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ON RECONSTRUCTING THE RIGHT-HAND PART OF A DISTRIBUTED DIFFERENTIAL EQUATION

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Abstract. A second order nonlinear differential equation is considered. An algorithm for reconstructing the right-hand part from inaccurate measurements of the solution at discrete times is designed. This algorithm based on constructions of feedback control theory is stable with respect to informational noises and computational errors. It is oriented to the sufficiently long time interval where the equation's solution is considered.

1. Introduction

Consider the differential equation

$$\ddot{x}(t,\eta) - \Delta x(t,\eta) + m x(t,\eta) + \gamma \dot{x}(t,\eta) = \tag{1.1}$$
$$= g(x(t,\eta)) + (Bv(t))(\eta) + f(t,\eta) \quad \text{for almost all (a.a.)} \quad T \times \Omega$$

with the boundary

$$x(t) = 0 \quad \text{for} \quad \mathrm{t} \in T$$

and the initial

$$x(0) = x_{10} \in V = H_0^1(\Omega), \quad \dot{x}(0) = x_0 \in H = L_2(\Omega) \tag{1.2}$$

conditions. Here, $T = [0, +\infty)$, Ω is an open bounded domain with the Lipschitz boundary G, $m = \text{const} > 0$, $\gamma = \text{const} > 0$, $g(\cdot) : R \to R$ is a Lipschitz function with some constant L, $g(0) = 0$, $f(\cdot) \in L_\infty(T; H)$ is a given function, the derivative $\dot{x}(\cdot)$ is treated in the sense of distributions, and B is a linear continuous operator acting from a Hilbert space U with norm $|\cdot|_U$ and inner product $(\cdot,\cdot)_U$ (the disturbance space) into V ($B \in L(U;V)$).

Equation (1.1) with initial conditions (1.2) is known as the Klein–Gordon equation. It was investigated by many authors (see, for example, monograph [5] and its bibliography). In all these works, the questions of existence and uniqueness of a solution, its continuity, smoothness and so on were considered. In this paper, we dwell on one reconstruction problem, which consists in the following. Values $v(t)$ of the disturbance are unknown and subject to a priori constraint $v(t) \in P \subset U$ ($t \geq 0$), where P is a convex, closed and bounded set. At times $\tau_i \in T$ ($i = 1, 2, \ldots$), a

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solution $x(\tau_i)$ of equation (1.1) is inaccurately measured. The measurement results are elements $\{\xi^h_{1i}, \xi^h_i\} \in V^* \times V^*$ such that

$$|\xi^h_{1i} - x(\tau_i)|_{V^*} \le \nu^h_i, \quad |\xi^h_i - \dot{x}(\tau_i)|_{V^*} \le \nu^h_i, \tag{1.3}$$

where $\nu^h_i \in (0,1)$ is the measurement error at the moment τ_i and the number $h \in (0,1)$ is the measurement accuracy. We assume that the initial condition is also known with some error. Namely, we know elements $\xi^h_{10} \in V$ and $\xi^h_0 \in H$ satisfying the inequalities

$$|\xi^h_{10} - x_{10}|_V \le h, \quad |\xi^h_0 - x_0|_H \le h. \tag{1.4}$$

The problem under consideration consists in constructing an algorithm of approximate reconstruction of the unknown disturbance $v(\cdot)$ through results of inaccurate measurements of the states $x(\cdot)$. We estimate the reconstruction quality by two criteria; first, by the deviation value for the solutions of equation (1.1) corresponding to the real disturbance $v(\cdot)$ and its approximation $v^h(\cdot)$, and second, by the difference of mean-square norms of the functions $v^h(\cdot)$ and $v(\cdot)$ on corresponding intervals. The choice of these two criteria is explained by the fact that if they are small (under appropriate conditions) then the approximation $v^h(\cdot)$ is close to the disturbance $v(\cdot)$ in the mean-square norm on every bounded time interval.

The problem described above belongs to the class of inverse problems [17]. The analogous problems for systems with distributed parameters attract a great attention in recent time (see, for example, [2, 3, 4, 25, 16] and their bibliography). In particular, the attention was paid to dynamical inverse problems [23, 24]. Note that there is a strong connection between inverse problems and optimal control problems [1, 6, 7]. The method of dynamic programming [9, 10], which is known in control theory, is also applied to inverse problems. One of approaches to solving dynamical reconstruction problems for systems described by ordinary differential equations was developed in [12, 15, 18]. (Here, we mention only monographs, where it is possible to find corresponding references.) This approach is based on methods of the theory of guaranteed control [11] and the method of smoothing functional [17]. Then, this approach was extended to systems with distributed parameters [8, 19, 21]. In these papers, algorithms oriented to reconstructing a disturbance on a bounded time interval were suggested. With increase in the length of this interval, computational and measurement errors are accumulated and, if the length tends to infinity then the approximation quality infinitely deteriorates. Algorithms that do not have this drawback were designed in [23, 24], where systems described by ordinary differential equations were considered. In this paper, we give a modification of algorithms from [13, 14] for differential equation (1.1).

In [22], the problem of reconstructing the right-hand part of an equation of form (1.1) was considered at a finite time interval. The measurements of the phase state were performed continuously, at every moment of time. In the present work, such measurements are carried out at discrete times. In addition, in contrast to [22], there are “instantaneous” constraints on disturbances, and the reconstruction process is considered at an infinite time interval.

2. Statement of the problem

Before the statement of the problem, we define a solution of equation (1.1). Any function $x(\cdot) \in C(T_\vartheta; V)$, $\dot{x}(\cdot) \in W(T_\vartheta; V) = \{y(\cdot) \in C(T_\vartheta; H) : \dot{y}(\cdot) \in L_2(T_\vartheta; V^*)\}$ satisfying the relation

$$\ddot{x}(t) - \Delta x(t) + mx(t) + \gamma \dot{x}(t) = g(x(t)) + (Bu)(t) + f(t) \quad \text{in} \quad V^* \quad \text{a.a.} \quad t \in T_\vartheta,$$

is called a solution of equation (1.1) on the time interval $T_\vartheta = [0, \vartheta]$, $\vartheta > 0$, and is denoted by $x(\cdot) = x(\cdot; x_{10}, x_0, v(\cdot))$. A function $x(t)$, $t \in T$, is called a solution of equation (1.1) on the interval T if $x(\cdot)$ is a solution of equation (1.1) on every interval T_ϑ, $\vartheta > 0$.

Let $P(\cdot)$ be the set of all Lebesgue measurable functions $v(\cdot) : [0, +\infty) \to P$; this set is called the set of admissible disturbances.

Condition 2.1. There exist numbers $K \geq 0$ and K_1 such that the inequalities $K_1 < \lambda + m - KL$ and $xg(x) - K\sigma(x) \leq K_1 x^2$ $\forall x \in R$ are fulfilled. Here,

$$\sigma(x) = \int\limits_0^x g(y)\, dy, \quad \lambda = \inf\{|\nabla x(\eta)|_H : x \in V, |x|_H = 1\}.$$

Condition 2.2. $2L < \lambda + m$.

The next lemma is a direct consequence of Theorem 8.4.5 [5, p. 139].

Lemma 2.3. *Let Conditions* 2.1 *and* 2.2 *be fulfilled. Then, for any* $v(\cdot) \in P(\cdot)$, *there exists a unique solution* $x(\cdot) = x(\cdot; x_{10}, x_0, v(\cdot))$ *of equation* (1.1) *on the time interval* T.

For every $h \in (0, 1)$, we fix a family $(\Delta_h)_{h>0}$ of uniform partitions of semiaxis $[0, +\infty)$ by times $\tau_{h,i}$:

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{\infty}, \quad \tau_{h,0} = 0, \quad \tau_{h,i+1} = \tau_{h,i} + \delta_i(h), \quad \delta_i(h) \in (0, 1). \tag{2.1}$$

Consider two cases. In the first case, we assume that noises implemented in the observation channel are subject to the constraints of “smallness” of their values at every time. In the second one, they are subject to the requirements of “smallness” of their mean values over the entire time interval (the “smallness” of their integral errors).

Condition 2.4. $\delta_i(h) = \delta(h)$, $\nu_i^h = h$ for all $i = 0, 1, \ldots$.

Condition 2.5. The family of partitions Δ_h and the values of measurement errors ν_i^h satisfy the relations

$$\nu_i^h \in [0, 1] \quad \text{for all } i \quad \text{and} \quad h \in (0, 1),$$
$$\sum_{i=0}^{+\infty} \delta_i(h) \nu_i^h \leq \varphi_1(h) \to 0+ \quad \text{as } h \to 0+\,.$$

Thus, under Condition 2.4, the partitions Δ_h are uniform.

Along with equation (1.1), we need one more equation of the form

$$\ddot{y}^h(t) - \Delta y^h(t) + my^h(t) + \gamma \dot{y}^h(t) = \tag{2.2}$$

$$= g(y^h(t)) + (Bv^h)(t) + f(t) \quad \text{in} \quad V^* \quad \text{for a.a.} \quad T,$$

with the initial condition $y^h(0) = \xi_{10}^h$, $\dot{y}^h(0) = \xi_0^h$ and a control $v^h(\cdot) \in P(\cdot)$. We call equation (2.2) a model. This equation is a “copy” of equation (1.1). The difference is only in the control $v^h(\cdot)$ in the right-hand part of (2.2); this control should be formed.

Note that, by virtue of the continuity of the embedding of the space V into the space H, the following inequalities

$$|x|_H \leq c_0 |x|_V \quad \forall x \in V, \tag{2.3}$$

$$|x|_{V^*} \leq c_1 |x|_H \ \forall x \in H \tag{2.4}$$

take place. Here, c_0 and c_1 are some positive constants.

Any piecewise constant function $\Xi^h(\cdot) : [0, +\infty) \mapsto V^* \times V^*$, $\Xi^h(t) = \Xi_i^h = \{\xi_{1i}^h, \xi_i^h\} \in V^* \times V^*$ for $t \in [\tau_{h,i}, \tau_{h,i+1})$, $\xi_{10}^h \in V$, $\xi_0^h \in H$, satisfying relations (1.3), (1.4) is called an *admissible measurement of* $x(\cdot)$ *with accuracy* h, and any Lebesque measurable function $v(\cdot) : [0, +\infty) \mapsto P$ is called an *admissible disturbance.* Analogously, we define an *admissible measurement of* $y^h(\cdot)$ *with accuracy* h. Let $\Psi^h(\cdot)$ be a piecewise constant function such that $\Psi^h(t) = \Psi_i^h = \{\psi_{1i}^h, \psi_i^h\} \in V^* \times V^*$ for $t \in [\tau_{h,i}, \tau_{h,i+1})$, where Ψ_i^h, $i \geq 1$, are results of measurements of $y^h(\tau_i)$ and $\dot{y}^h(\tau_i)$, respectively: $|\psi_{1i}^h - y^h(\tau_i)|_{V^*} \leq h$, $|\psi_i^h - \dot{y}^h(\tau_i)|_{V^*} \leq h$, $\tau_i = \tau_{h,i}$, $\Psi_0^h = \{\psi_{10}^h, \psi_0^h\}$, $\psi_{10}^h = \xi_{10}^h$, $\psi_0^h = \xi_0^h$.

We assume that the solution $y^h(t)$, $t \geq 0$, of equation (2.2) (solution of equation (1.1)) is inaccurately observed at the discrete times $\tau_{h,i}$ and is influenced by the action of some feedback $\mathcal{V}(t, \Psi^h, \Xi^h) \in P$. Therefore, the solution of equation (2.2) satisfies the following differential equations and initial conditions:

$$\ddot{y}^h(t) - \Delta y^h(t) + m y^h(t) + \gamma \dot{y}^h(t) = g(y^h(t)) + f(t) + \tag{2.5}$$
$$+ (B\mathcal{V}(\tau_i, \Xi_i^h, \Psi_i^h))(t) \quad \text{in} \quad V^* \quad \text{for a.a.} \quad t \in \delta_i = [\tau_i, \tau_{i+1}), \quad i \geq 0,$$
$$y^h(0) = \xi_{10}^h, \quad \dot{y}^h(0) = \xi_0^h.$$

For any $v(\cdot)$ and $v^h(\cdot)$ from $P(\cdot)$, we introduce the following two criteria for the deviation of $v^h(\cdot)$ from $v(\cdot)$ on some bounded time interval $[0, \vartheta]$:

$$\omega_1(v^h(\cdot), v(\cdot)|\vartheta) = \max_{t \in [0,\vartheta]} \Big\{ \Big| \dot{y}^h(t; \xi_{10}^h, \xi_0^h, v^h(\cdot)) - \dot{x}(t; x_{10}, x_0, v(\cdot)) \Big|_H +$$
$$+ \Big| y^h(t; \xi_{10}^h, \xi_0^h, v^h(\cdot)) - x(t; x_{10}, x_0, v(\cdot)) \Big|_V \Big\},$$

$$\omega_2(v^h(\cdot), v(\cdot), h|\vartheta) = \int_0^\vartheta |v^h(t)|_U^2 \, dt - \varrho_0(h) \int_0^\vartheta |v(t)|_U^2 \, dt.$$

Here, $\varrho_0(\cdot) : (0, 1) \to R^+ = \{r \in R : r > 0\}$ is a function with the property: $\varrho_0(h) \to 1$ as $h \to +0$, $x(\cdot; x_{10}, x_0, v(\cdot))$ and $y^h(\cdot; \xi_{10}^h, \xi_0^h, v^h(\cdot))$ are the solutions of equations (1.1) and (2.2) induced by the inputs $v(\cdot)$ and $v^h(\cdot)$, respectively.

Any function $\mathcal{V}(\cdot, \cdot, \cdot, \cdot, \cdot) : T \times V^* \times V^* \times V^* \times V^* \mapsto P.$ is called an *admissible feedback* (for model (2.2)). For any admissible feedback $\mathcal{V}(\cdot, \cdot, \cdot, \cdot, \cdot)$, any admissible measurements $\Xi^h(\cdot)$ of $x(\cdot)$ with accuracy h, and any admissible measurements $\Psi(\cdot)$ of $y^h(\cdot)$ with accuracy h, the solution $y^h(\cdot)$ of Cauchy problem (2.5) defined on

$[0,\infty)$ is called a *trajectory of the model* corresponding to the admissible feedback $\mathcal{V}(\cdot,\cdot,\cdot,\cdot,\cdot)$ and admissible measurements $\Xi^h(\cdot)$ and $\Psi^h(\cdot)$.

A *controlled process* corresponding to an admissible feedback $\mathcal{V}(\cdot,\cdot,\cdot,\cdot,\cdot)$, an admissible disturbance $v(\cdot)$, and a measurement accuracy h ($h > 0$) is any quintuple $(x(\cdot),\Xi^h(\cdot),y^h(\cdot),\Psi^h(\cdot),v^h(\cdot))$, where $x(\cdot) = x(\cdot;x_{10},x_0,v(\cdot))$ is the solution of equation (1.1), $\Xi^h(\cdot)$ is the admissible measurement of $x(\cdot)$ with accuracy h, $y^h(\cdot) = y^h(\cdot;\xi^h_{10},\xi^h_0,v^h(\cdot))$ is the solution of equation (2.5), $\Psi^h(\cdot)$ is the admissible measurement of $y^h(\cdot)$ with accuracy h, the function $v^h(\cdot) : [0,+\infty) \mapsto P$ is defined by the rule

$$v^h(t) = \mathcal{V}(\tau_i,\Xi^h_i,\Psi^h_i) \quad \text{for} \quad t \in \delta_i = \delta_{h,i} = [\tau_i,\tau_{i+1}), \quad \tau_i = \tau_{h,i}, \quad i \geq 0.$$

The function $v^h(\cdot)$ is called a *realization* of the admissible feedback $\mathcal{V}(\cdot,\cdot,\cdot,\cdot,\cdot)$ corresponding to the admissible disturbance $v(\cdot)$ and admissible measurement with accuracy h.

Following [13], the family of admissible feedbacks $(\mathcal{V}_h(\cdot,\cdot,\cdot,\cdot,\cdot))_{h>0}$ is called *stable with respect to time moment* ϑ if there exist functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot) : (0,+\infty) \mapsto [0,+\infty)$ such that $\gamma_1(h) \to 0$, $\gamma_2(h) \to 0$ as $h \to 0$ and for any admissible disturbance $v(\cdot)$, any number $h \in (0,1)$, any realization $v^h(\cdot)$ of the admissible feedback $\mathcal{V}_h(\cdot,\cdot,\cdot,\cdot,\cdot)$,

$$v^h(t) = \mathcal{V}_h(\tau_{h,i},\Xi^h_i,\Psi^h_i) \quad \text{for} \quad t \in \delta_{h,i}, \tag{2.6}$$

any trajectory of model (2.5) $y^h(\cdot) = y^h(\cdot;\xi^h_{10},\xi^h_0,v^h(\cdot))$ corresponding to the function $v^h(\cdot)$ of form (2.6), and any admissible measurements $\Xi^h(\cdot)$ and $\Psi^h(\cdot)$ of accuracy $h \in (0,1)$, the inequalities

$$\sup_{\vartheta\geq 0} \omega_1(v^h(\cdot),v(\cdot)|\vartheta) \leq \gamma_1(h), \tag{2.7}$$

$$\sup_{\vartheta\geq 0} \omega_2(v^h(\cdot),v(\cdot),h|\vartheta) \leq \gamma_2(h) \tag{2.8}$$

are fulfilled; i.e., inequalities (2.7) and (2.8) hold for the *controlled process* $(x(\cdot),\Xi^h(\cdot),y^h(\cdot),\Psi^h(\cdot),v^h(\cdot))$. Such a pair $(\gamma_1(\cdot),\gamma_2(\cdot))$ is called *an accuracy estimate* of the family $(\mathcal{V}_h(\cdot,\cdot,\cdot,\cdot,\cdot))_{h>0}$.

The problem considered in this paper consists in constructing a family of admissible feedbacks $\mathcal{V}_h$ that is stable with respect to the time ϑ.

3. Solution algorithm

We present two conditions to be used in what follows.

Condition 3.1. $\inf\{|u|_U : u \in P\} \geq 1$.

Condition 3.2. The family of partitions Δ_h has the property

$$\sum_{i=0}^{+\infty} \delta_i^2(h) \leq \varphi_2(h), \quad \varphi_2(h) \to 0 \quad \text{as} \quad h \to 0.$$

Remark 3.3. Conditions 2.5 and 3.2 are fulfilled if, for example,

$$\delta_i(h) = \nu_i^h = dh/(i+1)^\mu \le 1, \quad \mu \in (0.5;1], \quad i = 0,1,\dots, \quad d = \text{const} > 0.$$

In this case,

$$\varphi_1(h) = \varphi_2(h) = 2h^2 d^2 \sum_{i=1}^{\infty} i^{-2\mu}.$$

For any $\varepsilon > 0$, we introduce the functional

$$E_\varepsilon(x(t)-y^h(t),\dot{x}(t)-\dot{y}^h(t)) = \\ = E_1(x(t)-y^h(t),\dot{x}(t)-\dot{y}^h(t)) + \varepsilon(x(t)-y^h(t),\dot{x}(t)-\dot{y}^h(t)), \tag{3.1}$$

where

$$E_1(x(t)-y^h(t),\dot{x}(t)-\dot{y}^h(t)) = 0.5\{|x(t)-y^h(t)|_V^2 + \\ + m|x(t)-y^h(t)|_H^2 + |\dot{x}(t)-\dot{y}^h(t)|_H^2\}.$$

Hereinafter, the symbol $(\cdot,\cdot)$ stands for the inner product in the space H. Note that if $m > -\lambda$ then the space $X = V \times H$ is possible to be equipped by the following scalar product:

$$(\{x_1,y_1\},\{x_2,y_2\})_1 = \int\limits_\Omega \{\nabla x_1(\eta)\nabla x_2(\eta) + m.x_1(\eta)x_2(\eta) + y_1(\eta)y_2(\eta)\}\, d\eta. \tag{3.2}$$

This scalar product generates the norm, which is equivalent to the norm in the space $V \times H$ (see [5, p. 29])

$$2E_1(x,y) = |x,y|_X^2.$$

Here, the symbol $|\cdot|_X$ stands for the norm in X corresponding to scalar product (3.2).

Using Lemma 8.4.1 [5, p. 138], proposition 6.1.1 [5, p. 78], proposition 8.4.2 [5, p. 138] (see also proposition 6.2.3 [5, p. 83]), we derive

$$\frac{dE_1(x(t)-y^h(t),\dot{x}(t)-\dot{y}^h(t))}{dt} = -\gamma|\dot{x}(t)-\dot{y}^h(t)|_H^2 + \\ +(B(v(t)-v^h(t)),\dot{x}(t)-\dot{y}^h(t)) + (g(x(t))-g(y^h(t)),\dot{x}(t)-\dot{y}^h(t)). \tag{3.3}$$

As well, the equality

$$\frac{d(x(t)-y^h(t),\dot{x}(t)-\dot{y}^h(t))}{dt} = |\dot{x}(t)-\dot{y}^h(t)|_H^2 - |x(t)-y^h(t)|_V^2 - \\ - m|x(t)-y^h(t)|_H^2 - \gamma(x(t)-y^h(t),\dot{x}(t)-\dot{y}^h(t)) + (g(x(t)) - \\ - g(y^h(t)),x(t)-y^h(t)) + (B(v(t)-v^h(t)),x(t)-y^h(t)) \tag{3.4}$$

takes place for a.a. t. In this case, by virtue of (3.1), (3.3),(3.4), for a.a. $t \in T$, the equality

$$\frac{dE_\varepsilon(x(t)-y^h(t),\dot{x}(t)-\dot{y}^h(t))}{dt} = L_\varepsilon(x(t),y^h(t)) + \\ + (\varepsilon(x(t)-y^h(t)) + \dot{x}(t)-\dot{y}^h(t), B(v(t)-v^h(t))), \tag{3.5}$$

holds. Here,

$$L_\varepsilon(x(t),y^h(t)) = (-\gamma+\varepsilon)|\dot{x}(t)-\dot{y}^h(t)|_H^2 + (g(x(t)) -$$

$$- g(y^h(t)), \dot{x}(t) - \dot{y}^h(t) + \varepsilon(x(t) - y^h(t))) - \varepsilon|x(t) - y^h(t)|_V^2 -$$
$$- \varepsilon m|x(t) - y^h(t)|_H^2 - \varepsilon\gamma(x(t) - y^h(t), \dot{x}(t) - \dot{y}^h(t)).$$

Let symbol $\langle \cdot, \cdot \rangle$ denote the duality between V and V^*, $d(P) = \sup\{|u|_U : u \in P\}$.

Let us describe the algorithm for solving the problem under consideration. Before the algorithm starts, we fix a value $h \in (0, 1)$, a function

$$\alpha = \alpha(h) \in (0, 1), \quad \alpha(h) \to 0 \quad \text{as} \quad h \to 0,$$

and a partition $\Delta_h = \{\tau_{h,i}\}_{i=0}^{\infty}(2.1)$. The work of the algorithm is decomposed into identical steps. During the i-th step performed on the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following actions are fulfilled. First, at the moment τ_i, we find the element

$$\mathcal{V}_h(\tau_i, \Xi_i^h, \Psi_i^h) = \arg\min\{2(B^*[(\psi_i^h - \xi_i^h) + \\ + \varepsilon(\psi_{1i}^h - \xi_{1i}^h)], v)_U + \alpha(h)|v|_U^2 : v \in P\}, \tag{3.6}$$

where $\Xi_i^h = \{\xi_{1i}^h, \xi_i^h\}$, $\Psi_i^h = \{\psi_{1i}^h, \psi_i^h\}$. Then, control (2.6), (3.6) is fed onto the input of model (2.2) for all $t \in \delta_i$. As a result, under the action of this control, the model passes from the state $\{y^h(\tau_i), \dot{y}^h(\tau_i)\}$ to the state $\{y^h(\tau_{i+1}), \dot{y}^h(\tau_{i+1})\}$. In addition, under the action of some unknown disturbance $v(t)$, $t \in \delta_i$, in equation (1.1), the system described by this equation passes from the state $\{x(\tau_i), \dot{x}(\tau_i)\}$ to the state $\{x(\tau_{i+1}), \dot{x}(\tau_{i+1})\}$. Similar actions are repeated at the $(i+1)$th step.

Condition 3.4. There exist numbers $\varepsilon > 0$ and $c \in (0, \varepsilon)$ such that the inequality

$$L_\varepsilon(x(t), y^h(t)) \le -cE_\varepsilon(x(t) - y^h(t), \dot{x}(t) - \dot{y}^h(t)) \quad \text{for a.a.} \quad t \in T$$

is fulfilled.

Theorem 3.5. *Let Conditions* 2.1, 2.2, *and* 3.4 *be satisfied,* $\varepsilon < min\{1, m + c_0^{-1}\}$, $(h + \delta(h))\alpha(h)^{-1} \to 0$ *as* $h \to 0$, *and* $(x(\cdot), \Xi^h(\cdot), y^h(\cdot), \Psi^h(\cdot), v^h(\cdot))$, $h \in (0, 1)$, *be the controlled process corresponding to the admissible feedback* $\mathcal{V}_h(\cdot, \cdot, \cdot, \cdot, \cdot)$ *of form* (3.6)*, the admissible disturbance* $v(\cdot)$*, and the measurement accuracy* h*. Let also Conditions* 2.4, 3.1 *(in the first case) and* 2.5, 3.2 *(in the second one) be satisfied. Then, the inequality*

$$\int_0^t |v^h(\tau)|_U^2 \, d\tau \le \varrho_0(h) \int_0^t |v(\tau)|_U^2 \, d\tau + \varrho_1(h) \tag{3.7}$$

holds for all $t \in T$*, if* $h \in (0, h_*)$*. Here,*

$$\varrho_0(h) = \frac{\alpha(h) + b_1(h + \delta(h))}{\alpha(h) - b_1(h + \delta(h))}, \quad \varrho_1(h) = \frac{d_0 h^2}{\alpha(h) - b_1(h + \delta(h))}$$

in the first case and

$$\varrho_0(h) = 1, \qquad \varrho_1(h) = \frac{d_0 h^2 + b_3(\varphi_1(h) + \varphi_2(h))}{\alpha(h)}$$

in the second one. In addition, the inequality

$$|y^h(t) - x(t)|_V^2 + |\dot{y}^h(t) - \dot{x}(t)|_H^2 \le \nu(t, h, \alpha), \quad t \in T, \tag{3.8}$$

holds, where $d_0 = 1 + c_0 + 0.5\, mc_0^2$,

$$\nu(t,h,\alpha) = 2\max\{1,(m-\varepsilon)^{-1}\}\left[d_0 h^2 e^{-ct} + b_2(h+\delta(h)) + \frac{2d^2(P)}{c}\alpha(h)\right]$$

in the first case and

$$\nu(t,h,\alpha) = 2\max\{1,(m-\varepsilon)^{-1}\}\Big[d_0 h^2 e^{-ct} + \\ + \frac{2d^2(P)}{c}\alpha(h)) + b_4(\varphi_1(h)+\varphi_2(h))\Big]$$

in the second one.

Here, b_1, b_2, b_3, *and* b_4 *are some constants that can be written explicitly; in the first case, the number* $h_* \in (0,1)$ *is such that the inequality* $\alpha(h) - b_1(h+\delta(h)) > 0$ *holds for all* $h \in (0, h_*)$ *and, in the second case,* $h_* = 1$.

Proof. In the beginning, we consider the first case. By virtue of (1.1) and (2.2), we deduce that the difference

$$z(\cdot) = y^h(\cdot) - x(\cdot)$$

satisfies the relation

$$\ddot{z}(t) - \Delta z(t) + mz(t) + \gamma\dot{z}(t) = \\ = g(y^h(t)) - g(x(t)) + B(v^h(t) - v(t)) \quad \text{in } V^* \text{ for a.a. } t \in T, \tag{3.9}$$

where $z(0) = \xi_{10}^h - x_{10}$, $\dot{z}(0) = \xi_0^h - x_0$. In what follows, we need some estimates of the differences $|z(t) - z(\tau_i)|_H$ and $|\dot{z}(t) - \dot{z}(\tau_i)|_{V^*}$ for $t \in \delta_i$, $i = 0, 1, \ldots$. Due to Conditions 1 and 2 (see [5, Theorem 8.4.5, p.139]), there exists a number $c_1 \in (0, +\infty)$ such that the inequalities

$$\sup_{t\in T} |\{\dot{x}(t), x(t)\}|_{H\times V} \le c_1, \qquad \sup_{t\in T} |\{\dot{y}^h(t), y^h(t)\}|_{H\times V} \le c_1. \tag{3.10}$$

hold uniformly with respect to all $h \in (0,1)$, $v(\cdot) \in P(\cdot)$, and $v^h(\cdot) \in P(\cdot)$.

Let us fix some element $v \in V$. Then, using (3.9), (2.3) and (2.4), we obtain

$$|\langle \dot{z}(t+\Delta t) - \dot{z}(t), v\rangle| \le \int_t^{t+\Delta t} \{|z(\tau)|_V + c_0(m|z(\tau)|_H + \\ + \gamma|\dot{z}(\tau)|_H + L|z(\tau)|_H + c_2)|v|_V\}\, d\tau.$$

Therefore, by virtue of (3.10), we have the estimates

$$|\dot{z}(t+\Delta t) - \dot{z}(t)|_{V^*} \le c_3\Delta t, \qquad |z(t+\Delta t) - z(t)|_H \le c_4 \Delta t, \tag{3.11}$$

which hold for any t, $t+\Delta t \in T$, $\Delta t > 0$.

Consider the variation of the variable

$$\varepsilon_h(t) = E_\varepsilon(t) + \alpha\int_0^t \{|v^h(\tau)|_U^2 - |v(\tau)|_U^2\}\, d\tau, \quad \alpha = \alpha(h) \tag{3.12}$$

on the time interval T. Here, $E_\varepsilon(t) = E_\varepsilon(x(t) - y^h(t), \dot{x}(t) - \dot{y}^h(t))$. Finding the derivative of $\varepsilon_h(t)$, for a.a. $t \in [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, we have

$$\dot{\varepsilon}_h(t) \le (\dot{z}(t) + \varepsilon z(t), B(v^h(t) - v(t))) - cE_\varepsilon(t) + \alpha\{|v^h(t)|_U^2 - |v(t)|_U^2\} \tag{3.13}$$

(see (3.5) and Condition 3.4). Then, using (3.13), we obtain the estimate

$$\begin{aligned}(3.14)\quad \dot\varepsilon_h(t) \le -cE_\varepsilon(t) + (\dot z(t) - \dot z(\tau_i) + \varepsilon(z(t) - z(\tau_i)), B(v^h(t) - v(t))) + \chi_i^t(v^h, v) + \\ + \mu_i^t(v^h, v) + \langle \xi_i^h - \dot x(\tau_i), B(v^h(t) - v(t))\rangle + \langle \dot y^h(\tau_i) - \psi_i^h, B(v^h(t) - v(t))\rangle + \\ + \varepsilon(\xi_{1i}^h - x(\tau_i), B(v^h(t) - v(t))) + \varepsilon(y^h(\tau_i) - \psi_{1i}^h, B(v^h(t) - v(t))) \text{ for a.a. } t \in \delta_i,\end{aligned}$$

where

$$\chi_i^t(v^h, v) = -(v^h(t), B^*(\xi_i^h - \psi_i^h))_U + \alpha|v^h(t)|_U^2 + (v(t), B^*(\xi_i^h - \psi_i^h))_U - \alpha|v(t)|_U^2,$$

$$\mu_i^t(v^h, v) = -\varepsilon(v^h(t), B^*(\xi_{1i}^h - \psi_{1i}^h))_U + \varepsilon(v(t), B^*(\xi_{1i}^h - \psi_{1i}^h))_U.$$

Relations (2.6) and (3.6) imply the inequality

$$\chi_i^t(v^h, v) + \mu_i^t(v^h, v) \le 0.$$

In this case, for a.a. $t \in [\tau_i, \tau_{i+1})$, the following relation

$$\begin{aligned}(3.15)\qquad \dot\varepsilon_h(t) \le -cE_\varepsilon(t) + (\dot z(t) - \dot z(\tau_i), B(v^h(t) - v(t))) + \\ + \langle \xi_i^h - \dot x(\tau_i), B(v^h(t) - v(t))\rangle + \langle \dot y^h(\tau_i) - \psi_i^h, B(v^h(t) - v(t))\rangle + \\ + \varepsilon(\xi_{1i}^h - x(\tau_i), B(v^h(t) - v(t))) + \varepsilon(y^h(\tau_i) - \psi_{1i}^h, B(v^h(t) - v(t))) + \\ + \varepsilon(z(t) - z(\tau_i), B(v^h(t) - v(t)))\end{aligned}$$

is valid. By (1.3) and (3.11), we conclude that

$$\begin{aligned}(3.16)\qquad (\dot z(t) - \dot z(\tau_i), B(v^h(t) - v(t))) \le c_5\delta(h)\{|v^h(t)|_U + |v(t)|_U\}, \\ (z(t) - z(\tau_i), B(v^h(t) - v(t))) \le c_6\delta(h)\{|v^h(t)|_U + |v(t)|_U\}.\end{aligned}$$

In addition, by taking into account (1.3) for a.a. $t \in [\tau_i, \tau_{i+1})$, we have

$$\langle \xi_i^h - \dot x(\tau_i), B(v^h(t) - v(t))\rangle \le c_7 h\{|v^h(t)|_U + |v(t)|_U\},$$

$$\langle \dot y^h(\tau_i) - \psi_i^h, B(v^h(t) - v(t))\rangle \le c_7 h\{|v^h(t)|_U + |v(t)|_U\},$$

$$\varepsilon(\xi_{1i}^h - x(\tau_i), B(v^h(t) - v(t))) \le c_8 h\{|v^h(t)|_U + |v(t)|_U\},$$

$$\varepsilon(y^h(\tau_i) - \psi_{1i}^h), B(v^h(t) - v(t))) \le c_8 h\{|v^h(t)|_U + |v(t)|_U\}.$$

By using (3.15) and (3.16) for a.a. $t \in [\tau_i, \tau_{i+1})$, we derive the inequality

$$(3.17)\qquad \dot\varepsilon_h(t) \le b_1(h + \delta(h))\{|v^h(t)|_U + |v(t)|_U\} - cE_\varepsilon(t).$$

Then, by virtue of the inequality $\varepsilon < \min\{1, m + c_0^{-1}\}$, we get

$$0 \le E_\varepsilon(t) \quad \text{for} \quad t \in T.$$

From this inequality (see Condition 3.1) and (3.17), we obtain the inequality

$$(3.18)\qquad \varepsilon_h(t) \le \varepsilon_h(0) + b_1(h + \delta(h)) \int_0^t \{|v^h(\tau)|_U^2 + |v(\tau)|_U^2\}\, d\tau, \quad t \in T.$$

In its turn, from (3.18) we conclude that

$$\alpha(h) \int_0^t \{|v^h(\tau)|_U^2 - |v(\tau)|_U^2\}\, d\tau \le \varepsilon_h(0) +$$

$$+ b_1(h+\delta(h)) \int_0^t \{|v^h(\tau)|_U^2 + |v(\tau)|_U^2\}\, d\tau \quad \text{for} \quad \mathrm{t} \in T.$$

Thus, for all $t \in T$, the estimate
(3.19)

$$\{\alpha(h) - b_1(h+\delta(h))\} \int_0^t |v^h(\tau)|_U^2\, d\tau \le \varepsilon_h(0) + \{\alpha(h) + b_1(h+\delta(h))\} \int_0^t |v(\tau)|_U^2\, d\tau$$

takes place. Note that from (1.4), (2.3), and the inclusion $\varepsilon \in (0,1)$ the inequality

$$E_\varepsilon(0) \le (1 + c_0 + 0.5c_0^2 m)h^2 \tag{3.20}$$

follows. Inequality (3.19), together with the equality $\varepsilon_h(0) = E_\varepsilon(0)$, implies (for $h \in (0, h_a)$) the relation

$$\int_0^t |v^h(\tau)|_U^2\, d\tau \le \frac{\alpha(h) + b_1(h+\delta(h))}{\alpha(h) - b_1(h+\delta(h))} \int_0^t |v(\tau)|_U^2\, d\tau +$$

$$+ \frac{d_0 h^2}{\alpha(h) - b_1(h+\delta(h))}, \quad t \in T.$$

The existence of a number $h_a > 0$ with the property from theorem's conditions is evident. This implies inequality (3.7).

Now, let us show that inequality (3.8) is also true. By taking into account (3.6) for a.a. $t \in [\tau_i, \tau_{i+1})$, we obtain the estimate

$$(B^*[\psi_i^h - \xi_i^h + \varepsilon(\psi_{1i}^h - \xi_{1i}^h)], v^h(t))_U \le \tag{3.21}$$

$$\le \inf\{(B^*[\psi_i^h - \xi_i^h + \varepsilon(\psi_{1i}^h - \xi_{1i}^h)], v)_U : v \in P\} + d^2(P)\alpha(h).$$

By virtue of this estimate, by analogy with (3.17), we derive the inequality

$$\dot E_\varepsilon(t) \le -cE_\varepsilon(t) + c_9(h+\delta(h))\{|v^h(t)|_U + |v(t)|_U\} + 2d^2(P)\alpha(h) \text{ for a.a. } t \in T;$$

i.e.,

$$\dot E_\varepsilon(t) = -cE_\varepsilon(t) + c_{10}(h+\delta(h)) + 2d^2(P)\alpha(h) + \psi_0(t),$$

where $\psi_0(t) \le 0$, $t \in T$. In this case, we deduce that

$$E_\varepsilon(t) \le E_\varepsilon(0)e^{-ct} + 2d^2(P)\alpha(h) \int_0^t e^{-c(t-\tau)}\, d\tau + c_{10} \int_0^t e^{-c(t-\tau)}(h+\delta(h))\, d\tau.$$

Then, we have

$$\int_0^t e^{-c(t-\tau)}\, d\tau$$

From the last two inequalities and (3.20), it follows that

$$E_\varepsilon(t) \le d_0 h^2 e^{-ct} + \frac{2d^2(P)}{c}\alpha(h) + b_2(h+\delta(h))), \quad \mathrm{b}_2 = \frac{c_{10}}{c} \tag{3.22}$$

for all $t \in T$. Note that the following inequality

$$E_\varepsilon(t) \ge 0.5\{|z(t)|_V^2 + m|z(t)|_H^2 + |\dot z(t)|_H^2\} - 0.5\varepsilon\{|z(t)|_H^2 + |\dot z(t)|_H^2\} \ge \tag{3.23}$$

$$\geq 0.5\{c_0^{-1}|z(t)|_H^2 + 0.5(m-\varepsilon)|z(t)|_H^2 + (1-\varepsilon)|\dot{z}(t)|_H^2\}$$

is true. It follows from (3.22) and (3.23) that inequality (3.8) is satisfied.

Consider the second case. It is easily seen that in this case relations (3.9)–(3.15) also take place. Then, we get the estimates

$$\begin{aligned}
&(z(t)-z(\tau_i), B(v^h(t)-v(t))) \leq c_{11}\nu_i^h,\\
&(\dot{z}(t)-\dot{z}(\tau_i), B(v^h(t)-v(t))) \leq c_{11}\nu_i^h,\\
&\langle \xi_i^h - \dot{x}(\tau_i), B(v^h(t)-v(t))\rangle \leq c_{11}\nu_i^h,\\
&\langle \dot{y}^h(\tau_i) - \psi_i^h, B(v^h(t)-v(t))\rangle \leq c_{11}\nu_i^h,\\
&\varepsilon(\xi_{1i}^h - x(\tau_i), B(v^h(t)-v(t))) \leq c_{11}\nu_i^h,\\
&\varepsilon(y^h(\tau_i) - \psi_{1i}^h), B(v^h(t)-v(t))) \leq c_{11}\nu_i^h.
\end{aligned} \tag{3.24}$$

Using (3.15) and (3.24), we derive for a.a. $t \in [\tau_i, \tau_{i+1})$

$$\dot{\varepsilon}_h(t) \leq b_3(\nu_i^h + \delta_i(h)) - cE_\varepsilon(t). \tag{3.25}$$

From (3.25), we have the inequality

$$\varepsilon_h(t) \leq \varepsilon_h(0) + b_3\Sigma_{j=0}^{i}(\nu_j^h + \delta_j(h))\delta_j(h) \quad \text{for} \quad t \in [\tau_i, \tau_{i+1}). \tag{3.26}$$

In this case, by virtue of Conditions 2.5 and 3.2, inequalities (3.20) and (3.26), we obtain the estimate

$$\varepsilon_h(t) \leq d_0h^2 + b_3(\varphi_1(h) + \varphi_2(h)), \quad t \in T.$$

Inequality (3.7) follows from this estimate.

Let us verify inequality (3.8). Using inequality (3.21), by analogy with (3.25), we obtain

$$\dot{E}_\varepsilon(t) \leq -cE_\varepsilon(t) + b_4(\nu_i^h + \delta_i(h)) + 2d^2(P)\alpha(h) \quad \text{for a.a.} \quad t \in [\tau_i, \tau_{i+1});$$

i.e., for a.a. $t \in [\tau_i, \tau_{i+1})$ the equality

$$\dot{E}_\varepsilon(t) = -cE_\varepsilon(t) + b_4(\nu_i^h + \delta_i(h)) + 2d^2(P)\alpha(h) + \psi_1(t)$$

takes place. Here, $\psi_1(t) \leq 0$, $t \in T$. In this case, for a.a. $t \in [\tau_i, \tau_{i+1})$, the inequality

$$\begin{aligned}
E_\varepsilon(t) \leq E_\varepsilon(0)e^{-ct} + 2d^2(P)\alpha(h)\int_0^t e^{-c(t-\tau)}\,d\tau + \\
+ b_4\Sigma_{j=0}^{i-1}(\nu_j^h + \delta_j(h))\delta_j(h) + b_4(t-\tau_i)\delta_i(h)(\nu_i^h + \delta_i(h))
\end{aligned} \tag{3.27}$$

is fulfilled. From (3.27), we derive the inequality

$$E_\varepsilon(t) \leq d_0h^2e^{-ct} + \frac{2d^2(P)}{c}\alpha(h) + b_4(\varphi_1(h) + \varphi_3(h)). \tag{3.28}$$

Using (3.28) and (3.23), we get (3.8). The proof of the theorem is complete. □

We give one sufficient condition providing the fulfillment of Condition 3.4.

Theorem 3.6. *Let* $3L < m$, $\gamma > \{2(m-L)\}^{1/2}$ *and let Conditions* 2.1 *and* 2.2 *be fulfilled. Then, Condition* 3.4 *is also satisfied.*

Let $q_1 = \frac{m-3L}{m}$ and

$$\varphi_\gamma(q) = \frac{m(1-q) - L - \sqrt{(m(1-q)-L)^2 - 4L^2}}{2\gamma}.$$

Note that the radicand is nonnegative if $q \in (0, q_1)$.

The conclusion of Theorem 3.6 follows from Lemma 3.7 given below.

Lemma 3.7. *Let the conditions of Theorem* 3.6 *be fulfilled. Then, for any number* $q \in (0, q_1)$, *if* $c = 2q\varepsilon$ *and* $\varepsilon = \varphi_\gamma(q)$ *the inequality*

$$L_\varepsilon(x(t), y^h(t)) \le -cE_\varepsilon(x(t) - y^h(t), \dot{x}(t) - \dot{y}^h(t)) \tag{3.29}$$

takes place almost everywhere on T.

Proof. By virtue of the Lipschitz property of the function g, we have the inequality

$$\begin{aligned}(g(x(t)) - g(y^h(t)), \dot{x}(t) - \dot{y}^h(t) + \varepsilon(x(t) - y^h(t))) \le \\ \le L|x(t) - y^h(t)|_H \{|\dot{x}(t) - \dot{y}^h(t)|_H + \varepsilon|x(t) - y^h(t)|_H\} \le \\ \le (L\varepsilon + \frac{L^2}{2\gamma_1})|x(t) - y^h(t)|_H^2 + \frac{\gamma_1}{2}|\dot{x}(t) - \dot{y}^h(t)|_H^2\end{aligned}$$

for every $\gamma_1 > 0$. Moreover, for $c_* \in (0, \gamma)$, the inequality

$$\begin{aligned}-\varepsilon\gamma(x(t) - y(t), \dot{x}(t) - \dot{y}^h(t)) \le -\varepsilon c_*(x(t) - y^h(t), \dot{x}(t) - \dot{y}^h(t)) + \\ + \frac{\varepsilon^2(\gamma - c_*)^2}{2\gamma_1}|x(t) - y^h(t)|_H^2 + \frac{\gamma_1}{2}|\dot{x}(t) - \dot{y}^h(t)|_H^2\end{aligned}$$

is also fulfilled. We fix any numbers $\gamma_1 > 0$, $\varepsilon \in (0, \gamma)$, and $c_* \in (0, \varepsilon)$ in such a way that the relations

$$(-\gamma + \frac{\gamma_1}{2} + \frac{\gamma_1}{2} + \varepsilon)|\dot{x}(t) - \dot{y}^h(t)|_H^2 \le -c_*|\dot{x}(t) - \dot{y}^h(t)|_H^2, \tag{3.30}$$

$$-\varepsilon|x(t) - y^h(t)|_V^2 \le -c_*|x(t) - y^h(t)|_V^2, \tag{3.31}$$

$$(L\varepsilon + \frac{L^2}{2\gamma_1} - \varepsilon m + \frac{\varepsilon^2(\gamma - c_*)^2}{2\gamma_1})|x(t) - y^h(t)|_H^2 \le -mc_*|x(t) - y^h(t)|_H^2 \tag{3.32}$$

are valid for a.a. $t \in T$. Let $c_* = q\varepsilon$ and $q \in (0, q_1)$. Then, inequality (3.31) takes place. Moreover, inequality (3.30) is also fulfilled if the equality

$$-\gamma + \gamma_1 + \varepsilon = -c_*$$

takes place; i.e.,

$$\gamma_1 = \gamma - \varepsilon - c_* = \gamma - (1+q)\varepsilon > 0. \tag{3.33}$$

In its turn, inequality (3.32) is fulfilled if the inequalities

$$\frac{L^2 + \varepsilon^2(\gamma - q\varepsilon)^2}{2\gamma_1} \le [m(1-q) - L]\varepsilon \tag{3.34}$$

and

$$m(1-q) - L \ge 0 \tag{3.35}$$

hold. Note that, for $q \in (0, q_1)$, inequality (3.35) takes place. Let

$$(1+q)\varepsilon \le \gamma/2. \tag{3.36}$$

Then, relation (3.33) is fulfilled and

$$\frac{1}{2\gamma_1} \le \frac{1}{\gamma}.$$

In this case, inequality (3.34) is fulfilled if the inequality

$$\frac{L^2 + \varepsilon^2(\gamma - q\varepsilon)^2}{\gamma} \le [m(1-q) - L]\varepsilon \tag{3.37}$$

takes place. By virtue of (3.36), we have $q\varepsilon \in (0,\gamma)$ and

$$(\gamma - q\varepsilon)^2 \le \gamma^2.$$

Therefore, relation (3.37) and, consequently, (3.34) are valid if the number $\varepsilon \in (0,1)$ satisfies the inequality

$$\gamma^2\varepsilon^2 - \gamma[m(1-q) - L]\varepsilon + L^2 \le 0. \tag{3.38}$$

It is easily seen that the roots ε_{1q} and ε_{2q} of the quadratic equation

$$\gamma^2\varepsilon^2 - \gamma[m(1-q) - L]\varepsilon + L^2 = 0$$

are found by the formulas

$$\varepsilon_{1q} = \varphi_\gamma(q), \qquad \varepsilon_{2q} = \frac{m(1-q) - L + \sqrt{(m(1-q) - L)^2 - 4L^2}}{2\gamma}.$$

Therefore,

$$\varepsilon_{1q} < \gamma/4,$$

if $q \in (0, q_1)$. Consequently, inequality (3.38) is fulfilled if $\varepsilon = \varphi_\gamma(q)$ and $q \in (0, q_1)$. Note that, for $q \in (0, q_1)$, inequalities (3.33), (3.35), and (3.36) take place. So, for $\varepsilon = \varphi_\gamma(q)$, $q \in (0, q_1)$, $c_* = q\varepsilon$, and γ_1 of form (3.33), inequalities (3.30)–(3.32) are fulfilled. From these inequalities, we conclude that the inequality

$$L_\varepsilon(x(t), y^h(t)) \le -2c_* E_\varepsilon(x(t) - y^h(t), \dot{x}(t) - \dot{y}^h(t))$$

holds. This inequality implies inequality (3.29) if $c = 2c_*$. The lemma is proved. □

From Theorem 3.5 we obtain the main statement providing the solution of the above-posed problem of the stable dynamical inversion of system (1.1).

Theorem 3.8. *Let the conditions of Theorem* 3.5 *hold. Let also* $h^2/\alpha(h) \to 0$ *in the first case and* $(\varphi_1(h) + \varphi_2(h))/\alpha(h) \to 0$ *in the second one. Then, the family* $(\mathcal{V}_h(\cdot,\cdot,\cdot,\cdot,\cdot))_{h>0}$ *of admissible feedbacks of form* (2.6), (3.6) *is stable with respect to time moment* ϑ, *and the pair* $(\gamma_1(\cdot), \gamma_2(\cdot))$, *where*

$$\gamma_1(h) = \nu(0, h, \alpha(h)), \qquad \gamma_2(h) = \varrho_1(h),$$

is an accuracy estimate for this family.

Proof. First, note that $\gamma_1(h) \to 0$ and $\gamma_2(h) \to 0$ as $h \to 0$. Let $h \in (0,1)$ and $(x(\cdot), \Xi^h(\cdot), y^h(\cdot), \Psi^h(\cdot), v^h(\cdot))$ be the controlled process corresponding to the admissible feedback $\mathcal{V}_h(\cdot,\cdot,\cdot,\cdot,\cdot)$, the admissible disturbance $v(\cdot)$, and the measurement accuracy h. Then, by Theorem 3.5, inequalities (3.7) and (3.8) are true for all $t \ge 0$; this implies the statement of the theorem. The theorem is proved. □

Let the symbol $U_\vartheta(x(\cdot))$ mean the set of all admissible disturbances compatible with some output $x(t)$, $t \in T_\vartheta$; i.e.,

$$U_\vartheta(x(\cdot)) = \{u(\cdot) \in L_2(T_\vartheta; U) : u(t) \in P, \quad (Bu(t), z) = (\ddot{x}(t) + \gamma\dot{x}(t) -$$
$$- g(x(t)) + mx(t) - f(t), z) + (\nabla x(t), \nabla z) \text{ for a.a. } t \in T_\vartheta \text{ and all } z \in V\}.$$

Note that the set $U_\vartheta(x(\cdot))$ is convex, bounded, and closed in $L_2(T_\vartheta; U)$. Therefore, it contains a unique element $v_*^\vartheta(\cdot)$ of minimal $L_2(T_\vartheta; U)$-norm.

Remark 3.9. It is easily seen that, for any $\vartheta \in T$ and for all $t \in T_\vartheta$, inequality (3.7) is true if we replace the function $v(\cdot)$ by the function $v_*^\vartheta(\cdot)$.

Theorem 3.10. *Let the conditions of Theorem* 3.8 *hold. Then, for every* $\vartheta \in T$, *the convergence*

$$v^h(\cdot) \to v_*(\cdot) = v_*^\vartheta(\cdot) \text{ in } L_2 = L_2(T_\vartheta; U) \text{ as } h \to 0$$

takes place.

Proof. We establish that, for any sequence $h_j \to 0+$ as $j \to \infty$, any number $\vartheta \in T$, any family $\{\Delta_{h_j}\} = \{\tau_{h_j,i}\}_{i=0}^\infty$ of partitions of the interval T, and any admissible measurements $\Xi^{h_j}(\cdot)$ and $\Psi^{h_j}(\cdot)$ of accuracy h_j, the convergence

$$v^{h_j}(\cdot) \to v_*(\cdot) \text{ in } L_2 \text{ as } j \to \infty$$

takes place. Here and below, the controls $v^{h_j}(\cdot)$ are defined by rule (2.6), (3.6), where $h = h_j$. Assuming the contrary, we conclude that there exists a subsequence of the sequence $v^{h_j}(\cdot)$ (we denote it for simplicity by the same symbol $v^{h_j}(\cdot)$) such that

$$v^{h_j}(\cdot) \to v_0(\cdot) \text{ weakly in } L_2 \text{ as } j \to \infty, \tag{3.39}$$

$$v_0(\cdot) \neq v_*(\cdot). \tag{3.40}$$

Let $w^{h_j}(t) = y^{h_j}(t) - y_0(t)$, where $y^{h_j}(\cdot) = y^{h_j}(\cdot; \xi_{1i}^{h_j}, \xi_i^{h_j}, v^{h_j}(\cdot))$ and $y_0(\cdot)$ is the solution of the equation

$$\ddot{y}(t) - \Delta y(t) + my(t) + \gamma\dot{y}(t) =$$
$$= g(y(t)) + Bv_0(t) + f(t) \quad \text{in } V^* \quad \text{for a.a.} \quad t \in T, \quad y(0) = x_0, \quad \dot{y}(0) = x_{10}.$$

Then, we have

$$\ddot{w}^{h_j}(t) - \Delta w^{h_j}(t) + mw^{h_j}(t) + \gamma\dot{w}^{h_j}(t) = \tag{3.41}$$
$$= g(y^{h_j}(t)) - g(y_0(t)) + B(v^{h_j}(t) - v(t)) \quad \text{in } V^* \quad \text{for a.a.} \quad t \in T,$$
$$w^{h_j}(0) = \xi_0^{h_j} - x_0, \quad \dot{w}^{h_j}(0) = \xi_{10}^{h_j} - x_{10}.$$

Multiplying the right-hand and left-hand parts of equality (3.41) by $\dot{w}^{h_j}(t)$, integrating, and taking into account inequality (1.3), we conclude that

$$0.5\{|\dot{w}^{h_j}(t)|_H^2 + m|w^{h_j}(t)|_H^2 + |w^{h_j}(t)|_V^2\} + \gamma \int_0^t |\dot{w}^{h_j}(\tau)|_H^2 \, d\tau \leq$$
$$\leq 0.5\{|\dot{w}^{h_j}(0)|_H^2 + m|w^{h_j}(0)|_H^2 + |w^{h_j}(0)|_V^2\}+$$

$$+\int_0^t \{B(v^{h_j}(\tau)-v_0(\tau)),\dot w^{h_j}(\tau)) + (g(y^{h_j}(\tau))-g(y_0(\tau)),\dot w^{h_j}(\tau))\}\,d\tau.$$

From this inequality, using (1.3) and (1.4), we obtain

$$|\dot w^{h_j}(t)|_H^2 + m|w^{h_j}(t)|_H^2 + 2\gamma\int_0^t |w^{h_j}(\tau)|_H^2\,d\tau + |w^{h_j}(t)|_V^2 \le \tag{3.42}$$

$$\le \nu(h_j) + 2\int_0^t (B(v^{h_j}(\tau)-v_0(\tau)),\dot y^{h_j}(\tau)-\dot x(\tau))\,d\tau +$$

$$+2L\int_0^t |w^{h_j}(\tau)|_H|\dot w^{h_j}(\tau)|_H\,d\tau + \int_0^t (B(v^{h_j}(\tau)-v_0(\tau)),\dot x(\tau)-\dot y_0(\tau))\,d\tau,$$

where

$$\nu(h_j) = |\dot w^{h_j}(0)|_H^2 + |w^{h_j}(0)|_V^2 + m|w^{h_j}(0)|_H^2 \to 0 \quad \text{as} \quad j\to\infty. \tag{3.43}$$

The second term in the right-hand part of inequality (3.42) tends to zero as $j\to\infty$. This follows from Theorem 3.5. The convergence to zero of the latest term follows from the weak convergence of $v^{h_j}(\cdot)$ to $v_0(\cdot)$ (see (3.39)). Therefore, by virtue of (3.43) and Theorem 3.5 (see (3.8)), we deduce that

$$y_0(t) = x(t), \quad t\in T_\vartheta.$$

Hence, $v_0(\cdot)\in U_\vartheta(x(\cdot))$ and, consequently,

$$|v_0(\cdot)|_{L_2} \ge |v_*(\cdot)|_{L_2}. \tag{3.44}$$

The symbol $|\cdot|_{L_2}$ stands for the norm in the space $L_2(T_\vartheta;U)$. As well, by virtue of the known property of the weak limit, from (3.39) we derive

$$\varliminf_{j\to\infty} |v^{h_j}(\cdot)|_{L_2} \ge |v_0(\cdot)|_{L_2}. \tag{3.45}$$

In its turn, by virtue of (3.7), the following inequality

$$|v^{h_j}(\cdot)|_{L_2}^2 \le \varrho_0(h_j)|v_*(\cdot)|_{L_2}^2 + \varrho_1(h_j)$$

is valid. This implies

$$\varlimsup_{j\to\infty}|v^{h_j}(\cdot)|_{L_2} \le |v_*(\cdot)|_{L_2}, \tag{3.46}$$

and (see (3.44)–(3.46))

$$\varlimsup_{j\to\infty}|v^{h_j}(\cdot)|_{L_2} \le |v_*(\cdot)|_{L_2} \le |v_0(\cdot)|_{L_2} \le \varliminf_{j\to\infty}|v^{h_j}(\cdot)|_{L_2}. \tag{3.47}$$

The set $U_\vartheta(x(\cdot))$ possesses a unique element of minimal L_2-norm (namely, $v_*(\cdot)$). Therefore, from (3.47) we obtain

$$v_0(\cdot) = v_*(\cdot). \tag{3.48}$$

Using (3.39) and (3.48), we conclude that

$$v^{h_j}(\cdot) \to v_*(\cdot) \text{ in } L_2 \text{ as } j\to\infty. \tag{3.49}$$

Convergence (3.49) contradicts (3.39) and (3.40). The theorem is proved. □

4. Convergence rate of the algorithm

Under some additional conditions, on every bounded time interval $T_\vartheta = [0, \vartheta]$, one can rewrite the convergence rate of the algorithm (see Theorem 4.2 below). Let us obtain this estimate. In what follows, we need the following lemma.

Lemma 4.1. ([18, p. 47]) *Let $\tilde{u}(\cdot) \in L_\infty(T_*; V^*)$ and $\tilde{v}(\cdot) \in W(T_*; V)$, $T_* = [a, b]$, $-\infty < a < b < +\infty$,*

$$|\int_a^t \tilde{u}(\tau)\, d\tau|_{V^*} \le \varepsilon_*, \quad |\tilde{v}(t)|_V \le K \quad \forall\, t \in T_*.$$

Then, for all $t \in T_*$, the inequality

$$\Big| \int_a^t \langle \tilde{u}(\tau), \tilde{v}(\tau) \rangle\, d\tau \Big| \le \varepsilon_*(K + \mathrm{var}(T_*; v(\cdot)))$$

is valid. Here, the symbol $\mathrm{var}(T_*; v(\cdot))$ means the variation of the function $v(\cdot)$ over the segment T_*, and the symbol $W(T_*; V)$ means the set of functions $y(\cdot) : T_* \to V$ of bounded variation.

Theorem 4.2. *Suppose that the conditions of Theorem 3.8 hold. Let also $U = V$, B be the operator of canonical embedding of the space V into the space H and $v(\cdot) \in W(T_\vartheta; V)$. Then, the following estimate of the convergence rate of the algorithm is valid:*

$$|v(\cdot) - v^h(\cdot)|^2_{L_2(T_\vartheta; H)} \le K(\alpha, h)\{2d(P) + \mathrm{var}(T_\vartheta; v(\cdot))\} + \\ + \varrho_1(h) + |1 - \varrho_0(h)|\vartheta d^2(P),$$

where

$$K(\alpha, h) = c^{(0)} \nu^{1/2}(0, \alpha, h), \quad c^{(0)} = c^{(0)}_\vartheta \text{ is some constant.}$$

Proof. Note (see (3.9)) that, for every $t_1, t_2 \in T_\vartheta$, $t_1 < t_2$, the inequality

(4.1)
$$\Big| \int_{t_1}^{t_2} B(v(t) - v^h(t))\, dt \Big|_{V^*} = \sup_{|v|_V \le 1} \Big| \langle \int_{t_1}^{t_2} \{\ddot{x}(\tau) - \ddot{y}^h(\tau) + m(x(\tau) - y^h(\tau)) - g(x(\tau)) + \\ + g(y^h(\tau)) + \gamma(\dot{x}(\tau) - \dot{y}^h(\tau)) - \Delta(x(\tau) - y^h(\tau)))\, d\tau\}, v \rangle \Big| \le \\ \le |\dot{z}(t_2) - \dot{z}(t_1)|_{V^*} + c^{(1)} \int_{t_1}^{t_2} \{|z(\tau)|_V + |z(\tau)|_H\}\, d\tau + \gamma |z(t_2) - z(t_1)|_{V^*}$$

is fulfilled. Here, as above, $z(t) = y^h(t) - x(t)$, and $c^{(1)}$ is some constant. As well, by virtue of (3.8), for $t \in T_\vartheta$ the estimate

$$|\dot{z}(t)|_H + |z(t)|_V \le \nu^{1/2}(0, \alpha, h) \tag{4.2}$$

takes place. From (4.1) and (4.2), we get

$$\Big| \int_{t_1}^{t_2} B(v(t) - v^h(t))\,dt \Big|_{V^*} \leq c^{(2)} K(\alpha, h).$$

Using Lemma 4.1 and (3.7), we derive

$$|v(\cdot) - v^h(\cdot)|^2_{L_2(T_\vartheta;H)} \leq 2|v(\cdot)|^2_{L_2(T_\vartheta;H)} - 2\int_0^\vartheta (v(\tau), v^h(\tau))\,d\tau +$$

$$+ \varrho_1(h) + |1 - \varrho_0(h)|\vartheta d^2(P) \leq 2\int_0^\vartheta |B(v(\tau) - v^h(\tau))|_{V^*}|v(\tau)|_V\,d\tau +$$

$$+ \varrho_1(h) + |1 - \varrho_0(h)|\vartheta d^2(P) \leq K(\alpha, h)\{2d(P) + \mathrm{var}(T_\vartheta; v(\cdot))\} +$$
$$+ \varrho_1(h) + |1 - \varrho_0(h)|\vartheta d^2(P).$$

The theorem is proved. □

5. Conclusion

In the paper, the algorithm of stable reconstruction of the right-hand part of the dynamical system described by the Klein–Gordon differential equation is specified. In contrast to previously suggested algorithms with the property of accumulating numerical and informational errors when the time interval ($T_\vartheta = [0, \vartheta]$) of the system operation grows, the algorithm designed in the paper is free of such a lack. Its peculiarity is in the fact that the values of criteria for the deviation of the reconstructed right-hand part $v(\cdot)$ from its approximation $v^h(\cdot)$ do not depend on the value of ϑ (see (3.1) and (3.2)).

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