argument yields the estimate on each finite interval $J=[0,T]$. It follows that the solution-to-data operator $L:{}_0\mathbb{E}(J)\to{}_0\mathbb{F}(J)$, defined by the left hand side of (3.4) is injective with closed range. In particular, there exists a left inverse S for L, that is $SLz=z$ for all $z\in{}_0\mathbb{E}(J)$.

(II) Existence of a right inverse

It remains to prove the existence of a right inverse for L. To this end, let the data $F:=(f,f_d,g_v,g_w,g_1,g_2,g_3,g_4,u_\Sigma,g_h)\in\mathbb{F}(J)$, $(u_0,h_0)\in X_\gamma$, subject to the conditions in Theorem 3.2 be given. By the results in Subsection 3.2, we may assume without loss of generality that $u_0=h_0=0$. In particular this means that the time traces of all inhomogeneities at $t=0$ vanish if they exist.

Let $u_*,\nabla\psi\in{}_0\mathbb{E}_u(J)$ denote the unique solutions of (3.2) and (3.3), respectively, where now $q_*=\pi_*=h_*=0$. Set $\bar u:=u_*-\nabla\psi$, $\bar\pi:=\mu\Delta\psi-\rho\partial_t\psi$ and $\bar h=0$. Defining

$$\bar SF:=(\bar u,\bar\pi,[\![\bar\pi]\!],\bar h)$$

it holds that

$$L\bar SF=L(\bar u,\bar\pi,[\![\bar\pi]\!],\bar h)=\begin{pmatrix} f\\ f_d\\ g_v+G_v(\psi)\\ g_w+G_w(\psi)\\ u_\Sigma+G_\Sigma(\psi)\\ G_h(u_*,\psi)\\ g_1+G_1(\psi)\\ g_2\\ g_3+G_3(\psi)\\ 0\end{pmatrix},$$

where

$$G_v(\psi):=2[\![\mu(I-e_3\otimes e_3)(\nabla^2\psi e_3)]\!],$$

$$G_w(\psi) := 2[\![\mu(\nabla^2\psi e_3)\cdot e_3]\!] + [\![\mu\Delta\psi]\!],$$
$$G_\Sigma(\psi) := -[\![\nabla\psi]\!],\ G_h(u_*,\psi) := -m[u_*\cdot e_3 - \partial_3\psi],$$
$$G_1(\psi) := -2\mu(I-\nu_{S_1}\otimes\nu_{S_1})(\nabla^2\psi\nu_{S_1}),$$

and $G_3(\psi) := -\nabla\psi|_{S_2}$.

In a next step we consider the problems

$$\begin{aligned}
\partial_t(\rho\tilde u_k) - \mu\Delta\tilde u_k + \nabla\tilde\pi_k &= 0, \quad \text{in } \Omega^k\backslash\Sigma^k,\\
\operatorname{div}\tilde u_k &= 0, \quad \text{in } \Omega^k\backslash\Sigma^k,\\
-[\![\mu\partial_3\tilde v_k]\!] - [\![\mu\nabla_{x'}\tilde w_k]\!] &= G_v^k(\psi), \quad \text{on } \Sigma^k,\\
-2[\![\mu\partial_3\tilde w_k]\!] + [\![\tilde\pi_k]\!] - \sigma\Delta_{x'}\tilde h_k &= G_w^k(\psi), \quad \text{on } \Sigma^k,\\
[\![\tilde u_k]\!] &= G_\Sigma^k(\psi), \quad \text{on } \Sigma^k,\\
\partial_t\tilde h_k - m[\tilde w_k] &= G_h^k(u_*,\psi) - g_h^k, \quad \text{on } \Sigma^k,\\
P_{S_1^k}\Big(\mu(\nabla\tilde u_k + \nabla\tilde u_k^{\mathsf T})\nu_k\Big) &= G_1^k(\psi), \quad \text{on } S_1^k\backslash\partial\Sigma^k,\\
\tilde u_k\cdot\nu_k &= 0, \quad \text{on } S_1^k\backslash\partial\Sigma^k,\\
\tilde u_k &= G_3^k(\psi), \quad \text{on } S_2^k,\\
\partial_{\nu_k}\tilde h_k &= -g_4^k, \quad \text{on } \partial\Sigma^k,\\
\tilde u(0) &= 0, \quad \text{in } \Omega^k\backslash\Sigma^k\\
\tilde h(0) &= 0, \quad \text{on } \Sigma^k,
\end{aligned} \tag{3.10}$$

where

$$G_j^k(\psi) := G_j(\psi)\varphi_k,\ j\in\{v,w,\Sigma,1,3\},\ G_h^k(u_*,\psi) := G_h(u_*,\psi)\varphi_k,$$

and $g_m^k := g_m\varphi_k$, $m\in\{h,4\}$. Let us check whether the right hand side in (3.10) satisfies all relevant compatibility conditions at $\partial\Sigma^k$ and ∂S_2^k, $k\ge 7$. Consider first the case $x\in\partial S_2^k$, $k\in\{7,\dots,N_1\}$.

We have to show that the relations $G_3^k(\psi)\cdot\nu_k = 0$, $\mu\partial_{\nu_k}(G_3^k(\psi)\cdot e_3) = G_1^k(\psi)\cdot e_3$ and

$$P_{\partial G^k}[\mu(D'G_3^{k'}(\psi))\nu_k'] = -P_{\partial G^k}[\mu(D'\psi)\nu_k']\varphi_k$$

hold at ∂S_2^k, where

$$G_3^{k'}(\psi) := \begin{pmatrix} G_3^k(\psi)\cdot e_1\\ G_3^k(\psi)\cdot e_2\end{pmatrix}.$$

The first condition is equivalent to $\varphi_k(\nabla\psi\cdot\nu_k) = 0$ at ∂S_2^k. Since $\nu_k = \nu_{S_1} = (\nu',0)$ on $\operatorname{supp}\varphi_k$, the claim follows from the fact that $\partial_{\nu_k}\psi = \nabla\psi\cdot\nu_k = \nabla_{x'}\psi\cdot\nu' = 0$ at $x\in\partial S_2\cap\operatorname{supp}\varphi_k$, by construction of ψ. Next, we compute

$$\partial_{\nu_k}(G_3^k(\psi)\cdot e_3) = -\partial_{\nu_k}(\varphi_k\partial_3\psi) = -\partial_3\psi\partial_{\nu_k}\varphi_k - \varphi_k\partial_{\nu_k}\partial_3\psi = 0,$$

since $\partial_{\nu_k}\varphi_k = 0$ and

$$\partial_{\nu_k}\partial_3\psi = \partial_{\nu'}\partial_3\psi = \partial_3\partial_{\nu'}\psi = 0$$

at $\operatorname{supp}\varphi_k \cap \partial S_2$, since ν' does not depend on x_3 and $\partial_{\nu'}\psi(x_3) = 0$ for all $x_3 \in [H_1, H_2]\backslash\{0\}$ by construction of ψ. Furthermore we have

$$G_1^k\psi \cdot e_3 = \nu_1\partial_1\partial_3\psi + \nu_2\partial_2\partial_3\psi = \partial_3\partial_{\nu'}\psi = 0$$

at $\operatorname{supp}\varphi_k \cap \partial S_2$. Therefore, the second compatibility condition holds. Concerning the last compatibility condition, note that

$$D'G_3^{k'}(\psi) = -D'(\varphi_k\nabla_{x'}\psi) = -2\varphi_k\nabla^2\psi - \nabla_{x'}\varphi_k \otimes \nabla_{x'}\psi - \nabla_{x'}\psi \otimes \nabla_{x'}\varphi_k.$$

From this identity we obtain

$$\begin{aligned}(D'G_3^{k'}(\psi))\nu_k' &= -2\varphi_k\nabla^2\psi\nu_k' - \nabla_{x'}\varphi_k\partial_{\nu_k'}\psi - \nabla_{x'}\psi\partial_{\nu_k'}\varphi_k \\ &= -P_{\partial G^k}[\mu(D'\psi)\nu_k'],\end{aligned}$$

since $\nu_k' = \nu'$ on $\operatorname{supp}\varphi_k$ and therefore $\partial_{\nu_k'}\varphi_k = \partial_{\nu_k'}\psi = 0$ at $\partial S_2 \cap \operatorname{supp}\varphi_k$. It follows that all compatibility conditions at ∂S_2^k are satisfied.

The validity of the compatibility conditions at $\partial\Sigma^k$, $k \in \{N_1+1, \ldots, N\}$, can be checked in a very similar way, taking into account the properties of ψ and the fact that $\partial_{\nu_k'}\varphi_k = 0$ at $\partial\Sigma \cap \operatorname{supp}\varphi_k$, $k \in \{N_1+1, \ldots, N\}$.

Therefore, for each $k \in \{0, \ldots, N\}$, there exists a unique solution $(\tilde{u}_k, \tilde{\pi}_k, \tilde{h}_k)$ of (3.10). Let $\{\theta_k\}_{k=0}^N \subset C_c^\infty(U_k)$ such that $\theta_k|_{\operatorname{supp}\varphi_k} = 1$. Note that the function $(\nabla\theta_k \cdot \tilde{u}_k)|_\Omega$ is mean value free, since $\tilde{u}_k$ is a divergence free vector field and $[\![\tilde{u}_k]\!] \cdot e_3 = 0$ on $\Sigma \cap U_k$, $\tilde{u}_k \cdot \nu_k = 0$ at $(S_1\backslash\partial\Sigma) \cap U_k$ as well as $\tilde{u}_k \cdot e_3 = 0$ at $S_2 \cap U_k$. Therefore, we may solve the problems

$$\begin{aligned} \Delta\psi_k &= (\nabla\theta_k \cdot \tilde{u}_k)|_\Omega && \text{in} \quad \Omega\backslash\Sigma, \\ [\![\rho\psi_k]\!] &= 0 && \text{on} \quad \Sigma, \\ [\![\partial_{e_3}\psi_k]\!] &= 0 && \text{on} \quad \Sigma, \\ \partial_{\nu_{\partial\Omega}}\psi_k &= 0 && \text{on} \quad \partial\Omega\backslash\partial\Sigma = (S_1\backslash\partial\Sigma) \cup S_2, \end{aligned} \tag{3.11}$$

by Lemma 5.8. This yields unique solutions

$$\nabla\psi_k \in {}_0H_p^1(J; H_p^1(\Omega\backslash\Sigma)^3) \cap L_p(J; H_p^3(\Omega\backslash\Sigma)^3).$$

Finally, we define

$$\tilde{S}F := \sum_{k=0}^N (\theta_k\tilde{u}_k - \nabla\psi_k, \theta_k\tilde{\pi}_k + \rho\partial_t\psi_k - \mu\Delta\psi_k, \theta_k\tilde{h}_k),$$

and we observe that

$$L\tilde{S}F = \sum_{k=0}^{N} \begin{pmatrix} -\mu[\Delta,\theta_k]\tilde{u}_k + [\nabla,\theta_k]\tilde{\pi}_k \\ 0 \\ \theta_k G_v^k(\psi) + (I - e_3\otimes e_3)G(\tilde{u}_k,\tilde{h}_k) + G_v(\psi_k) \\ \theta_k G_w^k(\psi) + G(\tilde{u}_k,\tilde{h}_k)e_3 + G_w(\psi_k) \\ \theta_k G_\Sigma^k(\psi) + G_\Sigma(\psi_k) \\ \theta_k(G_h^k(u_*,\psi) - g_h^k) + m[\partial_3\psi_k] \\ \theta_k G_1^k(\psi) + P_{S_1^k}[\mu(\nabla\theta_k\otimes\tilde{u}_k + \tilde{u}_k\otimes\nabla\theta_k)\nu_k] + G_1(\psi_k) \\ 0 \\ \theta_k G_3^k(\psi) + G_3(\psi_k) \\ \tilde{h}_k\partial_{\nu_k}\theta_k - \theta_k g_4^k \end{pmatrix},$$

where

$$G(\tilde{u}_k,\tilde{h}_k) := [\![-\mu(\nabla\theta_k\otimes\tilde{u}_k + \tilde{u}_k\otimes\nabla\theta_k)]\!]e_3 - \sigma[\Delta_{x'},\theta_k]\tilde{h}_k e_3.$$

Since $\theta_k|_{\operatorname{supp}\varphi_k} = 1$ it follows that $\theta_k G_j^k(\psi) = G_j^k(\psi)$, $\theta_k g_m^k = g_m^k$ and $\theta_k G_h^k(u_*,\psi) = G_h^k(u_*,\psi)$ for $j\in\{v,w,\Sigma,1,3\}$, $m\in\{h,4\}$. Therefore we have

$$\sum_{k=0}^{N}\theta_k G_j^k(\psi) = G_j(\psi)$$

as well as $\sum_{k=0}^{N}\theta_k g_m^k = g_m$ and $\sum_{k=0}^{N}\theta_k G_h^k(u_*,\psi) = G_h(u_*,\psi)$ since $\sum_{k=0}^{N}\varphi_k = 1$. Setting $\hat{S}F := \bar{S}F - \tilde{S}F$, we obtain the identity

$$L\hat{S}F = L\bar{S}F - L\tilde{S}F = F - RF$$

where

$$RF := \sum_{k=0}^{N} \begin{pmatrix} -\mu[\Delta,\theta_k]\tilde{u}_k + [\nabla,\theta_k]\tilde{\pi}_k \\ 0 \\ (I - e_3\otimes e_3)G(\tilde{u}_k,\tilde{h}_k) + G_v(\psi_k) \\ G(\tilde{u}_k,\tilde{h}_k)e_3 + G_w(\psi_k) \\ G_\Sigma(\psi_k) \\ 0 \\ P_{S_1^k}[\mu(\nabla\theta_k\otimes\tilde{u}_k + \tilde{u}_k\otimes\nabla\theta_k)\nu_k] + G_1(\psi_k) \\ 0 \\ G_3(\psi_k) \\ \tilde{h}_k\partial_{\nu_k}\theta_k \end{pmatrix}.$$

If we can show that there exists a constant $C > 0$ being independent of $T > 0$ such that the estimate

$$\|RF\|_{\mathbb{F}(J)} \le CT^\gamma\|F\|_{\mathbb{F}(J)}$$

for some $\gamma > 0$ holds, then, if $T > 0$ is sufficiently small, the operator $(I - R)$ is invertible and the right inverse S for L is given by $S := \hat{S}(I - R)^{-1}$.

We remark that all terms which involve $\tilde{u}_k$ and $\tilde{h}_k$ are of lower order and therefore these terms carry additional (time-) regularity. Furthermore the terms involving ψ_k carry additional (time-) regularity as well, since $\nabla\psi_k$ is regular enough. The only difficulty that arises is the estimate of $\sum_{k=0}^{N}[\nabla,\theta_k]\tilde{\pi}_k$ in $L_p(J;L_p(\Omega)^3)$. However, by Lemma 3.1 we know that $\tilde{\pi}_k \in {}_0W_p^\alpha(0,T;L_{p,loc}(\Omega^k))$ for some $\alpha \in (0, 1/2-1/2p)$. Since θ_k has compact support, this yields the estimate

$$\|[\nabla,\theta_k]\tilde{\pi}_k\|_{W_p^\alpha(L_p)} \le C\left(\|\tilde{u}_k\|_{\mathbb{E}_u} + \|\tilde{h}_k\|_{\mathbb{E}_h} + \|\nabla\psi\|_{\mathbb{E}_u}\right)$$

for some constant $C > 0$ which does not depend on $T > 0$. In particular this implies

$$\begin{aligned}\|\sum_{k=0}^N [\nabla,\theta_k]\tilde{\pi}_k\|_{L_p(J;L_p(\Omega))} \le C_N T^\gamma \Big(&\|u_*\|_{\mathbb{E}_u(J)} + \|\nabla\psi\|_{\mathbb{E}_u(J)} \\ &+ \|g_h\|_{\mathbb{F}_6(J)} + \|g_4\|_{\mathbb{F}_{10}(J)}\Big) \le C_N T^\gamma \|F\|_{\mathbb{F}(J)},\end{aligned}$$

for some $\gamma > 0$. ☐

We shall also prove a result on well-posedness for the linear system

$$\begin{aligned}
\partial_t(\rho u) - \mu\Delta u + \nabla\pi &= f, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u &= f_d, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu\partial_3 v]\!] - [\![\mu\nabla_{x'} w]\!] &= g_v, && \text{on } \Sigma,\\
-2[\![\mu\partial_3 w]\!] + [\![\pi]\!] - \sigma\Delta_{x'} h - \gamma_a[\![\rho]\!] h &= g_w, && \text{on } \Sigma,\\
[\![u]\!] &= u_\Sigma, && \text{on } \Sigma,\\
\partial_t h - m[w] &= g_h, && \text{on } \Sigma,\\
P_{S_1}\left(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}\right) &= P_{S_1} g_1, && \text{on } S_1\backslash\partial\Sigma,\\
u\cdot\nu_{S_1} &= g_2, && \text{on } S_1\backslash\partial\Sigma,\\
u &= g_3, && \text{on } S_2,\\
\partial_{\nu_{\partial G}} h &= g_4, && \text{on } \partial\Sigma,\\
u(0) &= u_0, && \text{in } \Omega\backslash\Sigma\\
h(0) &= h_0, && \text{on } \Sigma.
\end{aligned} \tag{3.12}$$

Corollary 3.3. *Let $\gamma_a > 0$. Under the assumptions of Theorem 3.2, there exists a unique solution*

$$u \in H_p^1(J;L_p(\Omega)^3) \cap L_p(J;H_p^2(\Omega\backslash\Sigma)^3), \quad \pi \in L_p(J;\dot{H}_p^1(\Omega\backslash\Sigma)),$$
$$[\![\pi]\!] \in W_p^{1/2-1/2p}(J;L_p(\Sigma)) \cap L_p(J;W_p^{1-1/p}(\Sigma))$$
$$h \in W_p^{2-1/2p}(J;L_p(\Sigma)) \cap H_p^1(J;W_p^{2-1/p}(\Sigma)) \cap L_p(J;W_p^{3-1/p}(\Sigma)),$$

of (3.12) *if and only if the data are subject to the conditions (1)-(12) in Theorem 3.2.*

Proof. Necessity of the conditions follows from trace theory. To prove the sufficiency part, let

$$\mathbb{E}_1(J) := H_p^1(J;L_p(\Omega)^3) \cap L_p(J;H_p^2(\Omega\backslash\Sigma)^3), \quad \mathbb{E}_2(J) := L_p(J;\dot{H}_p^1(\Omega\backslash\Sigma)),$$
$$\mathbb{E}_3(J) := W_p^{1/2-1/2p}(J;L_p(\Sigma)) \cap L_p(J;W_p^{1-1/p}(\Sigma))$$
$$\mathbb{E}_4(J) := W_p^{2-1/2p}(J;L_p(\Sigma)) \cap H_p^1(J;W_p^{2-1/p}(\Sigma)) \cap L_p(J;W_p^{3-1/p}(\Sigma)),$$

and $\mathbb{E}(J) := \{(u, \pi, q, h) \in \times_{j=1}^4 \mathbb{E}_j(J) : q = [\![\pi]\!]\}$. We first solve (2.3) for the given data, to obtain a unique solution $(u_*, \pi_*, q_*, h_*) \in \mathbb{E}(J)$. Then we consider the problem

$$\begin{aligned}
\partial_t(\rho u) - \mu\Delta u + \nabla\pi &= 0, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u &= 0, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu\partial_3 v]\!] - [\![\mu\nabla_{x'} w]\!] &= 0, && \text{on } \Sigma,\\
-2[\![\mu\partial_3 w]\!] + [\![\pi]\!] - \sigma\Delta_{x'}h - \gamma_a[\![\rho]\!]h &= \gamma_a[\![\rho]\!]h_*, && \text{on } \Sigma,\\
[\![u]\!] &= 0, && \text{on } \Sigma,\\
\partial_t h - m[w] &= 0, && \text{on } \Sigma,\\
P_{S_1}\Big(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}\Big) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
u &= 0, && \text{on } S_2,\\
\partial_{\nu_{\partial G}} h &= 0, && \text{on } \partial\Sigma,\\
u(0) &= 0, && \text{in } \Omega\backslash\Sigma\\
h(0) &= 0, && \text{on } \Sigma.
\end{aligned}\tag{3.13}$$

Define $L : {}_0\mathbb{E}(J) \to {}_0\mathbb{F}(J)$ by the left side of (3.13) and $L_0 : {}_0\mathbb{E}(J) \to {}_0\mathbb{F}(J)$ by the left side of (2.3) without the initial conditions. We already know that $L_0 : {}_0\mathbb{E}(J) \to {}_0\mathbb{F}(J)$ is boundedly invertible, hence

$$L = L_0 + (L - L_0) = L_0(I + L_0^{-1}(L - L_0)).$$

This in turn yields that $L : {}_0\mathbb{E}(J) \to {}_0\mathbb{F}(J)$ is boundedly invertible, provided that $((I + L_0^{-1}(L - L_0)) : {}_0\mathbb{E}(J) \to {}_0\mathbb{E}(J)$ has this property. To this end it suffices to show that the norm of $L_0^{-1}(L - L_0)$ in $\mathbb{E}(J)$ is less than one. For $z \in {}_0\mathbb{E}(J)$ we obtain the estimate

$$\begin{aligned}
\|L_0^{-1}(L - L_0)z\|_{\mathbb{E}(J)} &\le M\gamma_a[\![\rho]\!]\|h\|_{\mathbb{F}_4(J)} \le\\
&\le T^\alpha M\gamma_a[\![\rho]\!]\|h\|_{\mathbb{E}_4(J)} \le T^\alpha M\gamma_a[\![\rho]\!]\|z\|_{\mathbb{E}(J)},
\end{aligned}$$

for some $\alpha > 0$. Here $M := \|L_0^{-1}\|_{\mathcal{B}({}_0\mathbb{F}(J_0);{}_0\mathbb{E}(J_0))}$ and $J = [0, T] \subset [0, T_0] =: J_0$. It follows that if $T > 0$ is sufficiently small, then $L : {}_0\mathbb{E}(J) \to {}_0\mathbb{F}(J)$ is boundedly invertible. The result extends to all $T > 0$ by a successive application of this argument. □

4. The nonlinear problem

It is the aim of this section to establish an existence and uniqueness result for the nonlinear problem (2.2).

4.1. **Function spaces and regularity.**

Before we go into details, a remark concerning the nonlinearity

$$H_2(u, h) = P_{S_1}\Big(\mu(M_0(h)\nabla u + \nabla u^{\mathsf{T}} M_0(h)^{\mathsf{T}})\nu_{S_1}\Big)$$

in (2.2) is in order. One readily computes

$$(M_0(h)\nabla u + \nabla u^{\mathsf{T}} M_0(h)^{\mathsf{T}})\nu_{S_1} = \frac{1}{1+h\varphi'}\begin{pmatrix}\varphi\partial_3 u_1\partial_{\nu_{\partial G}}h + \varphi\partial_1 h\partial_3(u\cdot\nu_{S_1})\\ \varphi\partial_3 u_2\partial_{\nu_{\partial G}}h + \varphi\partial_2 h\partial_3(u\cdot\nu_{S_1})\\ \varphi\partial_3 u_3\partial_{\nu_{\partial G}}h + \varphi' h\partial_3(u\cdot\nu_{S_1})\end{pmatrix},$$

where $\nu_{S_1} = (\nu_1, \nu_2, 0)^{\mathsf{T}}$. Therefore, since $u\cdot\nu_{S_1} = 0$ on $S_1\backslash\partial\Sigma$ and $\partial_{\nu_{\partial G}}h = 0$ on ∂G, it follows that $H_2(u,h) = 0$ at $S_1\backslash\partial\Sigma$ (note that the function h depends only on $x' = (x_1, x_2)$, wherefore it is constant with respect to x_3).

Define the solution spaces

$$\begin{aligned}\mathbb{E}_u(T) := \{u \in H_p^1(J; L_p(\Omega)^3) \cap L_p(J; H_p^2(\Omega\backslash\Sigma)^3) :&\\ [\![u]\!] = 0,\ u\cdot\nu_{S_1} = 0,\ P_{S_1}(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}) = 0,\ u|_{S_2} = 0\},&\end{aligned}$$

$$\mathbb{E}_\pi(T) := L_p(J; \dot{H}_p^1(\Omega\backslash\Sigma)),$$

$$\mathbb{E}_q(T) := W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma)),$$

$$\begin{aligned}\mathbb{E}_h(T) := \{h \in W_p^{2-1/2p}(J; L_p(\Sigma)) \cap H_p^1(J; W_p^{2-1/p}(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma)) :&\\ \partial_{\nu_{\partial G}}h = 0\},&\end{aligned}$$

and

$$\mathbb{E}(T) := \{(u, \pi, q, h) \in \mathbb{E}_u(T) \times \mathbb{E}_\pi(T) \times \mathbb{E}_q(T) \times \mathbb{E}_h(T) : q = [\![\pi]\!]\}.$$

Moreover, we define the data spaces as follows.

$$\mathbb{F}_1(T) := L_p(J; L_p(\Omega)^3),$$

$$\mathbb{F}_2(T) := H_p^1(J; \hat{H}_p^{-1}(\Omega)) \cap L_p(J; H_p^1(\Omega\backslash\Sigma)),$$

$$\mathbb{F}_3(T) := \{f_3 \in W_p^{1/2-1/2p}(J; L_p(\Sigma)^3) \cap L_p(J; W_p^{1-1/p}(\Sigma)^3) : P_\Sigma(f_3)\cdot\nu_{S_1} = 0\},$$

$$\mathbb{F}_4(T) := \{f_4 \in W_p^{1-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma)) : \partial_{\nu_{\partial G}}f_4 = 0\},$$

and $\mathbb{F}(T) := \times_{j=1}^4 \mathbb{F}_j(T)$.

Define an operator $L = (L_1, L_2, L_3, L_4)$ on $\mathbb{E}(T)$ by

$$\begin{aligned}L_1(u,\pi) &:= \rho\partial_t u - \mu\Delta u + \nabla\pi\\ L_2(u) &:= \operatorname{div} u\\ L_3(u,q,h) &:= [\![-\mu(\nabla u + \nabla u^{\mathsf{T}})]\!]e_3 + qe_3 - (\Delta_{x'}h)e_3 - \gamma_a[\![\rho]\!]he_3\\ L_4(u,h) &:= \partial_t h - (u|e_3)\end{aligned}$$

and a nonlinear mapping $N = (N_1, N_2, N_3, N_4)$ on $\mathbb{E}(T)$ by

$$\begin{aligned}N_1(u,\pi,h) &:= F(u,\pi,h)\\ N_2(u,h) &:= F_d(u,h) - \frac{1}{|\Omega|}\int_\Omega F_d(u,h)\,dx\\ N_3(u,h) &:= (G_v(u,h), 0)^{\mathsf{T}} + G_w(u,h)e_3\\ N_4(u,h) &:= H_1(u,h).\end{aligned}$$

It follows from Corollary 3.3 that for each fixed $T > 0$ the mapping $L : {}_0\mathbb{E}(T) \to {}_0\mathbb{F}(J)$ is an isomorphism, since all compatibility conditions at the contact line $\partial\Sigma$ are satisfied by construction.

Let $U_T := \{z = (u, \pi, q, h) \in \mathbb{E}(T) : \|h\|_{L_\infty(L_\infty)} < \eta\}$, where $\eta > 0$ is sufficiently small. Concerning the nonlinearity $N(z)$ we have the following result

Proposition 4.1. *Let $p > n + 2$. Then*

(1) $N \in C^2(U_T; \mathbb{F}(T))$ and $N(0) = 0$ as well as $DN(0) = 0$.

(2) $DN(w) \in \mathcal{B}(U_T; \mathbb{F}(T))$ for each $w \in \mathbb{E}(T)$.

Proof. We shall show that $N(z) \in \mathbb{F}(T)$ for each $z \in U_T$. Let $z = (u, \pi, q, h) \in U_T$. Then it is easily seen that $N_1(z) = F(u, \pi, h) \in \mathbb{F}_1(T)$. Concerning $N_2(z)$, we have

$$\|N_2(z)\|_{L_p(H^1_p)} \leq C(\|h\|_{L_\infty(W^2_\infty)}\|u\|_{L_p(H^1_p)} + \|h\|_{L_\infty(W^1_\infty)}\|u\|_{L_p(H^2_p)}),$$

since $\mathbb{E}_h(T) \hookrightarrow BUC([0, T]; C^2(\Sigma))$ for $p > n + 2$. Furthermore, for $\phi \in \dot{H}^1_p(\Omega)$ we obtain after integration by parts (h does not depend on x_3)

$$\begin{aligned}(N_2(z)|\phi)_2 = (N_2(z)|\phi - \bar{\phi})_2 = -\int_\Omega \Big[(u_1\partial_1 h + u_2\partial_2 h)\partial_3\Big((\phi - \bar{\phi})\frac{\varphi}{1 + h\varphi'}\Big) +& \\ + u_3 h\partial_3\Big((\phi - \bar{\phi})\frac{\varphi'}{1 + h\varphi'}\Big)\Big]dx&,\end{aligned}$$

where $\bar{\phi} := \frac{1}{|\Omega|}\int_\Omega \phi dx$. Since $\mathbb{E}_h(T) \hookrightarrow BUC^1([0, T]; C^1(\Sigma))$ for $p > n+2$, it follows from Poincaré's inequality for functions with mean value zero that $N_2(z) \in \mathbb{F}_2(T)$.

The desired regularity property of $N_3(z)$ can be readily checked. It remains to show that

$$P_\Sigma N_3(z) \cdot \nu_{S_1} = (G_v(u, h), 0)^\mathsf{T} \cdot \nu_{S_1} = 0.$$

Inserting the expression for $G_v(u, h)$ yields

$$\begin{aligned}P_\Sigma N_3(z) \cdot \nu_{S_1} = -\Big(&[\![\mu(\nabla_{x'}v + \nabla_{x'}v^\mathsf{T})]\!]\nabla_{x'}h|\nu_{\partial G}\Big) \\ &+ |\nabla_{x'}h|^2[\![\mu\partial_3(u|\nu_{S_1})]\!] + \big((1 + |\nabla_{x'}h|^2)[\![\mu\partial_3 w]\!] - (\nabla_{x'}h|[\![\mu\nabla w]\!])\big)\,\partial_{\nu_{\partial G}}h,\end{aligned}$$

where $\nu_{S_1} = (\nu_{\partial G}, 0)^\mathsf{T}$. The last term in this equation vanishes, since $\partial_{\nu_{\partial G}}h = 0$. Moreover, since $\mu(u \cdot \nu_{S_1})(x_3) = 0$ for each $x_3 \in (H_1, 0) \cup (0, H_2)$, the second term vanishes as well. Finally, since $P_{S_1}(\mu(\nabla u + \nabla u^\mathsf{T})\nu_{S_1}) = 0$, it holds that

$$\mu(\nabla u + \nabla u^\mathsf{T})\nu_{S_1} = \Big(\mu(\nabla u + \nabla u^\mathsf{T})\nu_{S_1}|\nu_{S_1}\Big)\,\nu_{S_1}$$

on $S_1\backslash\partial\Sigma$, hence also

$$[\![\mu(\nabla u + \nabla u^\mathsf{T})]\!]\nu_{S_1} = \Big([\![\mu(\nabla u + \nabla u^\mathsf{T}]\!])\nu_{S_1}|\nu_{S_1}\Big)\,\nu_{S_1}$$

at the contact line, since ν_{S_1} does not depend on x_3. Taking the inner product with $(\nabla_{x'}h, 0)^\mathsf{T}$ yields

$$([\![\mu(\nabla u + \nabla u^\mathsf{T})]\!]\nu_{S_1}|(\nabla_{x'}h, 0)^\mathsf{T}) = \Big(\mu(\nabla u + \nabla u^\mathsf{T})\nu_{S_1}|\nu_{S_1}\Big)\,\partial_{\nu_{\partial G}}h = 0,$$

since $\partial_{\nu_{\partial G}}h = 0$. But by symmetry of the stress tensor we also have

$$([\![\mu(\nabla u + \nabla u^\mathsf{T})]\!]\nu_{S_1}|(\nabla h, 0)^\mathsf{T}) = (\nu_{\partial G}|[\![\mu(\nabla_{x'}v + \nabla_{x'}v^\mathsf{T})]\!]\nabla h),$$

where $u = (v, w)$, hence $N_3(z) \in \mathbb{F}_3(T)$.

Finally, concerning $N_4(z)$, one has to observe that $(u|\nu_{S_1}) = 0$ and $P_{S_1}((\nabla u + \nabla u^\mathsf{T})\nu_{S_1}) = 0$ on $S_1\backslash\partial\Sigma$ if $u \in \mathbb{E}_u(T)$. For $\nu_{S_1} = (\nu_{\partial G}, 0)^\mathsf{T}$, this implies in particular that $(v|\nu_{\partial G}) = 0$ and $P_{\partial G}((\nabla_{x'}v + \nabla_{x'}v^{\mathsf{T}})\nu_{\partial G}) = 0$ on $S_1\backslash\partial\Sigma$. Since $[\![v]\!] = 0$ on $\overline{\Sigma}$, by continuity of v, we clearly have $[\![\nabla_{x'}v]\!] = 0$ on Σ, since the jump acts into the direction of x_3 which is perpendicular to both e_1 and e_2. In particular we have $(v|\nu_{\partial G}) = 0$ and $P_{\partial G}((\nabla_{x'}v + \nabla_{x'}v^{\mathsf{T}})\nu_{\partial G}) = 0$ at the contact line $\partial\Sigma$. Since in addition we know that $\partial_{\nu_{\partial G}}h = 0$ at $\partial\Sigma$, it follows from Proposition 5.12 that $\partial_{\nu_{\partial G}}(v|\nabla_{x'}h) = 0$ at $\partial\Sigma$.

The remaining assertions can be proved as in [24, Proposition 6.2]. $\square$

4.2. **Reduction to time trace zero.** Let $(u_0, h_0) \in W_p^{2-2/p}(\Omega\backslash\Sigma)^3 \times W_p^{3-2/p}(\Sigma)$ such that

$$\operatorname{div} u_0 = F_d(u_0, h_0), \quad -[\![\mu\partial_3 v_0]\!] - [\![\mu\nabla_{x'}w_0]\!] = G_v(v_0, h_0),$$

$[\![u_0]\!] = 0$ on Σ, $u_0 \cdot \nu_{S_1} = 0$, $P_{S_1}(\mu(\nabla u_0 + \nabla u_0^{\mathsf{T}})\nu_{S_1}) = 0$ on $S_1\backslash\partial\Sigma$, $u_0|_{S_2} = 0$ and $\partial_{\nu_{\partial G}}h_0 = 0$ on $\partial\Sigma$.

Let $H := \max\{H_1, -H_2\} < 0$ and $u_0^+ := u_0|_{x_3\in[0,H_2]}$. Define

$$\tilde{u}_0^+(x) := \begin{cases} u_0^+(x_1, x_2, x_3), & \text{if } x_3 \in [0, H_2), \\ -u_0^+(x_1, x_2, -2x_3) + 2u_0^+(x_1, x_2, -x_3/2), & \text{if } x_3 \in (H/2, 0) \end{cases}$$

as well as

$$\bar{u}_0^+(x) := \begin{cases} \tilde{u}_0^+(x_1, x_2, x_3), & \text{if } x_3 \in [0, H_2), \\ \tilde{u}_0^+(x_1, x_2, x_3)\psi(x_3), & \text{if } x_3 \in (H/2, 0), \\ 0, & \text{if } x_3 \in (H_1, H/2], \end{cases}$$

where $\psi \in C_c^\infty(\mathbb{R}; [0, 1])$ such that $\psi(s) = 1$ if $|s| < -H/6$ and $\psi(s) = 0$ if $|s| > -H/3$. It follows by construction that $\bar{u}_0^+ \in W_p^{2-2/p}(\Omega)^3 \hookrightarrow C^1(\overline{\Omega})^3$, if $p > n + 2$. We then solve the parabolic problem

$$\begin{aligned} \partial_t(u^+) - \mu^+\Delta u^+ &= 0, \quad \text{in } \Omega, \\ P_{S_1}\left(\mu^+(\nabla u^+ + \nabla(u^+)^{\mathsf{T}})\nu_{S_1}\right) &= 0, \quad \text{on } S_1, \\ u^+ \cdot \nu_{S_1} &= 0, \quad \text{on } S_1, \\ u^+ &= 0, \quad \text{on } S_2, \\ u^+(0) &= \bar{u}_0^+, \quad \text{in } \Omega, \end{aligned} \tag{4.1}$$

by Lemma 5.9, where $\mu^+ := \mu|_{x_3\in(0,H_2)} > 0$ is constant.

Let us check whether $\bar{u}_0^+$ satisfies the relevant compatibility conditions at S_1 and S_2. It is easy to see that $\bar{u}_0^+ = 0$ at S_2. Furthermore we have $u_0^+ \cdot \nu_{S_1} = 0$ for all $x_3 \in (0, H_2)$ by the assumption on u_0. From the definition of $\tilde{u}_0^+$ we obtain that $\tilde{u}_0^+ \cdot \nu_{S_1} = 0$ for all $x_3 \in (H/2, 0)$, hence also $\bar{u}_0^+ \cdot \nu_{S_1} = 0$ for $x_3 \in (H_1, 0)$ by the definition of $\bar{u}_0^+$. Since $\bar{u}_0^+ \in C^1(\overline{\Omega})^3$ we also have $\bar{u}_0^+ \cdot \nu_{S_1} = 0$ for $x_3 = 0$. It remains to prove that

$$P_{S_1}\left(\mu^+(\nabla\bar{u}_0^+ + \nabla(\bar{u}_0^+)^{\mathsf{T}})\nu_{S_1}\right) = 0 \tag{4.2}$$

on S_1. Again, this is true for $x_3 \in (0, H_2)$, by the assumption on u_0. Since the first two components of this tangential projection do only contain derivatives with respect to the (x_1, x_2)-variables, it follows from the definition of $\bar{u}_0^+$ that

$$P_{S_1}\left(\mu^+(\nabla\bar{u}_0^+ + \nabla(\bar{u}_0^+)^{\mathsf{T}})\nu_{S_1}\right)\cdot e_j = 0$$

for $j \in \{1,2\}$ and $x_3 \in (H_1, 0)$. The third component of the projection is given by

$$\partial_{\nu_{S_1}}(\bar{u}_0^+ \cdot e_3) + \partial_3(\bar{u}_0^+ \cdot \nu_{S_1}).$$

Evidently, it holds that $\partial_{\nu_{S_1}}(\bar{u}_0^+ \cdot e_3) = 0$ by the same reasons as above, since the last component of ν_{S_1} vanishes. Furthermore, we have

$$\partial_3(\bar{u}_0^+ \cdot \nu_{S_1}) = \begin{cases} \psi\partial_3(\tilde{u}_0^+ \cdot \nu_{S_1}) + \psi'(\tilde{u}_0^+ \cdot \nu_{S_1}), & \text{if } x_3 \in (H/2, 0), \\ 0, & \text{if } x_3 \in (H_1, H/2]. \end{cases}$$

Since $u_0^+ \cdot \nu_{S_1} = 0$ for all $x_3 \in (0, H_2)$ it follows that $\partial_3(u_0^+ \cdot \nu_{S_1}) = 0$ for $x_3 \in (0, H_2)$. From the identity

$$\partial_3(\tilde{u}_0^+ \cdot \nu_{S_1}) = -\partial_3[u_0^+(x_1, x_2, -2x_3)\cdot \nu_{S_1}] + 2\partial_3[u_0^+(x_1, x_2, -x_3/2)\cdot \nu_{S_1}]$$

for $x_3 \in (H/2, 0)$, we readily obtain that $\partial_3(\bar{u}_0^+ \cdot \nu_{S_1}) = 0$ for $x_3 \in (H_1, 0)$. Finally, since $\bar{u}_0^+ \in C^1(\overline{\Omega})^3$, it follows that (4.2) holds on all of S_1.

Solving (4.1) by Lemma 5.9 yields a unique solution

$$u^+ \in H_p^1(J; L_p(\Omega)^3) \cap L_p(J; H_p^2(\Omega)^3)$$

satisfying the estimate

$$\|u^+\|_{H_p^1(L_p)\cap L_p(H_p^2)} \le M\|\bar{u}_0^+\|_{W_p^{2-2/p}},$$

where $M > 0$ does not depend on u_0^+.

Applying the same procedure to $u_0^- := u_0|_{x_3\in[H_1,0]}$ (with a suitable cut-off function ψ) yields a C^1-extension $\bar{u}_0^-$ of u_0^-. Therefore, we obtain a unique solution

$$u^- \in H_p^1(J; L_p(\Omega)^3) \cap L_p(J; H_p^2(\Omega)^3)$$

of (4.1) with μ^+ and $\bar{u}_0^+$ replaced by μ^- and $\bar{u}_0^-$, respectively, satisfying the estimate

$$\|u^-\|_{H_p^1(L_p)\cap L_p(H_p^2)} \le M\|\bar{u}_0^-\|_{W_p^{2-2/p}},$$

where $M > 0$ does not depend on u_0^-. We then define

$$\bar{u} := \begin{cases} u^+, & \text{if } x_3 \in (0, H_2), \\ u^-, & \text{if } x_3 \in (H_1, 0). \end{cases}$$

Note that in general $\bar{u} \in H_p^1(J; L_p(\Omega)^3) \cap L_p(J; H_p^2(\Omega\backslash\Sigma)^3)$, since $[\![\bar{u}]\!]$ is not necessarily zero.

In a next step we solve the two phase problem

$$
\begin{aligned}
\partial_t(\rho\tilde{u}) - \mu\Delta\tilde{u} &= 0, \quad \text{in } \Omega\backslash\Sigma,\\
[\![\mu\partial_3\tilde{v}]\!] + [\![\mu\nabla_{x'}\tilde{w}]\!] &= [\![\mu\partial_3\bar{v}]\!] + [\![\mu\nabla_{x'}\bar{w}]\!], \quad \text{on } \Sigma,\\
[\![\mu\partial_3\tilde{w}]\!] &= [\![\mu\partial_3\bar{w}]\!], \quad \text{on } \Sigma,\\
[\![\tilde{u}]\!] &= 0, \quad \text{on } \Sigma,\\
P_{S_1}\left(\mu(\nabla\tilde{u} + \nabla\tilde{u}^{\mathsf{T}})\nu_{S_1}\right) &= 0, \quad \text{on } S_1\backslash\partial\Sigma,\\
\tilde{u}\cdot\nu_{S_1} &= 0, \quad \text{on } S_1\backslash\partial\Sigma,\\
\tilde{u} &= 0, \quad \text{on } S_2,\\
\tilde{u}(0) &= u_0, \quad \text{in } \Omega\backslash\Sigma,
\end{aligned}
\tag{4.3}
$$

by Lemma 5.10, where $\tilde{u} = (\tilde{v}, \tilde{w})$ and $\bar{u} = (\bar{v}, \bar{w})$. The compatibility conditions at $t = 0$ are satisfied, since $\bar{u}(0) = u_0$. Let us check that the compatibility condition

$$[\![\mu\partial_3(\bar{u}|\nu_{S_1})]\!] + [\![\mu\partial_{\nu_{S_1}}\bar{w}]\!] = 0$$

holds at the contact line $\partial\Sigma$. Since by construction of $\bar{u}$ we have

$$P_{S_1}\left(\mu(\nabla\bar{u} + \nabla\bar{u}^{\mathsf{T}})\nu_{S_1}\right) = 0,$$

at $S_1\backslash\partial\Sigma$, the third component yields $\mu\left(\partial_{\nu_{S_1}}\bar{w} + \partial_3(\bar{u}\cdot\nu_{S_1})\right) = 0$ at $S_1\backslash\partial\Sigma$. This in turn implies that $[\![\mu\partial_3(\bar{u}|\nu_{S_1})]\!] + [\![\mu\partial_{\nu_{S_1}}\bar{w}]\!] = 0$. Note that for the third equation in (4.3) there has no compatibility condition at $\partial\Sigma$ to be satisfied. Therefore we obtain a unique solution $\tilde{u} \in \mathbb{E}_u(T)$ by Lemma 5.10.

Define $f_d^* := \operatorname{div}\tilde{u} \in \mathbb{F}_2(T)$, $g^* := [\![-\mu(\nabla\tilde{u} + \nabla\tilde{u}^{\mathsf{T}})e_3]\!] \in \mathbb{F}_3(T)$ and $g_h^* := e^{-At}(v_0|_\Sigma\cdot\nabla h_0)$, with $A := (I - \Delta_N)$, where Δ_N is the Neumann-Laplacian and e^{-At} denotes the C_0-semigroup, generated by $-A$ in $L_p(\Sigma)$. Then, since $(v_0|_\Sigma\cdot\nabla h_0) \in W_p^{2-3/p}(\Sigma)$ with $\partial_{\nu_{\partial G}}(v_0|_\Sigma\cdot\nabla h_0) = 0$ by Proposition 5.12 at $\partial\Sigma$, it follows that $e^{-At}g_h \in \mathbb{F}_4(T)$. The fact that $P_\Sigma([\![-\mu(\nabla\tilde{u} + \nabla\tilde{u}^{\mathsf{T}})e_3]\!])\cdot\nu_{S_1} = 0$ holds by construction of $\tilde{u}$.

By Corollary 3.3 there exists a unique solution $z_* = (u_*, \pi_*, q_*, h_*) \in \mathbb{E}(T)$ of the initial value problem $Lz_* = (0, f_d^*, g^*, g_h^*)$, $(u_*, h_*)|_{t=0} = (u_0, h_0)$, since the compatibility conditions at $t = 0$ in the second and third component are satisfied by construction. We remark that z_* satisfies the estimate

$$\|z_*\|_{\mathbb{E}(T)} \le C_0\|(u_0, h_0)\|_{X_\gamma},$$

and $C_0 > 0$ does not depend on (u_0, h_0).

4.3. **Nonlinear well-posedness.** Define the mapping $K(z) := N(z + z_*) - Lz_*$, where $z \in {}_0\mathbb{E}(T)$. By Proposition 4.1 it holds that $K(z) \in {}_0\mathbb{F}(T)$ for each $z \in {}_0\mathbb{E}(T)$, wherefore, we may consider the mapping $\mathcal{K}(z) := L^{-1}K(z)$. We intend to show that this mapping has a fixed point in ${}_0\mathbb{E}(T)$.

The main result of this section reads as follows.

Theorem 4.2. *Let $n = 3$, $p > 5$. For each given $T > 0$ there exists a number $\eta = \eta(T) > 0$ such that for all initial values $(u_0, h_0) \in W_p^{2-2/p}(\Omega\backslash\Sigma)^3 \times W_p^{3-2/p}(\Sigma)$*

satisfying the compatibility conditions

$$\operatorname{div} u_0 = F_d(u_0, h_0), \quad -[\![\mu\partial_3 v_0]\!] - [\![\mu\nabla_{x'} w_0]\!] = G_v(v_0, h_0),$$

$[\![u_0]\!] = 0$, $u_0 \cdot \nu_{S_1} = 0$, $P_{S_1}(\mu(\nabla u_0 + \nabla u_0^{\mathsf{T}})\nu_{S_1}) = 0$, $u_0|_{S_2} = 0$ *and* $\partial_{\nu_{\partial G}} h_0 = 0$ *as well as the smallness condition*

$$\|u_0\|_{W_p^{2-2/p}(\Omega\setminus\Sigma)} + \|h_0\|_{W_p^{3-2/p}(\Sigma)} \le \eta,$$

there exists a unique solution $(u, \pi, q, h) \in \mathbb{E}(T)$ *of* (2.2).

Proof. For a given Banach space Z, let

$$\mathbb{B}_Z := \{z \in Z : \|z\|_Z \le 1\}.$$

Based on Proposition 4.1, for each $\varepsilon \in (0,1)$ there exists $\delta(\varepsilon) > 0$ such that

$$\|DN(z + z_*)\|_{\mathcal{B}(\mathbb{E}(T),\mathbb{F}(T))} \le \varepsilon$$

whenever $(z + z_*) \in \delta\mathbb{B}_{\mathbb{E}(T)} \subset U_T$. Let $M := \|L^{-1}\|_{\mathcal{B}({}_0\mathbb{F}(T);{}_0\mathbb{E}(T))} > 0$ and $C := \|L\|_{\mathcal{B}(\mathbb{E}(T);\mathbb{F}(T))} > 0$. We assume that $\varepsilon > 0$ from above is chosen sufficiently small, such that $\varepsilon \in (0, 1/(2M))$. Suppose furthermore that $z \in \frac{\delta}{2}\mathbb{B}_{{}_0\mathbb{E}(T)}$ and $(u_0, h_0) \in \frac{\delta}{4MC_0(1+C)}\mathbb{B}_{X_\gamma}$. This yields

$$\|z + z_*\|_{\mathbb{E}(T)} \le \delta/2 + \delta/(4M(1 + C)) < \delta$$

and therefore

$$\begin{aligned}
\|\mathcal{K}(z)\|_{\mathbb{E}(T)} &\le M\|K(z)\|_{\mathbb{F}(T)} \le M(\|N(z + z_*)\|_{\mathbb{F}(T)} + \|Lz_*\|_{\mathbb{F}(T)}) \\
&\le M[\varepsilon(\|z\|_{\mathbb{E}(T)} + \|z_*\|_{\mathbb{E}(T)}) + C\|z_*\|_{\mathbb{E}(T)}] \\
&\le M(\varepsilon\|z\|_{\mathbb{E}(T)} + C_0(1 + C)\|(u_0, h_0)\|_{X_\gamma}) \\
&\le M\varepsilon\frac{\delta}{2} + \frac{\delta}{4} \le \delta/2
\end{aligned}$$

hence $\mathcal{K} : \frac{\delta}{2}\mathbb{B}_{{}_0\mathbb{E}(T)} \to \frac{\delta}{2}\mathbb{B}_{{}_0\mathbb{E}(T)}$ is a self-mapping. Furthermore we obtain

$$\|\mathcal{K}(z_1) - \mathcal{K}(z_2)\|_{\mathbb{E}(T)} \le M\varepsilon\|z_1 - z_2\|_{\mathbb{E}(T)} \le \frac{1}{2}\|z_1 - z_2\|_{\mathbb{E}(T)},$$

valid for all $z_1, z_2 \in \frac{\delta}{2}\mathbb{B}_{{}_0\mathbb{E}(T)}$ and all initial values $(u_0, h_0) \in \frac{\delta}{4MC_0(1+C)}\mathbb{B}_{X_\gamma}$. The contraction mapping principle yields a unique fixed point $\tilde{z} \in \frac{\delta}{2}\mathbb{B}_{{}_0\mathbb{E}(T)}$ of $\mathcal{K}(z)$, i.e. $\tilde{z} = \mathcal{K}(\tilde{z})$. Equivalently this means $L\tilde{z} = N(\tilde{z} + z_*) - Lz_*$, hence $\bar{z} := \tilde{z} + z_*$ solves $L\bar{z} = N(\bar{z})$. To show that $\bar{z} = (\bar{u}, \bar{\pi}, \bar{q}, \bar{h})$ is a solution of (2.2), it remains to prove that $F_d(\bar{u}, \bar{h})$ is mean value free. Indeed, let $t \in [0, T]$ be fixed and set $\hat{u}(t, x) := \bar{u}(t, \Theta_{\bar{h}}^{-1}(t, x))$ it follows that $\hat{u} \in H_p^1(\Omega)$ with $(\hat{u}|\nu_{S_1}) = 0$ at $S_1\setminus\partial\Gamma(t)$, $\hat{u} = 0$ at S_2 and

$$\operatorname{div} \hat{u} = (\operatorname{div} \bar{u} - F_d(\bar{u}, \bar{h})) \circ \Theta_{\bar{h}}^{-1}.$$

The divergence theorem and the transformation formula yield

$$\begin{aligned}
0 &= \int_{\Omega\setminus\Gamma(t)} \operatorname{div} \hat{u}\; dx \\
&= \int_{\Omega\setminus\Sigma} \big(\operatorname{div} \bar{u} - F_d(\bar{u}, \bar{h})\big) \det \Theta_{\bar{h}}'\; d\bar{x}
\end{aligned}$$

$$= -\frac{1}{|\Omega|}\int_{\Omega\setminus\Sigma} F_d(\bar u, \bar h)\, d\bar x \int_{\Omega\setminus\Sigma} \det\Theta'_{\bar h}\, d\bar x,$$

where $\bar x := \Theta_{\bar h}^{-1}(x)$. Since $\det\Theta'_{\bar h} > 0$, the claim follows. □

5. Appendix

5.1. **Extension operators.**

Proposition 5.1. *Let $p > 2$. There exists a linear and bounded extension operator* ext *from*

$$ {}_0W_p^{1/2-1/p}(J; L_p(\mathbb{R})) \cap L_p(J; W_p^{1-2/p}(\mathbb{R}))$$

to

$$ {}_0W_p^{1/2-1/2p}(J; L_p(\mathbb{R}\times\mathbb{R}_+)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}\times\mathbb{R}_+))$$

such that $[\operatorname{ext} v]|_{\mathbb{R}\times\{0\}} = v$, *for all* $v \in {}_0W_p^{1/2-1/p}(J; L_p(\mathbb{R})) \cap L_p(J; W_p^{1-2/p}(\mathbb{R}))$.

Moreover, if

$$v = v(t,x,y) \in {}_0W_p^{1/2-1/2p}(J; L_p(\mathbb{R}\times\mathbb{R}_+)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}\times\mathbb{R}_+)) =: X,$$

then

$$\operatorname{tr}_{y=0} v \in {}_0W_p^{1/2-1/p}(J; L_p(\mathbb{R})) \cap L_p(J; W_p^{1-2/p}(\mathbb{R})) =: Y$$

and there exists a constant $C > 0$ such that

$$\|\operatorname{tr}_{y=0} v\|_Y \le C\|v\|_X$$

for all $v \in X$.

Proof. Let $X_0 = L_p(J; L_p(\mathbb{R}))$ and consider the operator $(\partial_t - \partial_x^2)$ in X_0 with domain

$$ {}_0W_p^1(J; L_p(\mathbb{R})) \cap L_p(J; W_p^2(\mathbb{R})).$$

The operator $-A := -(\partial_t - \partial_x^2)^{1/2}$ generates an analytic semigroup $\{e^{-Ay}\}_{y\ge 0}$ in X_0 with domain $D(A) = [X_0, D(A^2)]_{1/2}$. Since

$$D_A(1-2/p, p) = (X_0, D(A))_{1-2/p,p} = (X_0, D(A^2))_{1/2-1/p,p},$$

by [46, Theorem 1.15.2], we obtain

$$D_A(1-2/p,p) = {}_0W_p^{1/2-1/p}(J; L_p(\mathbb{R})) \cap L_p(J; W_p^{1-2/p}(\mathbb{R})).$$

Hence, if $v \in D_A(1-2/p,p)$, then

$$[y \mapsto e^{-Ay}v] \in W_p^{1-1/p}(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; D_A(1-1/p,p))$$

by [14, Theorems 3 & 8], where $D_A(1-1/p,p) = (X_0, D(A^2))_{1/2-1/2p,p}$, hence

$$D_A(1-1/p,p) = {}_0W_p^{1/2-1/2p}(J; L_p(\mathbb{R})) \cap L_p(J; W_p^{1-1/p}(\mathbb{R})).$$

Setting $[\operatorname{ext} v](y) = e^{-Ay}v$ yields the first claim, by the Fubini property of the spaces W_p^s.

For the proof of the second assertion, we consider $v(t,x,y)$ as a function $w(y)(t,x)$, i.e. $w(y)(t,x) := v(t,x,y)$. Then we have

$$w \in W_p^{1-1/p}(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; D_A(1-1/p,p)),$$

where X_0 and A are defined as above. By [22, Lemma 4.1, (4.4)] with $\alpha = 1 - 1/p$ and $\mu = 1$ it holds that $\operatorname{tr}|_{y=0}$ is a continuous mapping from

$$W_p^{1-1/p}(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; D_A(1-1/p,p))$$

to $D_A(1-1/2p,p) = D_{A^2}(1/2-1/p,p) = (X_0, D(A^2))_{1/2-1/p,p}$ with

$$(X_0, D(A^2))_{1/2-1/p,p} = {}_0W_p^{1/2-1/p}(J; L_p(\mathbb{R})) \cap L_p(J; W_p^{1-2/p}(\mathbb{R})).$$

The proof is complete. □

Proposition 5.2. *Let $p > 2$, $J = [0,T]$, $0 < T < \infty$ or $J = \mathbb{R}_+$ and*

$$g \in {}_0W_p^{3/2-1/p}(J; L_p(\mathbb{R})) \cap {}_0H_p^1(J; W_p^{1-2/p}(\mathbb{R})) \cap L_p(J; W_p^{2-2/p}(\mathbb{R})) =: Y.$$

Then there exists

$$h \in {}_0W_p^{2-1/2p}(J; L_p(\mathbb{R}^2_+)) \cap {}_0H_p^1(J; W_p^{2-1/p}(\mathbb{R}^2_+)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^2_+)) =: X,$$

such that $\partial_y h = g$ at $y = 0$.

Moreover, the mapping $(\operatorname{tr}|_{y=0} \circ \partial_y) : X \to Y$ is continuous.

Proof. (1) Consider the operator $(\partial_t - \partial_x^2)$ in $X_0 := L_p(J; L_p(\mathbb{R}))$ with domain

$${}_0W_p^1(J; L_p(\mathbb{R})) \cap L_p(J; W_p^2(\mathbb{R})).$$

Let $A := (\partial_t - \partial_x^2)^{1/2}$ with domain $D(A) = [X_0, D(A^2)]_{1/2}$. Denote by e^{-Ay} the analytic C_0-semigroup, generated by $-A$ in X_0 and set $h(y) := -e^{-Ay}A^{-1}g$. Since

$$g, \partial_t g, A^{-1}g, A^{-1}\partial_t g \in {}_0W_p^{1/2-1/p}(J; L_p(\mathbb{R})) \cap L_p(J; W_p^{1-2/p}(\mathbb{R}))$$

it follows from Proposition 5.1 that

$$h, \partial_t h, Ah, A\partial_t h \in W_p^{1-1/p}(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; D_A(1-1/p,p)).$$

The operator A^{-1} is an isomorphism from $(X_0, D(A^2))_{1/2-1/2p,p}$ to $(X_0, D(A^2))_{1-1/2p,p}$ by [46, Theorem 1.15.2], hence h as well as $\partial_t h$ belong to

$${}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^2_+)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^2_+))$$

by the Fubini property. Furthermore $\partial_t : {}_0W_p^s(J; X) \to {}_0W_p^{s-1}(J; X)$, $s \in [1,2)$ is an isomorphism, hence

$$h \in {}_0W_p^{2-1/2p}(J; L_p(\mathbb{R}^2_+)) \cap {}_0W_p^1(J; W_p^{2-1/p}(\mathbb{R}^2_+)). \tag{5.1}$$

(2) Next, we use the regularity

$$Ag \in {}_0W_p^{1-1/p}(J; L_p(\mathbb{R})) \cap L_p(J; W_p^{1-2/p}(\mathbb{R})),$$

to conclude

$$-\partial_y^2 h = A^2 e^{-Ay} A^{-1} g = e^{-Ay} Ag \in W_p^{1-1/p}(\mathbb{R}_+; X_0) \tag{5.2}$$

by [14, Theorem 8], since

$$Ag \in D_A(1-2/p,p) = {}_0W_p^{1/2-1/p}(J; L_p(\mathbb{R})) \cap L_p(J; W_p^{1-2/p}(\mathbb{R})).$$

In particular, this yields that

$$h \in W_p^{3-1/p}(\mathbb{R}_+; L_p(J; L_p(\mathbb{R}))).$$

(3) It remains to show that

$$h \in L_p(\mathbb{R}_+; L_p(J; W_p^{3-1/p}(\mathbb{R}))).$$

To this end we consider the semigroup $\{e^{-Ay}\}_{y\geq 0}$ in $\tilde{X}_0 := L_p(J; W_p^{1-1/p}(\mathbb{R}))$. The domain of the operator $A^2 := (\partial_t - \partial_x^2)$ in $\tilde{X}_0$ is given by

$${}_0W_p^1(J; W_p^{1-1/p}(\mathbb{R})) \cap L_p(J; W_p^{3-1/p}(\mathbb{R})).$$

Then we have

$$[y \mapsto e^{-Ay}g] \in L_p(\mathbb{R}_+; D(A)),$$

if

$$g \in D_A(1-1/p, p) = {}_0W_p^{1/2-1/2p}(J; W_p^{1-1/p}(\mathbb{R})) \cap L_p(J; W_p^{2-2/p}(\mathbb{R})).$$

Note that the assumption on g implies

$$g \in {}_0H_p^1(J; W_p^{1-2/p}(\mathbb{R})) \cap L_p(J; W_p^{2-2/p}(\mathbb{R})) \hookrightarrow {}_0W_p^{1/2-1/2p}(J; W_p^{1-1/p}(\mathbb{R})),$$

which follows from [22, Proposition 3.2]. Replacing g by $A^{-1}g$ it follows that

$$[y \mapsto e^{-Ay}A^{-1}g] \in L_p(\mathbb{R}_+; D(A^2)),$$

hence

$$[y \mapsto e^{-Ay}A^{-1}g] \in L_p(\mathbb{R}_+; L_p(J; W_p^{3-1/p}(\mathbb{R}))).$$

(4) For the proof of the second assertion, note first that ∂_y maps X continuously to

$${}_0W_p^{3/2-1/2p}(J; L_p(\mathbb{R}_+^2)) \cap {}_0H_p^1(J; W_p^{1-1/p}(\mathbb{R}_+^2)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}_+^2)),$$

since

$${}_0W_p^{2-1/2p}(J; L_p(\mathbb{R}_+^2)) \cap {}_0H_p^1(J; W_p^{2-1/p}(\mathbb{R}_+^2))$$

is continuously embedded into

$${}_0W_p^{3/2-1/2p}(J; H_p^1(\mathbb{R}_+^2)),$$

by [22, Proposition 3.2]. Then the assertion follows from similar arguments as in the proof of Proposition 5.1. □

5.2. **Partition of unity with vanishing Neumann trace.**

Proposition 5.3. *Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial G \in C^{m+1}$. Then for each finite open covering $\{U_k\}_{k=1}^N$ of ∂G in $\mathbb{R}^2$ there exists an open set $U_0 \subset G$ with $U_0 \cap \partial G = \emptyset$, $\bigcup_{k=0}^N U_k \supset \overline{G}$ and a subordinated partition of unity $\{\psi_k\}_{k=0}^N \subset C_c^m(\mathbb{R}^2)$ such that* $\operatorname{supp}\psi_k \subset U_k$ *and $\partial_\nu \psi_k = 0$ at ∂G.*

Proof. Let $\{U_j\}_{j=1}^N$ be a finite open cover of ∂G. Then there exist open sets V_j such that $K_j := \overline{V}_j \subset U_j$ and $\bigcup_{k=1}^N V_j \supset \partial G$. Moreover, there exist functions $\phi_j \in C_c^\infty(U_j)$ with $0 \leq \phi_j \leq 1$ such that $\phi_j|_{K_j} = 1$. It is well-known that for sufficiently small $a > 0$, the mapping $F : \partial G \times (-a, a) \to \mathbb{R}^n$, defined by $F(p, r) := p + r\nu(p)$, is a C^m-diffeomorphism onto its image $U_a := \operatorname{im} F$. The inverse mapping F^{-1} may be decomposed as $F^{-1} = (\Pi, d)$, where $\Pi \in C^m(U_a; \partial G)$ and $d \in C^m(U_a; (-a, a))$. Note that $\Pi(x)$ denotes the nearest point on ∂G to $x \in U_a$ and $d(x)$ stands for the signed distance from $x \in U_a$ to ∂G. It can be shown that

$$U_a = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial G) < a\}.$$

Choose $a > 0$ small enough such that $U_a \subset \bigcup_{j=1}^N K_j$ and define new functions $\bar{\phi}_j(x) := \phi_j(\Pi(x))$ for $x \in U_a$. It follows that $\nabla\bar{\phi}_j(x) = D\Pi^{\mathsf{T}}(x)\nabla\phi_j(\Pi(x))$, hence $\partial_\nu\bar{\phi}_j(x) = (\nabla\phi_j(\Pi(x))|D\Pi(x)\nu(x)) = 0$ for $x \in \partial G$, since $D\Pi(x)\nu(x)$, $x \in \partial G$. Let

$$\tilde{\phi}_j(x) := \begin{cases} \bar{\phi}_j(x)\varphi(d(x)), & x \in U_a, \\ 0, & x \notin U_a, \end{cases}$$

where $\varphi \in C_c^\infty(\mathbb{R};[0,1])$ such that $\varphi(s) = 1$ if $|s| < a/2$ and $\varphi(s) = 0$ if $|s| > 3a/4$. Then we still have $\partial_\nu\tilde{\phi}(x) = 0$ for $x \in \partial G$. Define $\tilde{K}_j := K_j \cap \partial G$. Then there exists some $\delta \in (0, a/2)$ such that $F_j := F(\tilde{K}_j, [-\delta,\delta])$ is compact, $F_j \subset U_j$ and $\bigcup_{j=1}^N F_j \supset \partial G$. It follows that $\phi_j|_{\tilde{K}_j} = 1$ and therefore $\tilde{\phi}_j|_{F_j} = 1$.

Consider the set $\mathcal{G} := G\backslash\bigcup_{j=1}^N F_j$. Then $\mathcal{G}$ is a proper open subset of G. Choose an open set $\mathcal{U}_0 \subset G$ that covers $\mathcal{G}$ and a set $\mathcal{F}_0 \supset \mathcal{G}$ that is compactly contained in $\mathcal{U}_0$. Define $F_0 := \overline{\mathcal{F}_0}$. Then there exists a smooth function $\tilde{\phi}_0 \in C_c^\infty(\mathcal{U}_0;[0,1])$ such that $\tilde{\phi}_0|_{F_0} = 1$. In particular it holds that $\bigcup_{j=0}^N F_j \supset \overline{G}$ and $\sum_{j=0}^N \tilde{\phi}_j(x) > 0$ for $x \in \overline{G}$. Finally, we set $\psi_k := \tilde{\phi}_k/\sum_{j=0}^N \tilde{\phi}_j$, $k = 0,\ldots,N$. Then $\sum_{k=0}^N \psi_k = 1$ and

$$\partial_\nu\psi_k = \frac{\partial_\nu\tilde{\phi}_k}{\sum_{j=0}^N \tilde{\phi}_j} - \frac{\tilde{\phi}_k\sum_{j=0}^N \partial_\nu\tilde{\phi}_j}{\left(\sum_{j=0}^N \tilde{\phi}_j\right)^2} = 0,$$

for $k \in \{0,\ldots,N\}$ at ∂G, since by construction also $\partial_\nu\tilde{\phi}_0 = 0$ at ∂G. The proof is complete. □

It is possible to extend the previous result to cylindrical domains $\Omega := G \times (H_1, H_2)$. To this end let $S_1 := \partial G \times (H_1, H_2)$,

$$S_2 := \bigcup_{j=1}^2 G \times \{H_j\},$$

and $\Sigma := G \times \{0\}$.

Proposition 5.4. *Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial G \in C^{m+1}$ and $\Omega := G \times (H_1, H_2)$, $H_1 < 0 < H_2$. Then for each finite open covering $\{U_k\}_{k=1}^N$ of $\partial S_2 \cup \partial\Sigma$ in $\mathbb{R}^n$ there exist open sets $U_j \subset \mathbb{R}^3$, $j \in \{N+1,\ldots,N+7\}$ such that*

- $U_{N+1} \subset G \times (H_1, 0)$, $U_{N+2} \subset G \times (0, H_2)$,
- $U_{N+3} \cap U_{N+1} \cap S_1 \neq \emptyset$, $U_{N+3} \cap (\Sigma \cup S_2) = \emptyset$,
- $U_{N+4} \cap U_{N+2} \cap S_1 \neq \emptyset$, $U_{N+4} \cap (\Sigma \cup S_2) = \emptyset$,
- $U_{N+5} \cap \Sigma \neq \emptyset$, $U_{N+5} \cap (S_1 \cup S_2) = \emptyset$,
- $U_{N+6} \cap U_{N+1} \cap S_2 \neq \emptyset$, $U_{N+6} \cap (S_1 \cup \Sigma) = \emptyset$,
- $U_{N+7} \cap U_{N+2} \cap S_2 \neq \emptyset$, $U_{N+7} \cap (S_1 \cup \Sigma) = \emptyset$,
- $\bigcup_{j=1}^{N+7} U_j \supset \overline{\Omega}$.

Furthermore, there exists a subordinated partition of unity $\{\phi_k\}_{k=1}^{N+7} \subset C_c^m(\mathbb{R}^3)$ such that $\operatorname{supp}\phi_k \subset U_k$ *and $\partial_{\nu_{\partial G}}\phi_k = \partial_{e_n}\phi_k = 0$ at $\partial S_2 \cup \partial\Sigma$.*

Proof. The idea of the proof is quite simple. Let $\{U_j\}_{j=1}^{N_1}$ be an open covering of $\partial\Sigma$ in $\mathbb{R}^n$ and define $\tilde{U}_j := U_j \cap \{\mathbb{R}^{n-1} \times \{0\}\}$. Let $V_j := \tilde{U}_j$, $j \in \{1,\ldots,N_1\}$, where we identify V_j with a set in $\mathbb{R}^{n-1}$. Then, of course, $\{V_j\}_{j=1}^{N_1}$ is an open covering of $\partial\Sigma$ in $\mathbb{R}^{n-1}$. Now we are in a position to apply Proposition 5.3 to find an open set $V_0 \subset \Sigma$ such that $\bigcup_{j=0}^{N_1} V_j \supset \overline{\Sigma}$. Furthermore, by Proposition 5.3, there exists a subordinated partition of unity $\{\psi_j^\Sigma\}_{j=0}^{N_1} \subset C_c^m(\mathbb{R}^{n-1})$ with $\operatorname{supp}\psi_j^\Sigma \subset V_j$ and $\partial_{\nu_{\partial G}}\psi_j^\Sigma = 0$ at $\partial\Sigma$.

Now we define $\phi_j^\Sigma(x', x_n) := \psi_j^\Sigma(x')\varphi(x_n)$, where $\varphi \in C_c^\infty(\mathbb{R}; [0, 1])$ such that $\varphi(s) = 1$ if $|s| < \delta$ and $\varphi(s) = 0$ if $|s| > 2\delta$, where $\delta > 0$ is sufficiently small. It follows that $\phi_j^\Sigma \in C_c^m(\mathbb{R}^n)$ and, if $\delta > 0$ is sufficiently small, then $\operatorname{supp}\phi_j^\Sigma \subset U_j$ for $j \in \{1, \ldots, N_1\}$. Furthermore we still have $\partial_{\nu_{\partial G}}\phi_k^\Sigma = 0$ and, in addition, $\partial_{e_n}\phi_j^\Sigma = 0$ at $\partial\Sigma$, since φ is constant in a neighborhood of $s = 0$.

The same procedure can be applied for the charts covering ∂S_2. The remaining set which is a proper subset of $\Omega\backslash(\overline{S_2} \cup \overline{\Sigma})$ can be covered by finitely many open charts. □

5.3. **Auxiliary elliptic and parabolic problems.**

5.3.1. *Elliptic problems.* The following result deals with the two-phase elliptic problem

$$\begin{aligned} \lambda u - \Delta u &= f && \text{in } \Omega\backslash\Sigma,\\ [\![\rho u]\!] &= g_1 && \text{on } \Sigma,\\ [\![\partial_{\nu_\Sigma} u]\!] &= g_2 && \text{on } \Sigma,\\ \partial_{\nu_{S_1}} u &= h_1 && \text{on } S_1\backslash\partial\Sigma,\\ \partial_{\nu_{S_2}} u &= h_2 && \text{on } S_2, \end{aligned} \tag{5.3}$$

where Ω and Σ satisfy one of the following conditions.
(a) Ω is either a full space, a (bent) half space or a (bent) quarter space and $\Sigma = \emptyset$,
(b) Ω is either a full space or a (bent) half space with outer unit normal $-e_{n-1}$ at $x = 0$ and $\Sigma = \{\mathbb{R}^{n-1} \times \{0\}\} \cap \Omega$,
(c) $\Omega = G \times (H_1, H_2)$, $H_1 < 0 < H_2$, is a cylindrical domain where G is a bounded domain with boundary $\partial G \in C^4$ and $\Sigma = G \times \{0\}$.
The sets S_1 and S_2 are the corresponding vertical and horizontal parts of the boundary of Ω, respectively.

Lemma 5.5. *Let $n = 2, 3$, $p \geq 2$ and assume that Ω and Σ are subject to one of the conditions in (a)-(c) above. Then there exists $\lambda_0 \geq 0$ such that for each $\lambda \geq \lambda_0$, problem* (5.3) *has a unique solution $u \in W_p^2(\Omega\backslash\Sigma)$ if and only if the data satisfy the following regularity and compatibility conditions.*

(1) $f \in L_p(\Omega)$,
(2) $g_1 \in W_p^{2-1/p}(\Sigma)$,
(3) $g_2 \in W_p^{1-1/p}(\Sigma)$,
(4) $h_1 \in W_p^{1-1/p}(S_1\backslash\partial\Sigma)$,
(5) $h_2 \in W_p^{1-1/p}(S_2)$,
(6) $[\![\rho h_1]\!] = \partial_{\nu_{\partial G}} g_1$ *on* $\partial\Sigma$.

Proof. For convenience we restrict ourselves to the case $n = 3$. The arguments for the case $n = 2$ are similar and even simpler.

(a) If Ω and Σ are subject to the first two conditions in (a), i.e. Ω is a full space or a half space, then the result is folklore. So let us consider the case where $\Sigma = \emptyset$ and Ω is a quarter space. To be precise, let $\Omega := \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ with $S_1 := \mathbb{R} \times \{0\} \times \mathbb{R}_+$ and $S_2 := \mathbb{R} \times \mathbb{R}_+ \times \{0\}$. Therefore we have to study the problem

$$
\begin{aligned}
\lambda u - \Delta u &= f, \quad x \in \Omega, \\
\partial_2 u &= h_1, \quad x \in S_1, \\
\partial_3 u &= h_2, \quad x \in S_2.
\end{aligned} \tag{5.4}
$$

Extend f and h_2 with respect to x_2 (by even reflection) to some functions $f \in L_p(\mathbb{R}^2 \times \mathbb{R}_+)$ and $\tilde{h}_2 \in W_p^{1-1/p}(\mathbb{R}^2)$ and solve the half space problem

$$
\begin{aligned}
\lambda \tilde{u} - \Delta \tilde{u} &= \tilde{f}, \quad x \in \mathbb{R}^2 \times \mathbb{R}_+, \\
\partial_3 \tilde{u} &= \tilde{h}_2, \quad x \in \mathbb{R}^2 \times \{0\},
\end{aligned}
$$

to obtain a unique solution $\tilde{u} \in W_p^2(\mathbb{R}^2 \times \mathbb{R}_+)$ for each $\lambda > 0$. Note that by symmetry, the function $[x \mapsto \tilde{u}(x_1, -x_2, x_3)]$ is a solution of this problem too. Therefore, by uniqueness, it holds that $\partial_2 \tilde{u}|_{S_1} = 0$.

In a next step, we extend h_1 by even reflection and with respect to the x_3 variable to some $\tilde{h}_1 \in W_p^{1-1/p}(\mathbb{R}^2)$ and solve the half space problem

$$
\begin{aligned}
\lambda \tilde{v} - \Delta \tilde{v} &= 0, \quad x \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}, \\
\partial_2 \tilde{v} &= \tilde{h}_2, \quad x \in \mathbb{R} \times \{0\} \times \mathbb{R},
\end{aligned}
$$

to obtain a unique solution $\tilde{v} \in W_p^2(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ for each $\lambda > 0$. As above, by symmetry and uniqueness, it holds that $\partial_3 \tilde{v}|_{S_2} = 0$. Therefore it follows that $u := (\tilde{u} + \tilde{v})|_\Omega$ is the unique solution of (5.4).

Finally, let Ω be a bent quarter space with S_2 as above and

$$
S_{1,\theta} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = \theta(x_1)\},
$$

where $\theta \in BC^3(\mathbb{R})$ with $\|\theta\|_\infty + \|\theta'\|_\infty \le \eta$ and $\eta > 0$ can be made as small as we wish. Then the corresponding result follows from change of coordinates (set $\bar{x}_2 := x_2 - \theta(x_1)$) and perturbation theory for elliptic problems. We will give a detailed proof for the case of a two-phase half space in part (b) below. The technique carries over to this case. Indeed, things are easier in (a) as there are no compatibility conditions, since $\Sigma = \emptyset$.

(b) Let $\Omega = \mathbb{R}^3$ and $\Sigma = \mathbb{R}^2 \times \{0\}$. Then we have to solve the problem

$$
\begin{aligned}
\lambda u - \Delta u &= f, \quad x \in \Omega \backslash \Sigma, \\
[\![\rho u]\!] &= g_1, \quad x \in \Sigma, \\
[\![\partial_3 u]\!] &= g_2, \quad x \in \Sigma,
\end{aligned} \tag{5.5}
$$

where $\rho = \rho_1 \chi_{x_3<0} + \rho_2 \chi_{x_3>0}$ and $\rho_j > 0$. Since $f \in L_p(\mathbb{R}^3)$ we may first solve the full space problem

$$
\lambda \tilde{u} - \Delta \tilde{u} = f, \quad x \in \mathbb{R}^n;
$$

to obtain a unique solution $\tilde{u} \in W_p^2(\mathbb{R}^n)$ for each $\lambda > 0$. Consider now the problem

$$\begin{aligned} \lambda\bar{u} - \Delta\bar{u} &= 0, \quad x \in \Omega\backslash\Sigma, \\ [\![\rho\bar{u}]\!] &= g_1 - [\![\rho\tilde{u}]\!] =: \bar{g}_1, \quad x \in \Sigma, \\ [\![\partial_3\bar{u}]\!] &= g_2, \quad x \in \Sigma. \end{aligned} \tag{5.6}$$

By semigroup theory, it is easy to see that the unique solution of (5.6) is explicitly given by

$$\bar{u}(x_3) := \frac{1}{\rho_1 + \rho_2} \begin{cases} e^{-Lx_3}a_+, & x_3 \geq 0, \\ e^{-L(-x_3)}a_-, & x_3 < 0, \end{cases}$$

where $L := (\lambda - \Delta_{x'})^{1/2}$ and

$$a_+ := \bar{g}_1 + \rho_2 L^{-1} g_2 - (\rho_1 + \rho_2)L^{-1}g_2,\ a_- = -(\bar{g}_1 + \rho_2 L^{-1})g_2.$$

Therefore the function $u := \tilde{u} + \bar{u}$ is the unique solution of (5.5) which exists for each $\lambda > 0$.

Let now $\Omega = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ and $\Sigma = \{\mathbb{R}^2 \times \{0\}\} \cap \Omega$, i.e. we consider the case of a two-phase half space. Now we have to solve the problem

$$\begin{aligned} \lambda u - \Delta u &= f \quad \text{in } \Omega\backslash\Sigma, \\ [\![\rho u]\!] &= g_1 \quad \text{on } \Sigma, \\ [\![\partial_3 u]\!] &= g_2 \quad \text{on } \Sigma, \\ \partial_2 u &= h_1 \quad \text{on } S_1\backslash\partial\Sigma, \end{aligned} \tag{5.7}$$

where $S_1 := \mathbb{R}\times\{0\}\times\mathbb{R}$. We will first reduce (5.7) to the case $h_1 = 0$. To this end we first extend $h_1^+ := h_1|_{x_3>0}$ with respect to the x_3 variable to some $\tilde{h}_1^+ \in W_p^{1-1/p}(\mathbb{R}^2)$ and solve the half space problem

$$\lambda u^+ - \Delta u^+ = 0,\ x_2 > 0,\ \partial_2 u^+ = \tilde{h}_1^+,\ x_2 = 0,$$

to obtain a unique solution $u^+ \in W_p^2(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$. Then we repeat the same procedure for $h_1^- := h_1|_{x_3<0}$ to obtain a unique solution $u^- \in W_p^2(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$. Define the function

$$\bar{u} := \begin{cases} u^+, & x_3 \geq 0, \\ u^-, & x_3 < 0, \end{cases}$$

and consider the problem

$$\begin{aligned} \lambda\tilde{u} - \Delta\tilde{u} &= \bar{f} \quad \text{in } \Omega\backslash\Sigma, \\ [\![\rho\tilde{u}]\!] &= \bar{g}_1 \quad \text{on } \Sigma, \\ [\![\partial_3\tilde{u}]\!] &= \bar{g}_2 \quad \text{on } \Sigma, \\ \partial_2\tilde{u} &= 0 \quad \text{on } S_1\backslash\partial\Sigma, \end{aligned} \tag{5.8}$$

where $\bar{f} := f$, $\bar{g}_1 := g_1 - [\![\rho\bar{u}]\!]$ and $\bar{g}_2 := g_2 - [\![\partial_3\bar{u}]\!]$. Note that by the compatibility condition on g_1 and h_1 at $\partial\Sigma$ it holds that $\partial_2\bar{g}_1 = 0$ at $\partial\Sigma$. Therefore it is possible to extend $\bar{f}, \bar{g}_j$ by even reflection in x_2 to some functions $\hat{f} \in L_p(\mathbb{R}^3)$, $\hat{g}_1 \in W_p^{2-1/p}(\mathbb{R}^2)$ and $\hat{g}_2 \in W_p^{1-1/p}(\mathbb{R}^2)$. Solve (5.5) with (f, g_j) replaced by $(\hat{f}, \hat{g}_j)$ to obtain a unique solution $\hat{u} \in W_p^2(\mathbb{R}^2 \times \dot{\mathbb{R}})$. Since the function $[x \mapsto \hat{u}(x_1, -x_2, x_3)]$ is a solution of this problem too, it follows by uniqueness that $\partial_2\hat{u} = 0$ at $S_1\backslash\partial\Sigma$, hence $\tilde{u} := \hat{u}|_\Omega$

is the unique solution of (5.8). Finally, $u := \tilde{u} + \bar{u}$ solves (5.7) for each $\lambda > 0$ and this solution is unique.

Consider now the case of a bent two-phase half space with outer unit normal $-e_2$ at $x = 0$. To be precise, let

$$\Omega_\theta := \{x \in \mathbb{R}^3 : x_2 > \theta(x_1)\},$$

where $\theta \in BC^3(\mathbb{R})$, with $\theta(0) = \theta'(0) = 0$ and $\|\theta'\|_\infty + \|\theta\|_\infty \le \eta$, where $\eta > 0$ can be made as small as we wish. Furthermore, let $S_{1,\theta} := \partial\Omega_\theta$ and $\Sigma_\theta := \{\mathbb{R}^2 \times \{0\}\} \cap \Omega_\theta$. We have to investigate the following problem.

$$\begin{aligned} \lambda u - \Delta u &= f && \text{in } \Omega_\theta \backslash \Sigma_\theta, \\ [\![\rho u]\!] &= g_1 && \text{on } \Sigma_\theta, \\ [\![\partial_3 u]\!] &= g_2 && \text{on } \Sigma_\theta, \\ \partial_{\nu_{\partial\Sigma_\theta}} u &= h_1 && \text{on } S_{1,\theta} \backslash \partial\Sigma_\theta. \end{aligned} \tag{5.9}$$

First of all we extend f , g_1 and g_2 to some functions $\tilde{f} \in L_p(\mathbb{R}^3)$, $\tilde{g}_1 \in W_p^{2-1/p}(\mathbb{R}^2)$ and $\tilde{g}_2 \in W_p^{1-1/p}(\mathbb{R}^2)$, respectively. Then we solve (5.5) with (f, g_1, g_2) replaced by $(\tilde{f}, \tilde{g}_1, \tilde{g}_2)$ to obtain a unique solution $\tilde{u} \in W_p^2(\mathbb{R}^2 \times \dot{\mathbb{R}})$. Let $\bar{h}_1 := h_1 - \partial_{\nu_{\partial\Sigma_\theta}} \tilde{u}|_{S_{1,\theta}}$ and note that $[\![\rho \bar{h}_1]\!] = 0$ at $\partial\Sigma_\theta$ by the compatibility condition on (g_1, h_1) at $\partial\Sigma_\theta$. We arrive at the problem

$$\begin{aligned} \lambda \bar{u} - \Delta \bar{u} &= 0 && \text{in } \Omega_\theta \backslash \Sigma_\theta, \\ [\![\rho \bar{u}]\!] &= 0 && \text{on } \Sigma_\theta, \\ [\![\partial_3 \bar{u}]\!] &= 0 && \text{on } \Sigma_\theta, \\ \partial_{\nu_{\partial\Sigma_\theta}} \bar{u} &= \bar{h}_1 && \text{on } S_{1,\theta} \backslash \partial\Sigma_\theta. \end{aligned} \tag{5.10}$$

Transforming Ω_θ, $S_{1,\theta}$ and Σ_θ to $\Omega = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$, $S_1 = \mathbb{R} \times \{0\} \times \mathbb{R}$ and $\Sigma = \{\mathbb{R}^2 \times \{0\}\} \cap \Omega$ via the diffeomorphism

$$\Omega \ni (\bar{x}_1, \bar{x}_2, \bar{x}_3) \mapsto (\bar{x}_1, \bar{x}_2 + \theta(\bar{x}_1), \bar{x}_3) \in \Omega_\theta$$

yields the transformed problem

$$\begin{aligned} \lambda \hat{u} - \Delta \hat{u} &= M_1(\theta, \hat{u}) && \text{in } \Omega \backslash \Sigma, \\ [\![\rho \hat{u}]\!] &= 0 && \text{on } \Sigma, \\ [\![\partial_3 \hat{u}]\!] &= 0 && \text{on } \Sigma, \\ \partial_2 \hat{u} &= M_2(\theta, \hat{u}) - \sqrt{1 + \theta'^2}\, \hat{h}_1 && \text{on } S_1 \backslash \partial\Sigma, \end{aligned} \tag{5.11}$$

where $\hat{u}(\bar{x}) := \bar{u}(\bar{x}_1, \bar{x}_2 + \theta(\bar{x}_1), \bar{x}_3)$, $\hat{h}_1(\bar{x}_1, \bar{x}_3) := \bar{h}(\bar{x}_1, \theta(\bar{x}_1), \bar{x}_3)$,

$$M_1(\theta, \hat{u}) := -2\theta'(\bar{x}_1)\partial_1\partial_2\hat{u} - \theta''(\bar{x}_1)\partial_2\hat{u} + \theta'(\bar{x}_1)^2\partial_2^2\hat{u},$$

and

$$M_2(\theta, \hat{u}) := \theta'(\bar{x}_1)\partial_1\hat{u}|_{S_1 \backslash \partial\Sigma} - \theta'(\bar{x}_1)^2\partial_2\hat{u}|_{S_1 \backslash \partial\Sigma}.$$

Observe that $[\![\rho \hat{h}_1]\!] = 0$ at $\partial\Sigma$.

Define the function spaces

$$\mathbb{E} := \{\hat{u} \in W_p^2(\Omega \backslash \Sigma) : [\![\rho \hat{u}]\!] = [\![\partial_3 \hat{u}]\!] = 0 \text{ on } \Sigma\},$$

equipped with the equivalent norm $\|\hat{u}\|_{\mathbb{E},\lambda} := \lambda\|\hat{u}\|_{L_p} + \|\hat{u}\|_{W_p^2}$, $\lambda > 0$ and

$$\mathbb{F} := \{(f_1, f_2) \in L_p(\Omega) \times W_p^{1-1/p}(S_1 \backslash \partial\Sigma) : [\![\rho f_2]\!] = 0 \text{ at } \partial\Sigma\}.$$

Furthermore, let a linear operator $L : \mathbb{E} \to \mathbb{F}$ be defined by

$$L\hat{u} := \begin{pmatrix} \lambda\hat{u} - \Delta\hat{u} \\ \partial_2\hat{u}|_{S_1\backslash\partial\Sigma} \end{pmatrix}.$$

It follows from our previous arguments that $L : \mathbb{E} \to \mathbb{F}$ is an isomorphism, provided $\lambda > 0$. Furthermore, by the same strategy as in [21, Section 3.1.1], there exists $\lambda_0 > 0$ and a constant $C > 0$ such that for all $\lambda \geq \lambda_0$ and $(f_1, f_2) \in \mathbb{F}$ the estimate

$$\|L^{-1}(f_1, f_2)\|_{\mathbb{E},\lambda} \leq C\left(\|f_1\|_{L_p(\Omega)} + |\lambda|^{1/2}\|\tilde{f}_2\|_{L_p(\Omega)} + \|\nabla\tilde{f}_2\|_{L_p(\Omega)}\right), \tag{5.12}$$

is valid, where $\tilde{f}_2$ is an extension of f_2 to $W_p^1(\Omega\backslash\Sigma)$.

Let now $F := (0, -\sqrt{1+\theta'^2}\hat{h}_1)$ and $M(\theta, \hat{u}) := (M_1, M_2)(\theta, \hat{u})$. Clearly, for each $\hat{u} \in \mathbb{E}$, it holds that $M(\theta, \hat{u}) \in \mathbb{F}$, since

$$[\![\rho\theta'(\bar{x}_1)\partial_1\hat{u}]\!] = \theta'(\bar{x}_1)\partial_1[\![\rho\hat{u}]\!] = 0$$

at $\partial\Sigma$. Furthermore it holds that

$$[\![\rho\sqrt{1+\theta'^2}\hat{h}_1]\!] = \sqrt{1+\theta'^2}[\![\rho\hat{h}_1]\!] = 0$$

at $\partial\Sigma$ as well, hence $F \in \mathbb{F}$. Therefore, for $\hat{u} \in \mathbb{E}$, the expressions $L^{-1}M(\theta, \hat{u})$, $L^{-1}F$ are well defined in $\mathbb{E}$ and we may rewrite (5.11) in the shorter form

$$\hat{u} = L^{-1}M(\theta, \hat{u}) + L^{-1}F. \tag{5.13}$$

We will now apply (5.12) to the term $L^{-1}M(\theta, \hat{u})$. To this end, note that

$$\tilde{M}_2(\theta, \hat{u}) := \theta'(\bar{x}_1)\partial_1\hat{u} - \theta'(\bar{x}_1)^2\partial_2\hat{u}$$

is a proper extension of $M_2(\theta, \hat{u})$ to $W_p^1(\Omega\backslash\Sigma)$. By (5.12), this yields the estimate

$$\begin{aligned} &\|L^{-1}M(\theta, \hat{u})\|_{\mathbb{E},\lambda} \leq \\ &\qquad \leq C\left(\|\theta'\|_{L_\infty(\Omega)}\|\hat{u}\|_{W_p^2(\Omega)} + [\|\theta''\|_{L_\infty(\Omega)} + \lambda^{1/2}\|\theta'\|_{L_\infty(\Omega)}]\|\hat{u}\|_{W_p^1(\Omega)}\right). \end{aligned}$$

Clearly, $\|\hat{u}\|_{W_p^2(\Omega)} \leq \|\hat{u}\|_{\mathbb{E},\lambda}$ and by complex interpolation we obtain furthermore

$$\|\hat{u}\|_{W_p^1(\Omega)} \leq C\|\hat{u}\|_{L_p(\Omega)}^{1/2}\|\hat{u}\|_{W_p^2(\Omega)}^{1/2} \leq \frac{1}{\lambda^{1/2}}\|\hat{u}\|_{\mathbb{E},\lambda}.$$

Choosing first $\|\theta'\|_\infty$ sufficiently small and then $\lambda > 0$ sufficiently large, it follows that for each $\varepsilon > 0$ there exist numbers $\eta_0 > 0$ and $\lambda_1 > 0$ such that $\|L^{-1}M(\theta, \hat{u})\|_{\mathbb{E},\lambda} \leq \varepsilon\|\hat{u}\|_{\mathbb{E},\lambda}$, whenever $\|\theta'\|_\infty \leq \eta \in (0, \eta_0)$ and $\lambda \geq \lambda_1$. Therefore, a Neumann series argument yields a unique solution of (5.13).

(c) The proof for this assertion uses the technique of localization. By Proposition 5.4 there exists a finite covering of $\overline{\Omega}$ and a subordinated partition of unity $\{\phi_k\}_{k=1}^N$ such that $\partial_{\nu_{\partial G}}\phi_k = 0$ at $(\partial\Sigma \cup \partial S_2) \cap \operatorname{supp}\phi_k$.

Multiplying each equation in (5.3) by ϕ_k, we obtain problems in local coordinates, which correspond to perturbed versions of one of the problems which have been treated in (a) & (b). Assume that u is a solution of (5.3), $u_k := u\phi_k$, $g_1^k := g_1\phi_k$ and $h_1^k := h_1\phi_k$, then $[\![\rho u_k]\!] = g_1^k$ and

$$\partial_{\nu_{S_1}} u_k = \phi_k \partial_{\nu_{S_1}} u + u\partial_{\nu_{S_1}} \phi_k = \phi_k h_1 = h_1^k,$$

since $\nu_{S_1} = (\nu_{\partial G}, 0)^\mathsf{T}$. In particular, the commutator term in the Neumann boundary condition is identically zero. By the same reason, one has

$$\partial_{\nu_{S_1}} g_{1,k} = \phi_k [\![\rho h_1]\!] = [\![\rho h_1^k]\!],$$

hence the local data (g_1^k, h_1^k) satisfy the compatibility condition at $\partial\Sigma \cap \operatorname{supp} \phi_k$.

The remaining localization procedure follows along standard arguments. We refrain from giving the details and refer the reader e.g. to [12]. $\square$

We shall also prove some results on the solvability of (5.3) in case $\lambda = 0$. If $\lambda = 0$ and Ω is unbounded, one cannot expect to obtain $u \in L_p(\Omega)$. Instead, we are looking for solutions $u \in \dot{W}_p^1(\Omega\backslash\Sigma) \cap \dot{W}_p^2(\Omega\backslash\Sigma)$, or equivalently $\nabla u \in W_p^1(\Omega\backslash\Sigma)$.

If $\nabla u \in W_p^1(\Omega\backslash\Sigma)$ is a solution of (5.3) with $g_1 = 0$, then, by trace theory, $f \in L_p(\Omega)$, $g_2 \in W_p^{1-1/p}(\Sigma)$, $h_1 \in W_p^{1-1/p}(S_1\backslash\partial\Sigma)$ and $h_2 \in W_p^{1-1/p}(S_2)$. There is some hidden compatibility/regularity condition for the data (f, g_2, h_1). To see this, let $\phi \in C_c^\infty(\overline{\Omega})$. We multiply $(5.3)_1$ by ϕ and integrate by parts, to obtain the identity

$$\begin{aligned}\langle (f, g_2, h_1, h_2), \phi\rangle := \int_\Omega f\phi\, dx + \int_{S_1} h_1\phi\, dS_1 + \int_{S_2} h_2\phi\, dS_2 - \int_\Sigma g_2\phi\, d\Sigma = \\ = \int_\Omega \nabla u \cdot \nabla\phi\, dx.\end{aligned}$$

It follows that the linear mapping $[\phi \mapsto \langle (f, g_2, h_1, h_2), \phi\rangle]$ is continuous on $C_c^\infty(\overline{\Omega})$ with respect to the norm $\|\nabla \cdot\|_{L_{p'}(\Omega)}$.

If Ω is a full space, a (bent) half space or a (bent) quarter space, then it is well known, that $C_c^\infty(\overline{\Omega})$ (hence also $W_{p'}^1(\Omega)$) is dense in $\dot{W}_{p'}^1(\Omega)$ with respect to the norm $\|\nabla \cdot\|_{L_{p'}(\Omega)}$. Therefore, since each functional in

$$\hat{W}_p^{-1}(\Omega) := \left(\dot{W}_{p'}^1(\Omega)\right)^*,$$

is uniquely determined by its restriction to $C_c^\infty(\overline{\Omega})$, it follows that (f, g_2, h_1, h_2) yields a well defined element of $\hat{W}_p^{-1}(\Omega)$ with norm given by

$$\begin{aligned}\|(f, g_2, h_1, h_2)\|_{\hat{W}_p^{-1}} &:= \sup\{\langle (f, g_2, h_1, h_2), \phi\rangle / \|\nabla\phi\|_{L_{p'}} : \phi \in C_c^\infty(\overline{\Omega})\} \\ &= \sup\{\langle (f, g_2, h_1, h_2), \phi\rangle / \|\nabla\phi\|_{L_{p'}} : \phi \in W_{p'}^1(\Omega)\}.\end{aligned}$$

Note that if Ω is bounded, then the above representation formula for (f, g_2, h_1, h_2) holds for each $\phi \in \dot{W}_{p'}^1(\Omega)$, since $\dot{W}_q^1(\Omega) \subset W_q^1(\Omega)$ if Ω is bounded. This follows for example from the Poincaré-Wirtinger inequality. However, if Ω is unbounded, then the above representation for (f, g_2, h_1, h_2) holds at least on the dense subspace $C_c^\infty(\overline{\Omega})$.

Furthermore, if $S_j = \emptyset$, $j \in \{1, 2\}$ and/or $\Sigma = \emptyset$, then we simply neglect h_j and/or g_2 in (f, g_2, h_1, h_2).

We are now in a position to state the next auxiliary lemma concerning the solvability of (5.3) with $\lambda = 0$.

Lemma 5.6. *Let $n = 2, 3$, $p \geq 2$ and $\lambda = 0$. Then the following assertions are valid.*

(1) *If Ω and Σ satisfy one of the conditions in $(a), (b)$ above, then there exists a unique solution $\nabla u \in W_p^1(\Omega\backslash\Sigma)$ of (5.3) with $g_1 = 0$ if and only if $f \in L_p(\Omega)$, $g_2 \in W_p^{1-1/p}(\Sigma)$, $h_1 \in W_p^{1-1/p}(S_1\backslash\partial\Sigma)$, $h_2 \in W_p^{1-1/p}(S_2)$, $[\![\rho h_1]\!] = 0$ on $\partial\Sigma$ and $(f, g_2, h_1, h_2) \in \hat{W}_p^{-1}(\Omega)$.*

(2) *If Ω and Σ are subject to the condition (c) above, then there exists a unique solution $u \in W_p^2(\Omega\backslash\Sigma)$ of (5.3) with $g_1 = h_1 = h_2 = 0$ if and only if*

$$f \in L_p^{(0)}(\Omega) := \{f \in L_p(\Omega) : \int_\Omega f dx = 0\}.$$

Proof. 1. (a) If $\Omega = \mathbb{R}^n$, then we have to solve $-\Delta u = f$ for f in $\hat{W}_p^{-1}(\Omega) \cap L_p(\Omega)$. It is a folkloristic result that whenever $f \in L_p(\mathbb{R}^n)$, then there is a unique solution $u \in \dot{W}_p^2(\mathbb{R}^n)$ of the equation $-\Delta u = f$. Multiplying $-\Delta u = f$ by $\phi \in C_c^\infty(\mathbb{R}^n)$ and integrating by parts, we obtain

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla\phi \; dx = -\int_{\mathbb{R}^n} \Delta u \phi \; dx = \int_{\mathbb{R}^n} f\phi \; dx.$$

Let us show that there exists a constant $C > 0$ such that the estimate

$$\|\nabla u\|_{L_p(\mathbb{R}^n)} \leq C \sup\left\{\frac{|\int_{\mathbb{R}^n} \nabla u \cdot \nabla\phi \; dx|}{\|\nabla\phi\|_{L_{p'}(\mathbb{R}^n)}} : \phi \in C_c^\infty(\mathbb{R}^n)\right\} \tag{5.14}$$

is valid. Indeed, it holds that

$$\begin{aligned} \sup\left\{\frac{|\int_{\mathbb{R}^n} \nabla u \cdot \nabla\phi \; dx|}{\|\nabla\phi\|_{L_{p'}(\mathbb{R}^n)}} : \phi \in C_c^\infty(\mathbb{R}^n)\right\} &\geq \frac{|\int_{\mathbb{R}^n} \nabla u \cdot \nabla\partial_j\varphi \; dx|}{\|\nabla\partial_j\varphi\|_{L_{p'}(\mathbb{R}^n)}} \\ &\geq \frac{1}{C}\frac{|\int_{\mathbb{R}^n} \partial_j u \cdot \Delta\varphi \; dx|}{\|\Delta\varphi\|_{L_{p'}(\mathbb{R}^n)}}, \end{aligned} \tag{5.15}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$, where we integrated by parts and applied the Caldéron-Zygmund inequality $\|\nabla^2\varphi\|_{L_{p'}(\mathbb{R}^n)} \leq C\|\Delta\varphi\|_{L_{p'}(\mathbb{R}^n)}$.

It is well-known that $\Delta C_c^\infty(\mathbb{R}^n)$ is dense in $L_{p'}(\mathbb{R}^n)$ with respect to the $L_{p'}$-norm. Taking the supremum on the right hand side of (5.15) over all functions $\varphi \in C_c^\infty(\mathbb{R}^n)$ we obtain the desired inequality (5.14). Evidently, for the solution $u \in \dot{W}_p^2(\mathbb{R}^n)$ of $-\Delta u = f$ it follows that

$$\|\nabla u\|_{L_p(\mathbb{R}^n)} \leq C \sup\left\{\frac{|\int_{\mathbb{R}^n} f\phi \; dx|}{\|\nabla\phi\|_{L_{p'}(\mathbb{R}^n)}} : \phi \in C_c^\infty(\mathbb{R}^n)\right\}.$$

hence, if $f \in L_p(\mathbb{R}^n) \cap \hat{W}_p^{-1}(\mathbb{R}^n)$, then

$$\|f\|_{\hat{W}_p^{-1}} = \sup\left\{\frac{|\int_{\mathbb{R}^n} f\phi \; dx|}{\|\nabla\phi\|_{L_{p'}(\mathbb{R}^n)}} : \phi \in C_c^\infty(\mathbb{R}^n)\right\} < \infty,$$

and we obtain the estimate $\|\nabla u\|_{L_p(\mathbb{R}^n)} \leq C\|f\|_{\hat{W}_p^{-1}}$. This shows that $u \in \dot{W}_p^1(\mathbb{R}^n) \cap \dot{W}_p^2(\mathbb{R}^n)$ is the unique solution.

Let $\Omega = \mathbb{R}^2 \times \mathbb{R}_+$ be a half space and consider the problem

$$\begin{aligned} -\Delta u &= f, \quad x \in \Omega, \\ \partial_3 u &= h, \quad x \in S, \end{aligned} \tag{5.16}$$

where $S := \partial\Omega = \mathbb{R}^2 \times \{0\}$. By Lemma 5.5 there exists some $\lambda_0 > 0$ such that the shifted problem

$$\begin{aligned} \lambda_0 \bar{u} - \Delta \bar{u} &= f, \quad x \in \Omega, \\ \partial_3 \bar{u} &= h, \quad x \in S, \end{aligned} \tag{5.17}$$

admits a unique solution $\bar{u} \in W_p^2(\Omega)$ satisfying the estimates

$$\|\bar{u}\|_{W_p^2(\Omega)} \leq C(\|f\|_{L_p(\Omega)} + \|h\|_{W_p^{1-1/p}(S)}),$$

and

$$\|\bar{u}\|_{W_p^1(\Omega)} \leq C\|(f,h)\|_{\hat{W}_p^{-1}(\Omega)}.$$

To see the validity of the second estimate we use the notation from [1, Chapter V] and let $A_0 := \lambda_0 - \Delta$ with domain

$$E_1 := D(A_0) = \{u \in W_p^2(\Omega) : \partial_3 u = 0 \text{ on } S\}$$

in $E_0 := L_p(\Omega)$. Then A_0 is a linear isomorphism from E_1 to E_0. Let $E_{1/2} := [E_0, E_1]_{1/2} = W_p^1(\Omega)$ and $E_{-1/2} := (E_{1/2}^\sharp)^* = (W_{p'}^1(\Omega))^*$, since $A_0^\sharp = (\lambda_0 - \Delta)|_{L_{p'}(\Omega)}$). Denote by $A_{-1/2}$ the $E_{-1/2}$-realization of A_0. By the results in [1, Chapter V] it follows that $A_{-1/2} : E_{1/2} \to E_{-1/2}$ is a linear isomorphism. Moreover, since E_1 is dense in $E_{1/2}$, it holds that

$$\langle A_{-1/2} u, \phi \rangle = \lambda_0 \int_\Omega u\phi \, dx + \int_\Omega \nabla u \cdot \nabla \phi \, dx$$

for all $\phi \in W_{p'}^1(\Omega)$ and each $u \in W_p^1(\Omega)$.

Multiply the first equation in (5.17) by $\phi \in W_{p'}^1(\Omega)$ and integrate by parts to the result

$$\lambda_0 \int_\Omega \bar{u}\phi \, dx + \int_\Omega \nabla \bar{u} \cdot \nabla \phi \, dx = \int_\Omega f\phi \, dx - \int_S h\phi|_S \, dS.$$

By assumption, the right side of the last equation determines a functional (f,h) on $\dot{W}_{p'}^1(\Omega)$, hence also on $W_{p'}^1(\Omega)$. Therefore it follows from the considerations above that

$$\begin{aligned} \|\bar{u}\|_{W_p^1(\Omega)} \leq C\|(f,h)\|_{(W_{p'}^1(\Omega))^*} &= C \sup_{0 \neq \phi \in W_{p'}^1(\Omega)} \frac{|\langle (f,h), \phi \rangle|}{\|\phi\|_{W_{p'}^1(\Omega)}} \\ &\leq C \sup_{0 \neq \phi \in W_{p'}^1(\Omega)} \frac{|\langle (f,h), \phi \rangle|}{\|\nabla \phi\|_{L_{p'}(\Omega)}} = C\|(f,h)\|_{\hat{W}_p^{-1}(\Omega)}. \end{aligned}$$

Therefore it suffices to study the problem

$$\begin{aligned} -\Delta u_* &= f_*, \quad x \in \Omega, \\ \partial_3 u_* &= 0, \quad x \in S, \end{aligned} \tag{5.18}$$

where $f_* := f + \Delta\bar{u}$. Observe that $f_* \in L_p(\Omega) \cap \hat{W}_p^{-1}(\Omega)$. We extend f_* with respect to x_3 by even reflection to some $\tilde{f}$ to obtain $\tilde{f} \in L_p(\mathbb{R}^3) \cap \hat{W}_p^{-1}(\mathbb{R}^3)$. Solve the full space problem $-\Delta\tilde{u} = \tilde{f}$ to obtain a unique solution $\tilde{u} \in \dot{W}_p^1(\mathbb{R}^3) \cap \dot{W}_p^2(\mathbb{R}^3)$. By uniqueness and symmetry, it follows that $\tilde{u}(x_1, x_2, x_3) = \tilde{u}(x_1, x_2, -x_3)$, hence $\partial_3\tilde{u} = 0$ on S. Since

$$\|\nabla u_*\|_{L_p(\Omega)} \leq \|\nabla\tilde{u}\|_{L_p(\mathbb{R}^3)} \leq C\|\tilde{f}\|_{\hat{W}_p^{-1}(\mathbb{R}^3)},$$

and $\|\tilde{f}\|_{\hat{W}_p^{-1}(\mathbb{R}^3)} \leq 2\|f_*\|_{\hat{W}_p^{-1}(\Omega)}$ ($\tilde{f}$ is the even extension of f_*) it follows that

$$\|\nabla u_*\|_{L_p(\Omega)} \leq C\|f_*\|_{\hat{W}_p^{-1}(\Omega)}.$$

The function $u := \bar{u} + \tilde{u}|_\Omega = \bar{u} + u_*$ is the desired unique solution of (5.19), satisfying the estimates

$$\|\nabla^2 u\|_{L_p(\Omega)} \leq C(\|f\|_{L_p(\Omega)} + \|h\|_{W_p^{1-1/p}(S)}),$$

and

$$\|\nabla u\|_{L_p(\Omega)} \leq C\|(f, h)\|_{\hat{W}_p^{-1}(\Omega)}.$$

Uniqueness follows by even reflection of the solution of (5.16) with $f = h = 0$ at S and the uniqueness result for the full space.

If $\Omega = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ is a quarter space, we have to solve

$$\begin{aligned} -\Delta u &= f, \quad x \in \Omega, \\ \partial_2 u &= h_1, \quad x \in S_1, \\ \partial_3 u &= h_2, \quad x \in S_2, \end{aligned} \tag{5.19}$$

where $S_1 = \mathbb{R} \times \{0\} \times \mathbb{R}_+$ and $S_2 = \mathbb{R} \times \mathbb{R}_+ \times \{0\}$. The data satisfy $f \in L_p(\Omega)$, $h_j \in W_p^{1-1/p}(S_j)$, $j = 1, 2$ and $(f, h_1, h_2) \in \hat{W}_p^{-1}(\Omega)$.

By Lemma 5.5 we first solve

$$\begin{aligned} \lambda_0\bar{u} - \Delta\bar{u} &= f, \quad x \in \Omega, \\ \partial_2\bar{u} &= h_1, \quad x \in S_1, \\ \partial_3\bar{u} &= h_2, \quad x \in S_2, \end{aligned} \tag{5.20}$$

for some sufficiently large $\lambda_0 > 0$ to obtain a unique solution $\bar{u} \in W_p^2(\Omega)$. Note that $\bar{u}$ satisfies the estimates

$$\|\bar{u}\|_{W_p^2(\Omega)} \leq C(\|f\|_{L_p(\Omega)} + \|h_1\|_{W_p^{1-1/p}(S_1)} + \|h_2\|_{W_p^{1-1/p}(S_2)}),$$

and

$$\|\bar{u}\|_{W_p^1(\Omega)} \leq C\|(f, h_1, h_2)\|_{\hat{W}_p^{-1}(\Omega)}.$$

We arrive at the problem

$$\begin{aligned} -\Delta u_* &= f_*, \quad x \in \Omega, \\ \partial_2 u_* &= 0, \quad x \in S_1, \\ \partial_3 u_* &= 0, \quad x \in S_2, \end{aligned} \tag{5.21}$$

where $f_* := f + \Delta\bar{u} \in \hat{W}_p^{-1}(\Omega) \cap L_p(\Omega)$, which follows from integration by parts. Extend f_* to the half space $\mathbb{R}^3_+$ by even reflection, i.e. we set

$$\tilde{f}(x) := \begin{cases} f_*(x_1, x_2, x_3), & x_2 \geq 0, \\ f_*(x_1, -x_2, x_3), & x_2 < 0. \end{cases}$$

Then $\tilde{f} \in \hat{W}_p^{-1}(\mathbb{R}^3_+) \cap L_p(\mathbb{R}^3_+)$. Next we extend $\tilde{f}$ by even reflection to the full space $\mathbb{R}^3$ by defining

$$\hat{f}(x) := \begin{cases} \tilde{f}(x_1, x_2, x_3), & x_3 \geq 0, \\ \tilde{f}(x_1, x_2, -x_3), & x_3 < 0, \end{cases}$$

This yields that $\hat{f} \in \hat{W}_p^{-1}(\mathbb{R}^3) \cap L_p(\mathbb{R}^3)$. Solve the full space problem $-\Delta\hat{u} = \hat{f}$ to obtain a unique solution $\hat{u} \in \dot{W}_p^1(\mathbb{R}^3) \cap \dot{W}_p^2(\mathbb{R}^3)$. Since with $\hat{u}$ also $\hat{u}(-x_3)$ and $\hat{u}(-x_2)$ are solutions of $-\Delta\hat{u} = \hat{f}$, it follows from the uniqueness of the solution that $\hat{u}(x_3) = \hat{u}(-x_3)$ and $\hat{u}(x_2) = \hat{u}(-x_2)$, hence $\partial_3\hat{u} = 0$ on S_2 as well as $\partial_2\hat{u} = 0$ on S_1. Since

$$\|\nabla u_*\|_{L_p(\Omega)} \leq \|\nabla\hat{u}\|_{L_p(\mathbb{R}^3)} \leq C\|\hat{f}\|_{\hat{W}_p^{-1}(\mathbb{R}^3)},$$

and $\|\hat{f}\|_{\hat{W}_p^{-1}(\mathbb{R}^3)} \leq C\|f_*\|_{\hat{W}_p^{-1}(\Omega)}$ it follows that

$$\|\nabla u_*\|_{L_p(\Omega)} \leq C\|f_*\|_{\hat{W}_p^{-1}(\Omega)}.$$

The function $u := \bar{u} + \hat{u}|_\Omega = \bar{u} + u_*$ is the desired unique solution of (5.19), satisfying the estimates

$$\|\nabla^2 u\|_{L_p(\Omega)} \leq C(\|f\|_{L_p(\Omega)} + \|h_1\|_{W_p^{1-1/p}(S_1)} + \|h_2\|_{W_p^{1-1/p}(S_2)}),$$

and

$$\|\nabla u\|_{L_p(\Omega)} \leq C\|(f, h_1, h_2)\|_{\hat{W}_p^{-1}(\Omega)}.$$

If Ω is a bent quarter space, we will use change of coordinates and perturbation theory to prove the assertion in this case. We will give a detailed proof for the case of a bent two-phase half space below. The technique from this case carries over to the bent quarter space case.

(b) Let $\Omega = \mathbb{R}^3$ and $\Sigma = \mathbb{R}^2 \times \{0\}$. Consider the problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega\backslash\Sigma, \\ [\![\rho u]\!] &= 0 \quad \text{on } \Sigma, \\ [\![\partial_3 u]\!] &= g_2 \quad \text{on } \Sigma, \end{aligned} \tag{5.22}$$

with $f \in L_p(\Omega)$, $g_2 \in W_p^{1-1/p}(\Sigma)$ and $(f, g_2) \in \hat{W}_p^{-1}(\Omega)$.

By Lemma 5.5 we may first solve the problem

$$\begin{aligned} \lambda_0\bar{u} - \Delta\bar{u} &= f \quad \text{in } \Omega\backslash\Sigma, \\ [\![\rho\bar{u}]\!] &= 0 \quad \text{on } \Sigma, \\ [\![\partial_3\bar{u}]\!] &= g_2 \quad \text{on } \Sigma, \end{aligned} \tag{5.23}$$

where $\lambda_0 > 0$ is sufficiently large but fixed. This yields a unique solution $\bar{u} \in W_p^2(\Omega\backslash\Sigma)$. Next we consider the equation $-\Delta\tilde{u} = \tilde{f}$ in $\mathbb{R}^3$, where $\tilde{f} := f + \Delta\bar{u} \in \hat{W}_p^{-1}(\mathbb{R}^3) \cap L_p(\mathbb{R}^3)$, since

$$\int_{\mathbb{R}^3} (f + \Delta\bar{u})\phi\, dx = -\int_\Sigma g_2\phi\, d\Sigma + \int_{\mathbb{R}^3} f\phi\, dx - \int_{\mathbb{R}^3} \nabla\bar{u}\cdot\nabla\phi\, dx.$$

by what we have already shown, this full space problem admits a unique solution $\tilde{u} \in \dot{W}_p^1(\Omega) \cap \dot{W}_p^2(\Omega)$. Finally we study the problem

$$\begin{aligned} -\Delta\hat{u} &= 0 \quad \text{in } \Omega\backslash\Sigma,\\ [\![\rho\hat{u}]\!] &= \hat{g}_1 \quad \text{on } \Sigma,\\ [\![\partial_3\hat{u}]\!] &= 0 \quad \text{on } \Sigma, \end{aligned} \tag{5.24}$$

with $\hat{g}_1 := -[\![\rho\tilde{u}]\!] \in \dot{W}_p^{1-1/p}(\Sigma) \cap \dot{W}_p^{2-1/p}(\Sigma)$. The solution is given in terms of the Poisson semigroup as follows.

$$\hat{u}(x_3) = \frac{1}{\rho_1+\rho_2}\begin{cases} e^{-Lx_3}\hat{g}_1, & x_3 \geq 0,\\ -e^{-L(-x_3)}\hat{g}_1, & x_3 < 0, \end{cases}$$

where $L := (-\Delta_{x'})^{1/2}$. By semigroup theory it follows that $\hat{u} \in \dot{W}_p^1(\Omega\backslash\Sigma) \cap \dot{W}_p^2(\Omega\backslash\Sigma)$. Here we use the fact that

$$\left(\int_0^\infty z^{(k-s)p}\|L^k e^{-Lz} g\|^p_{L_p(\Sigma)}\frac{dz}{z}\right)^{1/p} \tag{5.25}$$

defines an equivalent norm in $\dot{W}_p^s(\Sigma)$ for $s > 0$ and $k > s$ (if $s = j - 1/p$, $j \in \{1, 2\}$, we choose $k = j$). The function $u := \bar{u} + \tilde{u} + \hat{u}$ is the unique solution of (5.22), satisfying the estimates

$$\|\nabla^2 u\|_{L_p(\Omega)} \leq C(\|f\|_{L_p(\Omega)} + \|g_2\|_{W_p^{1-1/p}(\Sigma)}),$$

and

$$\|\nabla u\|_{L_p(\Omega)} \leq C\|(f, g_2)\|_{\hat{W}_p^{-1}(\Omega)}.$$

The uniqueness of the solution can be seen as follows. Let $u \in \dot{W}_p^1(\Omega\backslash\Sigma)\cap\dot{W}_p^2(\Omega\backslash\Sigma)$ be a solution of (5.22) with $f = g_2 = 0$. We want to show that $\nabla u = 0$ in $\Omega\backslash\Sigma$. To this end we define two functions

$$v(x_1, x_2, x_3) := \rho_2 u_+(x_1, x_2, x_3) - \rho_1 u_-(x_1, x_2, -x_3),\ (x_1, x_2) \in \mathbb{R}^2,\ x_3 > 0,$$

and

$$w(x_1, x_2, x_3) := u_+(x_1, x_2, x_3) + u_-(x_1, x_2, -x_3),\ (x_1, x_2) \in \mathbb{R}^2,\ x_3 > 0,$$

where $u_\pm := u|_{x_3 \gtrless 0}$. It follows that v and w solve the half space problems

$$\Delta v = 0,\ (x_1, x_2) \in \mathbb{R}^2,\ x_3 > 0, \quad v = 0,\ (x_1, x_2) \in \mathbb{R}^2,\ x_3 = 0,$$

and

$$\Delta w = 0,\ (x_1, x_2) \in \mathbb{R}^2,\ x_3 > 0, \quad \partial_3 w = 0,\ (x_1, x_2) \in \mathbb{R}^2,\ x_3 = 0,$$

respectively in $\dot{W}_p^1(\mathbb{R}^3_+) \cap \dot{W}_p^2(\mathbb{R}^3_+)$. Therefore $\nabla w = \nabla v = 0$ by even or uneven reflection at $\{x_3 = 0\}$. This yields

$$\begin{aligned} 0 &= \rho_2 \nabla u_+ + \rho_1 \nabla u_-, \\ 0 &= \nabla u_+ - \nabla u_-, \end{aligned}$$

wherefore $\nabla u_- = \nabla u_+ = 0$, hence $\nabla u = 0$ in $\Omega \backslash \Sigma$.

Let now $\Omega = \mathbb{R}^2_+ \times \mathbb{R}$ with $\Sigma = \{\mathbb{R}^2 \times \{0\}\} \cap \Omega$. Here we have to solve the problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \backslash \Sigma, \\ [\![\rho u]\!] &= 0 \quad \text{on } \Sigma, \\ [\![\partial_3 u]\!] &= g_2 \quad \text{on } \Sigma, \\ \partial_2 u &= h_1 \quad \text{on } S_1 \backslash \partial\Sigma, \end{aligned} \tag{5.26}$$

with $[\![\rho h_1]\!] = 0$ at $\partial\Sigma$. For some large $\lambda_0 > 0$, we first solve the problem

$$\begin{aligned} \lambda_0 \bar{u} - \Delta \bar{u} &= f \quad \text{in } \Omega \backslash \Sigma, \\ [\![\rho \bar{u}]\!] &= 0 \quad \text{on } \Sigma, \\ [\![\partial_3 \bar{u}]\!] &= g_2 \quad \text{on } \Sigma, \\ \partial_2 \bar{u} &= h_1 \quad \text{on } S_1 \backslash \partial\Sigma, \end{aligned} \tag{5.27}$$

by Lemma 5.5, to obtain a unique solution $\bar{u} \in W_p^2(\Omega \backslash \Sigma)$. Let $f_* := f + \Delta \bar{u}$ and note that $f_* \in \hat{W}_p^{-1}(\Omega) \cap L_p(\Omega)$, which follows from integration by parts and from the assumption on (f, g_2, h_1). We extend f_* with respect to x_2 by even reflection to some function

$$\tilde{f}(x) := \begin{cases} f_*(x_1, x_2, x_3), & x_2 \geq 0, \\ f_*(x_1, -x_2, x_3), & x_2 < 0. \end{cases}$$

Then $\tilde{f} \in \hat{W}_p^{-1}(\mathbb{R}^3) \cap L_p(\mathbb{R}^3)$ and we may solve the full space problem $-\Delta \tilde{u} = \tilde{f}$ to obtain a unique solution $\tilde{u} \in \dot{W}_p^1(\mathbb{R}^3) \cap \dot{W}_p^2(\mathbb{R}^3)$ with the property $\tilde{u}(x_2) = \tilde{u}(-x_2)$, hence $\partial_2 \tilde{u} = 0$ at $S_1 \backslash \partial\Sigma$. Consider next the problem

$$\begin{aligned} \Delta \hat{u} &= 0 \quad \text{in } \mathbb{R}^n \backslash \Sigma, \\ [\![\rho \hat{u}]\!] &= g \quad \text{on } \Sigma, \\ [\![\partial_3 \hat{u}]\!] &= 0 \quad \text{on } \Sigma, \end{aligned} \tag{5.28}$$

where $g := -[\![\rho \tilde{u}]\!] \in \dot{W}_p^{1-1/p}(\Sigma) \cap \dot{W}_p^{2-1/p}(\Sigma)$. As in the previous case, the unique solution $\hat{u}$ of (5.28) is given in terms of the Poisson semigroup.

Finally, since $\tilde{u}(x_2) = \tilde{u}(-x_2)$, it follows that $g(x_2) = g(-x_2)$, hence $\hat{u}(x_2) = \hat{u}(-x_2)$, by uniqueness, and therefore $\partial_2 \hat{u} = 0$ at $S_1 \backslash \partial\Sigma$. The function $u := \bar{u} + \tilde{u}|_\Omega + \hat{u}|_\Omega$ is the unique solution of (5.26), satisfying the estimates

$$\|\nabla^2 u\|_{L_p(\Omega)} \leq C(\|f\|_{L_p(\Omega)} + \|h_1\|_{W_p^{1-1/p}(S_1 \backslash \partial\Sigma)} + \|g_2\|_{W_p^{1-1/p}(\Sigma)}),$$

and

$$\|\nabla u\|_{L_p(\Omega)} \leq C\|(f, h_1, g_2)\|_{\hat{W}_p^{-1}(\Omega)}.$$

In a next step we consider the case of a bent two-phase half space. To be precise, we assume that

$$\Omega_\theta := \{x \in \mathbb{R}^3 : x_2 > \theta(x_1)\},$$

where $\theta \in BC^3(\mathbb{R})$ with $\|\theta\|_\infty + \|\theta'\|_\infty < \eta$, and $\eta > 0$ can be made as small as we wish. Let furthermore $S_{1,\theta} := \{x \in \mathbb{R}^3 : x_2 = \theta(x_1)\}$ and $\Sigma_\theta := \{\mathbb{R}^2 \times \{0\}\} \cap \Omega_\theta$. Consider the problem

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega_\theta \backslash \Sigma_\theta, \\
[\![\rho u]\!] &= 0 \quad \text{on } \Sigma_\theta, \\
[\![\partial_3 u]\!] &= g_2 \quad \text{on } \Sigma_\theta, \\
\partial_{\nu_{\partial\Sigma_\theta}} u &= h_1 \quad \text{on } S_{1,\theta} \backslash \partial\Sigma_\theta,
\end{aligned} \tag{5.29}
$$

where $f \in L_p(\Omega_\theta)$, $g_2 \in W_p^{1-1/p}(\Sigma_\theta)$, $h_1 \in W_p^{1-1/p}(S_{1,\theta} \backslash \partial\Sigma_\theta)$ and $(f, g_2, h_1) \in \hat{W}_p^{-1}(\Omega_\theta)$. Moreover, the compatibility condition $[\![\rho h_1]\!] = 0$ at $\partial\Sigma_\theta$ holds.

By means of Lemma 5.5, we may solve the problem

$$
\begin{aligned}
\lambda_0 \hat{u} - \Delta \hat{u} &= f \quad \text{in } \Omega_\theta \backslash \Sigma_\theta, \\
[\![\rho \hat{u}]\!] &= 0 \quad \text{on } \Sigma_\theta, \\
[\![\partial_3 \hat{u}]\!] &= g_2 \quad \text{on } \Sigma_\theta, \\
\partial_{\nu_{\partial\Sigma_\theta}} \hat{u} &= h_1 \quad \text{on } S_{1,\theta} \backslash \partial\Sigma_\theta,
\end{aligned} \tag{5.30}
$$

where $\lambda_0 > 0$ is large but fixed. This yields a unique solution $\hat{u} \in W_p^2(\Omega_\theta \backslash \Sigma_\theta)$. Let $\tilde{f} := f + \Delta\hat{u}$ and consider

$$
\begin{aligned}
-\Delta \tilde{u} &= \tilde{f} \quad \text{in } \Omega_\theta \backslash \Sigma_\theta, \\
[\![\rho \tilde{u}]\!] &= 0 \quad \text{on } \Sigma_\theta, \\
[\![\partial_3 \tilde{u}]\!] &= 0 \quad \text{on } \Sigma_\theta, \\
\partial_{\nu_{\partial\Sigma_\theta}} \tilde{u} &= 0 \quad \text{on } S_{1,\theta} \backslash \partial\Sigma_\theta.
\end{aligned} \tag{5.31}
$$

Observe that $\tilde{f} \in \hat{W}_p^{-1}(\Omega_\theta) \cap L_p(\Omega_\theta)$. We will now transform Ω_θ to Ω_0 by means of the coordinates $\bar{x}_1 := x_1$, $\bar{x}_2 := x_2 - \theta(x_1)$ and $\bar{x}_3 := x_3$. Assume that $\tilde{u}$ solves (5.31) and define $\bar{u}(\bar{x}) := \tilde{u}(\bar{x}_1, \bar{x}_2 + \theta(\bar{x}_1), \bar{x}_3)$. Then, the function $\bar{u}$ is a solution of the problem

$$
\begin{aligned}
-\Delta \bar{u} &= \bar{f} + M_1(\theta, \bar{u}) \quad \text{in } \Omega \backslash \Sigma, \\
[\![\rho \bar{u}]\!] &= 0 \quad \text{on } \Sigma, \\
[\![\partial_3 \bar{u}]\!] &= 0 \quad \text{on } \Sigma, \\
-\partial_2 \bar{u} &= M_2(\theta, \bar{u}) \quad \text{on } S_1 \backslash \partial\Sigma,
\end{aligned} \tag{5.32}
$$

where $\bar{f}$ is the transformation of $\tilde{f}$,

$$
M_1(\theta, \bar{u}) := -2\theta'(\bar{x}_1)\partial_1\partial_2\bar{u} - \theta''(\bar{x}_1)\partial_2\bar{u} + \theta'(\bar{x}_1)^2\partial_2^2\bar{u},
$$

and $M_2(\theta, \bar{u}) := -\theta'(\bar{x}_1)\partial_1\bar{u}|_{S_1\backslash\partial\Sigma} + \theta'(\bar{x}_1)^2\partial_2\bar{u}|_{S_1\backslash\partial\Sigma}$.

Define the function spaces

$$
\mathbb{E} := \{\nabla\bar{u} \in W_p^1(\Omega\backslash\Sigma) : [\![\rho\bar{u}]\!] = [\![\partial_3\bar{u}]\!] = 0 \text{ on } \Sigma\},
$$

with the equivalent norm $\|\bar{u}\|_{\mathbb{E},\lambda} := \lambda\|\nabla\bar{u}\|_{L_p} + \|\nabla^2\bar{u}\|_{L_p}$, $\lambda > 0$, and let

$$
\mathbb{F} := \{(f_1, f_2) \in L_p(\Omega) \times W_p^{1-1/p}(S_1\backslash\partial\Sigma) : [\![\rho f_2]\!] = 0 \text{ at } \partial\Sigma \text{ and } (f_1, f_2) \in \hat{W}_p^{-1}(\Omega)\}.
$$

Moreover, we define a linear operator $L : \mathbb{E} \to \mathbb{F}$ by

$$L\bar{u} := \begin{pmatrix} \Delta\bar{u} \\ \partial_2\bar{u}|_{S_1\backslash\partial\Sigma} \end{pmatrix}.$$

It follows from our previous considerations that $L : \mathbb{E} \to \mathbb{F}$ is an isomorphism. Let $F := (\bar{f}, 0)$ and $M(\theta, \bar{u}) := (M_1, M_2)(\theta, \bar{u})$. It follows that $F \in \mathbb{F}$, since

$$\int_{\Omega_\theta} \tilde{f}\phi \; dx = \int_\Omega \bar{f}\bar{\phi} \; d\bar{x},$$

with $\bar{\phi}(\bar{x}) := \phi(\bar{x}_1, \bar{x}_2 - \theta(\bar{x}_1), \bar{x}_3)$ and $\phi \in C_c^\infty(\overline{\Omega_\theta})$. Furthermore, for each $\bar{u} \in \mathbb{E}$, we have $M(\theta, \bar{u}) \in \mathbb{F}$. Indeed, as in the proof of Lemma 5.5, it can be readily checked that $M(\theta, \bar{u}) \in L_p(\Omega) \times W_p^{1-1/p}(S_1\backslash\partial\Sigma)$ and $[\![\rho M_2(\theta, \bar{u})]\!] = 0$ at $\partial\Sigma$. It remains to verify the condition $(M_1, M_2)(\theta, \bar{u}) \in \hat{W}_p^{-1}(\Omega)$. To this end, we integrate by parts to obtain the identity

$$\begin{aligned} \int_\Omega M_1(\theta, \bar{u})\phi \; dx &+ \int_{S_1} M_2(\theta, \bar{u})\phi \; dS_1 \\ &= \int_\Omega \big(\theta'(\bar{x}_1)\partial_2\bar{u}\partial_1\phi + \theta'(\bar{x}_1)\partial_1\bar{u}\partial_2\phi - \theta'(\bar{x}_1)^2\partial_2\bar{u}\partial_2\phi\big) \; dx. \end{aligned}$$

for each $\phi \in C_c^\infty(\overline{\Omega})$. This in turn yields the claim. We are now in a position to write (5.32) in the shorter form

$$\bar{u} = L^{-1}M(\theta, \bar{u}) + L^{-1}F.$$

We may now follow the lines of the proof of Lemma 5.5 to obtain a unique solution of (5.29).

2. It follows from Lemma 5.5 that the operator $Au := -\Delta u$ with domain

$$D(A) = \{u \in W_p^2(\Omega\backslash\Sigma) : [\![\rho u]\!] = [\![\partial_{\nu_\Sigma} u]\!] = 0, \; \partial_{\nu_{S_j}} u = 0\},$$

is closed. Since $D(A)$ is compactly embedded in $L_p(\Omega)$, the spectrum $\sigma(A)$ consists solely of isolated eigenvalues and $\lambda = 0$ is a simple eigenvalue of A. Indeed, $N(A) = \operatorname{span} \mathbb{1}_\rho$, with

$$\mathbb{1}_\rho := \chi_{\Omega_1} + \frac{\rho_1}{\rho_2}\chi_{\Omega_2}.$$

Furthermore, $N(A^2) \subset N(A)$, since if $u \in N(A^2)$, then $v := Au \in N(A)$. It follows that $v \in L_1(\Omega)$ and we may integrate $Au = v$ over Ω to obtain

$$\int_\Omega v \; dx = -\int_\Omega \Delta u \; dx = 0,$$

hence $v = 0$, since $\mathbb{1}_\rho$ has a non-vanishing mean value.

In particular this yields $L_p(\Omega) = N(A) \oplus R(A)$ and it holds that $R(A) = L_p^{(0)}(\Omega)$. This can be seen as follows. Obviously one has the inclusion

$$R(A) \subset L_p^{(0)}(\Omega).$$

So, let $f \in L_p^{(0)}(\Omega)$. Then there exist unique $f_1 \in N(A)$ and $f_2 \in R(A)$ such that $f = f_1 + f_2$. This in turn yields $f_1 \in L_p^{(0)}(\Omega)$. Since $f_1 = \alpha \mathbb{1}_\rho$ for some $\alpha \in \mathbb{R}$ with

$$\mathbb{1}_\rho := \chi_{\Omega_1} + \frac{\rho_1}{\rho_2}\chi_{\Omega_2},$$

it follows that

$$(f_1|\mathbb{1}) = \alpha\left(|\Omega_1| + \frac{\rho_1}{\rho_2}|\Omega_2|\right),$$

hence $\alpha = 0$ and therefore $f = f_2 \in R(A)$, hence $L_p^{(0)}(\Omega) \subset R(A)$. □

We will also need an existence and uniqueness result for the weak version of (5.3) with $\lambda = 0$. To be precise, we consider the problem

$$\begin{aligned} (\nabla u|\nabla\phi)_2 &= \langle f, \phi\rangle, \quad \phi \in W_{p'}^1(\Omega), \\ [\![\rho u]\!] &= g, \quad \text{on } \Sigma. \end{aligned} \tag{5.33}$$

Then we have the following result.

Lemma 5.7. *Let $\rho_j > 0$, $n = 2, 3$, $p \geq 2$ and let $\Omega \subset \mathbb{R}^n$ satisfy condition (c) from above. Then there exists a unique solution $u \in \dot{W}_p^1(\Omega\backslash\Sigma)$ of* (5.33) *if and only if $f \in \hat{W}_p^{-1}(\Omega)$ and $g \in W_p^{1-1/p}(\Sigma)$.*

Proof. Let $g \in W_p^{1-1/p}(\Sigma)$. The Neumann Laplacian Δ_N in $L_p(\Sigma)$ with domain

$$D(\Delta_N) = \{u \in W_p^2(\Sigma) : \partial_{\nu_{\partial G}} u = 0 \text{ on } \partial\Sigma\}$$

generates an analytic semigroup. In particular, $D(\Delta_N)$ is dense in

$$W_p^{1-1/p}(\Sigma) = (L_p(\Sigma), D(\Delta_N))_{1/2-1/2p} = D_{\Delta_N}(1/2 - 1/2p, p).$$

Therefore, there exists $(g_n)_{n\in\mathbb{N}} \subset W_p^{2-1/p}(\Sigma)$ such that $\partial_{\nu_{\partial G}} g_n = 0$ for each $n \in \mathbb{N}$ on $\partial\Sigma$ and $g_n \to g$ as $n \to \infty$ in $W_p^{1-1/p}(\Sigma)$. Denote by $u_n \in W_p^2(\Omega\backslash\Sigma)$ the solution of (5.3) with $f = g_2 = h_1 = h_2 = 0$, $g_1 = g_n$ and a fixed $\lambda \geq \lambda_0$. Making use of local coordinates one can show that the estimate

$$\|u_n - u_m\|_{W_p^1(\Omega\backslash\Sigma)} \leq C\|g_n - g_m\|_{W_p^{1-1/p}(\Sigma)}$$

is valid, with some constant $C > 0$ which does not depend on n. Indeed, each of the local charts yields a transformed problem which is subject to one of the conditions in (a) and (b) above. We have already seen in the proof of Lemma 5.5 that the two-phase half space and the quarter space can be pulled back to a two-phase full space and an ordinary half space, respectively, by means of reflection techniques. Making use of change of coordinates, perturbation theory and the results in [19, Section 8] one obtains the desired estimate.

In particular, (u_n) is a Cauchy sequence in $W_p^1(\Omega\backslash\Sigma)$ and therefore it has a limit point $u \in W_p^1(\Omega\backslash\Sigma)$. By trace theory it follows that u satisfies the weak problem

$$\begin{aligned} \lambda(u|\phi)_2 + (\nabla u|\nabla\phi)_2 &= 0, \quad \phi \in W_{p'}^1(\Omega), \\ [\![\rho u]\!] &= g, \quad \text{on } \Sigma, \end{aligned} \tag{5.34}$$

for some fixed $\lambda \geq \lambda_0$.

Next, let

$$a : \{u \in W_p^1(\Omega\backslash\Sigma) : [\![\rho u]\!] = 0 \text{ on } \Sigma\} \times W_{p'}^1(\Omega) \to \mathbb{R}, \quad a(u,\phi) := \int_\Omega \nabla u \cdot \nabla\phi dx,$$

and define an operator $B : W_p^1(\Omega\backslash\Sigma) \to (W_{p'}^1(\Omega))^*$ with domain

$$D(B) = \{u \in W_p^1(\Omega\backslash\Sigma) : [\![\rho u]\!] = 0 \text{ on } \Sigma\},$$

by means of $\langle Bu, \phi\rangle := a(u,\phi)$. It follows from integration by parts that the operator A from the proof of the second assertion of Lemma 5.6 is the part of B in $L_p(\Omega)$. As in the proof of Lemma 5.6 one can show that $\lambda = 0$ is a simple eigenvalue of B. It follows that $(W_{p'}^1(\Omega))^* = N(B) \oplus R(B)$ and $W_p^1(\Omega\backslash\Sigma) = N(B) \oplus Y$, where Y is a closed subspace of $W_p^1(\Omega\backslash\Sigma)$. Therefore there exists a unique solution $v \in Y$ of the equation $Bv = f$ if and only if $f \in R(B)$ or equivalently $\langle f, \mathbb{1}\rangle = 0$. It follows readily that $R(B) = \hat{W}_p^{-1}(\Omega)$. Indeed, the inclusion $\hat{W}_p^{-1}(\Omega) \subset R(B)$ is easy, since $\langle f, \mathbb{1}\rangle = 0$ for each $f \in \hat{W}_p^{-1}(\Omega)$ and the restriction of f to $W_{p'}^1(\Omega)$ belongs to $(W_{p'}^1(\Omega))^*$. Let now $f \in R(B)$, i.e. $f \in (W_{p'}^1(\Omega))^*$ and $\langle f, \mathbb{1}\rangle = 0$. This yields

$$|\langle f, \phi\rangle| = |\langle f, \phi - \bar{\phi}\rangle| \le C\|\phi - \bar{\phi}\|_{W_{p'}^1(\Omega)} \le C\|\nabla\phi\|_{L_{p'}(\Omega)},$$

by the Poincaré-Wirtinger inequality and therefore $[\phi \mapsto \langle f, \phi\rangle]$ is continuous on $C_c^\infty(\overline{\Omega})$ with respect to the norm $\|\nabla \cdot\|_{L_{p'}(\Omega)}$.

Let $u \in W_p^1(\Omega\backslash\Sigma)$ denote the unique solution of (5.34) and let $v \in \dot{W}_p^1(\Omega\backslash\Sigma)$ denote the unique solution of

$$\begin{aligned} (\nabla v|\nabla\phi)_2 &= \langle f, \phi\rangle - (\nabla u|\nabla\phi)_2, \quad \phi \in W_{p'}^1(\Omega), \\ [\![\rho v]\!] &= 0, \quad \text{on } \Sigma. \end{aligned}$$

It follows readily that the function $w := v + u \in \dot{W}_p^1(\Omega\backslash\Sigma)$ is the unique solution of (5.33). □

A final result considers the system (5.3) with $\lambda = g_1 = g_2 = h_1 = h_2 = 0$. We assume that the function f depends on the spatial variable x and on some parameter t, i.e. $f = f(t,x)$. In this case the solution $u = u(t,x)$ depends on t as well. The following result contains some information about the regularity of u with respect to t and x.

Lemma 5.8. *Let $n = 2, 3$, $p \ge 2$, $J = [0,T]$ or $J = \mathbb{R}_+$ and $\lambda = g_1 = g_2 = h_1 = h_2 = 0$. Then the following assertions are valid.*

(1) *If Ω and Σ satisfy one of the conditions in $(a), (b)$ above, then there exists a unique solution*

$$\nabla u \in {}_0W_p^1(J; W_p^1(\Omega\backslash\Sigma)) \cap L_p(J; W_p^3(\Omega\backslash\Sigma))$$

of (5.3) *if and only if*

$$f \in {}_0W_p^1(J; \hat{W}_p^{-1}(\Omega) \cap L_p(\Omega)) \cap L_p(J; W_p^2(\Omega\backslash\Sigma)).$$

(2) *If Ω and Σ are subject to the condition (c) above, then there exists a unique solution*

$$u \in {}_0W_p^1(J; W_p^1(\Omega\backslash\Sigma)) \cap L_p(J; W_p^3(\Omega\backslash\Sigma))$$

of (5.3) *if and only if*

$$f \in {}_0W^1_p(J; \hat{W}^{-1}_p(\Omega)) \cap L_p(J; W^1_p(\Omega\backslash\Sigma)).$$

Proof. (i) The regularity

$$\nabla u \in {}_0W^1_p(J; W^1_p(\Omega\backslash\Sigma))$$

in the first assertion and

$$u \in {}_0W^1_p(J; W^1_p(\Omega\backslash\Sigma))$$

in the second assertion is a direct consequence of Lemma 5.6 and Lemma 5.7, respectively.

(ii) Concerning the additional spatial regularity of u, one uses the fact that one already knows the unique solution u of (5.3) with the regularity stated in Lemma 5.6 and Lemma 5.7. By means of local coordinates, one reduces each of the local problems to one of the model probolems in (a) and (b) above. In particular, the two-phase half space and the quarter space can be pulled back to a two-phase full space and an ordinary half space, respectively, by reflection techniques. The mapping behavior of the Laplacian and the Poisson semigroup in homogeneous Sobolev-Slobodeckii spaces, see (5.25), yield the corresponding higher order estimates for the solution operators of the model problems. Therefore, the proof of the additional regularity of u with respect to x follows along the lines of [19, Proof of Theorem 8.6]. We will not repeat the arguments. □

5.3.2. *Parabolic problems.* The following auxiliary lemma is concerned with the parabolic one-phase problem

$$\begin{aligned} \partial_t u - \mu\Delta u &= f, \quad \text{in } \Omega,\\ P_{S_1}\left(\mu(\nabla u + \nabla u^{\mathsf{T}})\nu_{S_1}\right) &= P_{S_1} g_1, \quad \text{on } S_1,\\ u\cdot\nu_{S_1} &= g_2, \quad \text{on } S_1,\\ u &= g_3, \quad \text{on } S_2,\\ u(0) &= u_0, \quad \text{in } \Omega. \end{aligned} \tag{5.35}$$

Again, we will concentrate on the case $n = 3$. The results in this subsection remain true for the case $n = 2$.

Lemma 5.9. *Let $p > 2$, $p \neq 3$, $\mu > 0$, $T > 0$ and $J = [0, T]$. Then there exists a unique solution*

$$u \in H^1_p(J; L_p(\Omega)^3) \cap L_p(J; H^2_p(\Omega)^3)$$

of (5.35) *if and only if the data are subject to the following regularity and compatibility conditions*

(1) $f \in L_p(J; L_p(\Omega)^3)$,
(2) $g_1 \in W^{1/2-1/2p}_p(J; L_p(S_1)^3) \cap L_p(J; W^{1-1/p}_p(S_1)^3)$,
(3) $g_2 \in W^{1-1/2p}_p(J; L_p(S_1)) \cap L_p(J; W^{2-1/p}_p(S_1))$,
(4) $g_3 \in W^{1-1/2p}_p(J; L_p(S_2)^3) \cap L_p(J; W^{2-1/p}_p(S_2)^3)$,
(5) $u_0 \in W^{2-2/p}_p(\Omega)^3$,
(6) $P_{S_1}\left(\mu(\nabla u_0 + \nabla u_0^{\mathsf{T}})\nu_{S_1}\right) = P_{S_1} g_1|_{t=0}$ $(p > 3)$,
(7) $u_0|_{S_1} \cdot \nu_{S_1} = g_2|_{t=0}$, $u_0|_{S_2} = g_3|_{t=0}$,

(8) *$g_3 \cdot \nu_{S_1} = g_2$ at ∂S_2,*
(9) *$P_{\partial G}\left(\mu(\nabla_{x'} g_3' + \nabla_{x'}(g_3')^{\mathsf{T}})\nu_{\partial S_2}\right) = P_{\partial G} g_1'$ at ∂S_2,*
(10) *$\mu(\partial_{\nu_{S_1}}(g_3 \cdot e_3) + \partial_3 g_2) = g_1 \cdot e_3$ at ∂S_2,*

where $g_j' := \sum_{k=1}^2 (g_j \cdot e_k)e_k$ for $j \in \{1,3\}$.

The result remains true for the case $J = \mathbb{R}_+$ if ∂_t is replaced by $\partial_t + \omega$, with some sufficiently large $\omega > 0$.

Proof. 1. Extend u_0 to some function $\tilde{u}_0 \in W_p^{2-2/p}(\mathbb{R}^3)^3$ and solve the full space problem

$$\begin{aligned} \partial_t \tilde{u} - \mu\Delta\tilde{u} &= 0, \quad \text{in } \mathbb{R}^3, \\ \tilde{u}(0) &= \tilde{u}_0, \quad \text{in } \mathbb{R}^3, \end{aligned} \tag{5.36}$$

to obtain a unique solution

$$\tilde{u} \in H_p^1(J; L_p(\mathbb{R}^3)^3) \cap L_p(J; H_p^2(\mathbb{R}^3)^3).$$

If u is a solution of (5.35), then $u - \tilde{u}|_\Omega$ solves (5.35) with $u_0 = 0$ and some modified data (f, g_1, g_2, g_3) (not to be relabeled) having vanishing temporal trace at $t = 0$, whenever it exists. Therefore, we may w.l.o.g. assume that $u_0 = 0$ in (5.35).

Suppose that u is a solution of (5.35) with $u_0 = 0$. We cover ∂S_2 by finitely many open balls $U_k := B_r(x_k)$, $x_k \in \partial S_2$, $k = 1, \ldots, N$. This way, we obtain N bent quarter spaces with corresponding solution operators $\mathcal{S}_k$, which are well-defined, if $r > 0$ is sufficiently small. Furthermore, by the results in Subsection 5.2 there exist open sets U_{N+j}, $j = 1, \ldots, 3$ such that

- $U_{N+1} \subset \Omega$,
- $U_{N+2} \cap S_1 \neq \emptyset$, $U_{N+2} \cap S_2 = \emptyset$,
- $U_{N+3} \cap S_1 = \emptyset$, $U_{N+3} \cap S_2 \neq \emptyset$,
- $\overline{\Omega} \subset \bigcup_{k=1}^{N+3} U_k$,

and a subordinated partition of unity $\{\varphi_k\}_{k=0}^N \subset C_c^3(\mathbb{R}^3; [0,1])$ with $\partial_{\nu_{\partial G}} \varphi_k = \partial_3 \varphi_k = 0$ at ∂S_2. Let $u_k := u\varphi_k$, $f_k := f\varphi_k$ and $g_j^k := g_j \varphi_k$. Then u_k solves the problem

$$\begin{aligned} \partial_t u_k - \mu\Delta u_k &= F_k(u) + f_k, \quad \text{in } \Omega_k, \\ P_{S_1^k}\left(\mu(\nabla u_k + \nabla u_k^{\mathsf{T}})\nu_{S_1^k}\right) &= G_k(u) + P_{S_1^k} g_1^k, \quad \text{on } S_1^k, \\ u_k \cdot \nu_{S_1^k} &= g_2^k, \quad \text{on } S_1^k, \\ u_k &= g_3^k, \quad \text{on } S_2^k, \\ u_k(0) &= 0, \quad \text{in } \Omega_k, \end{aligned} \tag{5.37}$$

where $F_k(u) := -\mu[\Delta, \varphi_k]u$ and $G_k(u) := P_{S_1^k}\left(\mu(\nabla\varphi_k \otimes u + u \otimes \nabla\varphi_k)\nu_{S_1^k}\right)$.

Here $\Omega_{N+1} = \mathbb{R}^3$, Ω_{N+2} reduces to bent half-spaces with pure-slip boundary conditions, Ω_{N+3} is a half-space with Dirichlet boundary conditions and Ω_k, $k = 1, \ldots, N$ are bent quarter-spaces with pure-slip boundary conditions on one part of the boundary and Dirichlet boundary conditions on the other part. S_j^k denote the corresponding parts of the boundary $\partial\Omega_k$ and $S_j^{N+1} = S_1^{N+3} = S_2^{N+2} = \emptyset$.

Denoting by $\mathcal{S}_k$ the corresponding solution operators to each of the $N+3$ problems, we obtain the representation

$$u_k = \mathcal{S}_k \left((f_k, g_1^k, g_2^k, g_3^k) + (F_k(u), G_k(u), 0, 0) \right).$$

Let $\{\psi_k\}_{k=0}^N \subset C_c^\infty(\mathbb{R}^3; [0,1])$ such that $\psi_k \equiv 1$ on $\operatorname{supp}\varphi_k$ and $\operatorname{supp}\psi_k \subset U_k$. Multiplying u_k with ψ_k and summing from $k=0$ to N yields the identity

$$u = \sum_{k=0}^N \psi_k \mathcal{S}_k \left((f_k, g_1^k, g_2^k, g_3^k) + (F_k(u), G_k(u), 0, 0) \right). \tag{5.38}$$

Therefore, any solution to (5.35), with $u_0 = 0$, necessarily satisfies (5.38). The converse however is in general not true. This pathology stems from the compatibility conditions at ∂S_2^k for the commutator term $G_k(u)$ in (5.37). Thanks to Proposition 5.1 there exists an appropriate extension operator $\mathrm{ext}_{x_3,k}$ from

$${}_0W_p^{1/2-1/p}(J; L_p(\partial S_2^k)) \cap L_p(J; W_p^{1-2/p}(\partial S_2^k))$$

to

$${}_0W_p^{1/2-1/2p}(J; L_p(\partial S_2^k \times \mathbb{R}_+)) \cap L_p(J; W_p^{1-1/p}(\partial S_2^k \times \mathbb{R}_+)),$$

such that $[\mathrm{ext}_{x_3,k}\, v](0) = v$. Replace $G_k(u)$ by

$$\tilde{G}_k(u, g_3) := G_k(u) - \mathrm{ext}_{x_3,k} \left(G_k(u)|_{x_3=H_j} - G_k(g_3)|_{x_3=H_j} \right) = G_k^1(g_3) + G_k^2(u),$$

where $G_k^1(g_3) := \mathrm{ext}_{x_3,k}\, G_k(g_3)|_{x_3=H_j}$. We note on the go that $\tilde{G}_k(u, g_3) = G_k(u)$, if u is a solution of (5.35), since then $u = g_3$ at ∂S_2 and $g_3|_{S_1} \cdot \nu_{S_1} = g_2|_{S_2}$ at ∂S_2 by assumption.

Therefore we will henceforth work with the identity

$$u = \sum_{k=0}^N \psi_k \left(\mathcal{S}_k(f_k, g_1^k + G_k^1(g_3), g_2^k, g_3^k) + \mathcal{S}_k(F_k(u), G_k^2(u), 0, 0) \right). \tag{5.39}$$

Let ${}_0\mathbb{E}(T) := {}_0H_p^1(J; L_p(\Omega)^3) \cap L_p(J; H_p^2(\Omega)^3)$,

$$\mathbb{F}_1(T) := L_p(J \times \Omega)^3,$$
$${}_0\mathbb{F}_2(T) := {}_0W_p^{1/2-1/2p}(J; L_p(S_1)^3) \cap L_p(J; W_p^{1-1/p}(S_1)^3),$$
$${}_0\mathbb{F}_3(T) := {}_0W_p^{1-1/2p}(J; L_p(S_1)) \cap L_p(J; W_p^{2-1/p}(S_1)),$$
$${}_0\mathbb{F}_4(T) := {}_0W_p^{1-1/2p}(J; L_p(S_2)^3) \cap L_p(J; W_p^{2-1/p}(S_2)^3)$$

and

$${}_0\mathbb{F}(T) := \{(f, g_1, g_2, g_3) \in \mathbb{F}_1(T) \times_{j=2}^4 \{{}_0\mathbb{F}_j(T)\} : (8){-}(10) \text{ in Lemma 5.9 are satisfied}\}.$$

Since the terms involving u on the right side of (5.39) are of lower order, it follows that there exists $\gamma > 0$ such that the a priori estimate

$$\|u\|_{\mathbb{E}(T)} \le M \left(\|(f, g_1, g_2, g_3)\|_{\mathbb{F}(T)} + T^\gamma \|u\|_{\mathbb{E}(T)} \right),$$

holds for any solution u of (5.39). Therefore, if $T > 0$ is sufficiently small, it follows that the operator $L : {}_0\mathbb{E}(T) \to {}_0\mathbb{F}(T)$ defined by the left side of (5.35) without the initial condition is injective and has closed range. This in turn implies that L has a left-inverse.

Applying a Neumann series argument, we see that for each given set of data $(f, g_1, g_2, g_3) \in {}_0\mathbb{F}(T)$ there exists a unique solution u of (5.39) on a (possibly) small time interval $[0, T]$. This follows as above by taking into account that the terms involving u on the right side of (5.39) are linear and of lower order. Denote by $\mathcal{S} : {}_0\mathbb{F}(T) \to {}_0\mathbb{E}(T)$ the corresponding solution operator. It remains to prove the existence of a right inverse for L. Writing $u = \mathcal{S}(f, g_1, g_2, g_3)$, where $(f, g_1, g_2, g_3) \in {}_0\mathbb{F}(T)$, it follows that

$$\mathcal{S}(f, g_1, g_2, g_3) = \sum_{k=0}^{N} \psi_k \Big(\mathcal{S}_k(f_k, g_1^k + G_k^1(g_3), g_2^k, g_3^k) + \mathcal{S}_k(F_k(u), G_k^2(u), 0, 0) \Big). \tag{5.40}$$

Applying the operator L to (5.40) we obtain

$$L\mathcal{S}(f, g_1, g_2, g_3) = (f, g_1, g_2, g_3) + R(f, g_1, g_2, g_3),$$

where the linear operator R is given by

$$\begin{aligned} R(f, g_1, g_2, g_3) := &\sum_{k=0}^{N} [L, \psi_k] \Big(\mathcal{S}_k(f_k, g_1^k + G_k^1(g_3), g_2^k, g_3^k) + \mathcal{S}_k(F_k(u), G_k^2(u), 0, 0) \Big) \\ &+ \sum_{k=0}^{N} (F_k(u), G_k(u, g_3), 0, 0) \end{aligned}$$

Since the commutator $[L, \psi_k]$ as well as $F_k(u)$ and $G_k(u, g_3)$ are of lower order compared to L, it follows that there exists $\gamma > 0$ such that R satisfies the estimate

$$\|R(f, g_1, g_2, g_3)\|_{\mathbb{F}(T)} \le M T^{\gamma} \|(f, g_1, g_2, g_3)\|_{\mathbb{F}(T)},$$

where $M > 0$ does not depend on T. Therefore, a Neumann series argument implies that the right inverse for L is given by the linear operator $\mathcal{S}(I - R)^{-1}$, provided that $T > 0$ is sufficiently small. This implies that L is boundedly invertible and the proof of the first assertion is complete.

2. Concerning the second assertion, we use local coordinates and make use of the fact that the corresponding local solution operators are bounded by $1/\omega$ in the norm of $\mathbb{F}$. By means of interpolation we are able to control all lower order terms by C/ω^a for some uniform $a > 0$. Choosing $\omega > 0$ large enough, the norms of the lower order terms will become small. This yields the invertibility of L_ω as above, where L_ω results from L by replacing ∂_t with $\partial_t + \omega$. $\square$

We will also need a result on the well-posedness of the two-phase problem

$$\begin{aligned} \partial_t(\rho u) - \mu \Delta u &= f, \quad \text{in } \Omega \backslash \Sigma, \\ [\![\mu \partial_3 v]\!] + [\![\mu \nabla_{x'} w]\!] &= g_v, \quad \text{on } \Sigma, \\ [\![\mu \partial_3 w]\!] &= g_w, \quad \text{on } \Sigma, \\ [\![u]\!] &= u_\Sigma, \quad \text{on } \Sigma, \\ P_{S_1} \Big(\mu(\nabla u + \nabla u^{\mathsf{T}}) \nu_{S_1} \Big) &= P_{S_1} g_1, \quad \text{on } S_1 \backslash \partial\Sigma, \\ u \cdot \nu_{S_1} &= g_2, \quad \text{on } S_1 \backslash \partial\Sigma, \\ u &= g_3, \quad \text{on } S_2, \\ u(0) &= u_0, \quad \text{in } \Omega \backslash \Sigma. \end{aligned} \tag{5.41}$$

Lemma 5.10. *Let $p > 2$, $p \neq 3$, $\mu_j > 0$, $\rho_j > 0$, $T > 0$ and $J = [0, T]$. Then there exists a unique solution*

$$u \in H^1_p(J; L_p(\Omega)^3) \cap L_p(J; H^2_p(\Omega \backslash \Sigma)^3)$$

of (5.41) *if and only if the data are subject to the following regularity and compatibility conditions*

(1) $f \in L_p(J; L_p(\Omega)^3)$,
(2) $g_v \in W^{1/2-1/2p}_p(J; L_p(\Sigma)^2) \cap L_p(J; W^{1-1/p}_p(\Sigma)^2)$,
(3) $g_w \in W^{1/2-1/2p}_p(J; L_p(\Sigma)) \cap L_p(J; W^{1-1/p}_p(\Sigma))$,
(4) $u_\Sigma = (v_\Sigma, w_\Sigma) \in W^{1-1/2p}_p(J; L_p(\Sigma)^3) \cap L_p(J; W^{2-1/p}_p(\Sigma)^3)$,
(5) $g_1 \in W^{1/2-1/2p}_p(J; L_p(S_1)^3) \cap L_p(J; W^{1-1/p}_p(S_1 \backslash \partial\Sigma)^3)$,
(6) $g_2 \in W^{1-1/2p}_p(J; L_p(S_1)) \cap L_p(J; W^{2-1/p}_p(S_1 \backslash \partial\Sigma))$,
(7) $g_3 \in W^{1-1/2p}_p(J; L_p(S_2)^3) \cap L_p(J; W^{2-1/p}_p(S_2)^3)$,
(8) $u_0 = (v_0, w_0) \in W^{2-2/p}_p(\Omega \backslash \Sigma)^3$,
(9) $P_{S_1}\left(\mu(\nabla u_0 + \nabla u_0^{\mathsf{T}})\nu_{S_1}\right) = P_{S_1} g_1|_{t=0}$, $[\![\mu \partial_3 v_0]\!] + [\![\mu \nabla_{x'} w_0]\!] = g_v|_{t=0}$ $(p > 3)$,
(10) $u_0|_{S_1} \cdot \nu_{S_1} = g_2|_{t=0}$, $u_0|_{S_2} = g_3|_{t=0}$, $[\![\mu \partial_3 w_0]\!] = g_w|_{t=0}$, $[\![u_0]\!] = u_\Sigma|_{t=0}$,
(11) $g_3 \cdot \nu_{S_1} = g_2$ *at* ∂S_2, $u_\Sigma \cdot \nu_{S_1} = [\![g_2]\!]$ *at* $\partial\Sigma$,
(12) $P_{\partial\Sigma}\left((\nabla_{x'} v_\Sigma + \nabla_{x'} v_\Sigma^{\mathsf{T}})\nu_{\partial\Sigma}\right) = P_{\partial\Sigma}[\![g_1'/\mu]\!]$ *at* $\partial\Sigma$,
(13) $\partial_{\nu_{S_1}} w_\Sigma = [\![(g_1 \cdot e_3)/\mu - \partial_3 g_2]\!]$, $(g_v|\nu_{\partial\Sigma}) = [\![g_1 \cdot e_3]\!]$ *at* $\partial\Sigma$,
(14) $P_{\partial G}\left(\mu(\nabla_{x'} g_3' + \nabla_{x'}(g_3')^{\mathsf{T}})\nu_{\partial S_2}\right) = P_{\partial G} g_1'$ *at* ∂S_2
(15) $\mu(\partial_{\nu_{S_1}}(g_3 \cdot e_3) + \partial_3 g_2) = g_1 \cdot e_3$ *at* ∂S_2,

where $g_j' = \sum_{k=1}^2 (g_j \cdot e_k) e_k$ *for* $j \in \{1, 3\}$.

The result remains true for the case $J = \mathbb{R}_+$ *if* ∂_t *is replaced by* $\partial_t + \omega$, *with some sufficiently large* $\omega > 0$.

Proof. 1. Without loss of generality we may assume $u_0 = 0$. This can be seen as follows. Extend $u_0^+ := u_0|_{x_3 \in (0, H_2)} \in W^{2-2/p}_p(G \times (0, H_2))^3$ first w.r.t. x_3, then w.r.t. (x_1, x_2) to some $\tilde{u}_0^+ \in W^{2-2/p}_p(\mathbb{R}^3)^3$ and solve the full space problem

$$\begin{aligned} \partial_t \tilde{u}^+ - \Delta \tilde{u}^+ &= 0, \quad \text{in } \mathbb{R}^3, \\ \tilde{u}^+(0) &= \tilde{u}_0^+, \quad \text{in } \mathbb{R}^3, \end{aligned} \tag{5.42}$$

to obtain a unique solution

$$\tilde{u}^+ \in H^1_p(J; L_p(\mathbb{R}^3)^3) \cap L_p(J; H^2_p(\mathbb{R}^3)^3).$$

Then we extend $u_0^- := u_0|_{x_3 \in (H_1, 0)} \in W^{2-2/p}_p(G \times (H_1, 0))^3$ first w.r.t. x_3, then w.r.t. (x_1, x_2) to some $\tilde{u}_0^- \in W^{2-2/p}_p(\mathbb{R}^3)^3$ and solve (5.42) with $\tilde{u}_0^+$ replaced by $\tilde{u}_0^-$ to obtain a unique solution

$$\tilde{u}^- \in H^1_p(J; L_p(\mathbb{R}^3)^3) \cap L_p(J; H^2_p(\mathbb{R}^3)^3).$$

Define $\tilde{u} := \tilde{u}^+ \chi_{G \times (0, H_2)} + \tilde{u}^- \chi_{G \times (H_1, 0)}$. If u solves (5.41), then $u - \tilde{u}$ solves (5.41) with $u_0 = 0$ and with some modified data (f, g_j, u_Σ) (not to be relabeled). Note that the time traces of the modified data at $t = 0$ are zero by construction, whenever they exist.

Step 1: In a first step we consider the case $\mu_j = \rho_j = 1$. Extend

$$(g_v, g_w) \in {}_0W_p^{1/2-1/2p}(J; L_p(\Sigma)^3) \cap L_p(J; W_p^{1-1/p}(\Sigma)^3)$$

and

$$u_\Sigma \in {}_0W_p^{1-1/2p}(J; L_p(\Sigma)^3) \cap L_p(J; W_p^{2-1/p}(\Sigma)^3),$$

to some functions

$$(\tilde{g}_v, \tilde{g}_w) \in {}_0W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^2)^3) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^2)^3)$$

and

$$\tilde{u}_\Sigma \in {}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^2)^3) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^2)^3)$$

respectively. Then we solve the following two-phase problem in $\dot{\mathbb{R}}^3 := \mathbb{R}^2 \times \dot{\mathbb{R}}$.

$$\begin{aligned} \partial_t \tilde{u} - \Delta \tilde{u} &= 0, \quad \text{in } \dot{\mathbb{R}}^3, \\ [\![\partial_3 \tilde{v}]\!] + [\![\nabla_{x'} \tilde{w}]\!] &= \tilde{g}_v, \quad \text{on } \mathbb{R}^2 \times \{0\}, \\ [\![\partial_3 \tilde{w}]\!] &= \tilde{g}_w, \quad \text{on } \mathbb{R}^2 \times \{0\}, \\ [\![\tilde{u}]\!] &= \tilde{u}_\Sigma, \quad \text{on } \mathbb{R}^2 \times \{0\}, \\ \tilde{u}(0) &= 0, \quad \text{in } \dot{\mathbb{R}}^3. \end{aligned} \tag{5.43}$$

This yields the existence of a unique solution

$$\tilde{u} \in {}_0H_p^1(J; L_p(\mathbb{R}^3)^3) \cap L_p(J; H_p^2(\dot{\mathbb{R}}^3)^3).$$

If u solves (5.41) with $u_0 = 0$, then $u - \tilde{u}|_\Omega$ solves (5.41) with $u_0 = g_v = g_w = u_\Sigma = 0$ and some modified data $(\hat{f}, \hat{g}_1, \hat{g}_2, \hat{g}_3)$ in the right regularity classes and with vanishing trace at $t = 0$ whenever it exists. Observe that the compatibility conditions on the modified data at $\partial\Sigma$ read as follows.

$$[\![\hat{g}_2]\!] = [\![\partial_3 \hat{g}_2]\!] = 0, \text{ and } [\![P_{S_1} \hat{g}_1]\!] = P_{S_1} [\![\hat{g}_1]\!] = 0$$

Note that this is in general not the case if $[\![\mu]\!] \neq 0$. Therefore it follows that

$$P_{S_1} \hat{g}_1 \in {}_0W_p^{1/2-1/2p}(J; L_p(S_1)^3) \cap L_p(J; W_p^{1-1/p}(S_1)^3),$$

and

$$\hat{g}_2 \in {}_0W_p^{1-1/2p}(J; L_p(S_1)) \cap L_p(J; W_p^{2-1/p}(S_1)).$$

Since the modified data $\hat{g}_j$ also satisfy the compatibility conditions at ∂S_2, we may solve (5.35) by Lemma 5.9 with $\mu = 1$, $f = \hat{f}$, $g_1 = P_{S_1} \hat{g}_1$, $g_2 = \hat{g}_2$, $g_3 = \hat{g}_3$ and $u_0 = 0$. This in turn implies that problem (5.41) is well-posed, provided that $\mu_1 = \mu_2 = 1$.

Step 2: In the second step we consider the case $[\![\rho]\!] \neq 0$, $[\![\mu]\!] \neq 0$. Let us first reduce (5.41) with $u_0 = 0$ to the case $g_1 = g_2 = g_3 = 0$. To this end will apply Lemma 5.9 twice. First we extend $g_j^+ := g_j|_{x_3 \in (0,H_2)}$ by some (higher order) reflections at $\{x_3 = 0\}$ to some functions

$$\tilde{g}_1^+ \in {}_0W_p^{1/2-1/2p}(J; L_p(S_1)^3) \cap L_p(J; W_p^{1-1/p}(S_1)^3)$$

and

$$\tilde{g}_2^+ \in {}_0W_p^{1-1/2p}(J; L_p(S_1)) \cap L_p(J; W_p^{2-1/p}(S_1)),$$

such that $\tilde{g}_j^+|_{x_3=H_1}=0$. Then, we solve (5.35) with $\mu=\mu_2$, $f=0$, $g_1=P_{S_1}\tilde{g}_1^+$, $g_2=\tilde{g}_2^+$, $g_3|_{x_3=H_2}=g_3^+$ and $g_3|_{x_3=H_1}=0$ to obtain a unique solution

$$\tilde{u}^+\in{}_0H_p^1(J;L_p(\Omega)^3)\cap L_p(J;H_p^2(\Omega)^3).$$

Repeating the same procedure for $g_j^-:=g_j|_{x_3\in(H_1,0)}$ yields a unique solution

$$\tilde{u}^-\in{}_0H_p^1(J;L_p(\Omega)^3)\cap L_p(J;H_p^2(\Omega)^3).$$

Define $\tilde{u}:=\tilde{u}^+\chi_{G\times(0,H_2)}+\tilde{u}^-\chi_{G\times(H_1,0)}$. If u solves (5.41) with $u_0=0$, then $u-\tilde{u}$ solves (5.41) with $u_0=0$, $g_1=0$, $g_2=0$ $g_3=0$ and some modified data $(\hat{f},\hat{g}_v,\hat{g}_w,\hat{u}_\Sigma)$ which are subject to the following compatibility conditions at $\partial\Sigma$:

$$\hat{u}_\Sigma\cdot\nu_{S_1}=0,\ \partial_{\nu_{S_1}}\hat{w}_\Sigma=0,\ \hat{g}_v\cdot\nu_{\partial\Sigma}=0 \tag{5.44}$$

and

$$P_{\partial\Sigma}\left((\nabla_{x'}\hat{v}_\Sigma+\nabla_{x'}\hat{v}_\Sigma^{\mathsf{T}})\nu_{\partial\Sigma}\right)=0. \tag{5.45}$$

Step 3: Let ${}_0\mathbb{E}(T):={}_0H_p^1(J;L_p(\Omega)^3)\cap L_p(J;H_p^2(\Omega\backslash\Sigma)^3)$ and denote by ${}_0\mathbb{F}(T)$ the space of data (f,g_j,u_Σ), $j\in\{v,w,1,2,3\}$ such that the compatibility conditions (11)-(15) in Lemma 5.10 are satisfied. Define $L:{}_0\mathbb{E}(T)\to{}_0\mathbb{F}(T)$ by the left side of (5.41) without the initial condition. By means of a localization procedure one can show that L satisfies the a priori estimate

$$\|u\|_{{}_0\mathbb{E}(T)}\le M\|Lu\|_{{}_0\mathbb{F}(T)}. \tag{5.46}$$

This can be seen as in the proof of Lemma 5.9. Indeed, the charts which intersect ∂S_2 and $\partial\Sigma$ may be treated as in Subsections 2.3.1 & 2.3.3, respectively, while the treatment of the remaining charts is well-known. Note that there is no need to carry any correction terms as in the proof of Lemma 5.9, since for the proof of (5.46) one already starts with a solution of (5.41). Therefore, the compatibility conditions at ∂S_2 and $\partial\Sigma$ are necessarily satisfied.

Next, we set

$$\begin{aligned}{}_0\tilde{\mathbb{E}}(T):=\{u\in{}_0H_p^1(J;L_p(\Omega)^3)\cap L_p(J;H_p^2(\Omega\backslash\Sigma)^3):&\\ u|_{S_2}=0,\ u|_{S_1}\cdot\nu_{S_1}=0,\ P_{S_1}\left((\nabla u+\nabla u^{\mathsf{T}})\nu_{S_1}\right)=0\},&\end{aligned}$$

and denote by ${}_0\tilde{\mathbb{F}}(T)$ the space of data (f,g_v,g_w,u_Σ) together with the compatibility conditions (5.44) & (5.45) at $\partial\Sigma$. Note that

$$P_{S_1}\left((\nabla u+\nabla u^{\mathsf{T}})\nu_{S_1}\right)=0\Leftrightarrow P_{S_1}\left(\mu(\nabla u+\nabla u^{\mathsf{T}})\nu_{S_1}\right)=0$$

at $S_1\backslash\partial\Sigma$. Define $\tilde{L}:{}_0\tilde{\mathbb{E}}(T)\to{}_0\tilde{\mathbb{F}}(T)$ by

$$\tilde{L}u=\begin{pmatrix}\partial_t(\rho u)-\mu\Delta u\\ [\![\mu\partial_3 v]\!]+[\![\mu\nabla_{x'}w]\!]\\ [\![\mu\partial_3 w]\!]\\ [\![u]\!]\end{pmatrix}.$$

Since the norm in ${}_0\tilde{\mathbb{E}}(T)$ is the same as in ${}_0\mathbb{E}(T)$ and since

$$\|Lu\|_{{}_0\mathbb{F}(T)}=\|\tilde{L}u\|_{{}_0\tilde{\mathbb{F}}(T)}$$

for $u \in {}_0\tilde{\mathbb{E}}(T)$, it follows from (5.46) that $\tilde{L}$ is injective with closed range, i.e. $\tilde{L}$ is a semi-Fredholm operator. It is also crucial to observe that the constant $M > 0$ is uniform on compact sets of $\mu > 0$ and $\rho > 0$, by continuity.

We replace the coefficients $(\rho_1, \rho_2, \mu_1, \mu_2)$ by

$$(\rho_1^\tau, \rho_2^\tau, \mu_1^\tau, \mu_2^\tau) := \tau(\rho_1, \rho_2, \mu_1, \mu_2) + (1-\tau)(1,1,1,1), \quad \tau \in [0,1],$$

and denote by $\tilde{L}_\tau : {}_0\tilde{\mathbb{E}}(J) \to {}_0\tilde{\mathbb{F}}(J)$ the corresponding operator which is induced by replacing ρ and μ with ρ^τ and μ^τ, resectively. Note that $\tilde{L}_\tau$ satisfies the estimate

$$\|u\|_{{}_0\tilde{\mathbb{E}}(T)} \le M\|\tilde{L}_\tau u\|_{{}_0\tilde{\mathbb{F}}(T)},$$

with some constant $M > 0$ which is uniform with respect to $\tau \in [0,1]$. Hence $\tilde{L}_\tau$ is semi-Fredholm for each $\tau \in [0,1]$. By Step 1 of the proof, we already know that L_0 is a Fredholm operator with index zero. The continuity method for semi-Fredholm operators implies that L_1 is Fredholm with index zero as well. We remark that the reduction obtained in Step 2 of the proof is essential, since otherwise the viscosity coefficient μ appears in the definition of $\tilde{\mathbb{F}}(T)$. Replacing μ by μ^τ, it would follow that $\tilde{\mathbb{F}}(T)$ depends on τ as well.

2. The strategy for proof of the second assertion is the same as in the proof of Lemma 5.9. Will will not repeat the arguments. □

5.4. **Miscellaneous results.** Let $G \subset \mathbb{R}^{n-1}$, $n \in \{2,3\}$ be a bounded domain with boundary $\partial G \in C^1$ and define $\Omega := G \times (H_1, H_2)$, with $H_1 < 0 < H_2$. Furthermore, let $\Sigma := G \times \{0\}$, $S_1 := \partial G \times (H_1, H_2)$ and $S_2 := \bigcup_{j=1}^2 \{G \times \{H_j\}\}$. Define $x' = (x_1, \ldots, x_{n-1})^\mathsf{T}$ and $x = (x', x_n)^\mathsf{T}$. Assume that $h : G \to (H_1, H_2)$ is C^1 and set

$$\Gamma := \{x = (x', x_n) \in \Omega : x_n = h(x'),\ x' \in G\},$$

that is, Γ is an $(n-1)$-dimensional manifold in Ω which is given as the graph of the height function h over Σ.

Proposition 5.11 (Divergence theorem in cylindrical domains)**.** *For each $u \in H_2^1(\Omega\backslash\Sigma)^n$ the following identity holds.*

$$\int_\Omega \operatorname{div} u \, dx = \int_{S_1} u|_{S_1} \cdot \nu_{S_1} \, dS_1 + \int_{S_2} u|_{S_2} \cdot \nu_{S_2} \, dS_2 - \int_\Gamma [\![u]\!] \nu_\Gamma \, d\Gamma,$$

where ν_{S_j} are the outer unit normals on S_j and ν_Γ is the normal on Γ pointing from

$$\Omega_1 := \{x = (x', x_n) \in \Omega : x_n < h(x'),\ x' \in G\}$$

to $\Omega_2 := \Omega\backslash\overline{\Omega_1}$.

Proof. The proof follows from the fact that Ω_j are both Lipschitz domains. Indeed, it is well-known that the divergence theorem is valid for Lipschitz domains, see for example [17, Section 4.3]. □

Last but not least, we need an auxiliary result which is crucial for the proof of local well-posedness in Section 4. Here and in the sequel, Dv denotes the symmetric part of the gradient ∇v.

Proposition 5.12. *Let $p > 2$, $G \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial G \in C^2$ and outer unit normal vector field ν which is C^1 in a neighborhood of ∂G. If $v \in W_p^2(G;\mathbb{R}^2)$ and $h \in W_p^{3-1/p}(G)$ such that $(v|\nu) = \partial_\nu h = 0$ and $P_{\partial G}[(Dv)\nu] = 0$, then $\partial_\nu(v|\nabla h) = 0$.*

Proof. An easy computation shows that

$$\partial_\nu(v|\nabla h) = (\partial_\nu v|\nabla h) + (v|\nabla^2 h\nu),$$

where $\partial_\nu v := \nabla v^{\mathsf{T}}\nu$.

Note that $P_{\partial G}[(Dv)\nu] = 0$ implies that $((Dv)\nu|\nabla h) = 0$, since by assumption $\partial_\nu h = 0$. This in turn yields $(\partial_\nu v|\nabla h) = -(\nabla v\nu|\nabla h)$. Making use of the representation $\nabla h = \tau\partial_\tau h + \nu\partial_\nu h = \tau\partial_\tau h$, where $\tau \in \mathbb{R}^2$ with $|\tau| = 1$ and $(\tau|\nu) = 0$, we obtain

$$(\nabla v\nu|\nabla h) = ((\nabla h \cdot \nabla)v|\nu) = \partial_\tau h(\partial_\tau v|\nu) = -\partial_\tau h(v|\partial_\tau\nu).$$

Here we made use of the assumption $(v|\nu) = 0$ and $\nabla h \cdot \nabla := \sum_{j=1}^2 \partial_j h\partial_j$.

Concentrating on the term $(v|\nabla^2 h\nu)$, we obtain

$$\begin{aligned}(v|\nabla^2 h\nu) &= \sum_{i,j=1}^2 v_i\partial_i\partial_j h\nu_j = \sum_{i,j=1}^2 [v_i\partial_i(\partial_j h\nu_j) - v_i\partial_j h\partial_i\nu_j] \\ &= (v\cdot\nabla)\partial_\nu h - \sum_{i,j=1}^2 v_i\partial_j h\partial_i\nu_j = (v|\tau)\partial_\tau\partial_\nu h + (v|\nu)\partial_\nu^2 h - \sum_{i,j=1}^2 v_i\partial_j h\partial_i\nu_j \\ &= -\sum_{i,j=1}^2 v_i\partial_j h\partial_i\nu_j,\end{aligned}$$

since $(v|\nu) = \partial_\nu h = 0$. Here it is important to observe that $\partial_\tau\partial_\nu h = 0$, whenever $\partial_\nu h = 0$, since ∂_τ denotes the derivative in tangential direction.

Note that

$$\sum_{i,j=1}^2 v_i\partial_j h\partial_i\nu_j = ((v\cdot\nabla)\nu|\nabla h) = (v|\tau)(\partial_\tau\nu|\nabla h) = (v|\tau)\partial_\tau h(\partial_\tau\nu|\tau) = \partial_\tau h(\partial_\tau\nu|v),$$

since $v = \tau(v|\tau) + \nu(v|\nu) = \tau(v|\tau)$ and $\nabla h = \tau\partial_\tau h$. Finally, this yields

$$\partial_\nu(v|\nabla h) = \partial_\tau h[(\partial_\tau\nu|v) - (\partial_\tau\nu|v)] = 0.$$

The proof is complete. □

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