Pure and Applied Functional Analysis
Volume 5, Number 2, 2020, 287–294

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

THE NASH INEQUALITY IN GENERAL DOMAINS WITH APPLICATION TO THE LINEAR STOKES SYSTEM

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Abstract. The well-known Nash inequality is extended to functions satisfying a *weak Dirichlet condition* in a subset of $\mathbb{R}^n$. Two versions of the inequality are established, with constants independent of the domain. The inequality is applied to obtain an estimate for the sup-norm of a solution to the linearized Stokes system, independent of the velocity field.

1. Introduction

Let ϕ be an integrable Lipschitz function on $\mathbb{R}^n$, $n \geq 1$. The classical Nash's inequality [10] is

$$\Big(\int_{\mathbb{R}^n} |\phi|^2 dx\Big)^{\frac{n+2}{n}} \leq C_n \int_{\mathbb{R}^n} |\nabla\phi|^2 dx \cdot \Big(\int_{\mathbb{R}^n} |\phi| dx\Big)^{\frac{4}{n}}, \tag{1.1}$$

where $C_n > 0$ depends only on n.

Immediately after the publication of the paper by Nash, the whole array of the Gagliardo-Nirenberg-Sobolev (GNS) inequalities was established by Gagliardo [9] and Nirenberg [11]. We refer to [8, Part I, Theorem 9.3] for a full proof.

In fact, in the case of the *full space* $\mathbb{R}^n$, the Nash inequality is included (except for dimension $n = 2$) in the GNS inequalities [8, Part I, Theorem 9.3].

The significance of the Nash inequality is demonstrated by the variety of subsequent proofs; a proof based on the Fourier transform can be found in [7] and the best constant C_n was determined in [5]. A geometric proof, based on the logarithmic Sobolev inequality, was given in [1].

We recall the basic role played by this inequality in the study of the $2-D$ Navier-Stokes equations with singular initial vorticity [2] or the study of stability of travelling waves in conservation laws [12]. In Section 3 the inequality is used in the derivation of an estimate for the sup-norm of a solution to the linearized Stokes system. This estimate is independent of the velocity field.

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be any domain and let $\phi \in C_0^1(\Omega)$, the space of continuously differentiable, with compact support in Ω. Clearly, extending ϕ as zero outside Ω, the inequality (1.1) remains valid. However, consideration of more general situations needs a more detailed investigation. In a recent paper [4] Brezis and Mironescu

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2010 *Mathematics Subject Classification.* Primary 46E35; Secondary 35R45.

Key words and phrases. Nash inequality, weak Dirichlet condition, general domains, linearized Stokes system, blow-up.

I am grateful to the anonymous referee for very useful comments and suggestions.

study the GNS inequalities in a "standard domain" Ω, which is either the full or half-space $\mathbb{R}^n$, or a Lipschitz bounded domain. More specifically, they establish in these domains inequalities of the type

$$\|f\|_{W^{r,q}(\Omega)} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)} \|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}, \tag{1.2}$$

for suitable values of the various parameters.

Suppose that we want a "Nash inequality" of the type

$$\Big(\int_\Omega |\phi|^2 dx\Big)^{\frac{n+2}{n}} \le A_n \int_\Omega |\nabla\phi|^2 dx \cdot \Big(\int_\Omega |\phi| dx\Big)^{\frac{4}{n}}, \tag{1.3}$$

where $\Omega \subsetneqq \mathbb{R}^n$, and A_n may depend on Ω. Of course it fails if no additional conditions are imposed (for example, a constant function in a bounded domain). In [4] this inequality is derived from (1.2) when Ω is a bounded Lipschitz domain and $\phi = 0$ on $\partial\Omega$ (in a trace sense).

The purpose of this paper is to establish (1.3) in domains that are not necessarily bounded. We label the imposed boundary condition as "weak Dirichlet" (Definition 2.1). In particular, we shall not require that the functions vanish everywhere on $\partial\Omega$.

We derive two different estimates for $n \ge 3$ (see Theorems 2.2, 2.3). Only the second is valid in the case $n = 2$. The one-dimensional case is stated in Theorem 2.4.

2. Extended from of the Nash inequality

In the following definition, we let e_j be the unit vector in the direction of the x_j axis, $1 \le j \le n$, and denote by $l_j(y)$ the line $y + te_j$, $t \in \mathbb{R}$.

Definition 2.1. Let ϕ be an integrable , Lipschitz function defined on $\overline{\Omega}$. We say that ϕ satisfies the weak Dirichlet condition if for every point $y \in \Omega$ and any $j \in \{1, 2, ..., n\}$, there exists a point $z \in l_j(y) \cap \overline{\Omega}$, such that

(i) $\phi(z) = 0$.

(ii) The open segment $\{ty + (1-t)z,\ t \in (0,1)\}$ is contained in $\overline{\Omega}$.

This definition is modified in an obvious way if z is the "point at infinity", namely, if the full half line $\{y + te_j,\ t \in (-\infty, 0)\} \subseteq \Omega$, then $\lim_{t\to-\infty} \phi(y + te_j) = 0$.

Here are a few examples, where we take the plane ($n = 2$) for simplicity.

- Let Ω be the unit disk , then ϕ satisfies the weak Dirichlet condition if it vanishes on two orthogonal diameters.
- Let Ω be the upper half of the disk, then ϕ satisfies the weak Dirichlet condition if it vanishes on the curved part ($|x| = 1$) of the boundary.
- Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with a smooth boundary and let $\psi \in C^3(\overline{\Omega})$. Assume that $\nabla\psi$ vanishes on $\partial\Omega$ and let $\phi = \frac{\partial^2}{\partial x \partial y}\psi$. Then ϕ satisfies the weak Dirichlet condition. Indeed, $\phi(x, y)$ has zero mean value on every horizontal or vertical line, so it must vanish at a point on such a line.

Notation. We designate by $\|\phi\|_p$ the $L^p(\Omega)$ norm, $1 \le p \le \infty$. The set Ω will be understood from the context.

Theorem 2.2. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, be a domain and let $\phi \in C^1(\overline{\Omega})$ satisfy the weak Dirichlet condition. Then*

$$\Big(\int_\Omega |\phi|^2 dx\Big)^{\frac{n+2}{n}} \leq A_n \int_\Omega |\nabla\phi|^2 dx \cdot \Big(\int_\Omega |\phi| dx\Big)^{\frac{4}{n}}, \tag{2.1}$$

where $A_n = \Big(\frac{2(n-1)}{n-2}\Big)^2$, and, in particular, it does not depend on Ω.

Proof. Since ϕ can be replaced by $|\phi|$, there is no loss of generality in assuming $\phi \geq 0$.

The proof is actually a slight modification of the basic Gagliardo-Nirenberg-Sobolev inequality [6, Section 5.6, Theorem 1]. We bring it here for the convenience of the reader, and also because the inequality (2.3) below will be needed in the next theorem.

it is easily seen that if $\psi \in C^1(\overline{\Omega})$ satisfies the weak Dirichlet condition, then

$$\int_\Omega |\psi(x)|^{\frac{n}{n-1}} dx \leq \Big(\int_\Omega |\nabla\psi(x)| dx\Big)^{\frac{n}{n-1}}. \tag{2.2}$$

Indeed, this follows by integrating the derivative $\partial_{x_j}\psi$ on the segment of $l_j(x)$ connecting a point $x \in \Omega$ to a zero point of ψ.

Let ϕ be as in the statement and let $\psi = \phi^{q\frac{n-1}{n}}$, with $q > 1$ to be selected. Applying (2.2) , we have

$$\int_\Omega \phi(x)^q dx \leq \Big(\frac{q(n-1)}{n}\Big)^{\frac{n}{n-1}} \Big(\int_\Omega \phi^{\frac{q(n-1)-n}{n}} |\nabla\phi| dx\Big)^{\frac{n}{n-1}}. \tag{2.3}$$

Thus, by the Cauchy-Schwarz inequality,

$$\int_\Omega \phi(x)^q dx \leq \Big(\frac{q(n-1)}{n}\Big)^{\frac{n}{n-1}} \Big(\int_\Omega \phi(x)^{2\frac{q(n-1)-n}{n}} dx\Big)^{\frac{n}{2(n-1)}} \cdot \Big(\int_\Omega |\nabla\phi|^2 dx\Big)^{\frac{n}{2(n-1)}}. \tag{2.4}$$

Taking $q = \frac{2n}{n-2}$ we get,

$$\Big(\int_\Omega \phi(x)^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}} \leq \Big(\frac{2(n-1)}{n-2}\Big)^2 \int_\Omega |\nabla\phi|^2 dx. \tag{2.5}$$

Using the interpolation inequality

$$\|g\|_2 \leq \|g\|_{\frac{2n}{n-2}}^{\frac{n}{n+2}} \|g\|_1^{\frac{2}{n+2}},$$

the last estimate yields

$$\|\phi\|_2 \leq \Big(\frac{2(n-1)}{n-2}\Big)^{\frac{n}{n+2}} \|\phi\|_1^{\frac{2}{n+2}} \cdot \|\nabla\phi\|_2^{\frac{n}{n+2}}, \tag{2.6}$$

which is exactly (2.1). □

The case $n = 2$ is included in the following theorem.

Theorem 2.3. *Let Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a domain and let $\phi \in C^1(\overline{\Omega})$ satisfy the weak Dirichlet condition in Ω. Then*

$$\|\phi\|_2^2 \leq \frac{2n-1}{n} \|\phi\|_1 \|\nabla\phi\|_n. \tag{2.7}$$

Proof. As in the previous proof, we may assume $\phi \geq 0$. Applying Hölder's inequality to (2.3) yields

$$\int_\Omega \phi(x)^q dx \leq \Big(\frac{q(n-1)}{n}\Big)^{\frac{n}{n-1}} \Big(\int_\Omega \phi(x)^{\frac{q(n-1)-n}{n-1}} dx\Big) \cdot \Big(\int_\Omega |\nabla\phi|^n dx\Big)^{\frac{1}{n-1}}. \tag{2.8}$$

Taking $q = \frac{2n-1}{n-1}$ we get

$$\|\phi\|_{\frac{2n-1}{n-1}} \leq \Big(\frac{2n-1}{n}\Big)^{\frac{n}{2n-1}} \|\phi\|_1^{\frac{n-1}{2n-1}} \|\nabla\phi\|_n^{\frac{n}{2n-1}}. \tag{2.9}$$

Invoking the interpolation inequality

$$\|g\|_2 \leq \|g\|_{\frac{2n-1}{n-1}}^{\frac{2n-1}{2n}} \|g\|_1^{\frac{1}{2n}},$$

we infer from (2.9) that

$$\|\phi\|_2 \leq \Big(\frac{2n-1}{n}\Big)^{\frac{1}{2}} \|\phi\|_1^{\frac{n-1}{2n}} \|\nabla\phi\|_n^{\frac{1}{2}} \|\phi\|_1^{\frac{1}{2n}}, \tag{2.10}$$

which is precisely (2.7).

□

Finally, the one-dimensional case is given in the following theorem.

Theorem 2.4. *[The 1-D Nash Inequality] Let $\Omega \subseteq \mathbb{R}$, be an open set and let $\phi \in C^1(\overline{\Omega})$.*

Suppose that $\phi(x_0) = 0$ for some $x_0 \in \overline{\Omega}$.

Then,

$$\|\phi\|_2^3 \leq 2\|\phi\|_1^2 \|\phi'\|_2. \tag{2.11}$$

Proof. We have

$$\|\phi\|_\infty^2 \leq \|(\phi^2)'\|_1 \leq 2\|\phi\|_2\|\phi'\|_2,$$

hence

$$\|\phi\|_\infty^4 \leq 4\|\phi\|_2^2\|\phi'\|_2^2.$$

Since

$$\|\phi\|_2^2 = \|\phi^2\|_1 \leq \|\phi\|_1\|\phi\|_\infty,$$

we have

$$\|\phi\|_\infty^3 \leq 4\|\phi\|_1\|\phi'\|_2^2,$$

so that

$$\|\phi\|_2^6 \leq \|\phi\|_1^3\|\phi\|_\infty^3 \leq 4\|\phi\|_1^4\|\phi'\|_2^2,$$

as asserted. □

Remark 2.5. Obviously, the "best constant" in the above estimates cannot be smaller than that obtained in [5] for the case of the full space. However, for more general domains $\Omega \subseteq \mathbb{R}^n$, it is not clear if the best constant actually depends on Ω.

3. Application to the linearized stokes system

Consider the equation

$$\begin{aligned} -\Delta\phi + (\mathbf{u}\cdot\nabla)\phi &= f \\ \nabla\cdot\mathbf{u} &= 0, \end{aligned} \tag{3.1}$$

in a bounded smooth domain $\Omega \subseteq \mathbb{R}^n,\ n \geq 2$, subject to the boundary condition

$$\phi = 0, \quad \text{on } \partial\Omega. \tag{3.2}$$

It is assumed that $\mathbf{u}$ is a given smooth vectorfield. This is the well-known linearized Stokes problem. In addition, it plays a central role in the study of the explosion problem in a flow [3].

Assume first that

$$f \in L^p(\Omega), \quad p > \frac{n}{2}. \tag{3.3}$$

By standard elliptic estimates $\phi \in W^{2,p}(\Omega) \subseteq L^\infty(\Omega)$, so $\|\phi\|_\infty$ can be estimated in terms of $\|f\|_p$. From the general theory such an estimate depends on the velocity field $\mathbf{u}$. However, it was shown in [3, Lemma 1.3] that in the case at hand (namely, Equation (3.1)) one has , if $p > \frac{n}{2}$

$$\|\phi\|_\infty \leq C\|f\|_p, \tag{3.4}$$

where $C > 0$ is independent of $\mathbf{u}$.

Next consider the case $1 < p \leq \frac{n}{2}$. In what follows we focus on estimates that are independent of $\mathbf{u}$.

Multiplying Equation (3.1) by ϕ and integrating by parts we have

$$\int_\Omega |\nabla\phi|^2 dx \leq \|f\|_p\|\phi\|_{p'}, \quad p' = \frac{p}{p-1}. \tag{3.5}$$

On the other hand, the Sobolev embedding theorem implies that

$$\|\phi\|_q \leq C\|\nabla\phi\|_2, \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{n} \tag{3.6}$$

(take any $q < \infty$ if $n = 2$).

Here and below $C > 0$ denotes various constants depending on Ω, but not on $\mathbf{u}$.

Thus, in conjunction with (3.5) we get

$$\|\phi\|_q \leq C\|f\|_p^{\frac{1}{2}}\|\phi\|_{p'}^{\frac{1}{2}}, \quad p' = \frac{p}{p-1}, \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{n}. \tag{3.7}$$

If $\frac{2n}{n+2} \leq p \leq \frac{n}{2}$ then $1 < p' \leq q$ hence

$$\|\phi\|_{p'} \leq C\|\phi\|_q.$$

The estimate (3.7) now entails

$$\|\phi\|_q \leq C\|f\|_p, \quad \frac{2n}{n+2} \leq p \leq \frac{n}{2}. \tag{3.8}$$

In contrast to (3.4), the estimate (3.8) contains little information about the "size" of $|\phi|$. As an attempt to get a better idea of this size, note that $\frac{\|\phi\|_2^2}{\|\phi\|_1} \le \|\phi\|_\infty$. Thus $\frac{\|\phi\|_2^2}{\|\phi\|_1}$ can serve as such a measure.

In the following theorem we establish a result in this direction. It implies that the ratio $\frac{\|\phi\|_2^2}{\|\phi\|_1}$ can be large (for some vectorfield $\mathbf{u}$) only if $\|\phi\|_2$ gets small. In other words, the ratio can be large only if $|\phi|$ is everywhere small except for "narrow sharp spikes".

Theorem 3.1. *Let $n \ge 3$, $\frac{2n}{n+2} \le p \le \frac{n}{2}$, and assume that ϕ is a bounded Lipschitz continuous solution to* (3.1)-(3.2) *with $f \in L^p(\Omega)$.*

Then there exists a constant $C = C(\Omega, p)$ (in particular, independent of $\mathbf{u}$), such that

$$\|\phi\|_2^{\frac{n-2}{2}} \frac{\|\phi\|_2^2}{\|\phi\|_1} \le C\|f\|_p^{\frac{n}{2}}. \tag{3.9}$$

Proof. We use the form of the Nash inequality as given in Theorem 2.2.

$$\Big(\int_\Omega |\phi|^2 dx\Big)^{\frac{n+2}{n}} \le A_n \int_\Omega |\nabla\phi|^2 dx \cdot \Big(\int_\Omega |\phi| dx\Big)^{\frac{4}{n}}, \tag{3.10}$$

where $A_n = \Big(\frac{2(n-1)}{n-2}\Big)^2$.

Incorporating (3.5) we get

$$\Big(\int_\Omega |\phi|^2 dx\Big)^{\frac{n+2}{n}} \le A_n \|f\|_p \|\phi\|_{p'} \cdot \Big(\int_\Omega |\phi| dx\Big)^{\frac{4}{n}}. \tag{3.11}$$

Noting that $p' \le q$ we use repeatedly the estimates (see (3.6))

$$\|\phi\|_{p'} \le C_1 \|\phi\|_q, \quad \|\phi\|_q \le C_2 \|\nabla\phi\|_2,$$

and then again (3.5). Thus

$$\begin{aligned}
\Big(\int_\Omega |\phi|^2 dx\Big)^{\frac{n+2}{n}} &\le A_n C_1 \|f\|_p \|\phi\|_q \cdot \Big(\int_\Omega |\phi| dx\Big)^{\frac{4}{n}} \\
&\le A_n C_1 C_2 \|f\|_p \|\nabla\phi\|_2 \cdot \Big(\int_\Omega |\phi| dx\Big)^{\frac{4}{n}} \\
&\le A_n C_1 C_2 \|f\|_p \Big(\|f\|_p \|\phi\|_{p'}\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega |\phi| dx\Big)^{\frac{4}{n}} \\
&\le A_n C_1^{\frac{3}{2}} C_2 \|f\|_p \Big(\|f\|_p \|\phi\|_q\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega |\phi| dx\Big)^{\frac{4}{n}} \\
&\le A_n (C_1 C_2)^{\frac{3}{2}} \|f\|_p \Big(\|f\|_p \|\nabla\phi\|_2\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega |\phi| dx\Big)^{\frac{4}{n}}.
\end{aligned} \tag{3.12}$$

By induction, it follows that for $k = 3, 4, \dots$

$$\Big(\int_\Omega |\phi|^2 dx\Big)^{\frac{n+2}{n}} \le A_n (C_1 C_2)^{2-2^{-k}} \|f\|_p^{2-2^{-k}} \|\nabla\phi\|_2^{2^{-k}} \Big(\int_\Omega |\phi| dx\Big)^{\frac{4}{n}}. \tag{3.13}$$

Taking the limit as $k \to \infty$ we get

$$\Big(\int_\Omega |\phi|^2 dx\Big)^{\frac{n+2}{n}} \le A_n (C_1 C_2)^2 \|f\|_p^2 \Big(\int_\Omega |\phi| dx\Big)^{\frac{4}{n}}. \tag{3.14}$$

This estimate can be rewritten as

$$\|\phi\|_2^{\frac{2(n-2)}{n}} \Big(\frac{\|\phi\|_2^2}{\|\phi\|_1}\Big)^{\frac{4}{n}} \le A_n(C_1C_2)^2\|f\|_p^2, \tag{3.15}$$

which is (3.9). □

In the following theorem we obtain an estimate for $\|\phi\|_\infty$, valid for all $p > 1$. It is independent of **u** , but depends on the size of the subset where ϕ is "sufficiently large".

Theorem 3.2. *Let $n \ge 2,\ p > 1$, and assume that ϕ is a bounded Lipschitz continuous solution to* (3.1)-(3.2) *with $f \in L^p(\Omega)$.*
Let

$$\Omega_1 = \Big\{x \in \Omega,\ |\phi(x)| \ge \frac{1}{2}\|\phi\|_\infty\Big\},$$

and assume that $|\Omega_1| \ge \delta|\Omega|$ ($|B|$ is the Lebesgue measure of B) for some $\delta \in (0, 1)$.
Then there exists a constant $C = C(\Omega, p, \delta)$ (in particular, independent of **u***), such that the solution ϕ satisfies*

$$\|\phi\|_\infty \le C\|f\|_p. \tag{3.16}$$

Proof. We use the form of the Nash inequality as given in Theorems 2.2, 2.3 , valid for any Lipschitz continuous ϕ that vanishes on $\partial\Omega$.

$$\Big(\int_\Omega |\phi|^2 dx\Big)^{\frac{n+2}{n}} \le A_n \int_\Omega |\nabla\phi|^2 dx \cdot \Big(\int_\Omega |\phi| dx\Big)^{\frac{4}{n}}, \tag{3.17}$$

where $A_n = \Big(\frac{2(n-1)}{n-2}\Big)^2,\ n \ge 3$ and $A_2 = \frac{3}{2}$.

Incorporating the estimate (3.5) in (3.17) we get as in (3.11)

$$\Big(\int_\Omega |\phi|^2 dx\Big)^{\frac{n+2}{n}} \le A_n\|\phi\|_1^{\frac{4}{n}}\|f\|_p\|\phi\|_{p'}.$$

By assumption $|\Omega_1| \ge \delta|\Omega|$ so using the trivial estimate $\|\phi\|_r \le \|\phi\|_\infty|\Omega|^{\frac{1}{r}}$, the last estimate yields

$$\Big(\frac{\delta}{4}\|\phi\|_\infty^2|\Omega|\Big)^{\frac{n+2}{n}} \le A_n|\Omega|^{\frac{4}{n}+\frac{p-1}{p}}\|f\|_p\|\phi\|_\infty^{1+\frac{4}{n}},$$

hence

$$\|\phi\|_\infty \le \frac{1}{\delta}4^{\frac{n+2}{n}}A_n|\Omega|^{\frac{2}{n}-\frac{1}{p}}\|f\|_p. \tag{3.18}$$

□

As a corollary, we show that the "blow-up" set of a sequence of solutions is necessarily small.

Corollary 3.3. *Fix $p > 1$ and let $\{\mathbf{u}_k\}_{k=1}^\infty$ be a sequence of divergence-free vector-fields. Let $\{f_k\}_{k=1}^\infty \subseteq L^p(\Omega)$ be a uniformly bounded sequence:* $\sup\limits_{k=1,2,\ldots} \|f_k\|_p < \infty$.

Let $\{\phi_k\}_{k=1}^{\infty}$ *be the corresponding sequence of bounded Lipschitz continuous solutions to* (3.1)-(3.2). *Suppose that*

$$\|\phi_k\|_\infty \uparrow \infty.$$

Let

$$\Omega_{1,k} = \left\{ x \in \Omega,\ |\phi_k(x)| \geq \frac{1}{2}\|\phi_k\|_\infty \right\}, \quad k = 1, 2, \dots$$

Then

$$\lim_{k\to\infty} |\Omega_{1,k}| = 0. \tag{3.19}$$

Proof. Suppose to the contrary that (3.19) does not hold. Then there exists a $\delta > 0$ and a subsequence (without changing notation) such that

$$|\Omega_{1,k}| \geq \delta|\Omega|.$$

This is a contradiction in view of (3.18) and the assumption that $\|\phi_k\|_\infty \uparrow \infty$. □

Remark 3.4. Observe that we could modify the definition $\Omega_{1,k} = \{x \in \Omega,\ |\phi_k(x)| \geq \epsilon\|\phi_k\|_\infty\}$ for any $\epsilon > 0$ without changing the conclusion (3.19).

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Manuscript received April 12 2019
revised June 13 2019

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