Pure and Applied Functional Analysis
Volume 5, Number 1, 2020, 27–45

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

GLOBAL SOBOLEV PERSISTENCE FOR THE FRACTIONAL BOUSSINESQ EQUATIONS WITH ZERO DIFFUSIVITY

IGOR KUKAVICA AND WEINAN WANG

Abstract. We address the persistence of regularity for the 2D α-fractional Boussinesq equations with positive viscosity and zero diffusivity in general Sobolev spaces, i.e., for $(u_0, \rho_0) \in W^{s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)$, where $s > 1$ and $q \in (2, \infty)$. We prove that the solution $(u(t), \rho(t))$ exists and belongs to $W^{s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)$ for all positive time t for $q > 2$, where $\alpha \in (1, 2)$ is arbitrary.

1. Introduction

In this paper, we address the persistence of regularity for the 2D fractional Boussinesq equations

$$
\begin{aligned}
&u_t + \Lambda^\alpha u + u \cdot \nabla u + \nabla \pi = \rho e_2 \\
&\rho_t + u \cdot \nabla \rho = 0 \\
&\nabla \cdot u = 0
\end{aligned}
$$

in Sobolev spaces. Here, u is the velocity satisfying the 2D Navier-Stokes equations [8, 13, 17, 32, 35, 36] driven by ρ, which represents the density or temperature of the fluid, depending on the physical context. Also, $e_2 = (0, 1)$ is the unit vector in the vertical direction and $1 < \alpha < 2$.

The global existence and persistence of regularity has been a topic of high interest since the seminal works of Chae [5] and of Hou and Li [21], who proved the global existence of a unique solution in the case of Laplacian, $\alpha = 2$. Namely, the global persistence holds for (u_0, ρ_0) in $H^s \times H^{s-1}$ for integers $s \geq 3$ [21], while we have the global persistence in $H^s \times H^s$ for integers $s \geq 3$ by [5]. The global existence and uniqueness in the low regularity space $H^1 \times L^2$ was established by Lunasin et al in [30]. The persistence in $H^s \times H^{s-1}$ for the intermediate values $1 < s < 3$ was then settled in [22, 23]. For other results on the global existence and persistence of solutions, cf. [1, 3, 4, 7, 9, 10, 11, 12, 14, 15, 16, 18, 19, 24, 25, 28, 29, 31, 33, 34].

The main difficulty when studying the persistence of regularity in the Sobolev spaces $W^{s,q} \times W^{s-1,q}$ when $q > 2$ is the lack of availability of the energy equation, which is one of the essential features of the Boussinesq system. This problem was studied in [29], where it was proven that the persistence holds if $(s-1)q > 2$.

2010 *Mathematics Subject Classification.* 35Q35, 35Q30.
Key words and phrases. Boussinesq system, fractional dissipation, Sobolev persistence, Kato-Ponce inequality, global existence.

In the present paper, we consider the fractional dissipation in the range $1 < \alpha < 2$, addressing the persistence in $W^{s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)$. Namely, we prove that if $(u_0, \rho_0) \in W^{s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)$, then $(u(\cdot,t), \rho(\cdot,t)) \in W^{s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)$ for all $t \geq 0$. The main result is contained in Theorem 2.1 and asserts the global persistence for all $s > 1$. The main device in the proof is the generalized vorticity

$$\zeta = \omega - \partial_1 (I - \Delta)^{-\alpha/2} \rho. \tag{1.1}$$

This change of variable is inspired by the one introduced by Jiu et al in [25], (cf. also [33]), which in turn drew from the work of Hmidi, Keraani, and Rousset [20]. Here we need to modify it to avoid problems with low frequencies as our data are not square integrable. We show in (2.6) below that the modified vorticity ζ defined in (1.1) satisfies the equation

$$\zeta_t + u \cdot \nabla \zeta + \Lambda^\alpha \zeta = [S, u \cdot \nabla]\rho - (\tilde{\Lambda}^{-\alpha}\Lambda^\alpha - I)\partial_1 \rho \tag{1.2}$$

where $S = \partial_1 (I - \Delta)^{-\alpha/2}$ with $\Lambda = (-\Delta)^{1/2}$ and $\tilde{\Lambda} = (I - \Delta)^{1/2}$. Compared to the original change of variable in [25], we obtain a new term $N\rho = (\tilde{\Lambda}^{-\alpha}\Lambda^\alpha - I)\partial_1\rho$, for which however we show in Lemma 2.2 below that it is smoothing of degree 1. The reason why this change of variable is suitable for low frequencies is due to the inhomogeneity in the second term of (1.1).

Also, an important part of the proof of Sobolev persistence is based on the observation that a fractional derivative of the commutator term in (1.2) is a sum of two terms, which are also of commutator type and are thus suitable for the use of a Kato-Ponce type inequality; cf. (4.7) and Remark 4.1 below.

The paper is organized as follows. In Section 2, we state the main theorem on the persistence and introduce the change of the vorticity variable. We also prove the smoothing property of the operator N. The next section contains a variant of a Kato-Ponce lemma suitable for the operator S arising in (1.2). Lemma 3.3 contains the bounds for the vorticity and its modified version ζ. The proof of the main theorem for the case $s \leq \alpha$ is then provided in Section 4. Finally, the last section contains the proof of the main theorem for $s > \alpha$. This part of the proof requires the case $s \leq \alpha$ when we establish a bound on $\|\Lambda^{1/2} u\|_{L^\infty}$ in (5.8) below.

2. Notation and the main result on global persistence

We consider solutions of the Boussinesq system

$$u_t + \Lambda^\alpha u + u \cdot \nabla u + \nabla \pi = \rho e_2 \tag{2.1}$$

$$\rho_t + u \cdot \nabla \rho = 0 \tag{2.2}$$

$$\nabla \cdot u = 0, \tag{2.3}$$

where the operator Λ^α is defined by

$$\Lambda^\alpha = (-\Delta)^{\alpha/2}, \qquad 1 < \alpha < 2,$$

or, using the Fourier transform,

$$(\Lambda^\alpha f)\hat{}(\xi) = |\xi|^\alpha \hat{f}(\xi), \qquad \xi \in \mathbb{R}^2. \tag{2.4}$$

The following is the main result of the paper.

Theorem 2.1. *Let $q \in (2,\infty)$ and $s > 1$. Assume that $\|u_0\|_{W^{s,q}} < \infty$ with $\nabla \cdot u_0 = 0$ and $\|\rho_0\|_{W^{s,q}} < \infty$. Then there exists a unique solution (u, ρ) to the equations* (2.1)–(2.3) *such that $u \in C\left([0,T], W^{s,q}(\mathbb{R}^2)\right)$ and $\rho \in C\left([0,T], W^{s,q}(\mathbb{R}^2)\right)$ for all $T > 0$.*

Applying the curl operator to (2.1), we obtain the vorticity equation

$$\omega_t + \Lambda^\alpha \omega + u \cdot \nabla \omega = \partial_1 \rho. \tag{2.5}$$

Define $\tilde{\Lambda} = (I - \Delta)^{1/2}$ and set

$$\zeta = \omega - S\rho,$$

where

$$S = \partial_1 \tilde{\Lambda}^{-\alpha} = \partial_1 (I - \Delta)^{-\alpha/2}.$$

The equation satisfied by ζ is obtained by replacing ω with $\zeta + S\rho$ in (2.5) and combining the resulting equation with (2.2). We get

$$\begin{aligned} \zeta_t + \Lambda^\alpha \zeta + u \cdot \nabla \zeta &= -S\rho_t - u \cdot \nabla S\rho - \Lambda^\alpha S\rho + \partial_1 \rho \\ &= [S, u \cdot \nabla]\rho - (\tilde{\Lambda}^{-\alpha}\Lambda^\alpha - I)\partial_1 \rho. \end{aligned} \tag{2.6}$$

Therefore, the equation for the generalized vorticity ζ reads

$$\zeta_t + \Lambda^\alpha \zeta + u \cdot \nabla \zeta = [S, u \cdot \nabla]\rho - N\rho, \tag{2.7}$$

where we set

$$N = (\tilde{\Lambda}^{-\alpha}\Lambda^\alpha - I)\partial_1. \tag{2.8}$$

The operator N is a Fourier multiplier with the symbol

$$m(\xi) = \frac{|\xi|^\alpha \xi_1}{(1 + |\xi|^2)^{\alpha/2}} - \xi_1.$$

It is possible to check that the symbol satisfies the assumptions of the Hörmander-Mikhlin theorem and thus $\|N\rho\|_{L^{\bar{q}}} \le C\|\rho\|_{L^{\bar{q}}}$ for $1 < \bar{q} < \infty$. However, as asserted in the next lemma, a stronger statement holds. Namely, the operator N defined in (2.8) is smoothing of order 1.

Lemma 2.2. *Consider the Fourier multiplier $T_{\tilde{m}}$ with the symbol*

$$\tilde{m}(\xi) = (|\xi|^2 + 1)^{1/2} m(\xi).$$

Then $T_{\tilde{m}}$ is a Hörmander-Mikhlin operator satisfying

$$\|T_{\tilde{m}} f\|_{L^{\bar{q}}} \lesssim \|f\|_{L^{\bar{q}}}, \qquad f \in L^{\bar{q}}, \tag{2.9}$$

for $1 < \bar{q} < \infty$.

An equivalent way of stating (2.9) is

$$\|Nf\|_{L^{\bar{q}}} + \|\nabla N f\|_{L^{\bar{q}}} \lesssim \|f\|_{L^{\bar{q}}}, \qquad f \in L^{\bar{q}}, \qquad \bar{q} \in (1,\infty).$$

Proof of Lemma 2.2. It suffices to prove that the symbol

$$\tilde{m}(\xi) = \xi_1 \frac{(1 + |\xi|^2)^{\alpha/2} - |\xi|^\alpha}{(1 + |\xi|^2)^{\alpha - 1/2}}$$

satisfies the Hörmander-Mikhlin condition

$$|\partial^\alpha \tilde{m}(\xi)| \le \frac{C(|\alpha|)}{|\xi|^\alpha}, \qquad \alpha \in \mathbb{N}_0^2, \qquad \xi \in \mathbb{R}^2 \setminus \{0\}.$$

Since $\xi_1/(1+|\xi|^2)^{1/2}$ is of Hörmander-Mikhlin type, it is sufficient to prove that

$$\bar{m}(\xi) = (1+|\xi|^2)^{1-\alpha/2}((1+|\xi|^2)^{\alpha/2} - |\xi|^\alpha)$$

satisfies the Hörmander-Mikhlin condition. In order to check this, we write

$$\bar{m}(\xi) = \frac{\alpha}{2}\int_0^1 \frac{(1+|\xi|^2)^{1-\alpha/2}}{(t+|\xi|^2)^{1-\alpha/2}}\,dt$$

and then verify that the condition holds for the low and high frequencies, i.e., when $|\xi| \lesssim 1$ and $|\xi| \gtrsim 1$ respectively. □

Next, we recall a version of the Kato-Ponce inequality from [29].

Lemma 2.3 ([29]). *Let $s \in (0,1)$ and $f, g \in \mathcal{S}(\mathbb{R}^2)$. For $1 < q < \infty$ and $j \in \{1,2\}$, the inequality*

$$\|[\Lambda^s \partial_j, g]f\|_{L^q} \le C\|f\|_{L^{q_1}}\|\Lambda^{1+s} g\|_{L^{\tilde{q}_1}} + C\|\Lambda^s f\|_{L^{q_2}}\|\Lambda g\|_{L^{\tilde{q}_2}}$$

holds, where $q_1, \tilde{q}_1, \tilde{q}_2 \in [q,\infty]$ and $q_2 \in [q,\infty)$ satisfy $1/q = 1/q_1 + 1/\tilde{q}_1 = 1/q_2 + 1/\tilde{q}_2$ and $C = C(q_1, \tilde{q}_1, \tilde{q}_2, q_2, s)$.

Finally, we recall from [6, 26] an inequality useful for treating the fractional coercive term.

Lemma 2.4 ([6, 26]). *Consider the operator Λ defined in (2.4) on $\mathbb{R}^2$. If $\theta, \Lambda^s\theta \in L^p$, where $p \ge 2$, then*

$$\int_{\mathbb{R}^2} |\theta|^{p-2}\theta\Lambda^s\theta\,dx \ge \frac{2}{p}\int_{\mathbb{R}^2} (\Lambda^{s/2}(|\theta|^{p/2}))^2\,dx, \tag{2.10}$$

for all $s \in (0,2)$.

3. An L^q inequality for the vorticity and a Kato-Ponce type commutator estimate

The following lemma provides an L^q bound for the modified vorticity ζ.

Lemma 3.1. *Assume that $u_0, \rho_0 \in W^{s,q}(\mathbb{R}^2)$, where $s > 1$ and $q > 2$. Then we have*

$$\|\zeta\|_{L^q} \le Ce^{Ct}, \qquad t \ge 0 \tag{3.1}$$

and

$$\|\omega\|_{L^q} \le Ce^{Ct}, \qquad t \ge 0, \tag{3.2}$$

where $C = C(\|\omega_0\|_{L^q}, \|\rho_0\|_{L^q})$. Moreover, we have

$$\int_0^t \|\Lambda^{\alpha/2}(|\zeta|^{q/2})\|_{L^2}^2\,dx \le Ce^{Ct}, \tag{3.3}$$

for all $t \ge 0$.

Above and in the sequel, the exponent $q > 2$ and the parameter $s > 1$ are considered fixed, so we do not indicate dependence of constants on these parameters.

The main step in the proof of Lemma 3.1 and Theorem 2.1 is an inhomogeneous Kato-Ponce type commutator estimate, which is stated next.

Lemma 3.2. *Denote*

$$\bar{S} := |\nabla|(I-\Delta)^{-\alpha/2}.$$

Then, for $j \in \{1,2\}$ and $0 \le \mu \le \alpha$, we have

$$\|[\Lambda^\mu S\partial_j, g]f\|_{L^q} \le C\|\nabla g\|_{L^{r_1}}\|\Lambda^\mu \bar{S} f\|_{L^{\tilde{r}_1}} + C\|\Lambda^{\mu+1}\bar{S} g\|_{L^{r_2}}\|f\|_{L^{\tilde{r}_2}},$$

where $r_1, \tilde{r}_1, \tilde{r}_2 \in [q,\infty]$ and $r_2 \in [q,\infty)$ satisfy $1/q = 1/r_1 + 1/\tilde{r}_1 = 1/r_2 + 1/\tilde{r}_2$ and where $C = C(r_1, \tilde{r}_1, \tilde{r}_2, r_2, q)$.

Proof of Lemma 3.2 (sketch). We follow the strategy from [27] (cf. also [29]) and consider the commutator in three regions defined by the supports of Φ_k below. Namely, we write

$$\begin{aligned}
[\Lambda^\mu S\partial_j, g]f &= c_0 \sum_{k=1}^{3} \iint e^{ix(\xi+\eta)} \left(\frac{|\xi+\eta|^\mu(\xi_1+\eta_1)(\xi_j+\eta_j)}{(1+|\xi+\eta|^2)^{\alpha/2}} - \frac{|\xi|^\mu \xi_1\xi_j}{(1+|\xi|^2)^{\alpha/2}} \right) \\
&\qquad \times \hat{f}(\xi)\hat{g}(\eta)\Phi_k\left(\frac{|\xi|}{|\eta|}\right) d\eta\, d\xi \\
&= c_0 \sum_{k=1}^{3} \iint e^{ix(\xi+\eta)} A_k(\xi,\eta)\, d\eta\, d\xi,
\end{aligned} \tag{3.4}$$

where $\Phi_k \colon \mathbb{R} \to [0,1]$ are C^∞ cut-off functions such that

$$\sum_{k=1}^{3} \Phi_k = 1 \text{ on } [0,\infty)$$

with

$$\operatorname{supp}\Phi_1 \subseteq [-1/2, 1/2], \qquad \operatorname{supp}\Phi_2 \subseteq [1/4, 3], \qquad \operatorname{supp}\Phi_3 \subseteq [2,\infty]$$

and

$$A_k(\xi,\eta) = \left(\frac{|\xi+\eta|^\mu(\xi_1+\eta_1)(\xi_j+\eta_j)}{(1+|\xi+\eta|^2)^{\alpha/2}} - \frac{|\xi|^\mu \xi_1\xi_j}{(1+|\xi|^2)^{\alpha/2}} \right) \hat{f}(\xi)\hat{g}(\eta)\Phi_k\left(\frac{|\xi|}{|\eta|}\right).$$

Thus, the commutator (3.4) may be rewritten as

$$[\Lambda^{s-1} S\partial_j, u_j]\rho = \sum_{k=1}^{3} \iint e^{ix(\xi+\eta)} A_k(\xi,\eta)\, d\eta\, d\xi.$$

We write A_1 as

$$\begin{aligned}
A_1(\xi,\eta) &= \left(\frac{|\xi+\eta|^\mu(\xi_1+\eta_1)(\xi_j+\eta_j)(1+|\eta|^2)^{\alpha/2}}{(1+|\xi+\eta|^2)^{\alpha/2}|\eta|^{\mu+2}} - \frac{|\xi|^\mu \xi_1\xi_j(1+|\eta|^2)^{\alpha/2}}{(1+|\xi|^2)^{\alpha/2}|\eta|^{\mu+2}} \right) \\
&\qquad \times \hat{f}(\xi)(\Lambda^{\mu+1}\bar{S}g)\hat{\ }(\eta)\Phi_1\left(\frac{|\xi|}{|\eta|}\right) \\
&= \sigma_1(\xi,\eta)\hat{f}(\xi)(\Lambda^{\mu+1}\bar{S}g)\hat{\ }(\eta).
\end{aligned}$$

It is elementary to show that

$$|\sigma_1| \le C,$$

as well as more generally

$$|\partial^\alpha \partial^\beta \sigma_1| \le \frac{C(|\alpha|,|\beta|)}{(|\xi|+|\eta|)^{|\alpha|+|\beta|}}, \qquad \alpha,\beta \in \mathbb{N}_0^2.$$

By the Coifman-Meyer theorem, we get

$$\left\| \iint e^{ix(\xi+\eta)} A_1(\xi,\eta)\, d\eta\, d\xi \right\|_{L^q} \lesssim \|f\|_{L^{r_1}} \|\Lambda^{\mu+1}\bar{S}g\|_{L^{\tilde{r}_1}},$$

where $1/q = 1/r_1 + 1/\tilde{r}_1$. For A_3, we write

$$\begin{aligned}
A_3(\xi,\eta) &= \left(\frac{|\xi+\eta|^\mu(\xi_1+\eta_1)(\xi_j+\eta_j)}{(1+|\xi+\eta|^2)^{\alpha/2}} - \frac{|\xi|^\mu \xi_1\xi_j}{(1+|\xi|^2)^{\alpha/2}} \right) \hat{f}(\xi)\hat{g}(\eta)\Phi_3\left(\frac{|\xi|}{|\eta|}\right) \\
&= \frac{(1+|\xi|^2)^{\alpha/2}}{|\eta||\xi|^{\mu+1}} \left(\frac{|\xi+\eta|^\mu(\xi_1+\eta_1)(\xi_j+\eta_j)}{(1+|\xi+\eta|^2)^{\alpha/2}} - \frac{|\xi|^\mu \xi_1\xi_j}{(1+|\xi|^2)^{\alpha/2}} \right) \\
&\quad \times (\Lambda^\mu \bar{S} f)\hat{\ }(\xi)(\nabla g)\hat{\ }(\eta)\Phi_3\left(\frac{|\xi|}{|\eta|}\right) \\
&= \sigma_3(\xi,\eta)(\Lambda^\mu \bar{S} f)\hat{\ }(\xi)(\nabla g)\hat{\ }(\eta)\Phi_3\left(\frac{|\xi|}{|\eta|}\right).
\end{aligned}$$

Setting

$$\phi(t) = \frac{|\xi+t\eta|^\mu(\xi_1+t\eta_1)(\xi_j+t\eta_j)}{(1+|\xi+t\eta|^2)^{\alpha/2}}, \qquad t\in[0,1]$$

we have

$$\begin{aligned}
\phi'(t) &= \frac{\mu|\xi+t\eta|^{\mu-2}(\xi+t\eta)\eta(\xi_1+t\eta_1)(\xi_j+t\eta_j)}{(1+|\xi+t\eta|^2)^{\alpha/2}} + \frac{|\xi+t\eta|^\mu \eta_1(\xi_j+t\eta_j)}{(1+|\xi+t\eta|^2)^{\alpha/2}} \\
&\quad + \frac{|\xi+t\eta|^\mu \eta_j(\xi_1+t\eta_1)}{(1+|\xi+t\eta|^2)^{\alpha/2}} + \frac{\alpha|\xi+t\eta|^\mu(\xi_1+t\eta_1)(\xi_j+t\eta_j)(\xi+t\eta)\eta}{(1+|\xi+t\eta|^2)^{\alpha/2+1}}.
\end{aligned}$$

Note that in the region $\Phi_3 > 0$, we have $|\xi| \ge 2|\eta|$. Therefore,

$$|\sigma_3| \le C,$$

as well as more generally

$$|\partial^\alpha \partial^\beta \sigma_3| \le \frac{C(|\alpha|,|\beta|)}{(|\xi|+|\eta|)^{|\alpha|+|\beta|}}, \qquad \alpha,\beta \in \mathbb{N}_0^2.$$

By the Coifman-Meyer theorem, we get

$$\left\| \iint e^{ix(\xi+\eta)} A_3(\xi,\eta)\, d\eta\, d\xi \right\|_{L^q} \lesssim \|\nabla g\|_{L^{q_1}} \|\Lambda^\mu \bar{S} f\|_{L^{q_2}},$$

where $1/q = 1/q_1 + 1/q_2$. For A_2, we use the complex interpolation inequality. Since the argument is the same as in [27], we omit the proof. By combining the estimates for A_1, A_2, and A_3, we get

$$\|[\Lambda^\mu S\partial_j, g]f\|_{L^q} \lesssim \|\nabla g\|_{L^{q_1}} \|\Lambda^\mu \bar{S} f\|_{L^{q_2}} + \|\Lambda^\mu \bar{S}\nabla g\|_{L^{q_3}} \|f\|_{L^{q_4}},$$

where the parameters $q_1, q_2, q_3, q_4 \in [q,\infty]$ satisfy $1/q = 1/q_1 + 1/q_2 = 1/q_3 + 1/q_4$ and the implicit constant depends on q_1, q_2, q_3, q_4, and μ. $\square$

Proof of Lemma 3.1. Since $s > 1$, we have $W^{s,q}(\mathbb{R}^2) \subseteq L^\infty(\mathbb{R}^2)$, and thus

$$\rho_0 \in L^{\bar q}, \qquad \bar q \in [q, \infty].$$

Using the $L^{\bar q}$ conservation property for the density equation (2.2), we get

$$\|\rho(t)\|_{L^{\bar q}} \le \|\rho_0\|_{L^{\bar q}} \lesssim 1, \qquad \bar q \in [q, \infty], \tag{3.5}$$

where we assume that all the constants depend on $\|\rho_0\|_{L^q}$ and $\|\omega_0\|_{L^q}$. In order to estimate $\|\zeta\|_{L^q}$, we multiply the equation (2.7) with $|\zeta|^{q-2}\zeta$ and integrate obtaining

$$\begin{aligned} &\frac{1}{q}\frac{d}{dt}\|\zeta\|_{L^q}^q + \int (\Lambda^\alpha \zeta)|\zeta|^{q-2}\zeta\, dx \\ &\quad = -\int N\rho|\zeta|^{q-2}\zeta\, dx + \int [S, u\cdot\nabla]\rho|\zeta|^{q-2}\zeta\, dx = I_1 + I_2. \end{aligned} \tag{3.6}$$

For I_1, we have

$$I_1 \le \|N\rho\|_{L^q}\||\zeta|^{q-2}\zeta\|_{L^{q/(q-1)}} \lesssim \|\rho\|_{L^q}\|\zeta\|_{L^q}^{q-1} \lesssim \|\zeta\|_{L^q}^{q-1}, \tag{3.7}$$

where we used Hölder's inequality and (3.5). Since u is divergence-free, we may rewrite the commutator as

$$\begin{aligned} [S, u\cdot\nabla]\rho &= Su_j\partial_j\rho - u_j\partial_j S\rho = (\partial_j S)(u_j\rho) - u_j(\partial_j S)\rho \\ &= [\partial_j S, u_j]\rho. \end{aligned}$$

Observe that $\partial_j S$ is an operator of order $2-\alpha$. Thus, by Lemma 3.2 with $\mu = 0$, we have

$$\begin{aligned} I_2 &\le \|[S, u\cdot\nabla]\rho\|_{L^q}\|\zeta\|_{L^q}^{q-1} = \|[\partial_j S, u_j]\rho\|_{L^q}\|\zeta\|_{L^q}^{q-1} \\ &\lesssim (\|\bar S\rho\|_{L^{a_1}}\|\nabla u\|_{L^{b_1}} + \|\rho\|_{L^{a_2}}\|\bar S\nabla u\|_{L^{b_2}})\|\zeta\|_{L^q}^{q-1} \\ &\lesssim (\|\bar S\rho\|_{L^{a_1}}\|\omega\|_{L^{b_1}} + \|\rho\|_{L^{a_2}}\|\bar S\omega\|_{L^{b_2}})\|\zeta\|_{L^q}^{q-1}, \end{aligned}$$

with the Lebesgue exponents above satisfying $1/q = 1/a_1 + 1/b_1 = 1/a_2 + 1/b_2$ and $a_1, a_2, b_1, b_2 \in (q, \infty)$. Therefore, choosing $a_1 = a_2 = q/(\alpha-1)$ and $b_1 = b_2 = q/(2-\alpha)$,

$$I_2 \lesssim (\|\bar S\rho\|_{L^{q/(\alpha-1)}}\|\omega\|_{L^{q/(2-\alpha)}} + \|\rho\|_{L^{q/(\alpha-1)}}\|\bar S\omega\|_{L^{q/(2-\alpha)}})\|\zeta\|_{L^q}^{q-1}.$$

Now, by the fractional Gagliardo-Nirenberg inequality applied to $|\zeta|^{q/2}$, we have

$$\|\zeta\|_{L^r} \lesssim \|\zeta\|_{L^q}^{(r\alpha-2r+2q)/\alpha r}\|\Lambda^{\alpha/2}(|\zeta|^{q/2})\|_{L^2}^{4(r-q)/\alpha rq}, \qquad q \le r \le 2q/(2-\alpha), \tag{3.8}$$

from where

$$\|\zeta\|_{L^{q/(2-\alpha)}} \lesssim \|\zeta\|_{L^q}^{(2-\alpha)/\alpha}\|\Lambda^{\alpha/2}(|\zeta|^{q/2})\|_{L^2}^{4(\alpha-1)/\alpha q}.$$

Also, using the triangle inequality

$$\begin{aligned} \|\omega\|_{L^{q/(2-\alpha)}} &\le \|\zeta\|_{L^{q/(2-\alpha)}} + \|\bar S\rho\|_{L^{q/(2-\alpha)}} \lesssim \|\zeta\|_{L^{q/(2-\alpha)}} + \|\rho\|_{L^{q/(2-\alpha)}} \\ &\lesssim \|\zeta\|_{L^{q/(2-\alpha)}} + 1 \lesssim \|\zeta\|_{L^q}^{(2-\alpha)/\alpha}\|\Lambda^{\alpha/2}(|\zeta|^{q/2})\|_{L^2}^{4(\alpha-1)/\alpha q} + 1 \end{aligned}$$

we get

$$\begin{aligned} I_2 &\lesssim (\|\omega\|_{L^{q/(2-\alpha)}} + \|\bar S\omega\|_{L^{q/(2-\alpha)}})\|\zeta\|_{L^q}^{q-1} \\ &\lesssim \|\omega\|_{L^{q/(2-\alpha)}}\|\zeta\|_{L^q}^{q-1} \\ &\lesssim \|\zeta\|_{L^q}^{(2-\alpha)/\alpha+q-1}\|\Lambda^{\alpha/2}(|\zeta|^{q/2})\|_{L^2}^{4(\alpha-1)/\alpha q} + \|\zeta\|_{L^q}^{q-1}. \end{aligned} \tag{3.9}$$

Replacing (3.7) and (3.9) in (3.6) and using (2.10) on the coercive term, we obtain

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\zeta\|_{L^q}^q + \frac{2}{q}\int_\Omega (\Lambda^{\alpha/2}(|\zeta|^{q/2}))^2\,dx \\ \lesssim \|\zeta\|_{L^q}^{q-1} + \|\zeta\|_{L^q}^{(2-\alpha)/\alpha+q-1}\|\Lambda^{\alpha/2}(|\zeta|^{q/2})\|_{L^2}^{4(\alpha-1)/\alpha q}.\end{aligned}$$

Since $4(\alpha-1)/\alpha q < 2$, we may use Young's inequality with exponents $\alpha q/(\alpha q - 2\alpha + 2)$ and $\alpha q/2(\alpha-1)$ to get

$$\frac{d}{dt}\|\zeta\|_{L^q}^q + \int_\Omega (\Lambda^{\alpha/2}(|\zeta|^{q/2}))^2\,dx \lesssim \|\zeta\|_{L^q}^{q-1} + \|\zeta\|_{L^q}^q, \tag{3.10}$$

where the implicit constant depends on the initial data. The inequality (3.1) then follows by applying the Gronwall inequality, while (3.2) is a consequence of (3.1) and the triangle inequality. Finally, (3.3) holds by using (3.1) in (3.10) and integrating. □

It is important that we may bootstrap the above statement and obtain the conclusion on the behavior of the $L^{\bar q}$ norm of ζ, and thus of ω, for all $\bar q > q$.

Lemma 3.3. *Assume that $u_0, \rho_0 \in W^{s,q}(\mathbb{R}^2)$, where $s \geq 1$ and $q \in (2,\infty)$. Then for every $\bar q \in (q,\infty)$ and $t_0 > 0$ we have*

$$\|\zeta\|_{L^{\bar q}} \leq Ce^{Ct}, \qquad t \geq t_0$$

and

$$\|\omega\|_{L^{\bar q}} \leq Ce^{Ct}, \qquad t \geq t_0, \tag{3.11}$$

where $C = C(\|\omega_0\|_{L^q}, \|\rho_0\|_{L^q}, \bar q, t_0)$. Moreover, we have

$$\int_0^t \|\Lambda^{\alpha/2}(|\zeta^{\bar q/2}|)\|_{L^2}^2\,ds \leq Ce^{Ct},$$

for all $t \geq 0$ where $C = C(\|\omega_0\|_{L^q}, \|\rho_0\|_{L^q}, \bar q, t_0)$.

Proof of Lemma 3.2. We first prove that the statement holds for all $\bar q \in [q, 2q/(2-\alpha)]$, and the rest follows by an iteration argument. Using (3.3) with $t = t_0 = 1$, we obtain

$$\left|\left\{t \in (0,t_0] : \|\Lambda^{\alpha/2}(|\zeta^{\bar q/2}(t)|)\|_{L^2}^2 \leq C\right\}\right| \geq \frac{1}{C}$$

for $C > 0$ sufficiently large. It is easy to deduce then that there exists $\bar t \in (0,t_0)$ such that

$$\|\Lambda^{\alpha/2}(|\zeta^{\bar q/2}|)(\bar t)\|_{L^2} \leq C.$$

Since also

$$\|\zeta(\bar t)\|_{L^q} \leq C,$$

we get by (3.8)

$$\|\zeta(\bar t)\|_{L^{\bar q}} \leq C$$

since $q \leq \bar q \leq 2q/(2-\alpha)$. Applying Lemma 3.1 but with q replaced with $\bar q$, we obtain the statement for $\bar q$ in this range. Continuing by induction, we get then the conclusion for all $\bar q \in [q,\infty)$, and the lemma is established. □

4. The Sobolev persistence for $1 < s \leq \alpha$

In this section, we prove our main result, Theorem 2.1, in the case when $s \leq \alpha$.

Proof of Theorem 2.1 for $s \leq \alpha$. For $j = 1, 2$, we multiply the j-th velocity equation of (2.1) with $|u_j|^{q-2}u_j$, integrate the resulting equation with respect to x, and sum for $j = 1, 2$ obtaining

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\sum_{j=1}^{2}\|u_j\|_{L^q}^q &+ \sum_{j=1}^{2}\int(\Lambda^\alpha u_j)|u_j|^{q-2}u_j\,dx\\ &= -\sum_{j=1}^{2}\int\partial_j\pi\cdot|u_j|^{q-2}u_j\,dx + \sum_{j=1}^{2}\int\rho e_2\cdot|u_j|^{q-2}u_j\,dx\end{aligned}$$

since due to the divergence-free condition for u we have $\int(u\cdot\nabla u_j)|u_j|^{q-2}u_j\,dx = 0$ for $j = 1, 2$. By Lemma 2.4 and Hölder's inequality, we get

$$\frac{1}{q}\frac{d}{dt}\sum_{j=1}^{2}\|u_j\|_{L^q}^q + \frac{2}{q}\sum_{j=1}^{2}\|\Lambda^{\alpha/2}(|u_j|^{q/2})\|_{L^2}^2 \lesssim \|\nabla\pi\|_{L^q}\|u\|_{L^q}^{q-1} + \|u\|_{L^q}^{q-1} \tag{4.1}$$

where, as above, q is considered fixed (i.e., the constants are allowed to depend on q). Using the Calderón-Zygmund and Sobolev embedding theorems, we obtain

$$\|\nabla\pi\|_{L^q} \lesssim \|u\|_{L^{2q}}\|\omega\|_{L^{2q}} \lesssim \|u\|_{L^q}^{1-1/q}\|\omega\|_{L^q}^{1/q}\|\omega\|_{L^{2q}} \lesssim Ce^{Ct}\|u\|_{L^q}^{1-1/q}, \tag{4.2}$$

where we also used Lemma 3.3 in the last step. Applying (4.2) on the first term of the right hand side of (4.1) gives

$$\frac{d}{dt}\sum_{j=1}^{2}\|u_j\|_{L^q}^q + \sum_{j=1}^{2}\|\Lambda^{\alpha/2}(|u_j|^{q/2})\|_{L^2}^2 \lesssim e^{Ct}\|u\|_{L^q}^{q-1/q} + \|u\|_{L^q}^{q-1}$$

and thus

$$\|u\|_{L^q} \lesssim e^{Ct}$$

by the Gronwall inequality.

Next, we consider the L^q norm of higher order derivatives. Applying Λ^{s-1} to (2.7), multiplying the resulting equation by $|\Lambda^{s-1}\zeta|^{q-2}\Lambda^{s-1}\zeta$, and integrating, we get

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\Lambda^{s-1}\zeta\|_{L^q}^q &+ \int\Lambda^\alpha(\Lambda^{s-1}\zeta)|\Lambda^{s-1}\zeta|^{q-2}\Lambda^{s-1}\zeta\,dx\\ &= -\int\Lambda^{s-1}(u\cdot\nabla\zeta)|\Lambda^{s-1}\zeta|^{q-2}\Lambda^{s-1}\zeta\,dx\\ &\quad+\int\Lambda^{s-1}([S,u\cdot\nabla]\rho)|\Lambda^{s-1}\zeta|^{q-2}\Lambda^{s-1}\zeta\,dx\\ &\quad-\int\Lambda^{s-1}N\rho|\Lambda^{s-1}\zeta|^{q-2}\Lambda^{s-1}\zeta\,dx\\ &= J_1 + J_2 + J_3.\end{aligned} \tag{4.3}$$

By Lemma 2.3, we estimate

$$\begin{aligned} J_1 &= -\int \bigl(\Lambda^{s-1}(u\cdot\nabla\zeta) - u\cdot\Lambda^{s-1}\nabla\zeta\bigr)|\Lambda^{s-1}\zeta|^{q-2}\Lambda^{s-1}\zeta\,dx \\ &\le \|\Lambda^{s-1}(u\cdot\nabla\zeta) - u\cdot\Lambda^{s-1}\nabla\zeta\|_{L^q}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\ &\le \sum_{j=1}^{2}\|\partial_j\Lambda^{s-1}(u_j\zeta) - u_j\partial_j\Lambda^{s-1}\zeta\|_{L^q}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}, \end{aligned} \tag{4.4}$$

where we used the divergence-free condition and the triangle inequality in the last step. Therefore,

$$\begin{aligned} J_1 &\lesssim (\|\zeta\|_{L^{r_1}}\|\Lambda^{s-1}\omega\|_{L^{r_2}} + \|\Lambda u\|_{L^{r_3}}\|\Lambda^{s-1}\zeta\|_{L^{r_4}})\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\ &\lesssim \bigl(\|\zeta\|_{L^{r_1}}(\|\Lambda^{s-1}\zeta\|_{L^{r_2}} + \|\Lambda^{s-1}S\rho\|_{L^{r_2}}) + \|\omega\|_{L^{r_3}}\|\Lambda^{s-1}\zeta\|_{L^{r_4}}\bigr)\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}, \end{aligned}$$

for any $r_1, r_2, r_3, r_4 \in (q,\infty)$ such that $1/q = 1/r_1 + 1/r_2 = 1/r_3 + 1/r_4$. Choose $r_1 = r_2 = r_3 = r_4 = 2q$ and note that

$$\|\Lambda^{s-1}S\rho\|_{L^{2q}} \lesssim \|\rho\|_{L^{2q}} \lesssim 1$$

by $s \le \alpha$. Therefore, using Lemma 3.3,

$$J_1 \lesssim e^{Ct}(\|\Lambda^{s-1}\zeta\|_{L^{2q}} + 1)\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}.$$

By (3.8), have

$$\|\zeta\|_{L^{2q}} \lesssim \|\zeta\|_{L^q}^{(\alpha-1)/\alpha}\|\Lambda^{\alpha/2}(|\zeta|^{q/2})\|_{L^2}^{2/\alpha q}, \tag{4.5}$$

and thus we obtain

$$J_1 \lesssim e^{Ct}\|\Lambda^{s-1}\zeta\|_{L^q}^{(\alpha-1)/\alpha}\|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|)^{q/2}\|_{L^2}^{2/\alpha q} + e^{Ct}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}.$$

For J_2, we write

$$\begin{aligned} \Lambda^{s-1}\bigl([S, u\cdot\nabla]\rho\bigr) &= \Lambda^{s-1}\bigl(S\bigl((u\cdot\nabla)\rho\bigr)\bigr) - \Lambda^{s-1}\bigl((u\cdot\nabla)S\rho\bigr) \\ &= \Lambda^{s-1}\Bigl(S\bigl((u\cdot\nabla)\rho\bigr)\Bigr) - u\cdot\nabla\bigl(\Lambda^{s-1}(S\rho)\bigr) \\ &\quad + u\cdot\nabla\bigl(\Lambda^{s-1}(S\rho)\bigr) - \Lambda^{s-1}\bigl((u\cdot\nabla)S\rho\bigr) \\ &= \Lambda^{s-1}S\partial_j\bigl(u_j\rho\bigr) - u_j\partial_j\bigl(\Lambda^{s-1}(S\rho)\bigr) \\ &\quad + u_j\partial_j\bigl(\Lambda^{s-1}(S\rho)\bigr) - \Lambda^{s-1}\partial_j\bigl(u_jS\rho\bigr), \end{aligned} \tag{4.6}$$

where we used the divergence-free condition (2.3) in the last step. The first two and the last two terms on the far right side of (4.6) form commutators, as we may write

$$\Lambda^{s-1}\bigl([S, u\cdot\nabla]\rho\bigr) = [\Lambda^{s-1}S\partial_j, u_j]\rho - [\Lambda^{s-1}\partial_j, u_j]S\rho. \tag{4.7}$$

For the second commutator in (4.7), we apply Lemma 2.3 and obtain

$$\|[\Lambda^{s-1}\partial_j, u_j]S\rho\|_{L^q} \lesssim \|S\rho\|_{L^{p_1}}\|\Lambda^s u\|_{L^{p_2}} + \|\Lambda^{s-1}S\rho\|_{L^{p_3}}\ \|\nabla u\|_{L^{p_4}},$$

where $1/q = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$ and $p_i \in (q, \infty)$ for $i = 1, 2, 3, 4$. Thus, by Lemma 3.2,

$$
\begin{aligned}
J_2 &\le \|\Lambda^{s-1}[S, u\cdot\nabla]\rho\|_{L^q}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\
&\lesssim (\|\nabla u\|_{L^{2q}}\|\Lambda^{s-1}\bar{S}\rho\|_{L^{2q}} + \|\Lambda^{s-1}\bar{S}\nabla u\|_{L^{2q}}\|\rho\|_{L^{2q}})\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\
&\quad + (\|S\rho\|_{L^{2q}}\|\Lambda^{s} u\|_{L^{2q}} + \|\rho\|_{L^{2q}}\|S\Lambda^{s} u\|_{L^{2q}})\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\
&= J_{21} + J_{22}.
\end{aligned} \tag{4.8}
$$

Now, we use the conservation property (3.5) for the density and the fact that the operator $\Lambda^{s-1}\bar{S}$ is of Hörmander-Mikhlin type, and we get

$$
J_{21} \lesssim \|\nabla u\|_{L^{2q}}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \lesssim \|\omega\|_{L^{2q}}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \lesssim e^{Ct}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}, \tag{4.9}
$$

where we applied Lemma 3.1 in the last step. For J_{22}, we choose $p_2 = p_4 = 2q$. Then by the conservation of density and (4.5) we have

$$
\begin{aligned}
J_{22} &\lesssim (\|\Lambda^{s-1}\omega\|_{L^{2q}} + \|\omega\|_{L^{2q}})\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\
&\lesssim \big(\|\Lambda^{s-1}\zeta\|_{L^{2q}} + \|\Lambda^{s-1}S\rho\|_{L^{2q}} + e^{Ct}\big)\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\
&\lesssim \big(\|\Lambda^{s-1}\zeta\|_{L^{2q}} + e^{Ct}\big)\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\
&\lesssim \|\Lambda^{s-1}\zeta\|_{L^q}^{(\alpha-1)/\alpha+q-1}\|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|^{q/2})\|_{L^2}^{2/\alpha q} + e^{Ct}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}.
\end{aligned} \tag{4.10}
$$

From (4.8), (4.9), and (4.10), we conclude

$$
J_2 \le \frac{1}{C}\|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|^{q/2})\|_{L^2}^2 + C\|\Lambda^{s-1}\zeta\|_{L^q}^q + Ce^{Ct}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}.
$$

For J_3, we use Lemma 2.2 and obtain

$$
J_3 \le \|\Lambda^{s-1}N\rho\|_{L^q}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \lesssim \|\rho\|_{L^q}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \lesssim \|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}.
$$

Combining the estimates of J_1, J_2, and J_3, using Young's inequality, we get

$$
\frac{d}{dt}\|\Lambda^{s-1}\zeta\|_{L^q}^q + \frac{1}{C}\|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|^{q/2})\|_{L^2}^2 \lesssim e^{Ct}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} + e^{Ct}. \tag{4.11}
$$

Setting

$$
\begin{aligned}
X &= \|\Lambda^{s-1}\zeta\|_{L^q}^q, \\
\bar{X} &= \|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|^{q/2})\|_{L^2}^2,
\end{aligned}
$$

we may rewrite (4.11) as

$$
\frac{d}{dt}X + \frac{1}{C}\bar{X} \lesssim e^{Ct}(X^{1-1/q} + 1).
$$

Therefore, by the Gronwall lemma,

$$
\|\Lambda^{s-1}\zeta\|_{L^q} \lesssim e^{Ct}, \qquad t \ge 0.
$$

Similarly to Lemma 3.3, we also obtain

$$
\|\Lambda^{s-1}\zeta\|_{L^{\bar{q}}} \lesssim e^{Ct}, \qquad t \ge 0,
$$

for all $\bar{q} \in [q, \infty)$, where the constant C depends on $\bar{q}$. Consequently, we get

$$
\|\Lambda^{s-1}\omega\|_{L^{\bar{q}}} \le \|\Lambda^{s-1}\zeta\|_{L^{\bar{q}}} + \|\Lambda^{s-1}S\rho\|_{L^{\bar{q}}} \lesssim e^{Ct}, \qquad \bar{q} \in [q, \infty). \tag{4.12}
$$

Next, we consider the evolution of $\|\Lambda^s\rho\|_{L^q}$. We apply Λ^s to the equation (2.2), multiply it by $|\Lambda^s\rho|^{q-2}\Lambda^s\rho$, and integrate obtaining

$$\frac{1}{q}\frac{d}{dt}\|\Lambda^s\rho\|_{L^q}^q + \int \Lambda^s(u\cdot\nabla\rho)|\Lambda^s\rho|^{q-2}\Lambda^s\rho\,dx = 0.$$

Therefore, using Lemma 2.3,

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\Lambda^s\rho\|_{L^q}^q &= -\int \Lambda^s(u\cdot\nabla\rho)|\Lambda^s\rho|^{q-2}\Lambda^s\rho\,dx\\ &= -\int \left(\Lambda^s(u\cdot\nabla\rho) - u\cdot\Lambda^s\nabla\rho\right)|\Lambda^s\rho|^{q-2}\Lambda^s\rho\,dx\\ &\lesssim \|\Lambda^s(u\cdot\nabla\rho) - u\cdot\Lambda^s\nabla\rho\|_{L^q}\|\Lambda^s\rho\|_{L^q}^{q-1}\\ &\lesssim \left(\|\Lambda^s u\|_{L^{s_1}}\|\nabla\rho\|_{L^{s_2}} + \|\Lambda u\|_{L^\infty}\|\Lambda^s\rho\|_{L^q}\right)\|\Lambda^s\rho\|_{L^q}^{q-1},\end{aligned}$$

under the conditions $s_1, s_2 \in (q,\infty)$ and $1/q = 1/s_1 + 1/s_2$. Now, choose

$$s_2 = q + \frac{s-1}{C_0},$$

where C_0 is a positive constant. Note that $s_1, s_2 \in (q,\infty)$. If C_0 is sufficiently large, we may use the fractional Gagliardo-Nirenberg inequality to write

$$\|\nabla\rho\|_{L^{s_2}} \lesssim \|\rho\|_{L^q}^{1-\lambda}\|\Lambda^s\rho\|_{L^q}^{\lambda}$$

with $\lambda \in (0,1)$. Therefore, using (4.12),

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\Lambda^s\rho\|_{L^q}^q &\lesssim \left(\|\Lambda^{s-1}\omega\|_{L^{s_1}}\|\rho\|_{L^q}^{1-\lambda}\|\Lambda^s\rho\|_{L^q}^{\lambda} + \|\Lambda u\|_{L^\infty}\|\Lambda^s\rho\|_{L^q}\right)\|\Lambda^s\rho\|_{L^q}^{q-1}\\ &\lesssim e^{Ct}\|\Lambda^s\rho\|_{L^q}^{q+\lambda-1} + \|\Lambda u\|_{L^\infty}\|\Lambda^s\rho\|_{L^q}^q.\end{aligned} \tag{4.13}$$

Let $\bar q \in [q,\infty)$ be sufficiently large so that we have

$$\|\Lambda u\|_{L^\infty} \lesssim \|\Lambda u\|_{L^{\bar q}}^{1-\mu}\|\Lambda^{s-1}(\Lambda u)\|_{L^{\bar q}}^{\mu}$$

where $\mu \in (0,1)$. Then we get

$$\|\Lambda u\|_{L^\infty} \lesssim \|\Lambda u\|_{L^{\bar q}}^{1-\mu}\|\Lambda^{s-1}(\Lambda u)\|_{L^{\bar q}}^{\mu} \lesssim \|\omega\|_{L^{\bar q}}^{1-\mu}\|\Lambda^{s-1}\omega\|_{L^{\bar q}}^{\mu} \lesssim e^{Ct}$$

by (3.11) and (4.12). Hence, continuing from (4.13), we get

$$\frac{d}{dt}\|\Lambda^s\rho\|_{L^q}^q \lesssim e^{Ct}(1 + \|\Lambda^s\rho\|_{L^q}^q).$$

The proof of persistence for $s \in (1,\alpha]$ is then concluded by an application of the Gronwall lemma.

It remains to prove the uniqueness of solutions. Consider two solutions $(u^{(1)}, p^{(1)}, \rho^{(1)})$ and $(u^{(2)}, p^{(2)}, \rho^{(2)})$ of the system (2.1)–(2.3), and set

$$\begin{aligned}U &= u^{(1)} - u^{(2)}\\ R &= \rho^{(1)} - \rho^{(2)}\\ P &= p^{(1)} - p^{(2)}.\end{aligned}$$

Subtracting the equations for $(u^{(1)}, p^{(1)}, \rho^{(1)})$ and $(u^{(2)}, p^{(2)}, \rho^{(2)})$, we get

$$U_t + \Lambda^\alpha U + U\cdot\nabla u^{(1)} + u^{(2)}\cdot\nabla U + \nabla P = Re_2 \tag{4.14}$$

and

$$R_t + U \cdot \nabla \rho^{(1)} + u^{(2)} \cdot \nabla R = 0. \tag{4.15}$$

We shall establish uniqueness in the space $L^2(\mathbb{R}^2) \times L^r(\mathbb{R}^2)$ where

$$r = \frac{4q}{2q - q\alpha + 4}.$$

Note that $1 < r < \infty$ and $(U(0), R(0)) = (0,0) \in L^2(\mathbb{R}^2) \times L^r(\mathbb{R}^2)$. From (4.14), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 + \|\Lambda^{\alpha/2} U\|_{L^2}^2 &\lesssim \|\nabla u^{(1)}\|_{L^\infty} \|U\|_{L^2}^2 + \|U\|_{L^{r/(r-1)}} \|R\|_{L^r} \\ &\lesssim \|\nabla u^{(1)}\|_{L^\infty} \|U\|_{L^2}^2 + \|\Lambda^{\alpha/2} U\|_{L^2} \|R\|_{L^r} + \|U\|_{L^2} \|R\|_{L^r}, \end{aligned} \tag{4.16}$$

where we used

$$\frac{r}{r-1} \leq \frac{4}{2-\alpha},$$

which follows from $r \geq 4/(2+\alpha)$ and this holds by $q\alpha \geq 2$. Also, (4.15) implies

$$\frac{d}{dt}\|R\|_{L^r}^r \lesssim \|U\|_{L^{4/(2-\alpha)}} \|\nabla \rho^{(1)}\|_{L^q} \|R\|_{L^r}^{r-1},$$

from where

$$\frac{d}{dt}\|R\|_{L^r}^2 \lesssim \|U\|_{L^{4/(2-\alpha)}} \|\nabla \rho^{(1)}\|_{L^q} \|R\|_{L^r} \lesssim \|\Lambda^{\alpha/2} U\|_{L^2} \|\rho^{(1)}\|_{W^{s,q}} \|R\|_{L^r}. \tag{4.17}$$

Now, $u^{(1)} \in L^\infty_{\text{loc}}((0,\infty)([0,T], W^{s,\bar{q}}(\mathbb{R}^2))$ for all $\bar{q} \in [q,\infty)$, and $W^{s,\bar{q}}(\mathbb{R}^2) \subseteq W^{1,\infty}(\mathbb{R}^2)$ for all $\bar{q}$ sufficiently large. Thus (4.16) and (4.17) imply $U(t) = 0$ and $R(t) = 0$ for all $t \geq 0$. ☐

Remark 4.1. Note that the identity (4.7) only uses the additivity of Λ^{s-1} and the fact that it commutes with the differential operators. Thus, for any multiplier operator T, we have

$$T\big([S, u \cdot \nabla]\rho\big) = [TS\partial_j, u_j]\rho - [T\partial_j, u_j]S\rho.$$

The proof of this identity uses the fact that u is divergence-free.

5. The Sobolev persistence for $s > \alpha$

We now consider the persistence of regularity when $s > \alpha$.

Proof of Theorem 2.1 for the case $s > \alpha$. Let J_1, J_2, and J_3 be as in (4.3). For J_1, (4.4) and Lemma 2.3 imply

$$\begin{aligned} J_1 &\lesssim (\|\Lambda^{s-1}\zeta\|_{L^{r_2}} \|\Lambda u\|_{L^{r_1}} + \|\zeta\|_{L^{r_1}} \|\Lambda^{s-1}\omega\|_{L^{r_2}}) \|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\ &\lesssim \big(\|\Lambda^{s-1}\zeta\|_{L^{r_2}} \|\omega\|_{L^{r_1}} + \|\zeta\|_{L^{r_1}} \|\Lambda^{s-1}\zeta\|_{L^{r_2}} + \|\zeta\|_{L^{r_1}} \|\Lambda^{s-1} S\rho\|_{L^{r_2}}\big) \|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\ &\lesssim e^{Ct} (\|\Lambda^{s-1}\zeta\|_{L^{r_2}} + \|\Lambda^{s-\alpha}\rho\|_{L^{r_2}}) \|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}, \end{aligned} \tag{5.1}$$

for any $r_1, r_2 \in (q, \infty)$ such that $1/q = 1/r_1 + 1/r_2$. We restrict

$$r_2 \in (q, 2q/(2-\alpha))$$

so that we may use the inequality (3.8) obtaining

$$\|\Lambda^{s-1}\zeta\|_{L^{r_2}} \lesssim \|\Lambda^{s-1}\zeta\|_{L^q}^{1-\theta_1} \|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|^{q/2})\|_{L^2}^{2\theta_1/q} \tag{5.2}$$

where $\theta_1 = 2(r_2 - q)/\alpha r_2$. Also, we have

$$\|\Lambda^{s-\alpha}\rho\|_{L^{r_2}} \lesssim \|\rho\|_{L^q}^{1-\theta_2} \|\Lambda^s\rho\|_{L^q}^{\theta_2} \lesssim \|\Lambda^s\rho\|_{L^q}^{\theta_2} \tag{5.3}$$

with $\theta_2 = (2/q - 2/r_2 + s - \alpha)/s$. Thus, by (5.1) and (5.2), we obtain

$$J_1 \lesssim e^{Ct}\|\Lambda^{s-1}\zeta\|_{L^q}^{1-\theta_1} \|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|^{q/2})\|_{L^2}^{2\theta_1/q} + e^{Ct}\|\Lambda^s\rho\|_{L^q}^{\theta_2} \|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}.$$

The term J_2 is rewritten using (4.7) as

$$\begin{aligned} J_2 &= \int [\Lambda^{s-1}S\partial_j, u_j]\rho|\Lambda^{s-1}\zeta|^{q-2}\Lambda^{s-1}\zeta\,dx \\ &\quad - \int [\Lambda^{s-1}\partial_j, u_j]S\rho|\Lambda^{s-1}\zeta|^{q-2}\Lambda^{s-1}\zeta\,dx \\ &= J_{21} + J_{22}. \end{aligned}$$

For the first term, we have

$$\begin{aligned} J_{21} &\lesssim (\|\nabla u\|_{L^{r_1}}\|\Lambda^{s-1}\bar{S}\rho\|_{L^{r_2}} + \|\Lambda^{s-1}\bar{S}\nabla u\|_{L^{r_3}}\|\rho\|_{L^{r_4}})\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\ &\lesssim e^{Ct}(\|\Lambda^{s-\alpha}\rho\|_{L^{r_2}} + \|\Lambda^{s-\alpha}\omega\|_{L^{r_3}})\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}, \end{aligned} \tag{5.4}$$

where $r_3, r_4 \in (q, \infty)$ are such that $1/r_3 + 1/r_4 = 1/q$. For $\|\Lambda^{s-\alpha}\rho\|_{L^{r_2}}$, we use (5.3), while for $\|\Lambda^{s-\alpha}\omega\|_{L^{r_3}}$, we have by the triangle inequality

$$\begin{aligned} \|\Lambda^{s-\alpha}\omega\|_{L^{r_3}} &\lesssim \|\Lambda^{s-\alpha}\zeta\|_{L^{r_3}} + \|\Lambda^{s-\alpha}S\rho\|_{L^{r_3}} \\ &\lesssim \|\zeta\|_{L^q}^{1-\theta_3}\|\Lambda^{s-1}\zeta\|_{L^q}^{\theta_3} + \|\Lambda^{(s-2\alpha+1)_+}\rho\|_{L^{r_3}} \\ &\lesssim e^{Ct}\|\Lambda^{s-1}\zeta\|_{L^q}^{\theta_3} + \|\Lambda^{(s-2\alpha+1)_+}\rho\|_{L^{r_3}}, \end{aligned}$$

where $\theta_3 = (2/q - 2/r_3 + s - \alpha)/(s-1)$, as long as r_3 is sufficiently close to q. From (5.4) we thus obtain

$$J_{21} \le e^{Ct}\left(\|\Lambda^s\rho\|_{L^q}^{\theta_2} + \|\Lambda^{s-1}\zeta\|_{L^q}^{\theta_3} + 1\right)\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \tag{5.5}$$

if $s \le 2\alpha - 1$, and

$$J_{21} \le e^{Ct}\left(\|\Lambda^s\rho\|_{L^q}^{\theta_2} + \|\Lambda^{s-1}\zeta\|_{L^q}^{\theta_3} + \|\Lambda^s\rho\|_{L^q}^{(s-2\alpha+1)/s+\epsilon_0}\right)\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1},$$

if $s > 2\alpha - 1$, where $\epsilon_0 > 0$ is arbitrarily small if r_3 is sufficiently close to q. Since $\theta_2 > (s - 2\alpha + 1)/s$, we obtain that (5.5) holds even if $s > 2\alpha - 1$ as long as $r_3 > q$ is sufficiently close to q. For J_{22}, we recall (4.8), by which

$$\begin{aligned} J_{22} &\lesssim \big(\|\bar{S}\rho\|_{L^{r_1}}\|\Lambda^s u\|_{L^{r_2}} + \|\rho\|_{L^\infty}\|\bar{S}\Lambda^s u\|_{L^q}\big)\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\ &\lesssim e^{Ct}(\|\Lambda^s u\|_{L^{r_2}} + \|\bar{S}\Lambda^s u\|_{L^q})\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\ &\lesssim e^{Ct}(\|\Lambda^{s-1}\zeta\|_{L^{r_2}} + \|\Lambda^{s-1}S\rho\|_{L^{r_2}} + \|\bar{S}\Lambda^s u\|_{L^q})\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \end{aligned}$$

and thus

$$\begin{aligned} J_{22} &\lesssim e^{Ct}(\|\Lambda^{s-1}\zeta\|_{L^{r_2}} + \|\Lambda^{s-\alpha}\rho\|_{L^{r_2}} + \|\bar{S}\Lambda^s u\|_{L^q})\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\ &\lesssim e^{Ct}(\|\Lambda^{s-1}\zeta\|_{L^{r_2}} + \|\Lambda^{s-\alpha}\rho\|_{L^{r_2}} + \|\bar{S}\Lambda^{s-1}\omega\|_{L^q})\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\ &\lesssim e^{Ct}(\|\Lambda^{s-1}\zeta\|_{L^{r_2}} + \|\Lambda^{s-\alpha}\rho\|_{L^{r_2}} \\ &\qquad + \|\bar{S}\Lambda^{s-1}\zeta\|_{L^q} + \|\bar{S}\Lambda^{s-1}S\rho\|_{L^q})\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}. \end{aligned}$$

Note that the last two terms inside the parentheses are lower order compared to the first two. Therefore,

$$\begin{aligned} J_{22} &\lesssim e^{Ct}(\|\Lambda^{s-1}\zeta\|_{L^{r_2}} + \|\Lambda^{s-\alpha}\rho\|_{L^{r_2}} + \|\zeta\|_{L^{r_2}} \\ &\qquad + \|\zeta\|_{L^q} + \|\rho\|_{L^{r_2}} + \|\rho\|_{L^q})\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\ &\lesssim e^{Ct}(\|\Lambda^{s-1}\zeta\|_{L^{r_2}} + \|\Lambda^{s-\alpha}\rho\|_{L^{r_2}} + 1)\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}. \end{aligned}$$

The right hand side does not lead to any new terms compared to the estimate for J_1 in (5.1), except for the lower order third term inside the parentheses.

Next, we treat J_3. When $s \le 2$, we have

$$J_3 \lesssim \|\Lambda^{s-1}N\rho\|_{L^q}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \lesssim \|\rho\|_{L^q}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \lesssim \|\Lambda^{s-1}\zeta\|_{L^q}^{q-1},$$

while if $s \ge 2$,

$$\begin{aligned} J_3 &\lesssim \|\Lambda^{s-1}N\rho\|_{L^q}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \lesssim \|\Lambda^{s-2}\rho\|_{L^q}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\ &\lesssim \|\rho\|_{L^q}^{2/s}\|\Lambda^s\rho\|_{L^q}^{(s-2)/s}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \lesssim \|\Lambda^s\rho\|_{L^q}^{(s-2)/s}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}. \end{aligned}$$

We thus conclude

$$\begin{aligned} &\frac{d}{dt}\|\Lambda^{s-1}\zeta\|_{L^q}^q + \frac{1}{C}\|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|^{q/2})\|_{L^2}^2 \\ &\quad\lesssim e^{Ct}\|\Lambda^{s-1}\zeta\|_{L^q}^q + e^{Ct}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} + e^{Ct}\|\Lambda^s\rho\|_{L^q}^{\theta_2}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1} \\ &\qquad + \|\Lambda^s\rho\|_{L^q}^{((s-2)/s)_+}\|\Lambda^{s-1}\zeta\|_{L^q}^{q-1}. \end{aligned} \tag{5.6}$$

Next, we consider $\|\Lambda^s\rho\|_{L^q}$. First, we have by Sobolev embedding, with $q^* = \max\{2/(\alpha-1), q\} + 1$,

$$\|\Lambda u\|_{L^\infty} \lesssim \|\Lambda^{\alpha-1}\Lambda u\|_{L^{q^*}} + \|\Lambda u\|_{L^{q^*}} \tag{5.7}$$

$$\lesssim \|\Lambda^{\alpha-1}\zeta\|_{L^{q^*}} + \|\Lambda^{\alpha-1}S\rho\|_{L^{q^*}} + \|\omega\|_{L^{q^*}} \lesssim \phi(t), \tag{5.8}$$

where we used Theorem 2.1 in the third inequality and where

$$\phi(t) = C\exp\left(C\exp(Ct)\right)$$

with a sufficiently large C. (The dependence on t can be improved, but we do not optimize the dependence in this paper.) Thus, by Lemma 2.3,

$$\begin{aligned} \frac{1}{q}\frac{d}{dt}\|\Lambda^s\rho\|_{L^q}^q &= -\int\left(\Lambda^s(u\cdot\nabla\rho) - u\cdot\Lambda^s\nabla\rho\right)|\Lambda^s\rho|^{q-2}\Lambda^s\rho\,dx \\ &\lesssim \|\Lambda^s(u\cdot\nabla\rho) - u\cdot\Lambda^s\nabla\rho\|_{L^q}\|\Lambda^s\rho\|_{L^q}^{q-1} \\ &\lesssim (\|\Lambda^s u\|_{L^{s_1}}\|\nabla\rho\|_{L^{s_2}} + \|\Lambda u\|_{L^\infty}\|\Lambda^s\rho\|_{L^q})\,\|\Lambda^s\rho\|_{L^q}^{q-1} \\ &\lesssim \left(\|\Lambda^{s-1}\omega\|_{L^{s_1}}\|\nabla\rho\|_{L^{s_2}} + \phi(t)\|\Lambda^s\rho\|_{L^q}\right)\|\Lambda^s\rho\|_{L^q}^{q-1} \end{aligned} \tag{5.9}$$

where $s_1, s_2 \in (q, \infty)$ are such that $1/s_1 + 1/s_2 = 1/q$. At this point, we employ an inequality from [2], which gives

$$\|\nabla\rho\|_{L^{s_2}} \lesssim \|\rho\|_{L^{\bar{s}_2}}^{1-1/s}\|\Lambda^s\rho\|_{L^q}^{1/s} \lesssim \|\Lambda^s\rho\|_{L^q}^{1/s} \tag{5.10}$$

where $1/s_2 = 1/sq + (1/\bar{s}_2)(1-1/s)$, assuming that $s_2 \le qs$, which is equivalent to

$$s_1 \ge \frac{qs}{qs-1}.$$

From (5.9) and (5.10) we then obtain

$$\frac{d}{dt}\|\Lambda^s\rho\|_{L^q} \lesssim \|\Lambda^{s-1}\omega\|_{L^{s_1}}\|\Lambda^s\rho\|_{L^q}^{1/s} + \phi(t)\|\Lambda^s\rho\|_{L^q}. \tag{5.11}$$

If

$$s_1 \le \frac{2q}{2-\alpha} \tag{5.12}$$

we may apply (3.8) and obtain

$$\begin{aligned}
\|\Lambda^{s-1}\omega\|_{L^{s_1}} &\le \|\Lambda^{s-1}\zeta\|_{L^{s_1}} + \|\Lambda^{s-1}S\rho\|_{L^{s_1}} \\
&\lesssim \|\Lambda^{s-1}\zeta\|_{L^q}^{1-\theta_3}\|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|^{q/2})\|_{L^2}^{2\theta_3/q} + \|\Lambda^{s-\alpha}\rho\|_{L^{s_1}} \\
&\lesssim \|\Lambda^{s-1}\zeta\|_{L^q}^{1-\theta_3}\|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|^{q/2})\|_{L^2}^{2\theta_3/q} + \|\rho\|_{L^q}^{1-\theta_4}\|\Lambda^s\rho\|_{L^q}^{\theta_4} \\
&\lesssim \|\Lambda^{s-1}\zeta\|_{L^q}^{1-\theta_3}\|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|^{q/2})\|_{L^2}^{2\theta_3/q} + \|\Lambda^s\rho\|_{L^q}^{\theta_4},
\end{aligned}$$

where $\theta_3 = 2(s_1-q)/\alpha s_1$ and $\theta_4 = (s-\alpha-2/s_1+2/q)/s$. Therefore, by (5.11),

$$\begin{aligned}
\frac{d}{dt}\|\Lambda^s\rho\|_{L^q} &\lesssim \|\Lambda^{s-1}\zeta\|_{L^q}^{1-\theta_3}\|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|^{q/2})\|_{L^2}^{2\theta_3/q}\|\Lambda^s\rho\|_{L^q}^{1/s} \\
&\quad + \|\Lambda^s\rho\|_{L^q}^{\theta_4+1/s} + \phi(t)\|\Lambda^s\rho\|_{L^q}.
\end{aligned} \tag{5.13}$$

Now, in order to conclude the proof, let $\gamma > 0$, and denote

$$\begin{aligned}
X &= \|\Lambda^{s-1}\zeta\|_{L^q}^q, \\
Y &= (\|\Lambda^s\rho\|_{L^q} + 1)^{q/\gamma}, \\
Z &= \|\Lambda^{\alpha/2}(|\Lambda^{s-1}\zeta|^{q/2})\|_{L^2}^2.
\end{aligned}$$

Then (5.6) and (5.13) may be rewritten as

$$\begin{aligned}
\frac{d}{dt}X + \frac{1}{C}Z &\lesssim e^{Ct}X + e^{Ct}X^{(q-1)/q} + e^{Ct}Y^{\theta_2\gamma/q}X^{(q-1)/q} \\
&\quad + Y^{((s-2)/s)_+\gamma/q}X^{(q-1)/q}
\end{aligned} \tag{5.14}$$

and

$$\frac{d}{dt}Y \lesssim X^{(1-\theta_3)/q}Z^{\theta_3/q}Y^{1+\gamma/sq-\gamma/q} + Y^{\gamma\theta_4/q+\gamma/sq+1-\gamma/q} + \phi(t)Y, \tag{5.15}$$

respectively. (We use here that if $(d/dt)\|\Lambda^s\rho\|_{L^q} \le f$, then $\dot{Y} \le fY^{1-\gamma/q}$.) Adding (5.14) and (5.15), we obtain

$$\begin{aligned}\frac{d}{dt}(X+Y)+\frac{1}{C}Z \lesssim e^{Ct}X &+ e^{Ct}X^{(q-1)/q} + e^{Ct}Y^{\theta_2\gamma/q}X^{(q-1)/q}\\ &+ Y^{((s-2)/s)_+\gamma/q}X^{(q-1)/q} + X^{(1-\theta_3)/q}Z^{\theta_3/q}Y^{1+\gamma/sq-\gamma/q}\\ &+ Y^{\gamma\theta_4/q+\gamma/sq+1-\gamma/q} + \phi(t)Y.\end{aligned}$$

In order to apply the Gronwall lemma, it is sufficient that the conditions

$$\begin{aligned}&\frac{\theta_2\gamma}{q}+\frac{q-1}{q}\le 1\\ &\left(\frac{s-2}{s}\right)_+\frac{\gamma}{q}+\frac{q-1}{q}\le 1\\ &\frac{1}{q}+\frac{\gamma}{sq}-\frac{\gamma}{q}\le 0\\ &\frac{\gamma\theta_4}{q}+\frac{\gamma}{sq}-\frac{\gamma}{q}\le 0\end{aligned} \tag{5.16}$$

hold. The first three conditions may be summarized as

$$\frac{s}{s-1}\le\gamma\le\min\left\{\frac{1}{\theta_2},\frac{s}{s-2}\right\}$$

if $s>2$ and as

$$\frac{s}{s-1}\le\gamma\le\frac{1}{\theta_2}$$

if $s\le 2$. The last condition in (5.16) is equivalent to

$$s\ge\frac{1}{1-\theta_4}.$$

Setting $s_1 = qs/(qs-1)$, it is easy to verify that we may simply take $\gamma = s/(s-1)$ as we have $s/(s-1)\le 1/\theta_2$. The condition (5.12) can also be checked easily. The proof is concluded by a simple application of a Gronwall lemma. □

Acknowledgments

The authors were supported in part by the NSF grants DMS-1615239 and DMS-1907992.

References

[1] D. Adhikari, C. Cao, H. Shang, J. Wu, X. Xu and Z. Ye, *Global regularity results for the 2D Boussinesq equations with partial dissipation*, J. Differential Equations **260** (2016), 1893–1917.

[2] H. Brezis and P. Mironescu, *Gagliardo-Nirenberg inequalities and non-inequalities: the full story*, Ann. Inst. H. Poincaré Anal. Non Linéaire **35** (2018), 1355–1376.

[3] L.C. Berselli and S. Spirito, *On the Boussinesq system: regularity criteria and singular limits*, Methods Appl. Anal. **18** (2011), 391–416.

[4] L. Brandolese and M.E. Schonbek, *Large time decay and growth for solutions of a viscous Boussinesq system*, Trans. Amer. Math. Soc. **364** (2012), 5057–5090.

[5] D. Chae, *Global regularity for the 2D Boussinesq equations with partial viscosity terms*, Adv. Math. **203** (2006), 497–513.
[6] A. Córdoba and D. Córdoba, *A pointwise estimate for fractionary derivatives with applications to partial differential equations*, Proc. Natl. Acad. Sci. USA **100** (2003), 15316–15317.
[7] J.R. Cannon and E. DiBenedetto, *The initial value problem for the Boussinesq equations with data in L^p*, Approximation methods for Navier-Stokes problems (Proc. Sympos., Univ. Paderborn, Paderborn, 1979), Lecture Notes in Math., vol. 771, Springer, Berlin, 1980, pp. 129–144.
[8] P. Constantin and C. Foias, *Navier-Stokes Equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.
[9] M. Chen and O. Goubet, *Long-time asymptotic behavior of two-dimensional dissipative Boussinesq systems*, Discrete Contin. Dyn. Syst. Ser. S **2** (2009), 37–53.
[10] P. Constantin, M. Lewicka and L. Ryzhik, *Travelling waves in two-dimensional reactive Boussinesq systems with no-slip boundary conditions*, Nonlinearity **19** (2006), 2605–2615.
[11] D. Chae and H.-S. Nam, *Local existence and blow-up criterion for the Boussinesq equations*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), 935–946.
[12] C. Cao and J. Wu, *Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation*, Arch. Ration. Mech. Anal. **208** (2013), 985–1004.
[13] C.R. Doering and J.D. Gibbon, *Applied Analysis of the Navier-Stokes Equations*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1995.
[14] R. Danchin and M. Paicu, *Les théorèmes de Leray et de Fujita-Kato pour le système de Boussinesq partiellement visqueux*, Bull. Soc. Math. France **136** (2008), 261–309.
[15] R. Danchin and M. Paicu, *Les théorèmes de Leray et de Fujita-Kato pour le système de Boussinesq partiellement visqueux*, Bull. Soc. Math. France **136** (2008), 261–309.
[16] W. E and C.-W. Shu, *Small-scale structures in Boussinesq convection*, Phys. Fluids **6** (1994), 49–58.
[17] C. Foias, O. Manley, and R. Temam, *Modelling of the interaction of small and large eddies in two-dimensional turbulent flows*, RAIRO Modél. Math. Anal. Numér. **22** (1988), 93–118.
[18] T. Hmidi and S. Keraani, *On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity*, Adv. Differential Equations **12** (2007), 461–480.
[19] T. Hmidi and S. Keraani, *On the global well-posedness of the Boussinesq system with zero viscosity*, Indiana Univ. Math. J. **58** (2009), 1591–1618.
[20] T. Hmidi, S. Keraani and F. Rousset, *Global well-posedness for Euler-Boussinesq system with critical dissipation*, Comm. Partial Differential Equations **36** (2011), 420–445.
[21] T.Y. Hou and C. Li, *Global well-posedness of the viscous Boussinesq equations*, Discrete Contin. Dyn. Syst. **12** (2005), 1–12.
[22] W. Hu, I. Kukavica and M. Ziane, *On the regularity for the Boussinesq equations in a bounded domain*, J. Math. Phys. **54** (2013), 081507, 10.
[23] W. Hu, I. Kukavica and M. Ziane, *Persistence of regularity for the viscous Boussinesq equations with zero diffusivity*, Asymptot. Anal. **91** (2015), 111–124.
[24] F. Hadadifard and A. Stefanov, *On the global regularity of the 2D critical Boussinesq system with $\alpha > 2/3$*, Comm. Math. Sci. **15** (2017), 1325–1351.
[25] Q. Jiu, C. Miao, J. Wu and Z. Zhang, *The two-dimensional incompressible Boussinesq equations with general critical dissipation*, SIAM J. Math. Anal. **46** (2014), 3426–3454.
[26] N. Ju, *The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations*, Comm. Math. Phys. **255** (2005), 161–181.
[27] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), 891–907.
[28] J.P. Kelliher, R. Temam, and X. Wang, *Boundary layer associated with the Darcy-Brinkman-Boussinesq model for convection in porous media*, Phys. D **240** (2011), 619–628.
[29] I. Kukavica, F. Wang and M. Ziane, *Persistence of regularity for solutions of the Boussinesq equations in Sobolev spaces*, Adv. Differential Equations **21** (2016), , 85–108.
[30] A. Larios, E. Lunasin, and E.S. Titi, *Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion*, J. Differential Equations **255** (2013), 2636–2654.

[31] M.-J. Lai, R. Pan, and K. Zhao, *Initial boundary value problem for two-dimensional viscous Boussinesq equations*, Arch. Ration. Mech. Anal. **199** (2011), 739–760.

[32] J.C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.

[33] A. Stefanov and J. Wu, *A global regularity result for the 2D Boussinesq equation with critical dissipation*, to appear, Journal d Analyse Mathématique.

[34] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, second ed., Applied Mathematical Sciences, vol. 68, Springer-Verlag, New York, 1997.

[35] R. Temam, *Navier-Stokes Equations*, AMS Chelsea Publishing, Providence, RI, 2001, Theory and numerical analysis, Reprint of the 1984 edition.

[36] R. Temam, *Navier-Stokes equations and nonlinear functional analysis*, second ed., CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 66, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.

Manuscript received March 8 2019
revised June 7 2019

I. Kukavica
Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA
E-mail address: `kukavica@usc.edu`

W. Wang
Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA
E-mail address: `wangwein@usc.edu`