



## SOLVING THE REGULATOR PROBLEM FOR THE ONE-DIMENSIONAL SCHRÖDINGER EQUATION VIA BACKSTEPPING

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**ABSTRACT.** We investigate the regulator problem (tracking and disturbance rejection) for a system (plant) described by a boundary controlled anti-stable linear one-dimensional Schrödinger equation, using the backstepping approach. The output to be controlled is not required to be measurable and its observation operator is assumed to be admissible for a certain operator semigroup that is related to the operator semigroup of the original plant. We consider both the state feedback and the output feedback regulator problem. In the latter case, the measurement from the Schrödinger equation is taken at the boundary. First we show that the open-loop system is well-posed. We design a state feedback control law that solves the regulator problem by the backstepping method. Then, a finite-dimensional reference observer and an infinite-dimensional disturbance observer are designed. Putting these together, we obtain an output feedback controller with internal loop that achieves output regulation.

### 1. INTRODUCTION AND PROBLEM FORMULATION

The *regulator problem* is one of the fundamental issues in control theory. It concerns tracking a reference signal  $r$  with a certain output  $y$  of the plant, while rejecting a disturbance signal  $d$ , where both  $r$  and  $d$  are generated by a marginally stable finite-dimensional exosystem. In the *state feedback regulator problem*, the controller has access to the state of the exosystem and also to the state of the plant. In the *output feedback regulator problem*, the controller has access to a measurement output  $y_m$  (that may be different from  $y$ ) and also to  $r$ . (This is related to the more often encountered *error feedback regulator problem*, where the signal available to the controller is  $e_y = r - y$ .) It is also required that the closed-loop system (not including the exosystem) should be stable, in some suitable sense (e.g., exponentially).

The main approach to the output (or error) feedback regulator problem is the *internal model principle* [5, 8]. Research in this branch of control theory has been

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active for over 30 years [1, 2, 11, 12, 13, 14, 15, 24, 22, 25]. The first results concerning the regulator problem were developed for lumped parameter linear systems, see [5, 8]. These results were extended to distributed parameter systems in [2], where the control and observation operators are bounded, and in [22, 24, 25], where the control and observation operators are unbounded but admissible. In all these references, the exosystem is assumed to be finite-dimensional, while in [10, 11, 23], it is infinite-dimensional. Another powerful method in dealing with the regulator problem is the backstepping approach. In [6], the regulator problem for a boundary controlled parabolic PDEs is solved using the backstepping approach. This method is again used for the robust output regulation of parabolic PDEs in [7]. An interesting recent work is [16], where based on backstepping, the output tracking problem is considered for a general  $2 \times 2$  system of first order linear hyperbolic PDEs, but no disturbances are taken into consideration. Adaptive control is used for output tracking for the Schrödinger equation in [17], where the system is exponentially stable and the disturbance acts at the boundary. For the optimal regularity, sharp uniform decay rates and observability of Schrödinger equations in several space dimensions, we refer to the work of Irena Lasiecka and collaborators [18, 19, 20, 21].

We consider the following one-dimensional Schrödinger equation with Neumann boundary control and both distributed and boundary disturbance, with  $t \geq 0$ :

$$(1.1) \quad \begin{cases} z_t(x, t) = -iz_{xx}(x, t) + h(x)z(x, t) + g(x)d_1(t), & 0 < x < 1, \\ z_x(0, t) = -iqz(0, t) + d_2(t), & z_x(1, t) = u(t), \\ z(x, 0) = z_0(x), & 0 \leq x \leq 1, \\ y(t) = C_e[z(\cdot, t)], & y_m(t) = z(1, t). \end{cases}$$

We denote by  $z'(x, t)$  or  $z_x(x, t)$  the derivative of  $z(x, t)$  with respect to  $x$  and by  $\dot{z}(x, t)$  or  $z_t(x, t)$  the derivative of  $z(x, t)$  with respect to  $t$ .  $u(t)$  is the control input signal,  $y$  is the output signal to be controlled,  $y_m$  is the measurement (the information available to the controller),  $d_1(t), d_2(t)$  are the disturbances,  $z_0$  is the initial state,  $q > 0$  and  $h, g \in C[0, 1]$  are known. The system (1.1) is a typical unmatched boundary control problem: the control  $u$  acts on one end of the domain and one disturbance  $d_2$  acts on the other end (while the other disturbance  $d_1$  acts distributed).

We consider the system (1.1) in the energy state space  $\mathbb{H} = L^2[0, 1]$  with the usual inner product and norm. We will also use the Sobolev spaces  $H^1(0, 1)$  and  $H^2(0, 1)$ , with their usual norms. If  $z \in C([0, \infty), \mathbb{H})$ , then instead of  $[z(t)](x)$  we write  $z(x, t)$ . The observation operator  $C_e$  in (1.1) is a bounded linear functional on  $H^2(0, 1)$  (not specified). We call  $C_e$  *bounded* if it has a continuous extension to  $\mathbb{H}$  and *unbounded* otherwise.

We will often need to refer to the *unperturbed system* (perhaps not the best name) that is obtained from (1.1) by setting  $d_1(t) = 0$  (for all  $t \geq 0$ ), and also  $h(x) = 0$

(for all  $x \in [0, 1]$ ):

$$(1.2) \quad \begin{cases} z_t(x, t) = -iz_{xx}(x, t) & 0 < x < 1, \\ z_x(0, t) = -iqz(0, t) + d_2(t), & z_x(1, t) = u(t), \\ z(x, 0) = z_0(x), & 0 \leq x \leq 1, \\ y(t) = C_e[z(\cdot, t)], & y_m(t) = z(1, t). \end{cases}$$

We introduce the operator  $A$  as the generator of the operator semigroup  $\mathbb{T}$  that describes the evolution of the state  $z(\cdot, t)$  of (1.2) in  $\mathbb{H}$  if the inputs are  $d_2 = 0$  and  $u = 0$ :

$$(1.3) \quad Af = -if'', \quad D(A) = \{f \in H^2(0, 1) \mid f'(0) = -iqf(0), f'(1) = 0\}.$$

We shall investigate this semigroup in Lemma 3.1. **We assume that  $C_e$  (restricted to  $D(A)$ ) is an admissible observation operator for the operator semigroup  $\mathbb{T}$  generated by  $A$ .** The concept of admissible observation operator will be recalled at the beginning of Sect. 2.

For instance, the above assumption is true if  $C_e$  is the sum of a point observation operator and a distributed observation operator, which means that

$$(1.4) \quad y(t) = C_e[z(\cdot, t)] = \theta z(x_0) + \int_0^1 c(x)z(x, t)dx,$$

where  $\theta \in \mathbb{C}$ ,  $x_0 \in [0, 1]$  and  $c \in L^2[0, 1]$  (the proof of this is similar to the proof of Lemma 3.1).

A triple  $(z, [\frac{d_2}{u}], y)$  is called a *classical solution* of (1.2) on  $[0, \infty)$  if:

- (a)  $z \in C^1([0, \infty); \mathbb{H})$ ,
- (b)  $d_2, u, y \in C[0, \infty)$ ,
- (c)  $z(t) \in H^2(0, 1)$  holds for all  $t \geq 0$ ,
- (d) (1.2) holds for all  $t \geq 0$ .

The system (1.2) has many classical solutions. Indeed, we show in Proposition 3.3 that if  $d_2, u \in H_{loc}^1(0, \infty)$  and  $z_0 \in H^2(0, 1)$  are such that  $z'_0(0) = -iqz_0(0) + d_2(0)$  and  $z'_0(1) = u(0)$ , then (1.2) has a corresponding classical solution on  $[0, \infty)$ . A similar statent holds for (1.1), see Corollary 3.4. Moreover, the systems (1.1) and (1.2) are well-posed, see Proposition 3.5.

We suppose, as is common in regulator theory, that there exists a linear system with no input, referred to as the *exosystem* (sometimes called the exogenous system), that generates both the disturbances  $d_1, d_2$  and the reference  $r$  (these are all scalar signals):

$$(1.5) \quad \begin{aligned} \dot{w}(t) &= Sw(t), \quad t > 0, \quad w(0) = w_0 \in \mathbb{R}^{n_w}, \\ d_1(t) &= p_1^\top w(t) = q_{d_1}^\top w_d(t), \quad t \geq 0, \\ d_2(t) &= p_2^\top w(t) = q_{d_2}^\top w_d(t), \quad t \geq 0, \\ r(t) &= p_r^\top w(t) = q_r^\top w_r(t), \quad t \geq 0. \end{aligned}$$

Here,  $S$  is a block diagonal matrix  $S = \text{diag}(S_d, S_r)$ , which leads with  $w = [\frac{w_d}{w_r}]$  to the signal models  $\dot{w}_d = S_d w_d$ ,  $w_d(0) = w_{d_0} \in \mathbb{C}^{n_d}$ , and  $\dot{w}_r = S_r w_r$ ,  $w_r(0) = w_{r_0} \in \mathbb{C}^{n_r}$ ,  $n_d + n_r = n_w$ . Clearly  $q_{d_1}, q_{d_2} \in \mathbb{C}^{n_d}$ . **We assume that  $S$  is a diagonalizable matrix, all its eigenvalues are on the imaginary axis, the**

**eigenvalues of  $S_d$  are distinct and  $(q_r^\top, S_r)$  is observable.** The disturbances cannot be measured and the reference signal is available to the controller.

**Our objective** is to design an output feedback regulator such that for all initial states of the systems (1.1) and (1.5), the following requirements are satisfied: (i) All the internal signals are bounded. (ii) If the observation operator  $C_e$  is bounded, then we design a state feedback control law, using the state  $z(\cdot, t)$  of (1.1) as well as the state  $w(t)$  of (1.5), such that the tracking error  $e_y = y - r$  is exponentially vanishing: there exist constants  $m_0, \mu_0 > 0$  such that

$$(1.6) \quad |e_y(t)| \leq m_0 e^{-\mu_0 t} \quad \forall t \geq 0.$$

Based on this, we also design an output feedback controller, a dynamical system with inputs  $y_m(t)$  and  $r(t)$ , such that in the closed-loop system, (1.6) holds.

Alternatively, if  $C_e$  is unbounded but admissible, then we design a state feedback controller and an output feedback controller (with internal loop), such that for some  $\alpha < 0$ ,

$$(1.7) \quad e_y \in L_\alpha[0, \infty),$$

where  $L_\alpha[0, \infty)$  is a weighted function space defined by

$$L_\alpha^2[0, \infty) := \left\{ f \in L_{loc}^2[0, \infty) \mid \int_0^\infty e^{-2\alpha t} |f(t)|^2 dt < \infty \right\}.$$

For the concept of stabilizing controller with internal loop we refer to [32, 4]. Essentially it means that the controller is well-posed and to create the well-posed and stable closed-loop system, we have to close two feedback loops: one involving the plant and the controller and another one (called the internal loop) involving the controller only. Closing the internal loop on the controller only (without the plant) may lead to a non-well-posed system.

The outline of the paper is as follows: In Sect. 2 we give a bit of mathematical background on compatible system nodes, admissibility and well-posedness. In Sect. 3 we derive various properties of the Schrödinger equation system (1.1), which we reformulate in the operator theoretic language. In Sect. 4 we solve the state feedback regulator problem, while using backstepping for the stabilization. Sect. 5 is devoted to the design of an observer for the combined system (1.1) and (1.5), using again a backstepping transformation. In Sect. 6, based on the estimated state from the observer, we show how to solve the output feedback regulator problem.

## 2. SOME BACKGROUND ON WELL-POSED SYSTEM NODES

In this section we recall some general facts on admissible control and observation operators, compatible system nodes, classical solutions, well-posedness, transfer functions, feedback and closed-loop systems, following [27], [28], [29] and [31]. For a better understanding of these topics and for the proofs, the reader is advised to look up the mentioned references.

Let  $X, U$  and  $Y$  be Hilbert spaces, let  $\mathbb{T}_t$  be a strongly continuous semigroup of operators on  $X$  with generator  $A$ , let  $X_1$  be the space  $D(A)$  with the norm  $\|x\|_1 = \|(\beta I - A)x\|$  and let  $X_{-1}$  be the completion of  $X$  with respect to the norm  $\|x\|_{-1} = \|(\beta I - A)^{-1}x\|$ , where  $\beta$  is an arbitrary (but fixed) element in the resolvent

set  $\rho(A)$ . An operator  $B \in \mathcal{L}(U, X_{-1})$  is called an *admissible control operator* for  $\mathbb{T}$  if for some (hence, for every)  $\tau > 0$  and for every  $u \in L^2([0, \infty); U)$ ,

$$\int_0^\tau \mathbb{T}_{\tau-s} B u(s) ds \in X.$$

In this case, for any  $x_0 \in X$  and any  $u \in L^2_{loc}([0, \infty); U)$  the equation  $\dot{x} = Ax + Bu$  has a unique solution in  $X_{-1}$  that satisfies  $x(0) = x_0$ , and moreover we have  $x \in C([0, \infty); X)$ .

An operator  $C \in \mathcal{L}(X_1, Y)$  is an *admissible observation operator* for  $\mathbb{T}$  if for some (hence, for every)  $\tau > 0$  there exists  $m_\tau > 0$  such that

$$\int_0^\tau \|C \mathbb{T}_t x\|^2 dt \leq m_\tau \|x\|^2 \quad \forall x \in D(A).$$

The  $\Lambda$ -extension of an operator  $C \in \mathcal{L}(X_1, Y)$  (with respect to  $A$ ), denoted  $C_\Lambda$ , is defined as follows:

$$(2.1) \quad C_\Lambda x = \lim_{\lambda \rightarrow \infty} C \lambda (\lambda I - A)^{-1} x$$

and its domain  $D(C_\Lambda)$  consists of those  $x \in X$  for which the above limit exists. In this case, by [28, Proposition 4.3.6], for every  $x \in X$ , the output  $y(t) = C_\Lambda \mathbb{T}_t x$  exists for almost every  $t \geq 0$  and

$$(2.2) \quad y(t) = C_\Lambda \mathbb{T}_t x \Rightarrow y \in L^2_\alpha([0, \infty); Y) \text{ for all } \alpha > \omega_{\mathbb{T}},$$

where  $\omega_{\mathbb{T}}$  is the growth bound of the semigroup  $\mathbb{T}$ . We have that  $C$  is an admissible observation operator for  $\mathbb{T}$  if and only if  $C^*$  is an admissible control operator for  $\mathbb{T}^*$ .

Let  $U, X, Y$  and  $A$  be as above, and let  $B \in \mathcal{L}(U, X_{-1})$ . We introduce the space

$$D(S) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in X \times U \mid Ax + Bu \in X \right\}.$$

We also define the space  $Z \subset X$  that consists of all the vectors  $z \in X$  that can be the first component of a vector in  $D(S)$ :

$$(2.3) \quad Z = D(A) + (\beta I - A)^{-1} B U,$$

which is independent of the choice of  $\beta \in \rho(A)$ . This is a Hilbert space with the norm

$$\|z\|_Z^2 = \inf \left\{ \|x\|_1^2 + \|v\|^2 \mid x \in X_1, v \in U, z = x + (\beta I - A)^{-1} B v \right\}.$$

Let  $C : D(C) \rightarrow Y$  be such that  $Z \subset D(C)$  and the restriction of  $C$  to  $Z$  is in  $\mathcal{L}(Z, Y)$ . Finally, let  $D \in \mathcal{L}(U, Y)$ . Then  $(A, B, C, D)$  is called a *compatible system node* on  $(U, X, Y)$ . (We mention that we took a short-cut here: in the cited references, and several others, the more general and complicated concept of system node is introduced first, and compatible system nodes are introduced later as a special case. It is easy to show that our definition above is equivalent to the one in [27, 29]. In the cited references, the notation  $\overline{C}$  appears instead of  $C$ , and  $C$  is  $\overline{C}$  restricted to  $D(A)$ .)

To a compatible system node as above we associate its *system operator*  $S : D(S) \rightarrow X \times Y$ :

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The compatible system node is usually associated with the equation

$$(2.4) \quad \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \quad \forall t \geq 0,$$

where  $u, x$  and  $y$  have the meaning of input, state and output functions.  $B$  and  $C$  are called the *control operator* and the *observation operator* of the system node, respectively.

In the spirit of [27, Sect. 3], [29, Sect. 4], we define the following concept:

**Definition 2.1.** Let  $S$  be the system operator of a compatible system node  $(A, B, C, D)$  on  $(U, X, Y)$ . A triple  $(x, u, y)$  is called a *classical solution* of (2.4) on  $[0, \infty)$  if:

- (a)  $x \in C^1([0, \infty); X)$ ,
- (b)  $u \in C([0, \infty); U)$ ,  $y \in C([0, \infty); Y)$ ,
- (c)  $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in D(S)$  for all  $t \geq 0$ ,
- (d) (2.4) holds.

**Proposition 2.2.** With the notation of the last definition, if  $u \in C^2([0, \infty); U)$  and  $\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in D(S)$ , then the equation (2.4) has a unique classical solution  $(x, u, y)$  satisfying  $x(0) = x_0$ .

For the proof we refer to Proposition 4.2.11 in [28] (it also appears in various other references). Under the conditions of the above proposition, we have

$$x(t) = \mathbb{T}_t x(0) + \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma \quad \forall t \geq 0.$$

**Definition 2.3.** With the notation of the previous definition,  $(A, B, C, D)$  is *well-posed* if for some (hence, for every)  $\tau > 0$  there is a  $K_\tau > 0$  such that for every classical solution  $(x, u, y)$  of (2.4),

$$\|z(\tau)\|^2 + \int_0^\tau |y(t)|^2 dt \leq K_\tau \left( \|z(0)\|^2 + \int_0^\tau |u(t)|^2 dt \right).$$

Here,  $\mathbb{T}$  is the operator semigroup generated by  $A$ .

We will use the term “well-posed system node” instead of the cumbersome “well-posed compatible system node”. There is a good justification for this, see [29, Proposition 4.5].

**Proposition 2.4.** We use the notation of Definition 2.1 and we denote again by  $\mathbb{T}$  the operator semigroup generated by  $A$ .

If  $(A, B, C, D)$  is well-posed, then it follows that  $B$  is an admissible control operator for  $\mathbb{T}$ ,  $C$  (restricted to  $D(A)$ ) is an admissible observation operator for  $\mathbb{T}$ , and the transfer function of  $(A, B, C, D)$ , defined by

$$(2.5) \quad \mathbf{G}(s) = C(sI - A)^{-1}B + D \quad \forall s \in \mathbb{C} \text{ with } \operatorname{Re} s > \omega_{\mathbb{T}},$$

is bounded on any half-plane  $\mathbb{C}_\alpha = \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}$ , if  $\alpha > \omega_{\mathbb{T}}$ .

Conversely, if  $B$  is an admissible control operator for  $\mathbb{T}$ ,  $C$  is an admissible observation operator for  $\mathbb{T}$  and  $\mathbf{G}$  is bounded on some right half-plane, then it follows that  $(A, B, C, D)$  is well-posed.

The following simple perturbation result will be useful.

**Proposition 2.5.** *Let  $(A, B, C, D)$  be a well-posed system node on  $(U, X, Y)$  and let  $P \in \mathcal{L}(X)$ . Then  $(A + P, B, C, D)$  is again a well-posed system node on  $(U, X, Y)$ .*

*Proof.* Assume that  $(A, B, C, D)$  is well-posed, hence (according to the previous proposition)  $A$  and  $B$  are admissible for  $\mathbb{T}$ , the semigroup generated by  $A$ . It follows from [28, Theorem 5.4.2 and Corollary 5.5.1] that  $B$  and  $C$  are admissible also for the semigroup generated by  $A + P$ , and the spaces  $X_1$  and  $X_{-1}$  remain the same for  $A + P$ . According to Proposition 2.4, the transfer function  $\mathbf{G}$  from (2.5) is bounded on some right half-plane. Denoting the transfer function of the compatible system node  $(A + P, B, C, D)$  by  $\mathbf{G}_P$ , we have the elementary identity

$$(2.6) \quad \mathbf{G}_P(s) - \mathbf{G}(s) = C(sI - A)^{-1}P(sI - A - P)^{-1}B.$$

The functions  $C(sI - A)^{-1}$  and  $(sI - A - P)^{-1}B$  are bounded on some right half-plane, according to [28, Theorem 4.3.7 and Proposition 4.4.6]. Thus, it follows that  $\mathbf{G}_P$  is bounded on some right half-plane. Now it follows from Proposition 2.4 that  $(A + P, B, C, D)$  is well-posed.  $\square$

We mention that the above proposition remains valid for a time varying  $P : [0, \infty) \rightarrow \mathcal{L}(X)$ , as long as it is strongly continuous. This is much harder to prove, see [3, Theorems 4.2 and 5.3].

We introduce a special class of well-posed systems, following the terminology in [29], [31], [32], [34] and many other papers. We do this because our systems (1.1) and (1.2) fall into this category (as we shall see), and we will use tools developed for such systems.

**Definition 2.6.** Let  $(A, B, C, D)$  be a well-posed system node on  $(U, X, Y)$ , with transfer function  $\mathbf{G}$  (see (2.5)). We say that this system is *regular* if the limit  $D_0v = \lim_{\lambda \rightarrow \infty, \lambda \in \mathbb{R}} \mathbf{G}(\lambda)v$  exists, for each  $v \in U$ . In this case,  $D_0 \in \mathcal{L}(U, Y)$  is called the *feedthrough operator* of the system.

**Proposition 2.7.** *Suppose that the compatible system node  $(A, B, C, D)$  on  $(U, X, Y)$  is well-posed, and let  $\mathbf{G}$  be its transfer function. Recall  $C_\Lambda$  from (2.1) and the space  $Z$  introduced in (2.3). We have  $Z \subset D(C_\Lambda)$  if and only if the system is regular.*

*If the system is regular, then the quadruple  $(A, B, C, D)$  may be replaced with the equivalent quadruple  $(A, B, C_\Lambda, D_0)$  (where  $D_0$  is the feedthrough operator of the system), in the sense that this new quadruple has the same system operator  $S$  and the same transfer function.*

The following proposition recalls some properties of output feedback for regular linear systems (for the proof see [31]). In the proposition we make the simplifying assumption  $K_f D = 0$  (true in our application in Sect. 4) that greatly simplifies the formulas.

**Proposition 2.8.** *Let  $(A, B, C, D)$  be a regular linear system on  $(U, X, Y)$ , with transfer function  $\mathbf{G}$ . Assume that the feedthrough operator of this system is  $D$ , and let  $K_f \in \mathcal{L}(Y, U)$ . We assume that the function  $I - K_f \mathbf{G}(s)$  has a uniformly bounded inverse for all  $s$  in some right half-plane, and  $K_f D = 0$ . Then  $(A_{cl}, B, (I + DK_f)C_\Lambda, D)$  is a regular linear system on  $(U, X, Y)$ , called the closed-loop system corresponding to  $(A, B, C, D)$  with the output feedback operator  $K_f$ . Here*

$$A_{cl} = A + BK_f C_\Lambda, \quad D(A_{cl}) = \{x \in Z \mid Ax + BK_f C_\Lambda x \in X\}.$$

(The sum  $Ax + BK_f C_\Lambda x$  is computed in  $X_{-1}$ .) In particular,  $(I + DK_f)C_\Lambda$  is an admissible observation operator for the semigroup generated by  $A_{cl}$ .

Intuitively, the closed-loop system  $(A_{cl}, B, (I + DK_f)C_\Lambda, D)$  is obtained from the original system  $(A, B, C, D)$  via the output feedback  $u = K_f y + \rho$  (where  $\rho$  is the new input function). The transfer function of the closed-loop system is  $\mathbf{G}_{cl} = \mathbf{G}(I - K_f \mathbf{G})^{-1} = (I - \mathbf{G}K_f)^{-1}\mathbf{G}$ .

Let  $\mathbf{G}$  be a function defined on some domain in  $\mathbb{C}$  that contains a right half-plane, with values in a normed space. Following [34], we say that  $\mathbf{G}$  is *strictly proper* if

$$\lim_{\operatorname{Re} s \rightarrow \infty} \|\mathbf{G}(s)\| = 0, \quad \text{uniformly with respect to } \operatorname{Im} s.$$

In other words, there exists an  $\alpha \in \mathbb{R}$  and a continuous function  $\beta : (\alpha, \infty) \rightarrow (0, \infty)$  such that

$$(2.7) \quad \|\mathbf{G}(s)\| \leq \beta(\operatorname{Re} s) \quad \forall s \in \mathbb{C}_\alpha \quad \text{and} \quad \lim_{\xi \rightarrow \infty} \beta(\xi) = 0.$$

The notation  $\mathbb{C}_\alpha$  has been introduced in Proposition 2.4. The above concept generalizes the well-known one of strictly proper rational transfer function. A well-posed system node is called *strictly proper* if its transfer function is strictly proper. Clearly such systems are regular and their feedthrough operator is zero.

The following proposition shows a curious property of certain semigroup generators  $A$ : if  $B$ ,  $C$  and  $D$  are such that  $(A, B, C, D)$  is a compatible system node, then the admissibility of  $B$  and  $C$  for the semigroup generated by  $A$  implies the well-posedness of  $(A, B, C, D)$ . Moreover, it turns out that the compatible system node  $(A, B, C_\Lambda, 0)$  is strictly proper.

**Proposition 2.9.** *Let  $X = l^2$ ,  $a > 0$  and let the operator  $A : D(A) \rightarrow X$  be defined on sequences  $x = (x_k)$  ( $k \in \mathbb{N}$ ) by*

$$(Ax)_k = iak^2 x_k, \quad D(A) = \left\{ x \in l^2 \mid \sum_{k \in \mathbb{N}} k^4 |x_k|^2 < \infty \right\}.$$

*Then  $A$  is the generator of the diagonal unitary operator group*

$$(\mathbb{T}_t x)_k = e^{iak^2 t} x_k \quad \forall x \in X, \quad t \geq 0.$$

*Let  $B \in X_{-1}$  be an admissible control operator for  $\mathbb{T}$  (for the input space  $\mathbb{C}$ ) and let the bounded linear functional  $C : X_1 \rightarrow \mathbb{C}$  be an admissible observation operator for  $\mathbb{T}$  (for the output space  $\mathbb{C}$ ).*

*Then  $(A, B, C_\Lambda, 0)$  is a compatible system node that is well-posed and strictly proper.*

*Proof.* The fact that  $A$  generates the indicated operator group  $\mathbb{T}$  is easy and standard material in semigroup theory, see e.g. [28, Proposition 2.6.5]. Let  $\{e_1, e_2, e_3, \dots\}$  be the standard orthonormal basis of  $l^2$ . We denote by  $b_k$  and  $c_k$  the components of  $B$  and  $C$ , respectively:

$$b_k = \langle B, e_k \rangle, \quad c_k = C e_k \quad \forall k \in \mathbb{N}.$$

It follows from the Carleson measure criterion for admissibility (see e.g. [28, Proposition 5.3.5]) that the sequences  $(b_k)$  and  $(c_k)$  are bounded. We want to check that for some (hence for every)  $s \in \mathbb{C}_0$  we have  $(sI - A)^{-1}B \in D(C_\Lambda)$ . For this, we compute



$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} C\lambda(\lambda I - A)^{-1}(sI - A)^{-1}B \\ &= \lim_{\lambda \rightarrow \infty} \sum_{k \in \mathbb{N}} \frac{b_k c_k \lambda}{(\lambda - iak^2)(s - iak^2)} = \sum_{k \in \mathbb{N}} \frac{b_k c_k}{s - iak^2}. \end{aligned}$$

This shows that indeed  $(sI - A)^{-1}B \subset D(C_\Lambda)$ , which implies that  $Z \subset D(C_\Lambda)$ , and

$$C_\Lambda(sI - A)^{-1}B = \sum_{k \in \mathbb{N}} \frac{b_k c_k}{s - iak^2}.$$

Hence, for any  $s \in \mathbb{C}_0$  we have, denoting  $\theta = \operatorname{Re} s/a$  and  $\omega = \operatorname{Im} s$ ,

$$|C_\Lambda(sI - A)^{-1}B| \leq \sum_{k \in \mathbb{N}} |b_k c_k| \frac{1}{|\theta a + i(\omega - ak^2)|}.$$

Using the elementary inequality  $|\tilde{a} + i\tilde{b}| \geq (|\tilde{a}| + |\tilde{b}|)/\sqrt{2}$  (for any  $\tilde{a}, \tilde{b} \in \mathbb{R}$ ), we get

$$(2.8) \quad |C_\Lambda(sI - A)^{-1}B| \leq m\sqrt{2} \sum_{k \in \mathbb{N}} \frac{1}{\theta a + |\omega - ak^2|} = \frac{m\sqrt{2}}{a} \sum_{k \in \mathbb{N}} \frac{1}{\theta + |\mu - k^2|},$$

where  $m = \sup |b_k c_k|$  and  $\mu = \omega/a$ . Considering the case  $\mu \leq 0$ , we get

$$(2.9) \quad |C_\Lambda(sI - A)^{-1}B| \leq \frac{m\sqrt{2}}{a} \sum_{k \in \mathbb{N}} \frac{1}{\theta + k^2} \quad \text{for } \operatorname{Re} s = \theta a, \quad \operatorname{Im} s < 0.$$

Now consider the case  $\mu > 0$ , and denote by  $k_\mu$  the largest integer  $k$  satisfying  $k^2 \leq \mu$ . We decompose

$$(2.10) \quad \sum_{k \in \mathbb{N}} \frac{1}{\theta + |\mu - k^2|} = \sum_{1 \leq k \leq k_\mu} \frac{1}{\theta + \mu - k^2} + \sum_{k > k_\mu} \frac{1}{\theta + k^2 - \mu}.$$

It is very easy to see that the second sum on the right side above is bounded by  $\sum_{k \in \mathbb{N}} 1/(\theta + k^2)$ . For the first sum we do the change of discrete variables  $j = k_\mu - k$ , obtaining

$$\begin{aligned} \sum_{1 \leq k \leq k_\mu} \frac{1}{\theta + \mu - k^2} &= \sum_{1 \leq j \leq k_\mu} \frac{1}{\theta + \mu - k_\mu^2 + 2k_\mu j - j^2} \\ &\leq \sum_{1 \leq j \leq k_\mu} \frac{1}{\theta + 2k_\mu j - j^2} \leq \sum_{1 \leq j \leq k_\mu} \frac{1}{\theta + j^2}. \end{aligned}$$

Combining this with our earlier estimate for the second sum in (2.10), it follows that

$$\sum_{k \in \mathbb{N}} \frac{1}{\theta + |\mu - k^2|} \leq 2 \sum_{k \in \mathbb{N}} \frac{1}{\theta + k^2} \quad \text{for } \mu > 0.$$

This, together with (2.8) and (2.9) implies that, for any  $\theta > 0$ ,

$$(2.11) \quad |C_\Lambda(sI - A)^{-1}B| \leq \frac{2\sqrt{2}m}{a} \sum_{k \in \mathbb{N}} \frac{1}{\theta + k^2} \quad \text{for } \operatorname{Re} s = \theta a.$$

If we denote the righ-hand side of (2.11) with  $\beta(\operatorname{Re} s)$  and compare with (2.7), we see that  $C_\Lambda(sI - A)^{-1}B$  is strictly proper. In particular, this transfer function is bounded on any half-plane  $\mathbb{C}_\alpha$  with  $\alpha > 0$ . According to the last part of Proposition 2.4,  $(A, B, C_\Lambda, 0)$  is well-posed.  $\square$

**Corollary 2.10.** *Let  $X$  be a Hilbert space, let  $A : D(A) \rightarrow X$  be the generator of an operator semigroup  $\mathbb{T}$  on  $X$ , let  $B \in \mathcal{L}(\mathbb{C}^m, X_{-1})$  be an admissible control operator for  $\mathbb{T}$  and let  $C \in \mathcal{L}(X_1, \mathbb{C}^p)$  be an admissible observation operator for  $\mathbb{T}$ . Assume that  $A$  is diagonalizable, meaning that there is a Riesz basis  $(\phi_k)$  in  $X$  ( $k \in \mathbb{N}$ ) consisting of eigenvectors of  $A$ , and the corresponding eigenvalues  $\mu_k$  satisfy*

$$\mu_k = iak^2 + \mathcal{O}(1), \quad \text{where } a > 0.$$

*Then  $(A, B, C_\Lambda, 0)$  is a compatible system node that is well-posed and strictly proper.*

Indeed, this follows from Propositions 2.5 and 2.9.

### 3. PROPERTIES OF THE SYSTEM TO BE CONTROLLED

We want to reformulate the equations (1.1) and (1.2) in the abstract operator theory framework. For this, first we introduce a semigroup generator on  $\mathbb{H}$ , a bounded perturbation of  $A$  from (1.3):

$$(3.1) \quad A_h f = A f + h f \quad \forall f \in D(A_h) = D(A).$$

We define the operators  $B_l, B_r$  as follows:

$$B_l = i\delta(\cdot), \quad B_r = -i\delta(\cdot - 1).$$

Here  $\delta$  is the Dirac mass. We denote the adjoints of  $A$ ,  $B_l$  and  $B_r$  by  $A^*$ ,  $B_l^*$  and  $B_r^*$ , respectively, and it is easy to check that

$$\begin{aligned} A^* f &= i f'', \quad D(A^*) = \{f \in H^2(0, 1) \mid f'(0) = i q f(0), f'(1) = 0\}, \\ B_l^* f &= -i f(0), \quad B_r^* f = i f(1) \quad \forall f \in D(A^*). \end{aligned}$$

The operators  $B_l$  and  $B_r$  are the control operators that correspond to the inputs  $d_2$  and  $u$  in the boundary control systems (1.1) as well as (1.2). This can be checked using [28, Remark 10.1.6].

Define  $C_m \in \mathcal{L}(H^1(0, 1), \mathbb{C})$  by  $C_m f = f(1)$ . Then (1.1) can be rewritten in the abstract form

$$(3.2) \quad \begin{cases} \dot{z}(\cdot, t) = A_h z(\cdot, t) + g(\cdot) d_1(t) + B_l d_2(t) + B_r u(t), \\ y(t) = C_e[z(\cdot, t)], \quad y_m(t) = C_m[z(\cdot, t)], \end{cases}$$

which corresponds to the compatible system node  $(A_h, [g(\cdot) \ B_l \ B_r], [\begin{smallmatrix} C_e \\ C_m \end{smallmatrix}], 0)$  on  $(\mathbb{C}^3, \mathbb{H}, \mathbb{C}^2)$ . It is easy to check that for this system node, the space  $Z$  from (2.3) is given by

$$(3.3) \quad Z = H^2(0, 1).$$

The equivalence between (1.1) and (3.2) means that they have the same classical solutions, and this equivalence can be checked using the techniques in [28, Sect. 10.1].

Similarly, the system (1.2) can be rewritten in the abstract form

$$(3.4) \quad \begin{cases} \dot{z}(\cdot, t) = A z(\cdot, t) + B_l d_2(t) + B_r u(t), \\ y(t) = C_e[z(\cdot, t)], \quad y_m(t) = C_m[z(\cdot, t)], \end{cases}$$

which corresponds to the compatible system node  $(A, [B_l \ B_r], [\begin{smallmatrix} C_e \\ C_m \end{smallmatrix}], 0)$  on  $(\mathbb{C}^2, \mathbb{H}, \mathbb{C}^2)$ . For this system node, the space  $Z$  is again given by (3.3).

**Lemma 3.1.** *Let  $A$  be defined by (1.3). Then  $A^{-1}$  exists and it is compact. Hence,  $\sigma(A)$ , the spectrum of  $A$ , consists of isolated eigenvalues of finite algebraic multiplicity. All eigenvalues of  $A$  are located in a vertical strip, they have positive real parts and there exists a sequence of eigenfunctions of  $A$ , which forms a Riesz basis for  $\mathbb{H}$ . Therefore,  $A$  generates an operator group  $\mathbb{T}$  on  $\mathbb{H}$ .*

*The observation operator  $C_m$  is admissible for the group  $\mathbb{T}$ .*

*Proof.* A straightforward computation shows that  $A$  has a bounded inverse on  $\mathbb{H}$  and

$$(A^{-1}\phi)(x) = \frac{-(i+qx) \int_0^1 \phi(y)dy}{iq} - i \int_0^x (x-y)\phi(y)dy.$$

Since the embedding of  $H^1(0,1)$  into  $L^2[0,1]$  is compact, it follows that  $A^{-1}$  is compact. This implies that  $\sigma(A)$  consists of isolated eigenvalues of finite algebraic multiplicity. It is easy to verify that  $\operatorname{Re} \langle Af, f \rangle = q|f(0)|^2 \geq 0$ , which implies that all the eigenvalues of  $A$  have non-negative real parts. Next, we show that there is no eigenvalue on the imaginary axis. Otherwise, suppose that  $Af = i\beta f$  with  $\beta \in \mathbb{R}$  has a nonzero solution, i.e.,

$$(3.5) \quad f''(x) = -\beta f(x), \quad f'(0) = -iqf(0), \quad f'(1) = 0.$$

Multiplying the first equation of (3.5) with  $\overline{f(x)}$  (the conjugate of  $f(x)$ ) and integrating over  $[0,1]$ , it follows from the boundary condition that

$$iq|f(0)|^2 - \int_0^1 |f'(x)|^2 dx = -\beta \int_0^1 |f(x)|^2 dx,$$

which, jointly with  $q > 0$  and taking imaginary part, gives  $f(0) = 0$ . By the second equation of (3.5),  $f'(0) = 0$ . Thus, (3.5) has only the zero solution, a contradiction. Therefore, all the eigenvalues of  $A$  have positive real parts.

Now we consider the eigenvalue problem  $Af = \mu f$  and let  $\mu = -i\lambda^2$ , that is

$$\phi''(x) = \lambda^2 \phi(x), \quad \phi'(0) = -iq\phi(0), \quad \phi'(1) = 0,$$

to yield

$$(3.6) \quad \phi(x) = \frac{\lambda - iq}{\lambda + iq} e^{\lambda x} + e^{-\lambda x},$$

where  $\lambda \in \mathbb{C}$  satisfies

$$(3.7) \quad e^{2\lambda} = \frac{\lambda + iq}{\lambda - iq} = 1 + \frac{2iq}{\lambda - iq} = 1 + \frac{2iq}{\lambda} + \mathcal{O}(|\lambda|^{-2}) \text{ as } |\lambda| \rightarrow \infty.$$

Thus, we have

$$\lambda_n = n\pi i + \mathcal{O}(n^{-1}), \quad n \in \mathbb{N}.$$

Substituting this into (3.7), we get that for this specific case,  $\mathcal{O}(n^{-1}) = q/(n\pi) + \mathcal{O}(n^{-2})$ , hence

$$\lambda_n = n\pi i + \frac{q}{n\pi} + \mathcal{O}(n^{-2}), \quad n \in \mathbb{N}.$$

It follows from here and (3.6) that the asymptotic expressions for eigenpairs of  $A$  are

$$(3.8) \quad \begin{cases} \mu_n = 2q + i(n\pi)^2 + \mathcal{O}(n^{-2}), \\ \phi_n(x) = \cos(n\pi x) + \mathcal{O}(n^{-1}), \end{cases}$$

which implies that all the eigenvalues  $\mu_n$  of  $A$  are located in a vertical strip and the corresponding eigenvectors  $\phi_n$  are quadratically close to an orthonormal basis. By a theorem known as “Bari’s theorem”, see [9, Theorem 6.3] or [33, Theorem 2.4],  $\{\phi_n\}$  forms a Riesz basis for  $\mathbb{H}$ . This shows that the spectrum-determined growth

condition holds for  $A$ . Thus,  $A$  generates an operator semigroup  $\mathbb{T}$  and  $\|\mathbb{T}_t\| \leq Le^{\omega t}$  with  $\omega = \sup\{\operatorname{Re} \lambda \mid \lambda \in \sigma(A)\}$  for some  $L \geq 1$ . Similarly, we can show that  $-A$  generates an operator semigroup. By [28, Proposition 2.7.8],  $\mathbb{T}$  can be extended to a group.

We show that  $C_m$  is admissible for the group  $\mathbb{T}$ . Denote  $c_n = C_m \phi_n$ ,  $n \in \mathbb{N}$ , then according to (3.8) we have  $c_n = \cos(n\pi) + \mathcal{O}(n^{-1})$ . The eigenvalues  $\mu_n$  are in a vertical strip and for large enough  $n$ , the distance between their imaginary parts is bounded from below by a positive number. Thus, the admissibility of  $C_m$  follows from the simple version of the Carleson measure criterion applicable for diagonal operator groups, see [28, Proposition 5.3.5].  $\square$

**Remark 3.2.** By Lemma 3.1, all the eigenvalues of  $A$  have positive real parts. So, if the values  $h(x)$  in (1.1) are non-negative or if its sup norm is sufficiently small, then also the eigenvalues of  $A_h$  have positive real parts. This is why we call the system (1.1) anti-stable. This situation is different from the unstable case in [6], where there are at most finitely many unstable eigenvalues for the system to be controlled. The system (1.1) is different also from the one in [17], where the Schrödinger equation is essentially exponentially stable when the disturbance vanishes.

**Proposition 3.3.** *The operators  $B_l, B_r$  are admissible control operators for  $\mathbb{T}$ . Therefore, for any initial state  $z(0) = z_0 \in \mathbb{H}$  and any  $d_2, u \in L_{loc}^2[0, \infty)$ , the first equation in (3.4) admits a unique solution in  $\mathbb{H}_{-1}$  (in the sense of [28, Definition 4.1.1]) and  $z \in C([0, \infty); \mathbb{H})$ .*

*Moreover, if  $d_2, u \in H_{loc}^1(0, \infty)$  are such that  $Az_0 + B_l d_2(0) + B_r u(0) \in \mathbb{H}$ , then the solution  $z$  satisfies*

$$(3.9) \quad z \in C([0, \infty); Z) \cap C^1([0, \infty); \mathbb{H}).$$

*In this case, the functions  $y$  and  $y_m$  can be defined by the second equation in (3.4) and  $(z, [\frac{d_2}{u}], [\frac{y}{y_m}])$  is a classical solution of (3.4) and also of (1.2).*

Recall that  $Z$  appearing above is given by (3.3). We remark that the condition  $A_h z_0 + B_l d_2(0) + B_r u(0) \in \mathbb{H}$  appearing above is equivalent to

$$z_0 \in H^2(0, 1), \quad \frac{d}{dx} z_0(0) = -iqz(0) + d_2(0), \quad \frac{d}{dx} z_0(1) = u(0).$$

This can be verified using the techniques of boundary control systems in [28, Sect. 10.1].

*Proof.* We prove the admissibility of  $B_l$  for  $\mathbb{T}$ . For this, recall from [28, Theorem 4.4.3] that it suffices to show that  $B_l^*$  is an admissible observation operator for the adjoint semigroup  $\mathbb{T}^*$ . This is equivalent to showing that (i)  $B_l^* A^{*-1}$  is a bounded operator on  $\mathbb{H}$  and (ii) for each  $T > 0$  there exists  $M_T > 0$  such that for every initial state, the output signal  $\eta$  of the system (defined for  $t \geq 0$ )

$$(3.10) \quad \begin{cases} z_t(x, t) = iz_{xx}(x, t), & x \in (0, 1), \\ z_x(0, t) = iqz(0, t), & z_x(1, t) = 0, \\ \eta(t) = z(0, t), \end{cases}$$

satisfies

$$\int_0^T |\eta(t)|^2 dt \leq M_T E(0), \quad \text{where} \quad E(t) = \frac{1}{2} \|z(\cdot, t)\|_{\mathbb{H}}^2.$$

A simple computation shows that  $A^*$  has bounded inverse on  $\mathbb{H}$  and

$$\begin{cases} A^{*-1}\phi = \frac{(-i + qx) \int_0^1 \phi(y) dy}{-iq} - i \int_0^x (x-y)\phi(y) dy, \\ B_l^* A^{*-1}\phi = -\frac{i}{q} \int_0^1 \phi(y) dy. \end{cases}$$

Hence  $B_l^* A^{*-1}$  is bounded on  $\mathbb{H}$ . We differentiate  $E$  with respect to  $t$  along the solution of (3.10) to obtain  $\dot{E}(t) = q|\eta(t)|^2$ , which, together with Lemma 3.1, gives

$$\int_0^T |\eta(t)|^2 dt = \frac{1}{q} [E(T) - E(0)] \leq \frac{1}{q} [1 + Le^{\omega T}] E(0),$$

where  $\omega, L$  are as in the proof of Lemma 3.1. Thus,  $B_l$  is admissible.

The proof of the fact that  $B_r$  is an admissible control operator for  $\mathbb{T}$  is similar. The statement about unique and continuous solutions of (3.4) follows from [28, Proposition 4.2.5]. Finally, the statement for  $d_2, u \in H_{loc}^1(0, \infty)$  follows from [28, Proposition 4.2.10].  $\square$

There is a similar statement for the original system (1.1), formulated abstractly in (3.2):

**Corollary 3.4.** *The operator  $A_h$  from (3.1) generates an operator group  $(e^{A_h t})_{t \in \mathbb{R}}$  on  $\mathbb{H}$  and  $B_l, B_r$  are admissible control operators for this operator group. Therefore, for any initial state  $z(0) = z_0 \in \mathbb{H}$  and any  $d_1, d_2, u \in L_{loc}^2[0, \infty)$ , the first equation in (3.2) admits a unique solution in  $\mathbb{H}_{-1}$  (in the sense of [28, Definition 4.1.1]) and  $z \in C([0, \infty); \mathbb{H})$ .*

*Moreover, if  $d_1, d_2, u \in H_{loc}^1(0, \infty)$  are such that  $A_h z_0 + B_l d_2(0) + B_r u(0) \in \mathbb{H}$ , then the solution  $z$  satisfies (3.9). In this case, the functions  $y$  and  $y_m$  can be defined by the second equation in (3.2) and  $(z, \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \begin{bmatrix} y \\ y_m \end{bmatrix})$  is a classical solution of (3.2), and also of (1.1).*

*Proof.* By Lemma 3.1 and the boundedness of  $h$ , it is clear that  $A_h$  generates a strongly continuous operator group on  $\mathbb{H}$  (this follows, for instance, by applying [28, Theorem 2.11.2] to  $A_h$  and also to  $-A_h$ ). Since  $A_h$  is a bounded perturbation of  $A$ , according to [28, Corollary 5.5.1],  $B_l$  and  $B_r$  are admissible control operators also for  $(e^{A_h t})_{t \geq 0}$ . The end of the proof is now the same as for Proposition 3.3.  $\square$

**Proposition 3.5.** *The compatible system node  $(A_h, [g(\cdot) \ B_l \ B_r], \begin{bmatrix} C_e \\ C_m \end{bmatrix}, 0)$  (which corresponds to the equations (3.2)) is well-posed. Similarly, the compatible system node  $(A, [B_l \ B_r], \begin{bmatrix} C_e \\ C_m \end{bmatrix}, 0)$  (which corresponds to the equations (3.4)) is well-posed. If we replace  $C_e$  with  $C_{e\Lambda}$  (defined as in (2.1)), then both of these system nodes become strictly proper (hence, all these systems are regular).*

*Proof.* We start with the compatible system node  $(A, [B_l \ B_r], \begin{bmatrix} C_e \\ C_m \end{bmatrix}, 0)$ , whose control operator  $B = [B_l \ B_r]$  is known to be admissible from Proposition 3.3 and whose observation operator  $C = \begin{bmatrix} C_e \\ C_m \end{bmatrix}$  is known to be admissible from our assumption on  $C_e$  in Sect. 1 and from Lemma 3.1. We know from (3.8) that  $A$  satisfies the assumptions of Corollary 2.10. Hence, according to this corollary,  $(A, [B_l \ B_r], \begin{bmatrix} C_e \\ C_m \end{bmatrix}, 0)$  is well-posed. According to Proposition 2.5,  $(A_h, [B_l \ B_r], \begin{bmatrix} C_e \\ C_m \end{bmatrix}, 0)$  is also well-posed.

The well-posedness of this system node will not be affected if we add another bounded component to its control operator, changing it to  $[g(\cdot) \ B_l \ B_r]$ .

For the operator  $C_m$  it is not difficult to show that its extension  $C_{m\Lambda}$ , when restricted to  $Z$ , is again  $C_m$ . However for  $C_e$ , which has not been specified, we do not know if this is the case. However, after having replaced  $C_e$  with  $C_{e\Lambda}$ , we can apply Corollary 2.10 to  $(A, [B_l \ B_r], \begin{bmatrix} C_{e\Lambda} \\ C_m \end{bmatrix}, 0)$  to conclude that its transfer function  $\mathbf{G}$  is strictly proper. For the transfer function  $\mathbf{G}_P$  of  $(A_h, [B_l \ B_r], \begin{bmatrix} C_{e\Lambda} \\ C_m \end{bmatrix}, 0)$  we use the identity (2.6), with  $P$  being the operator of pointwise multiplication with the function  $h$ , so that  $A_h = A + P$ . Since the functions  $C(sI - A)^{-1}$  and  $(sI - A - P)^{-1}B$  (with  $C = \begin{bmatrix} C_e \\ C_m \end{bmatrix}$  and  $B = [B_l \ B_r]$ ) are known to be strictly proper, see for instance [28, Theorem 4.3.7 and Proposition 4.4.6], it follows that  $\mathbf{G}_P$  is strictly proper. Finally, when adding the extra component to  $B$ , replacing the earlier  $B$  with  $[g(\cdot) \ B_l \ B_r]$ , then the transfer function remains strictly proper, because the new component  $g(\cdot)$  is a bounded control operator.  $\square$

#### 4. STATE FEEDBACK REGULATION

In this section we will construct a state feedback operator that solves the regulator problem. We denote  $\Omega = \{(x, \xi) \in \mathbb{R}^2 \mid 0 \leq \xi \leq x \leq 1\}$ . First we introduce the *backstepping transformation*

$$(4.1) \quad v(x, t) = \mathcal{F}[z(\cdot, t)](x, t) := z(x, t) - \int_0^x k(x, \xi) z(\xi, t) d\xi,$$

where the kernel function  $k : \Omega \rightarrow \mathbb{R}$  satisfies, for some fixed  $c_s > 0$ ,

$$(4.2) \quad \begin{cases} k_{xx}(x, \xi) - k_{\xi\xi}(x, \xi) = (h(\xi) + c_s)ik(x, \xi), \\ k_\xi(x, 0) + qik(x, 0) = 0, \\ k(x, x) = -\frac{i}{2} \int_0^x (h(\xi) + c_s) d\xi - qi. \end{cases}$$

By [26, Theorem 2.1], the above system of equations has a unique solution  $k \in C^2(\Omega)$ . It can be shown [26, Theorem 2.2] that this transformation is boundedly invertible, and

$$\mathcal{F}^{-1}[v(\cdot, t)](x, t) = v(x, t) + \int_0^x K(x, \xi) v(\xi, t) d\xi,$$

where the kernel function  $K$  is also in  $C^2(\Omega)$ . It is easy to see from (4.1) and the above formula that  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  leave  $C^1$  and  $H^2$  functions invariant:

$$\mathcal{F}C^1[0, 1] \subset C^1[0, 1], \quad \mathcal{F}^{-1}C^1[0, 1] \subset C^1[0, 1],$$

$$\mathcal{F}H^2(0, 1) \subset H^2(0, 1), \quad \mathcal{F}^{-1}H^2(0, 1) \subset H^2(0, 1).$$

The proposed *state feedback law* (applied to classical solutions of (1.1)) is given by a continuous linear functional  $F$  defined on  $H^2(0, 1)$  plus a term applied to the exosystem state  $w$ :

$$(4.3) \quad u(t) = F[z(\cdot, t)] + m_w^\top w(t) = k(1, 1)z(1, t) + \int_0^1 k_x(1, \xi) z(\xi, t) d\xi + m_w^\top w(t),$$

where  $m_w^\top$  is a constant vector to be determined later. With this feedback, the first equation in (3.2) becomes

$$(4.4) \quad \dot{z}(\cdot, t) = (A_h + B_r F)z(\cdot, t) + g(\cdot)d_1(t) + B_l d_2(t) + B_r m_w^\top w(t).$$

Under the state feedback (4.3), the classical solutions of (4.4) must satisfy the following equations (which are obtained by substituting (4.3) into (1.1)):

$$(4.5) \quad \begin{cases} z_t(x, t) = -iz_{xx}(x, t) + h(x)z(x, t) + g(x)d_1(t), \\ z_x(0, t) = -iqz(0, t) + d_2(t), \\ z_x(1, t) = k(1, 1)z(1, t) + \int_0^1 k_x(1, \xi)z(\xi, t)d\xi + m_w^\top w(t), \\ z(x, 0) = z_0(x), \quad y(t) = C_e[z(\cdot, t)], \quad y_m(t) = z(1, t). \end{cases}$$

Using the transformation (4.1) and omitting  $y_m$ , the system (4.5) becomes

$$(4.6) \quad \begin{cases} v_t(x, t) = -iv_{xx}(x, t) - c_s v(x, t) + \mathcal{F}[g](x)d_1(t) - k(x, 0)d_2(t), \\ v_x(0, t) = d_2(t), \quad v_x(1, t) = m_w^\top w(t), \\ v(x, 0) = z_0(x) - \int_0^x k(x, \xi)z_0(\xi)d\xi, \quad y(t) = C_e \mathcal{F}^{-1}[v(\cdot, t)]. \end{cases}$$

In order to find the constant vector  $m_w$  in (4.3), we introduce the *error transformation*

$$(4.7) \quad \tilde{v}(x, t) = v(x, t) - m(x)^\top w(t).$$

We are searching for a function  $m \in C^2([0, 1]; \mathbb{R}^{n_w})$  for the transformation (4.7) so that the first three equations in (4.6) can be converted into the following (with  $x \in (0, 1)$  and  $t \geq 0$ ):

$$(4.8) \quad \begin{cases} \tilde{v}_t(x, t) = -i\tilde{v}_{xx}(x, t) - c_s \tilde{v}(x, t), \\ \tilde{v}_x(0, t) = 0, \quad \tilde{v}_x(1, t) = 0. \end{cases}$$

In other words,  $\dot{\tilde{v}} = (\mathbb{A} - c_s I)\tilde{v}$ , where  $\mathbb{A}$  is the following skew-adjoint operator:

$$(4.9) \quad \mathbb{A}f = -if'' \quad \text{with} \quad D(\mathbb{A}) = \{f \in H^2(0, 1) \mid f'(0) = f'(1) = 0\}.$$

This  $\mathbb{A}$  is a simplified version of  $A$  from (1.3) that corresponds to  $q = 0$ . Thus, the differential equation of  $\tilde{v}$  is exponentially stable in  $\mathbb{H}$ .

Substituting (4.7) into the first part of (4.8), we get

$$(4.10) \quad \begin{aligned} 0 &= \tilde{v}_t(x, t) + i\tilde{v}_{xx}(x, t) + c_s \tilde{v}(x, t) \\ &= v_t(x, t) - m(x)^\top S w(t) + iv_{xx}(x, t) - im''(x)^\top w(t) + c_s v(x, t) - c_s m(x)^\top w(t) \\ &= - \left[ im''(x)^\top + m(x)^\top S + c_s m(x)^\top - \mathcal{F}[g](x)p_1^\top + k(x, 0)p_2^\top \right] w(t). \end{aligned}$$

Here we have used  $p_1, p_2$  from (1.5). Substituting (4.7) into the second part of (4.8), we get (using (4.6))

$$(4.11) \quad 0 = \tilde{v}_x(0, t) = v_x(0, t) - m'(0)^\top w(t) = \left[ p_2^\top - m'(0)^\top \right] w(t).$$

Substituting (4.7) into the third part of (4.8), we have (using (4.6))

$$(4.12) \quad 0 = \tilde{v}_x(1, t) = v_x(1, t) - m'(1)^\top w(t) = \left[ m_w^\top - m'(1)^\top \right] w(t).$$

Recall from Sect. 1 that  $D(C_e) = H^2(0, 1)$ . By (4.1) and (4.7), for any classical solution of the closed-loop system, the output tracking error is, for every  $t \geq 0$ ,

$$(4.13) \quad \begin{aligned} e_y(t) &= y(t) - r(t) = C_e[z(\cdot, t)] - p_r^\top w(t) = C_e \mathcal{F}^{-1}[v(\cdot, t)] - p_r^\top w(t) \\ &= C_e \mathcal{F}^{-1}[\tilde{v}(\cdot, t)] + \left( C_e \mathcal{F}^{-1}[m] - p_r^\top \right) w(t). \end{aligned}$$

It follows from (4.10)-(4.13) that if the function  $m$  satisfies the following *regulator equations*:

$$(4.14) \quad \begin{cases} im''(x)^\top + m(x)^\top S + c_s m(x)^\top = \mathcal{F}[g](x) p_1^\top - k(x, 0) p_2^\top, \\ m'(0)^\top = p_2^\top, \quad C_e \mathcal{F}^{-1}[m] = p_r^\top, \end{cases}$$

and we choose  $m_w$  in (4.3) so that  $m_w = m'(1)$ , provided that the equation (4.14) is solvable, then the system (4.6) is reduced to (4.8), and the output tracking error for classical solutions of the closed-loop system becomes, according to (4.13),

$$(4.15) \quad e_y(t) = y(t) - r(t) = C_e \mathcal{F}^{-1}[\tilde{v}(\cdot, t)].$$

**Remark 4.1.** The state feedback operator from (4.3) can be written in the form

$$(4.16) \quad F = k(1, 1) C_m + \mathcal{K},$$

where  $\mathcal{K}$  is a bounded linear functional on  $\mathbb{H}$ . This shows (using Proposition 3.5) that  $F$  is an admissible observation operator for the semigroups generated by  $A$  and  $A_h$ . We have from (4.2)

$$k(1, 1) = -\frac{i}{2} \int_0^1 h(\xi) d\xi - i \left[ \frac{c_s}{2} + q \right],$$

and clearly  $C_m = -iB_r^*$ . Thus, we can write

$$F = -\left[ \frac{c_s}{2} + q + \int_0^1 h(\xi) d\xi \right] B_r^* + \mathcal{K},$$

which shows that the dominant component of this feedback is collocated.

**Remark 4.2.** The compatible system node  $\Sigma = (A_h, [g(\cdot) \ B_l \ B_r], \begin{bmatrix} C_e \\ C_m \end{bmatrix}, 0)$  represents the systems (3.2) and also (1.1), see Corollary 3.4. This is a regular linear system, according to Proposition 3.5. Since  $F$  satisfies (4.16), it follows that also the system node  $\Sigma_f = (A_h, [g(\cdot) \ B_l \ B_r], \begin{bmatrix} C_e \\ C_m \\ F \end{bmatrix}, 0)$  is regular (with input and output space  $\mathbb{C}^3$ ). This  $\Sigma_f$  has been obtained by adding a third output to  $\Sigma$ , namely,  $u_f(t) = F_\Lambda z(t)$  (for classical solutions we may write  $u_f(t) = Fz(t)$ ). Now the state feedback law (4.3) can be written in the abstract output feedback form that fits Proposition (2.8):

$$(4.17) \quad \begin{bmatrix} d_1(t) \\ d_2(t) \\ u(t) \end{bmatrix} = K_f \begin{bmatrix} y(t) \\ y_m(t) \\ u_f(t) \end{bmatrix} + \rho(t), \quad \text{where} \quad K_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \\ m_w^\top w(t) \end{bmatrix},$$

and  $\rho$  is the new input signal of the closed-loop system.

**Proposition 4.3.** *With the notation of Remark 4.2, define  $A_{cl} : D(A_{cl}) \rightarrow \mathbb{H}$  as follows:*

$$A_{cl} = A_h + B_r F_\Lambda, \quad D(A_{cl}) = \{x \in H^2(0, 1) \mid A_h x + B_r F_\Lambda x \in X\}.$$



The closed-loop system  $\Sigma_{cl}$  obtained from  $\Sigma$  with the feedback law (4.3) (described by the equations (4.4) and the second line of (3.2)) is a regular linear system  $\Sigma_{cl}$  with semigroup generator  $A_{cl}$ , control operator  $B = [g \ B_l \ B_r]$ , observation operator  $C = \begin{bmatrix} C_e \\ C_m \end{bmatrix}$  (restricted to  $D(A_{cl})$ ) and its feedthrough operator  $D$  is the same as for the open-loop system  $\Sigma$ .

*Proof.* We know from Proposition 3.5 that the compatible system node  $(A_h, [g(\cdot) \ B_l \ B_r], \begin{bmatrix} C_{e\Lambda} \\ C_m \end{bmatrix}, 0)$  is well-posed and strictly proper. Since  $F$  satisfies (4.16), it follows that also

$$\Sigma_{f0} = \left( A_h, [g(\cdot) \ B_l \ B_r], \begin{bmatrix} C_{e\Lambda} \\ C_m \\ F \end{bmatrix}, 0 \right)$$

is well-posed and strictly proper. This regular system node differs from  $\Sigma_f$  in Remark 4.2 only in its feedthrough operator: the feedthrough operator of  $\Sigma_{f0}$  is zero, while for  $\Sigma_f$  it is of the form

$$D_0 = \begin{bmatrix} 0 & D_1 & D_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{where} \quad \begin{aligned} D_1 &= \lim_{\lambda \rightarrow \infty, \lambda \in \mathbb{R}} C_e(\lambda I - A_h)^{-1} B_l, \\ D_2 &= \lim_{\lambda \rightarrow \infty, \lambda \in \mathbb{R}} C_e(\lambda I - A_h)^{-1} B_r. \end{aligned}$$

The limits  $D_1$  and  $D_2$  could be any numbers in  $\mathbb{C}$ , because  $C_e$  has not been specified. According to the last part of Proposition 2.7, the system  $\Sigma_f$  is equivalent to

$$\Sigma_f = \left( A_h, [g(\cdot) \ B_l \ B_r], \begin{bmatrix} C_{e\Lambda} \\ C_m \\ F \end{bmatrix}, D_0 \right)$$

in the sense that these systems have the same system operator and the same transfer function. (According to the theory of system nodes, having the same system operator means that they are the same system.) We denote by  $\mathbf{G}$  and  $\mathbf{G}_0$  the transfer functions of  $\Sigma_f$  and  $\Sigma_{f0}$  respectively, so that  $\mathbf{G}(s) = \mathbf{G}_0(s) + D_0$ . We see that  $I - K_f \mathbf{G}(s)$  has a uniformly bounded inverse on some right half-plane, because  $K_f \mathbf{G}(s) = K_f \mathbf{G}_0(s)$  and  $\mathbf{G}_0$  is strictly proper. Note that  $K_f D_0 = 0$ . Thus, we can apply Proposition 2.8 to conclude that  $\Sigma_f$  with the feedback law (4.3), which is equivalent to (4.17), leads to a well-posed and regular closed-loop system  $\Sigma_{cl,f}$ .

According to Proposition 2.8, after a little computation, we find that

$$\Sigma_{cl,f} = \left( A_{cl}, [g(\cdot) \ B_l \ B_r], \begin{bmatrix} C_{e\Lambda} + D_2 F \\ C_m \\ F \end{bmatrix}, D_0 \right),$$

where  $A_{cl}$  is defined in the proposition. Another short computation shows that the above system node  $\Sigma_{cl,f}$  is equivalent to

$$\Sigma_{cl,f} = \left( A_{cl}, [g(\cdot) \ B_l \ B_r], \begin{bmatrix} C_e \\ C_m \\ F \end{bmatrix}, 0 \right).$$

If we ignore the third output of this system,  $u_f$  introduced in Remark 4.2, then we obtain the closed-loop system  $\Sigma_{cl}$  stated in the proposition. We remark that  $D$  consists of the first two lines of  $D_0$  and that the restrictions of  $C_{e\Lambda} + D_2 F$  and of  $C_e$  to  $D(A_{cl})$  are equal.  $\square$

**Proposition 4.4.** *We use the notation of Proposition 4.3. Assume that the regulator equations (4.14) have a solution  $m$  and  $m_w = m'(1)$ , so that (4.8) and (4.15) hold.*

*Then  $C_e \mathcal{F}^{-1}$  is an admissible observation operator for the group generated by  $\mathbb{A}$  from (4.9).*

*Proof.* Consider the cascade connection of the closed-loop system  $\Sigma_{cl}$  with the exosystem from (1.5) according to (4.4), so that all three inputs of  $\Sigma_{cl}$  come from the finite-dimensional exosystem. Since  $\Sigma_{cl}$  is well-posed, it follows that this cascade connection is again well-posed, implying that for any  $T > 0$  there exists an  $m_T > 0$  such that

$$\int_0^T \|y(t)\|^2 \leq m_T \left\| \begin{bmatrix} z(\cdot, 0) \\ w(0) \end{bmatrix} \right\|^2.$$

Clearly a similar estimate holds for the signal  $r$ , and using (4.15) it follows that a similar estimate holds for  $e_y$ : for some  $\tilde{m}_T > 0$ ,

$$\int_0^T \|e_y(t)\|^2 \leq \tilde{m}_T \left\| \begin{bmatrix} z(\cdot, 0) \\ w(0) \end{bmatrix} \right\|^2.$$

Now consider the special case  $w(0) = 0$ . Then according to (4.1) and (4.7), we have  $\tilde{v} = \mathcal{F}z$  and according to (4.8) and (4.15) we have  $\tilde{v}(t) = (\mathbb{A} - c_s I)\tilde{v}(t)$  and  $e_y(t) = C_e \mathcal{F}^{-1}\tilde{v}(t)$ . From

$$\int_0^T \|e_y(t)\|^2 \leq \tilde{m}_T \|z(0)\|^2 \leq \tilde{m}_T \|\mathcal{F}^{-1}\|^2 \|\tilde{v}(0)\|^2.$$

This shows that  $C_e \mathcal{F}^{-1}$  is an admissible observation operator for the group generated by  $\mathbb{A} - c_s I$  (equivalently, for the group generated by  $\mathbb{A}$ ).  $\square$

**Remark 4.5.** Let  $\tilde{v}$  satisfy (4.8) and denote  $\hat{v}(\cdot, t) := \mathcal{F}^{-1}[\tilde{v}(\cdot, t)]$ . Then  $\hat{v}(x, t)$  is governed by

$$\begin{cases} \hat{v}_t(x, t) = -i\hat{v}_{xx}(x, t) + h(x)\hat{v}(x, t), \\ \hat{v}_x(0, t) = -iq\hat{v}(0, t), \\ \hat{v}_x(1, t) = K(1, 1)\tilde{v}(1, t) + \int_0^1 K_x(1, \xi)\tilde{v}(\xi, t)d\xi = K(1, 1)\left[\hat{v}(1, t) - \int_0^1 k(1, \xi)\hat{v}(\xi, t)d\xi\right] + \int_0^1 K_x(1, \xi)\left(\hat{v}(\xi, t) - \int_0^1 k(\xi, \zeta)\hat{v}(\zeta, t)d\zeta\right)d\xi. \end{cases}$$

We state a lemma which describes the solvability condition of the regulator equation (4.14). This lemma is related to [22, Theorem 5.2].

**Lemma 4.6.** *The regulator equation (4.14) has a unique solution if and only if  $C_e \mathcal{F}^{-1}[\cosh(\sqrt{-i(\lambda + c_s)} \cdot)] \neq 0$ , for all  $\lambda \in \sigma(S)$ .*

*Proof.* Since  $S$  is diagonalizable, there exists a square matrix

$$V = [v_1, v_2, \dots, v_{n_w}], \quad v_j \in \mathbb{R}^{n_w},$$

such that  $V^{-1}SV = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n_w})$ , where  $\lambda_j, j = 1, 2, \dots, n_w$  are the eigenvalues of  $S$ . Multiply with  $v_j$  from the right in (4.14) to obtain

$$(4.18) \quad \begin{cases} \bar{m}_j''(x) - i\lambda_j \bar{m}_j(x) - ic_s \bar{m}_j(x) = -i[\mathcal{F}[g](x)p_1^\top v_j - k(x, 0)p_2^\top v_j], \\ \bar{m}_j'(0) = p_2^\top v_j, \quad C_e \mathcal{F}^{-1}[\bar{m}_j] = p_r^\top v_j, \quad j = 1, 2, \dots, n_w, \end{cases}$$

where  $\bar{m}_j = m(x)^\top v_j, j = 1, 2, \dots, n_w$ . If  $\lambda_j + c_s \neq 0$ , the general solution of the first equation of (4.18) is of the following form (with the coefficients  $\gamma_1, \gamma_2$  to be

determined):

$$\begin{aligned} \bar{m}_j(x) = & \gamma_1 \cosh(\sqrt{-i(\lambda_j + c_s)}x) + \gamma_2 \frac{\sinh(\sqrt{-i(\lambda_j + c_s)}x)}{\sqrt{-i(\lambda_j + c_s)}} \\ & - i \int_0^x [\mathcal{F}[g](\xi)p_1^\top v_j - k(\xi, 0)p_2^\top v_j] \frac{\sinh(\sqrt{-i(\lambda_j + c_s)}(x - \xi))}{\sqrt{-i(\lambda_j + c_s)}} d\xi. \end{aligned}$$

Substituting this into the boundary conditions in (4.18), we get

$$(4.19) \quad \begin{cases} \gamma_2 = p_2^\top v_j, \\ \gamma_1 C_e \mathcal{F}^{-1}[\cosh(\sqrt{-i(\lambda_j + c_s)}\cdot)] + \gamma_2 C_e \mathcal{F}^{-1} \left[ \frac{\sinh(\sqrt{-i(\lambda_j + c_s)}x)}{\sqrt{-i(\lambda_j + c_s)}} \right] \\ + C_e \mathcal{F}^{-1} \left[ -i \int_0^\cdot [\mathcal{F}[g](\xi)p_1^\top v_i - k(\xi, 0)p_2^\top v_j] \right. \\ \quad \left. \times \frac{\sinh(\sqrt{-i(\lambda_j + c_s)}(\cdot - \xi))}{\sqrt{-i(\lambda_j + c_s)}} d\xi \right] = p_r^\top v_j. \end{cases}$$

It is obvious that the coefficients  $\gamma_1, \gamma_2$  can be uniquely determined by equation (4.19) if and only if  $C_e \mathcal{F}^{-1}[\cosh(\sqrt{-i(\lambda_j + c_s)}\cdot)] \neq 0$ .

If  $\lambda_j + c_s = 0$ , then the solutions of the first equation in (4.18) are of the form

$$(4.20) \quad \bar{m}_j(x) = \gamma_1 + \gamma_2 x - i \int_0^x (x - \xi)[\mathcal{F}[g](\xi)p_1^\top v_j - k(\xi, 0)p_2^\top v_j] d\xi,$$

where  $\gamma_1, \gamma_2$  are the coefficients to be determined. Substituting (4.20) into the boundary conditions in (4.18), we get  $\gamma_2 = p_2^\top v_i$  and, denoting by  $\eta$  the identity function,  $\eta(x) = x$ ,

$$(4.21) \quad \begin{aligned} \gamma_1 C_e \mathcal{F}^{-1}[1] = & -\gamma_2 C_e \mathcal{F}^{-1}[\eta] + p_r^\top v_j \\ & - C_e \mathcal{F}^{-1} \left[ -i \int_0^\cdot (\cdot - \xi)[\mathcal{F}[g](\xi)p_1^\top v_j - k(\xi, 0)p_2^\top v_j] d\xi \right]. \end{aligned}$$

It is clear that  $\gamma_1$  can be uniquely determined from this equation if and only if  $C_e \mathcal{F}^{-1}[1] \neq 0$ .  $\square$

Now, with the state feedback, we turn to the closed-loop system which is composed of (1.1), (1.5), (4.3) and (4.15), that is

$$(4.22) \quad \begin{cases} z_t(x, t) = -iz_{xx}(x, t) + h(x)z(x, t) + g(x)p_1^\top w(t), \\ z_x(0, t) = -iqz(0, t) + p_2^\top w(t), \\ z_x(1, t) = k(1, 1)z(1, t) + \int_0^1 k_x(1, \xi)z(\xi, t)d\xi + m_w^\top w(t), \\ z(\cdot, 0) = z_0(\cdot) \in L^2[0, 1], \\ \dot{w}(t) = Sw(t), \quad w(0) = w_0 \in \mathbb{R}^{n_w}, \\ e_y(t) = y(t) - r(t) = C_e[z(\cdot, t)] - p_r(t)^\top w(t). \end{cases}$$

The following is the main result of this section.

**Theorem 4.7.** *Let  $c_s > 0$  and let the functions  $k$  and  $m$  be solutions of (4.2) and (4.14). Suppose that*

$$C_e \mathcal{F}^{-1}[\cosh(\sqrt{-i(\lambda + c_s)} \cdot)] \neq 0 \quad \forall \lambda \in \sigma(S).$$

*Then the state feedback law (4.3) with  $m_w^\top = m'(1)^\top$  solves the output regulation problem for the system (4.22), i.e.,  $e_y \in L_\alpha[0, \infty)$  for some  $\alpha < 0$ . If  $C_e$  is bounded, then there exist  $M, \mu > 0$  such that  $|e_y(t)| \leq M e^{-\mu t}$  holds for all  $t \geq 0$ .*

*Proof.* We have seen after (4.8) that  $\dot{\tilde{v}}(\cdot, t) = (\mathbb{A} - c_s I) \tilde{v}(\cdot, t)$ , where  $\mathbb{A}$  is skew-adjoint. Clearly  $\mathbb{A} - c_s I$  generates an exponential stable operator group, which, jointly with the admissibility of the observation operator  $C_e \mathcal{F}^{-1}$  (see Proposition 4.4) implies that  $e_y = C_e \mathcal{F}^{-1}[\tilde{v}] \in L_\alpha^2[0, \infty)$  with  $\alpha \in (-c_s, 0)$ , see [28, Proposition 4.3.6]. If the observation operator  $C_e$  is bounded, then by the boundedness of the transformation  $\mathcal{F}^{-1}$ , there exist three constants  $C_0, M, \mu > 0$  such that

$$|e_y(t)| = |C_e \mathcal{F}^{-1}[\tilde{v}](\cdot, t)| \leq C_0 \|\tilde{v}(\cdot, t)\| \leq C_0 M e^{-\mu t} \|\tilde{v}(\cdot, 0)\|. \quad \square$$

## 5. OBSERVER DESIGN

The full states  $w(t)$  and  $z(\cdot, t)$  used in (4.3) are not always available (as measurements) to the controller. Thus, to implement the feedback law (4.3), we need to design an observer for the combined system (1.1) and (1.5), to recover its state from the output measurement  $y_m(t) = z(1, t)$  and from the reference  $r(t)$ . Since  $(q_r^\top, S_r)$  is observable, there exists an observer gain  $l_r \in \mathbb{R}^{n_r}$  such that  $S_r + l_r q_r^\top$  is Hurwitz. So, we can use the finite dimensional reference observer

$$(5.1) \quad \dot{\hat{w}}_r(t) = S_r \hat{w}_r(t) + l_r (q_r^\top \hat{w}_r(t) - r(t)),$$

where  $\hat{w}_r(t)$  is the estimate of  $w_r(t)$  in (1.5). In order to estimate  $z(\cdot, t)$  and  $w_d$  in (1.1) and (1.5), we design the following observer:

$$(5.2) \quad \begin{cases} \dot{\hat{w}}_d(t) = S_d \hat{w}_d(t) + l_d (\hat{z}(1, t) - y_m(t)), \\ \hat{z}_t(x, t) = -i \hat{z}_{xx}(x, t) + h(x) \hat{z}(x, t) + g(x) q_{d1}^\top \hat{w}_d(t) + l(x) [\hat{z}(1, t) - y_m(t)], \\ \hat{z}_x(0, t) = -i q \hat{z}(0, t) + q_{d2}^\top \hat{w}_d(t), \\ \hat{z}_x(1, t) = u(t) + l_0 (\hat{z}(1, t) - y_m(t)), \end{cases}$$

where  $l(\cdot)$ ,  $l_0$  are observer gains, to be designed later. It should be noted that the above observer (5.2) is implemented based on the boundary measurement  $y_m(t)$  and the input signal  $u(t)$ . Let

$$\tilde{w}_r(t) = \hat{w}_r(t) - w_r(t), \quad \tilde{w}_d(t) = \hat{w}_d(t) - w_d(t), \quad \tilde{z}(x, t) = \hat{z}(x, t) - z(x, t)$$

be the observer errors. Then, by (1.1), (5.1) and (5.2),  $\tilde{w}_d(t)$ ,  $\tilde{w}_r(t)$  and  $\tilde{z}(x, t)$  satisfy

$$(5.3) \quad \begin{cases} \dot{\tilde{w}}_r(t) = (S_r + l_r q_r^\top) \tilde{w}_r(t), & \dot{\tilde{w}}_d(t) = S_d \tilde{w}_d(t) + l_d \tilde{z}(1, t), \\ \tilde{z}_t(x, t) = -i \tilde{z}_{xx}(x, t) + h(x) \tilde{z}(x, t) + g(x) q_{d1}^\top \tilde{w}_d(t) + l(x) \tilde{z}(1, t), \\ \tilde{z}_x(0, t) = -i q \tilde{z}(0, t) + q_{d2}^\top \tilde{w}_d(t), & \tilde{z}_x(1, t) = l_0 \tilde{z}(1, t), \end{cases}$$

which has to be exponentially stabilized. In order to find the observer gains  $l(\cdot)$ ,  $l_0$  that ensure that (5.3) is exponentially stable, we look for the *backstepping transformation*

$$(5.4) \quad \tilde{z}(x, t) = \mathcal{F}_o[e](x, t) := e(x, t) - \int_x^1 p(x, \xi) e(\xi, t) d\xi,$$

that transforms (5.3) into the following system:

$$(5.5) \quad \begin{cases} \dot{\tilde{w}}_r(t) = (S_r + l_r q_r^\top) \tilde{w}_r(t), & \dot{\tilde{w}}_d = S_d \tilde{w}_d(t) + l_d e(1, t), \\ e_t(x, t) = -ie_{xx}(x, t) - c_o e(x, t) + \tilde{g}(x)^\top \tilde{w}_d(t) + \tilde{l}(x) e(1, t), & x \in (0, 1), \\ e_x(0, t) = q_{d_2}^\top \tilde{w}_d(t), & e_x(1, t) = 0, \\ e(x, 0) = e_0(x) = \mathcal{F}_o^{-1}[\tilde{z}_0](x), \end{cases}$$

where  $\tilde{g}(x)^\top$  is given by  $\tilde{g}(x)^\top = \mathcal{F}_o^{-1}[g](x) q_{d_1}^\top$  and  $\tilde{l}(x)$  is needed as an additional degree of freedom for the subsequent design.

By the third equations of (5.3) and (5.5), and the transformation (5.4), through integration by parts we obtain

$$(5.6) \quad \begin{aligned} & g(x) q_{d_1}^\top \tilde{w}_d(t) + l(x) e(1, t) = g(x) q_{d_1}^\top \tilde{w}_d(t) + l(x) \tilde{z}(1, t) \\ & = \tilde{z}_t(x, t) + i \tilde{z}_{xx}(x, t) - h(x) \tilde{z}(x, t) \\ & = e_t(x, t) - \int_x^1 p(x, \xi) e_t(\xi, t) d\xi + i \left[ e(x, t) - \int_x^1 p(x, \xi) e(\xi, t) d\xi \right]_{xx} \\ & \quad - h(x) \left[ e(x, t) - \int_x^1 p(x, \xi) e(\xi, t) d\xi \right] \\ & = -ie_{xx}(x, t) - c_o e(x, t) + \tilde{g}(x)^\top \tilde{w}_d(t) + \tilde{l}(x) e(1, t) \\ & \quad - h(x) \left[ e(x, t) - \int_x^1 p(x, \xi) e(\xi, t) d\xi \right] + \int_x^1 p(x, \xi) c_o e(\xi, t) d\xi \\ & \quad + i \int_x^1 p_{\xi\xi}(x, \xi) e(\xi, t) d\xi - i \int_x^1 p_{xx}(x, \xi) e(\xi, t) d\xi \\ & \quad - \int_x^1 p(x, \xi) \tilde{g}(\xi)^\top d\xi \tilde{w}_d(t) - \int_x^1 p(x, \xi) \tilde{l}(x) d\xi e(1, t) \\ & \quad + i \left[ e_{xx}(x, t) + \frac{d}{dx} p(x, t) e(x, t) + p(x, x) e_x(x, t) + p_x(x, x) e(x, t) \right] \\ & \quad + i [p(x, 1) e_x(1, t) - p(x, x) e_x(x, t) - p_\xi(x, 1) e(1, t) + p_\xi(x, x) e(x, t)] \\ & = \left[ 2i \frac{d}{dx} p(x, t) e(x, t) - h(x) - c_o \right] e(x, t) \\ & \quad + [\mathcal{F}_o[\tilde{l}(x)] - i p_\xi(x, 1)] e(1, t) + \mathcal{F}_o[\tilde{g}(x)^\top] \tilde{w}_d(t) \\ & \quad + \int_x^1 [-i p_{xx}(x, \xi) + i p_{\xi\xi}(x, \xi) + h(x) p(x, \xi) + c_o p(x, \xi)] e(\xi, t) d\xi. \end{aligned}$$

By the fourth equations of (5.3) and (5.5), and the transformation (5.4), we obtain

$$\begin{aligned}
0 &= e_x(0, t) - q_{d_2}^\top \tilde{w}_d(t) = \tilde{z}_x(0, t) - p(0, 0)e(0, t) + \int_0^1 p_\xi(0, \xi)e(\xi, t)d\xi \\
&\quad - q_{d_2}^\top \tilde{w}_d(t) \\
&= -iq[e(0, t) - \int_x^1 p(0, \xi)e(\xi, t)d\xi] - p(0, 0)e(0, t) + \int_0^1 p_\xi(0, \xi)e(\xi, t)d\xi \\
&= -[p(0, 0) + qi]e(0, t) + \int_0^1 [p_\xi(0, \xi) + qip(0, \xi)]e(\xi, t)d\xi.
\end{aligned}$$

By the fifth equations of (5.3) and (5.5) and the transformation (5.4),

$$\begin{aligned}
(5.7) \quad 0 &= e_x(1, t) = \tilde{z}_x(1, t) - p(1, 1)e(1, t) = l_0 \tilde{z}(1, t) - p(1, 1)e(1, t) \\
&= [l_0 - p(1, 1)]e(1, t).
\end{aligned}$$

It follows from (5.6)-(5.7) that the kernel function  $p(x, \xi)$  in (5.4) should satisfy

$$\begin{cases} p_{\xi\xi}(x, \xi) - p_{xx}(x, \xi) = (h(x) + c_o)ip(x, \xi), & c_o > 0, \\ p_x(0, \xi) + qip(0, \xi) = 0, \\ p(x, x) = -\frac{i}{2} \int_0^x (h(\xi) + c_o)d\xi - qi, \end{cases}$$

and that we should choose the observer gains  $l(\cdot)$  and  $l_0$  in (5.2) so that

$$l(x) = \mathcal{F}_o[\tilde{l}](x) - ip_\xi(x, 1), \quad l_0 = p(1, 1).$$

By [26, Theorem 2.2], the above equations in  $p$  have a unique solution  $p \in C^2(\Omega)$ . We note that we have still not obtained the final expression of the observer gain  $l(\cdot)$ , because  $\tilde{l}(\cdot)$  is a new design function. In order to find  $\tilde{l}(\cdot)$  in (5.5) so that the “ $e$ -part” of the system (5.5) is exponentially stable in  $L^2[0, 1]$ , we further introduce the *error transformation*

$$(5.8) \quad \tilde{e}(x, t) = e(x, t) - n(x)^\top \tilde{w}_d(t).$$

It is expected that under the above transformation, the system (5.5) can be transformed into

$$(5.9) \quad \begin{cases} \dot{\tilde{w}}_r(t) = (S_r + l_r q_r^\top) \tilde{w}_r(t), \\ \dot{\tilde{w}}_d = (S_d + l_d n(1)^\top) \tilde{w}_d(t) + l_d \tilde{e}(1, t), \\ \tilde{e}_t(x, t) = -i\tilde{e}_{xx}(x, t) - c_o \tilde{e}(x, t), \\ \tilde{e}_x(0, t) = 0, \quad \tilde{e}_x(1, t) = 0. \end{cases}$$

Substituting (5.8) into the third equation of (5.9), we derive

$$\begin{aligned}
(5.10) \quad 0 &= \tilde{e}_t(x, t) + i\tilde{e}_{xx}(x, t) + c_o \tilde{e}(x, t) \\
&= e_t(x, t) - n(x)^\top S_r \tilde{w}_d(t) - n(x)^\top l_d e(1, t) \\
&\quad + i[e_{xx}(x, t) - n''(x)^\top \tilde{w}_d(t)] + c_o e(x, t) - c_o n(x)^\top \tilde{w}_d(t) \\
&= [\tilde{l}(x) - n(x)^\top l_d]e(1, t) + [\tilde{g}(x)^T - n(x)^\top S_d - in''(x)^\top - c_o n(x)^\top] \tilde{w}_d(t) = 0.
\end{aligned}$$

Substituting (5.8) into the fourth equation of (5.9), we have

$$(5.11) \quad 0 = \tilde{e}_x(0, t) = e_x(0, t) - n'(0)\tilde{w}_d(t) = [q_{d_2}^\top - n'(0)^\top]\tilde{w}_d(t) = 0.$$

Substituting (5.8) into the fifth equation of (5.9), we obtain

$$(5.12) \quad 0 = \tilde{e}_x(1, t) = e_x(0, t) - n'(1)^\top \tilde{w}_d(t) = -n'(1)^\top \tilde{w}_d(t) = 0.$$

It follows from (5.10)-(5.12) that  $n(\cdot)$  must satisfy the following equations:

$$(5.13) \quad \begin{cases} in''(x)^\top + n(x)^\top S_d + c_o n(x)^\top = \tilde{g}(x)^\top, \\ n'(0)^\top = q_{d_2}^\top, \quad n'(1)^\top = 0. \end{cases}$$

If we choose  $\tilde{l}$  so that  $\tilde{l}(x) = n(x)^\top l_d$ , provided that the equation (5.13) is solvable, then the system (5.5) becomes (5.9). Thus, the observer gains  $l(\cdot)$  and  $l_0$  in (5.2) are designed as follows:

$$(5.14) \quad l(x) = \mathcal{F}_o(n(x)^\top l_d) - ip_\xi(x, 1), \quad l_0 = p(1, 1),$$

provided that the equation (5.13) has a solution.

**Lemma 5.1.** *The equations (5.13) have a unique solution if and only if  $\sigma_o \cap \sigma(S_d) = \emptyset$ , where  $\sigma_o = \{-j^2\pi^2 i - c_o\}$  is the eigenvalue set of the “ $\tilde{e}$ -part” of (5.9).*

*Proof.* Since  $S$  is diagonalizable, there exists a matrix  $V = [v_1, v_2, \dots, v_{n_d}]$ ,  $v_j \in \mathbb{R}^{n_w}$ ,  $j = 1, 2, \dots, n_d$ , such that  $V^{-1}SV = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n_d})$ , where  $\lambda_j$ ,  $j = 1, 2, \dots, n_d$  are the eigenvalues of  $S_d$ . Multiply by  $v_j$  from the right in (4.14) to obtain

$$(5.15) \quad \begin{cases} i\bar{n}_j''(x) + \lambda_j \bar{n}_j(x) + c_o \bar{n}_j(x) = \tilde{g}(x)^\top v_j, \\ \bar{n}_j'(0) = q_{d_2}^\top v_j, \quad \bar{n}_j'(1) = 0, \quad j = 1, 2, \dots, n_d, \end{cases}$$

where  $\bar{n}_j = n(x)^\top v_j$ ,  $j = 1, 2, \dots, n_d$ . If  $\lambda_j + c_o \neq 0$ , the solutions of the first equation in (5.15) are of the form

$$(5.16) \quad \begin{aligned} \bar{n}_j(x) = & \gamma_1 \cosh\left(\sqrt{-i(\lambda_j + c_o)}x\right) + \gamma_2 \frac{\sinh(\sqrt{-i(\lambda_j + c_o)}x)}{\sqrt{-i(\lambda_j + c_o)}} \\ & + \int_0^x [-i\tilde{g}(\xi)^\top v_j] \frac{\sinh(\sqrt{-i(\lambda_j + c_o)}(x - \xi))}{\sqrt{-i(\lambda_j + c_o)}} d\xi, \end{aligned}$$

where  $\gamma_1, \gamma_2$  are coefficients to be determined. Substituting (5.16) into the boundary conditions in (5.15) yields

$$\begin{cases} \gamma_2 = q_{d_2}^\top v_j, \\ \gamma_1 \sqrt{-i(\lambda_j + c_o)} \sinh \sqrt{-i(\lambda_j + c_o)} + \gamma_2 \cosh \sqrt{-i(\lambda_j + c_o)} \\ = \int_0^1 i\tilde{g}(\xi)^\top v_j \cosh(\sqrt{-i(\lambda_j + c_o)}(1 - \xi)) d\xi. \end{cases}$$

It is obvious that the coefficients  $\gamma_1, \gamma_2$  are uniquely determined if and only if  $\sinh \sqrt{-i(\lambda_j + c_o)} \neq 0$ . It is easy to see that  $\sinh \sqrt{-i(\lambda_j + c_o)} \neq 0$  is equivalent to  $\sigma_o \cap \sigma(S_d) = \emptyset$ .

If  $\lambda_j + c_o = 0$ , the solutions of the first equation in (5.15) are of the form

$$(5.17) \quad \bar{n}_j(x) = \gamma_1 + \gamma_2 x + \int_0^x (x - \xi) [-i\tilde{g}(\xi)^\top v_j] d\xi,$$

where  $\gamma_1, \gamma_2$  are the coefficients to be determined. Substituting (5.17) into the boundary conditions in (5.15), we get  $\gamma_2 = q_{d_2}^\top v_j$  and  $\gamma_2 = \int_0^1 i\tilde{g}(\xi)^\top v_j d\xi$ . It is obvious that the  $\gamma_1$  cannot be uniquely determined and that there is no solution  $\gamma_2$  if  $q_{d_2}^\top v_j \neq \int_0^1 i\tilde{g}(\xi)^\top v_j d\xi$ . Therefore, (5.13) admits a unique solution if and only if  $\sigma_o \cap \sigma(S_d) = \emptyset$ .  $\square$

The next result confirms the existence, uniqueness and the exponential stability of the solutions of the observer error system (5.3). Rewrite the system (5.9) in the form

$$\frac{d}{dt}(\tilde{w}_r(t), \tilde{w}_d(t), \tilde{e}(\cdot, t))^\top = \mathbf{A}(\tilde{w}_r(t), \tilde{w}_d(t), \tilde{e}(\cdot, t))^\top$$

where the operator  $\mathbf{A} : D(\mathbf{A}) \rightarrow$  is defined as follows:

$$\begin{cases} \mathbf{A}(X_r, X_d, \phi(x)) \\ = ((S_r + l_r q_r^\top)X_r, (S_d + l_d n(1)^\top)X_d + l_d \phi(1), -i\phi''(x) - c_o \phi(x)), \\ D(\mathbf{A}) = \{(X_r, X_d, \phi(x)) \in \mathbb{R}^{n_r} \times \mathbb{R}^{n_d} \times H^2(0, 1) \mid \phi'(0) = 0, \phi'(1) = 0\}. \end{cases}$$

**Theorem 5.2.** *Let  $\sigma_o \cap \sigma(S_d) = \emptyset$ . Suppose that the observer gains  $l(x), l_0$  are given by (5.14) and the gain  $l_r \in \mathbb{R}^{n_r}$  is chosen so that  $S_r + l_r q_r^\top$  is Hurwitz. Suppose that  $S_d + l_d n(1)^\top$  is also Hurwitz. Moreover, assume that  $S_r + l_r q_r^\top$  has simple and stable eigenvalues  $\lambda_{rj}$  with the corresponding eigenvectors  $X_{rj} \in \mathbb{R}^{n_r}$ ,  $j = 1, 2, \dots, n_r$ , and  $S_d + l_d n(1)^\top$  has simple and stable eigenvalues  $\lambda_{dj}$  with the corresponding eigenvectors  $X_{dj} \in \mathbb{R}^{n_d}$ ,  $j = 1, 2, \dots, n_d$ , and  $\lambda_{rj_1} \neq \lambda_{dj_2}$  for  $1 \leq j_1 \leq n_r, 1 \leq j_2 \leq n_d$ . Let  $c_s, c_o > 0$ . Then (5.1) with (5.2) is an observer for the system (1.1). Moreover, the observer error dynamics (5.3) is exponentially stable in the sense that for some  $M \geq 1, \mu > 0$ ,*

$$(5.18) \quad \|(\tilde{w}_d(t), \tilde{w}_r(t), \tilde{z}(\cdot, t))\| \leq M e^{-\mu t} \|(\tilde{w}_d(0), \tilde{w}_r(0), \tilde{z}(\cdot, 0))\|.$$

*Proof.* We compute the eigenvalues and the corresponding eigenfunctions of  $\mathbf{A}$ . We solve  $\mathbf{A}(X_r, X_d, \phi(x)) = \lambda(X_r, X_d, \phi(x))$ , where  $\lambda \in \sigma(\mathbf{A})$  and  $(X_r, X_d, \phi(x)) \in D(\mathbf{A})$ , to obtain

$$(5.19) \quad \begin{cases} (S_r + l_r q_r^\top)X_r = \lambda X_r, & (S_d + l_d n(1)^\top)X_d + l_d \phi(1) = \lambda X_d, \\ -i\phi''(x) - c_o \phi(x) = \lambda \phi(x), & \phi'(0) = 0, \quad \phi'(1) = 0. \end{cases}$$

There are two cases:

*Case I:*  $\phi \equiv 0$ . In this case (5.19) becomes

$$(S_r + l_r q_r^\top)X_r = \lambda X_r, \quad (S_d + l_d n(1)^\top)X_d = \lambda X_d,$$

which has nontrivial solutions  $(\lambda_{rj}, [X_{rj}, 0_{n_d \times 1}])$ ,  $j = 1, 2, \dots, n_r$  and  $(\lambda_{dj}, [0_{n_r \times 1}, X_{dj}])$ ,  $j = 1, 2, \dots, n_d$ . Hence,  $(\lambda_{rj}, F_{1j}) = (\lambda_{rj}, [X_{rj}, 0_{n_d \times 1}, 0])$ ,  $j = 1, 2, \dots, n_r$ , together with  $(\lambda_{dj}, F_{1(j+n_r)}) = (\lambda_{dj}, [0_{n_r \times 1}, X_{dj}, 0])$ ,  $j = 1, 2, \dots, n_d$  are eigen-pairs of  $\mathbf{A}$ .

*Case II:*  $\phi \neq 0$ . Now

$$-i\phi''(x) - c_o \phi(x) = \lambda \phi(x), \quad \phi'(0) = 0, \quad \phi'(1) = 0,$$

which has nontrivial solutions  $(\lambda_j, \phi_j(x))$ :

$$\lambda_j = j^2 \pi^2 i - c_o, \quad \phi_j(x) = \cos(j\pi x), \quad j = 0, 1, 2, \dots$$

Substituting  $(\lambda_j, \phi_j(x))$  into the first and the second equation of (5.19), we get

$$X_r^j = 0_{n_r \times 1}, \quad X_d^j = -[(S_d + l_d n(1)^\top) - (j^2 \pi^2 i - c_o)I_{n_d \times n_d}]^{-1} l_d \cos(j\pi).$$



Thus we have found for  $\mathbf{A}$  the eigen-pairs  $(\lambda_j, F_{2j})$ , for  $j = 0, 1, 2, \dots$ , where

$$F_{2j} = [0_{n_r \times 1}, -[(S_d + l_d n(1)^\top) - (j^2 \pi^2 i - c_o) I_{n_d \times n_d}]^{-1} l_d \cos(j\pi), \cos(j\pi)].$$

Now we prove that the set  $\{F_{1j_1}(x), F_{2j_2}(x) \mid j_1 = 1, 2, \dots, n_w, j_2 = 0, 1, 2, \dots\}$  is a Riesz basis for  $\mathbb{R}^{n_w} \times L^2[0, 1]$ . Indeed, let us denote by  $G_j$  the first part of  $F_{1j}$ , so that  $G_j \in \mathbb{R}^{n_w}$  and  $F_{1j} = [G_j, 0]$ . Since the set  $\{G_j \mid j = 1, 2, \dots, n_w\}$  and the set  $\{\cos(j\pi \cdot) \mid j = 0, 1, 2, \dots\}$  form Riesz bases for  $\mathbb{R}^{n_w}$  and for  $L^2[0, 1]$ , respectively,  $\{F_{1j} \mid j = 1, 2, \dots, n_w\} \cup \{F_{2j}^* = [0_{n_w \times 1}, \cos(j\pi \cdot)] \mid j = 1, 2, \dots\}$  is a Riesz basis in  $\mathbb{R}^{n_w} \times \mathbb{H}$ . Moreover, the set that we want to prove to be a Riesz basis is quadratically close to the Riesz basis that we have just found:

$$\begin{aligned} (5.20) \quad & \sum_{j=0}^{\infty} \|F_{2j} - F_{2j}^*\|_{\mathbb{R}^{n_w} \times \mathbb{H}}^2 \\ &= \sum_{j=0}^{\infty} \|[(S_d + l_d n(1)^\top) - (j^2 \pi^2 i - c_o) I_{n_d \times n_d}]^{-1} l_d\|_{\mathbb{R}^{n_d}}^2 \\ &= \sum_{j=0}^{\infty} \frac{1}{|j^2 \pi^2 i - c_o|^2} \|[(S_d + l_d n(1)^\top) / (j^2 \pi^2 i - c_o) - I_{n_d \times n_d}]\|_{\mathbb{R}^{n_d}}^2. \end{aligned}$$

Since

$$\lim_{j \rightarrow \infty} \|[(S_d + l_d n(1)^\top) / (j^2 \pi^2 i - c_o) - I_{n_d \times n_d}]\|_{\mathbb{R}^{n_d}}^2 = \|l_d\|_{\mathbb{R}^{n_d}}^2,$$

it follows from (5.20) that  $\sum_{j=0}^{\infty} \|F_{2j} - F_{2j}^*\|_{\mathbb{R}^{n_w} \times \mathbb{H}}^2 < \infty$ . By the classical theorem of Bari,  $\{F_{1j}\}_{j=1}^{n_w} \cup \{F_{2j}\}_{j=0}^{+\infty}$  forms a Riesz basis for  $\mathbb{R}^{n_w} \times \mathbb{H}$ . This shows that  $\mathbf{A}$  generates an operator semigroup on  $\mathbb{R}^n \times \mathbb{H}$ , for which the spectrum determined growth assumption holds. As a consequence, the system (5.9) admits a unique solution. Since  $\sup\{\operatorname{Re} \lambda \mid \lambda \in \sigma(\mathbf{A})\} < 0$ ,  $e^{\mathbf{A}t}$  is an exponentially stable operator semigroup, which, together with the boundedness of the transformations (5.4) and (5.8), implies (5.18).  $\square$

## 6. OUTPUT FEEDBACK REGULATION

By Theorem 5.2 we have obtained the estimated states  $\hat{w}$  and  $\hat{z}(x, t)$  for  $w$  and  $z(x, t)$ , respectively. Since the state feedback control (4.3) achieves the output regulation, we naturally propose the following output feedback control law:

$$(6.1) \quad u(t) = k(1, 1)\hat{z}(1, t) + \int_0^1 k_x(1, \xi)\hat{z}(\xi, t)d\xi + m_w^\top \hat{w}(t).$$

Here we can see that the terms  $k(1, 1)\hat{z}(1, t) + \int_0^1 k_x(1, \xi)\hat{z}(\xi, t)d\xi$  are to stabilize the system (1.1) and the term  $m_w^\top \hat{w}(t)$  is to track the reference signal  $r(t) = p_r^\top w(t)$ . Now we turn to the closed-loop system composed of (1.1), (1.5), (5.1), (5.2) and (6.1), that is

$$(6.2) \quad \begin{cases} z_t(x, t) = -iz_{xx}(x, t) + h(x)z(x, t) + g(x)d_1(t), \\ z_x(0, t) = -iqz(0, t) + d_2(t), \\ z_x(1, t) = k(1, 1)\hat{z}(1, t) + \int_0^1 k_x(1, \xi)\hat{z}(\xi, t)d\xi + m_w^\top \hat{w}(t), \end{cases}$$

$$(6.3) \quad \begin{cases} \dot{w}(t) = Sw(t), \\ \dot{\hat{w}}_r(t) = S_r \hat{w}_r(t) + l_r(q_r^\top \hat{w}_r(t) - r(t)), \\ \dot{\hat{w}}_d(t) = S_d \hat{w}_d(t) + l_d(\hat{z}(1, t) - y_m(t)), \\ \hat{z}_t(x, t) = -i\hat{z}_{xx}(x, t) + h(x)\hat{z}(x, t) + g(x)q_{d1}^\top \hat{w}_d(t) \\ \quad + l(x)[\hat{z}(1, t) - y_m(t)], \\ \hat{z}_x(0, t) = -iq\hat{z}(0, t) + q_{d2}^\top \hat{w}_d(t), \\ \hat{z}_x(1, t) = k(1, 1)\hat{z}(1, t) + \int_0^1 k_x(1, \xi)\hat{z}(\xi, t)d\xi + m_w^\top \hat{w}(t) \\ \quad + l_0[\hat{z}(1, t) - y_m(t)], \end{cases}$$

where the gains  $l(\cdot)$ ,  $l_0$  are given by (5.14) and the gain  $l_r$  is chosen so that  $S_r + l_r q_r^\top$  is Hurwitz. The following is the main result of this section.

**Theorem 6.1.** *Suppose that the conditions in Theorems 4.7 and 5.2 hold.*

*Then for any initial state  $(z_0(x), w(0), \hat{z}_0(x), \hat{w}(0)) \in \mathbb{H} \times \mathbb{R}^{n_w} \times \mathbb{H} \times \mathbb{R}^{n_w}$ , the closed-loop system (6.2)-(6.3) admits a unique solution  $(z(\cdot, t), w(t), \hat{z}(\cdot, t), \hat{w}(t)) \in C([0, \infty); \mathbb{H} \times \mathbb{R}^{n_w} \times \mathbb{H} \times \mathbb{R}^{n_w})$ . Moreover, there exist  $M \geq 1$ ,  $\mu > 0$  such that*

$$\begin{aligned} & \|(\hat{w}_r(t) - w_r(t), \hat{w}_d(t) - w_d(t), \hat{z}(\cdot, t) - z(\cdot, t))\| \\ & \leq M e^{-\mu t} \|(\hat{w}_r(0) - w_r(0), \hat{w}_d(0) - w_d(0), \hat{z}_0 - z_0)\|. \end{aligned}$$

*The observer based controller (with internal loop) (5.1), (5.2) and (6.1) solves the output feedback regulator problem for the plant (1.1) with the exosystem (1.5). This means that the output error  $e_y(t) = y(t) - r(t) = C_{e\Lambda}[z(\cdot, t)] - p_r^\top w(t)$  for the closed-loop system (6.2)-(6.3) satisfies  $e_y \in L_\alpha^2[0, \infty)$  for some  $\alpha < 0$ . If  $C_e$  is bounded, then there exist  $m_0, \mu_0 > 0$  ( $m_0$  depends on the initial state mentioned above) such that we have  $|e_y(t)| \leq m_0 e^{-\mu_0 t}$  for all  $t \geq 0$ .*

*Proof.* Using the error variables  $\tilde{w}$  and  $\tilde{z}$  defined before (5.3), we can write an equivalent system to (6.2)-(6.3) as follows:

$$(6.4) \quad \begin{cases} z_t(x, t) = -iz_{xx}(x, t) + h(x)z(x, t) + g(x)d_1(t), \\ z_x(0, t) = -iqz(0, t) + d_2(t), \\ z_x(1, t) = k(1, 1)[z(1, t) + \tilde{z}(1, t)] + m_w^\top[w(t) + \tilde{w}(t)] \\ \quad + \int_0^1 k_x(1, \xi)[z(\xi, t) + \tilde{z}(\xi, t)]d\xi, \\ \dot{w}(t) = Sw(t), \quad w(0) = w_0 \in \mathbb{R}^{n_w}, \end{cases}$$

$$(6.5) \quad \begin{cases} \dot{\tilde{w}}_r(t) = (S_r + l_r q_r^\top) \tilde{w}_r(t), \quad \dot{\tilde{w}}_d = S_d \tilde{w}_d(t) + l_d \tilde{z}(1, t), \\ \tilde{z}_t(x, t) = -i\tilde{z}_{xx}(x, t) + h(x)\tilde{z}(x, t) + g(x)q_{d1}^\top \tilde{w}_d(t) + l(x)\tilde{z}(1, t), \\ \tilde{z}_x(0, t) = -iq\tilde{z}(0, t) + q_{d2}^\top \tilde{w}_d(t), \quad \tilde{z}_x(1, t) = l_0 \tilde{z}(1, t). \end{cases}$$

The “ $(\tilde{w}_r, \tilde{w}_d, \tilde{z})$ -part” in (6.5) has been shown to be exponentially stable in Theorem 5.2. Now we only need to consider the “ $(w, z)$ -part” in (6.4), which we rewrite as

$$(6.6) \quad \begin{cases} z_t(x, t) = -iz_{xx}(x, t) + h(x)z(x, t) + g(x)d_1(t), \\ z_x(0, t) = -iqz(0, t) + d_2(t), \\ z_x(1, t) = k(1, 1)[z(1, t) + \tilde{z}(1, t)] + m_w^\top[w(t) + \tilde{w}(t)] \\ \quad + \int_0^1 k_x(1, \xi)[z(\xi, t) + \tilde{z}(\xi, t)]d\xi, \\ \dot{w}(t) = Sw(t), \quad w(0) = w_0 \in \mathbb{R}^{n_w}. \end{cases}$$

Under the backstepping transformation (4.1), the “ $z$ -part” of system (6.6) can be converted into the following equivalent system:

$$\begin{cases} v_t(x, t) = -iv_{xx}(x, t) - c_s v(x, t) + \mathcal{F}[g](x)d_1(t) - k(x, 0)d_2(t), \\ v_x(0, t) = d_2(t), \\ v_x(1, t) = k(1, 1)\tilde{z}(1, t) + \int_0^1 k_x(1, \xi)\tilde{z}(\xi, t)d\xi + m_w^\top[w(t) + \tilde{w}(t)]. \end{cases}$$

Further, by the transformation (4.7), the above system is equivalent to

$$(6.7) \quad \begin{cases} \tilde{v}_t(x, t) = -i\tilde{v}_{xx}(x, t) - c_s \tilde{v}(x, t), \\ \tilde{v}_x(0, t) = 0, \\ \tilde{v}_x(1, t) = k(1, 1)\tilde{z}(1, t) + \int_0^1 k_x(1, \xi)\tilde{z}(\xi, t)d\xi + m_w^\top \tilde{w}(t). \end{cases}$$

Note that this is different from the system (4.8), that was derived for the case of state feedback.

Now we show that  $\|\tilde{v}(\cdot, t)\| \leq M_0 e^{-\mu_0 t}$  for some  $M_0, \mu_0 > 0$ . To do this, first we show that  $\tilde{e}(1, \cdot)$  in (5.9) belongs to  $L^2_{-\alpha_o}[0, \infty)$  for some  $\alpha_o \in (0, c_o/2)$ , where  $L^2_{-\alpha_o}[0, \infty)$  is defined after (1.7). Define the sequence  $(c_j)_{j \in \mathbb{N}}$  by  $c_j = \cos(j\pi)$ . Obviously, this sequence satisfies the Carleson measure criterion, see [28, Definition 5.3.1]. Define the observation operator  $\mathcal{C}z = \sum_{j=0}^{\infty} c_j z_j$ , where  $z = \sum_{j=0}^{\infty} z_j \cos(j\pi x)$  with  $(z_j)_{j=0}^{\infty} \in l^2$ . From the proof of Theorem 5.2 (Case II) we have that (i) system (5.9) is associated with a diagonal group  $\mathbb{T}$  with  $(\mathbb{T}_t z)_j = z_j e^{(j\pi^2 i - c_o)t}$  ( $\forall j \in \mathbb{N}$ ) on  $l^2$ ; (ii) the generator  $A_0$  of the diagonal group  $\mathbb{T}$  satisfies  $A_0 \cos(j\pi x) = \lambda_j \cos(j\pi x)$  with  $\lambda_j = j\pi^2 i - c_o$ ; (iii)  $\tilde{e}(1, t) = \mathcal{C}\tilde{e}(x, t)$ . Moreover, it is easy to verify that

$$\sum_{\operatorname{Im} \lambda_j \in [n, n+1)} |c_j|^2 \leq 1 \quad \forall n \in \mathbb{Z}.$$

It follows from [28, Proposition 5.3.5.] that  $\mathcal{C}$  is an admissible observation operator for  $\mathbb{T}$ . With [28, Proposition 4.3.6] we get that  $\tilde{e}(1, \cdot) \in L^2_{-\alpha_o}[0, \infty)$  for some  $\alpha_o \in (0, c_o/2)$ :

$$(6.8) \quad \int_0^\infty |e^{\alpha_o s} \tilde{e}(1, s)|^2 ds := C_1 < \infty.$$

By (5.4) and (5.8), we get  $\tilde{z}(1, t) = \tilde{e}(1, t) + n(1)^\top \tilde{w}_d(t)$ . From Theorem 5.2 and (6.8) we know that

$$k(1, 1)\tilde{z}(1, t) + \int_0^1 k_x(1, \xi)\tilde{z}(\xi, t)d\xi + m_w^\top \tilde{w}(t) := \eta_1(t) + \eta_2(t).$$

with

$$\eta_1(t) = k(1, 1)\tilde{e}(1, t),$$

$$\eta_2(t) = k(1, 1)n(1)^\top \tilde{w}_d(t) + \int_0^1 k_x(1, \xi)\tilde{z}(\xi, t)d\xi + m_w^\top \tilde{w}(t),$$

which satisfies

$$(6.9) \quad \int_0^\infty |e^{\alpha s} \eta_1(s)|^2 ds := k^2(1, 1)C_1 < \infty, \quad |\eta_2(t)| \leq M_1 e^{-\mu_1 t} \quad \forall t \geq 0,$$

for some  $M_1, \mu_2 > 0$ . Using the operator  $\mathbb{A}$  from (4.9), we write the system (6.7) as

$$\frac{d}{dt} \tilde{v}(\cdot, t) = (\mathbb{A} - c_s I) \tilde{v}(\cdot, t) + B [\eta_1(t) + \eta_2(t)],$$

where  $B = i\delta(\cdot - 1)$ . Clearly  $e^{(\mathbb{A} - c_s I)t}$  is exponentially stable. As in the proof of Proposition 3.3, we have that  $B$  is an admissible control operator for  $e^{(\mathbb{A} - c_s I)t}$ . Thus, it follows from [28, Proposition 4.2.5] that the solution  $\tilde{v}$  is a continuous  $L^2[0, 1]$ -valued function of  $t$  given by

$$(6.10) \quad \tilde{v}(\cdot, t) = e^{(\mathbb{A} - c_s I)t} \tilde{v}(\cdot, 0) + \int_0^t e^{(\mathbb{A} - c_s I)(t-s)} B [\eta_1(s) + \eta_2(s)] ds.$$

Moreover, from the exponential stability of  $e^{(\mathbb{A} - c_s I)t}$  and [35, Lemma 2.1], we have that  $\|\tilde{v}(\cdot, t)\| \leq M_0 e^{-\mu_0 t}$  for some  $M_0, \mu_0 > 0$ . Noting the formula (4.15) for  $e_y$  and Proposition 4.4, the admissibility of the observation operator  $C_e$  implies that the tracking error system is exponentially stable in the sense that  $e_y = C_e \mathcal{F}^{-1}[\tilde{v}] \in L_\alpha^2[0, \infty)$  with  $\alpha \in (-\mu_1, 0)$ . In particular, if the observation operator  $C_e$  is bounded, then by the boundedness of the transformation  $\mathcal{F}^{-1}$  there exists a constant  $C_2 > 0$  such that  $|e_y(t)| = |C_e \mathcal{F}^{-1}[\tilde{v}](\cdot, t)| \leq C_2 \|\tilde{v}(\cdot, t)\| \leq C_2 M_0 e^{-\mu_0 t}$ , so that (1.6) holds.

The inequality in this theorem follows from Theorem 5.2. By (4.1) and (4.7), we have

$$z(x, t) = \mathcal{F}^{-1}[\tilde{v} + mw](x, t).$$

Since  $\lim_{t \rightarrow \infty} \|\tilde{v}(\cdot, t)\| = 0$ ,  $w(t)$  is bounded for all  $t \geq 0$  and the transformation  $\mathcal{F}^{-1}$  is bounded, we know that  $\|z(\cdot, t)\|$  is bounded for all  $t \geq 0$ . It follows from the inequality in this theorem that all internal signals  $z(\cdot, t), w(t), \hat{z}(\cdot, t), \hat{w}(t)$  are bounded.  $\square$

**Remark 6.2.** A very concise version of this paper, with weaker results and missing proofs, was presented at a conference [36].

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