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SOME NEW ASPECTS OF PERTURBATION THEORY OF POSITIVE SOLUTIONS OF SECOND-ORDER LINEAR ELLIPTIC EQUATIONS

DEBDIP GANGULY AND YEHUDA PINCHOVER

Dedicated to the memory of Aizik I. Volpert

Abstract. We present some new results concerning perturbation theory for positive solutions of second-order linear elliptic operators, including further study of the equivalence of positive minimal Green functions and the validity of a Liouville comparison principle for nonsymmetric operators.

1. Introduction

Let M be a smooth, connected, and noncompact Riemannian manifold of dimension $N \geq 2$. We consider a second-order elliptic operator P with real coefficients in the divergence form

$$Pu := -\mathrm{div}\Big[A(x)\nabla u + u\tilde{b}(x)\Big] + b(x)\cdot\nabla u + c(x)u \qquad x \in M. \tag{1.1}$$

More precisely, let $m > 0$ be a strictly positive measurable function in M such that m and m^{-1} are bounded on any compact subset of M, and denote $\mathrm{d}m := m(x)\mathrm{d}x$, where $\mathrm{d}x$ is the Riemannian volume form of M (which is just the Lebesgue measure in the case of Schrödinger operators on domains of $\mathbb{R}^N$).

We denote by T_xM and TM the tangent space to M at $x \in M$ and the tangent bundle, respectively. Let $\mathrm{End}(T_xM)$ and $\mathrm{End}(TM)$ be the set of endomorphisms in T_xM and the corresponding bundle, respectively. The gradient with respect to the Riemannian metric is denoted by ∇, and $-\mathrm{div}$ is the formal adjoint of the gradient with respect to the measure $\mathrm{d}m$. The inner product and the induced norm on TM are denoted by $\langle X, Y\rangle$ and $|X|$, respectively, where $X, Y \in TM$.

We assume that A is a symmetric measurable section on M of $\mathrm{End}(TM)$ such that for any compact set K in M there exists a positive constant $\lambda_K \geq 1$ satisfying

$$\lambda_K^{-1}|\xi|^2 \leq |\xi|^2_{A(x)} := \langle A(x)\xi, \xi\rangle \leq \lambda_K|\xi|^2 \quad \forall x \in K \text{ and } (x,\xi) \in TM. \tag{1.2}$$

We also assume that the coefficients b and $\tilde{b}$ are measurable vector fields in M of class $L^p_{\mathrm{loc}}(M)$ and c is a measurable function in M of class $L^{p/2}_{\mathrm{loc}}(M)$ for some $p > N$.

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We denote by $P^\star$ the formal adjoint operator of P on its natural space $L^2(M, \mathrm{d}m)$. When P is in divergence form (1.1) and $b = \tilde{b}$, then the operator

$$Pu = -\mathrm{div}\left[\big(A\nabla u + ub\big)\right] + b\cdot\nabla u + cu, \tag{1.3}$$

is *symmetric* in the space $L^2(M, \mathrm{d}m)$. Throughout the paper, we call this setting the *symmetric case*. We note that if P is symmetric and b is smooth enough, then P is in fact a Schrödinger-type operator of the form

$$Pu = -\mathrm{div}\big(A\nabla u\big) + \tilde{c}u, \tag{1.4}$$

where $\tilde{c} = c - \mathrm{div}\, b$.

By a *solution* v of the equation $Pu = 0$, we mean $v \in W^{1,2}_{\mathrm{loc}}(M)$ that satisfies the equation in the *weak sense*. Subsolutions and supersolutions are defined similarly.

Denote the cone of all positive solutions of the equation $Pu = 0$ in M by $\mathcal{C}_P(M)$. Let V be a real valued potential. The *generalized principal eigenvalue* of the operator P and a potential $V \in L^q_{\mathrm{loc}}(M)$, $q > N/2$, is defined by

$$\lambda_0(P, V, M) := \sup\{\lambda \in \mathbb{R} \mid \mathcal{C}_{P-\lambda V}(M) \neq \emptyset\}.$$

We say that P is *nonnegative in* M (and we denote it by $P \geq 0$ in M) if $\lambda_0 := \lambda_0(P, \mathbf{1}, M) \geq 0$, where $\mathbf{1}$ is the constant function on M taking at any point $x \in M$ the value 1. Throughout the paper we always assume that $\lambda_0 \geq 0$, that is, $P \geq 0$ in M.

The main purpose of the paper is to present some new results concerning perturbation theory of the cone $\mathcal{C}_P(M)$. Perturbation theory of positive solutions was studied extensively in the past few decades. S. Agmon in [1, 2] studied positivity and decay properties of solutions of second-order elliptic equations using the notion of *Agmon ground state*. His results turned out to be highly influential in the study of the structure of $\mathcal{C}_P(M)$ and its behaviour under certain types of perturbations (the so-called *criticality theory*). Without any claim of completeness, we refer to some relevant papers studying criticality theory [3, 4, 14, 17, 18, 19, 22, 23, 24, 26, 30] and references therein.

The perturbation that we consider here is of the form $P_\lambda := P - \lambda V$, where $P \geq 0$ in M, $\lambda \in \mathbb{R}$ and $V \in L^q_{\mathrm{loc}}(M)$, $q > N/2$. We study, in particular, the maximal interval such that the Green function of P_λ is equivalent to the Green function of P, certain classes of 'big' and 'small' perturbations, compactness properties of weighted Green operators for certain classes of 'small' weights, and a new Liouville comparison principle for nonsymmetric operators. See Section 3 for more details.

The outline of our paper is as follows. In Section 2 we recall some definitions and basic known results concerning criticality theory, and in Section 3 we discuss the problems that we study in the present paper. Section 4 is devoted to our results concerning the equivalence of positive minimal Green functions of second-order elliptic operators under nonnegative perturbation. In Section 5 we prove that *optimal* Hardy-weights are *h-big* perturbations in the sense of [14], while in Section 6 we present a large family of 'small' Hardy-weights W_μ, given by a simple explicit formula, such that $P - W_\mu$ is positive-critical. In Section 7 we prove that for symmetric operators, the assumption of finite torsional rigidity implies that the spectrum of P

on $L^2(M, \mathrm{d}m)$ is discrete. Section 8 is devoted to a Liouville comparison principle for *nonsymmetric*, nonnegative, elliptic operators. We conclude our paper in Section 9 where we apply perturbation theory to study the asymptotic of the positive minimal Green function of the shifted Laplace-Beltrami operator on the hyperbolic space $\mathbb{H}^N$.

2. Preliminaries

In the present section we fix our setting and notation, and recall some basic definitions and results concerning criticality theory.

Let M be a smooth, connected, and noncompact Riemannian manifold of dimension $N \geq 2$, and P an elliptic operator of the form (1.1). Throughout the paper we use the following notation.

- We denote by ∞ the ideal point which is added to M to obtain the one-point compactification of M.
- We write $X_1 \Subset X_2$ if the set X_2 is open in M, the set $\overline{X_1}$ is compact and $\overline{X_1} \subset X_2$.
- Let g_1, g_2 be two positive functions defined in a domain D. We say that g_1 is *equivalent* to g_2 in D (and use the notation $g_1 \asymp g_2$ in D) if there exists a positive constant C such that
$$C^{-1} g_2(x) \leq g_1(x) \leq C g_2(x) \qquad \text{for all } x \in D.$$
- We fix a *compact exhaustion* of M, i.e., a sequence of smooth relatively compact domains in M such that $M_1 \neq \emptyset$, $M_j \Subset M_{j+1}$ and $\cup_{j=1}^{\infty} M_j = M$. We denote $M_j^* := M \setminus \overline{M_j}$.
- We denote the restriction of a function $f : M \to \mathbb{R}$ to $A \subset M$ by $f\restriction_A$.

We first recall the definitions of critical and subcritical operators and of a ground state (for more details on criticality theory, see [18, 19, 22, 23, 24] and references therein).

Definition 2.1. Let $K \Subset M$. We say that $u \in \mathcal{C}_P(M \setminus K)$ is a *positive solution of the operator P of minimal growth in a neighborhood of infinity in M*, if for any compact set $K \Subset K_1 \Subset M$ with a smooth boundary and any positive supersolution v of the equation $Pw = 0$ in $M \setminus K_1$, $v \in C((M \setminus K_1) \cup \partial K_1)$, the inequality $u \leq v$ on ∂K_1 implies that $u \leq v$ in $M \setminus K_1$.

A positive solution $u \in \mathcal{C}_P(M)$ which has minimal growth in a neighborhood of infinity in M is called the *(Agmon) ground state* of P in M (see [2]).

Definition 2.2. The operator P is said to be *critical* in M if P admits a ground state in M. The operator P is called *subcritical* in M if $P \geq 0$ in M but P is not critical in M. If $P \not\geq 0$ in M, then P is said to be *supercritical* in M.

If $W \in L^q_{\mathrm{loc}}(M; \mathbb{R}_+)$ with $q > N/2$ is a nonzero nonnegative potential, then $P - \lambda W$ is subcritical for every $\lambda \in (-\infty, \lambda_0(P, W, M))$, and supercritical for $\lambda > \lambda_0(P, W, M)$. Furthermore, if P is critical in M, then $\lambda_0(P, W, M) = 0$.

Remark 2.3. Let $P \geq 0$ in M. It is well known that the operator P is critical in M if and only if the equation $Pu = 0$ in M has a unique (up to a multiplicative constant) positive supersolution (see [19, 22]). In particular, if P is critical in M, then $\dim \mathcal{C}_P(M) = 1$. Further, in the critical case, the unique positive supersolution (up to a multiplicative positive constant) is a ground state of P in M.

On the other hand, P is subcritical in M if and only if P admits a (unique) positive minimal Green function $G_P^M(x,y)$ in M. Moreover, for any fixed $y \in M$, the function $G_P^M(\cdot,y)$ is a positive solution of minimal growth in a neighborhood of infinity in M. Since, $G_{P^\star}^M(x,y) = G_P^M(y,x)$, it follows that P is critical (respectively, subcritical) in M if and only if $P^\star$ is critical (respectively, subcritical) in M.

Remark 2.4. In the critical case there exists a (sign-changing) *Green function* which is bounded above by the corresponding ground state away from the singularity, see [11].

Definition 2.5. 1. We say that $W \gneqq 0$ is a *Hardy-weight* of P in M if $P - W \geq 0$ in M.

2. Assume that $W \gneqq 0$ is a Hardy-weight of P in M, and that $P - W$ is critical in M. Let ϕ and $\phi^\star$ be the ground states of $P - W$ and $P^\star - W$, respectively. The operator $P - W$ is said to be *null-critical* (respectively, *positive-critical*) in M with respect to W if $\phi\phi^\star \notin L^1(M, W\mathrm{d}x)$ (respectively, $\phi\phi^\star \in L^1(M, W\mathrm{d}x)$).

Fix a potential $V \in L^q_{\mathrm{loc}}(M;\mathbb{R})$, where $q > N/2$. Set $S := S_+ \cup S_0$, where

$$S_+ := S_+(P,V,M) = \{t \in \mathbb{R} : P - tV \text{ is subcritical in } M\},$$

$$S_0 := S_0(P,V,M) = \{t \in \mathbb{R} : P - tV \text{ is critical in } M\}.$$

Then S is a closed interval and $S_0 \subset \partial S$ [24]. Moreover, if V has compact support in M, then $S_0 = \partial S$. In particular, subcriticality is stable under compact perturbation, i.e., if P is subcritical and V is a nonzero potential with compact support in M, then there exists $\varepsilon > 0$ such that $P - \varepsilon V$ is subcritical for $|\varepsilon| < \varepsilon_0$ (see [23, 24]).

The above stability property of subcritical operators and other positivity properties are preserved under a larger (and in fact maximal) class of potentials V called *small perturbations* [23]. We recall below the definition of small perturbation and other types of perturbations by a potential V and discuss briefly some of their properties.

Definition 2.6 ([19, 23])**.** Let P be a subcritical operator in M and let $V \in L^q_{\mathrm{loc}}(M)$ for some $q > N/2$ be a real valued potential. We say that V is a *small (semismall) perturbation* of P in M if

$$\lim_{n\to\infty} \left\{ \sup_{x,y\in M_n^*} \int_{M_n^*} \frac{G_P^M(x,z)|V(z)|G_P^M(z,y)\,\mathrm{d}m(z)}{G_P^M(x,y)} \right\} = 0,$$

$$\left(\lim_{n\to\infty} \left\{ \sup_{y\in M_n^*} \int_{M_n^*} \frac{G_P^M(x_0,z)|V(z)|G_P^M(z,y)\mathrm{d}m(z)}{G_P^M(x_0,y)} \right\} = 0, \text{ where } x_0 \in M \text{ is fixed} \right).$$

Definition 2.7. We say that V is a *G-(semi)bounded perturbation* of P in M if there exists a positive constant C_0 such that

$$C_0 := \sup_{x,y\in M} \int_M \frac{G_P^M(x,z)|V(z)|G_P^M(z,y)\,\mathrm{d}m(z)}{G_P^M(x,y)} < \infty, \tag{2.1}$$

$$\left(\sup_{y\in M} \int_M \frac{G_P^M(x_0,z)|V(z)|G_P^M(z,y)\mathrm{d}m(z)}{G_P^M(x_0,y)} < \infty, \text{ where } x_0 \in M \text{ is fixed}\right).$$

Remark 2.8. A small perturbation is semismall and G-bounded [19]. On the other hand, if V is G-bounded perturbation of P in M, and f is an arbitrary bounded function vanishing at infinity in Ω (i.e. with respect of the one-point compactification of M), then clearly, fV is a small perturbation of P in M.

Definition 2.9. Let P_i, $i = 1, 2$ be two subcritical operators in M. We say that the Green functions $G_{P_1}^M(x,y)$ and $G_{P_2}^M(x,y)$ are *equivalent* (respectively, *semiequivalent*) if $G_{P_1}^M \asymp G_{P_2}^M$ on $M \times M \setminus \{(x,x) : x \in M\}$ (respectively, if for a fixed $y \in M$, we have $G_{P_1}^M(\cdot,y) \asymp G_{P_2}^M(\cdot,y)$ on $M \setminus \{y\}$).

In the sequel we use the notation

$$E_+ = E_+(P,V,M) := \{t \in \mathbb{R} \,|\, G_{P-tV}^M \asymp G_P^M \quad \text{on } M \times M \setminus \{(x,x) : x \in M\},$$

$$SE_+ = SE_+(P,V,M) := \{t \in \mathbb{R} \,|\, G_{P-tV}^M \text{ is semiequivalent to } G_P^M\}.$$

Remark 2.10. Clearly, $E_+ \subseteq S_+$. It is known that if the operator P is subcritical and V is a small perturbation of P in M, then $E_+ = S_+$, $\partial S = S_0$, and the corresponding ground states are equivalent to $G_P^M(x,x_0)$ in $M \setminus B(x_0,\varepsilon)$ for sufficiently small $\varepsilon > 0$.

On the other hand, If V is a G-bounded perturbation of P in M, then $G_P^M \asymp G_{P-tV}^M$ on $M \times M \setminus \{(x,x) : x \in M\}$ provided $|t|$ is small enough [19, 22, 23]. Furthermore, if $G_P^M(x,y)$ and $G_{P-V}^M(x,y)$ are equivalent and V has a definite sign, then V is a G-bounded perturbation of P in M. Moreover, in this case, E_+ is an open half-line which is contained in $S_+ \setminus \{\lambda_0\}$ [24, Corollary 3.6].

Finally, we discuss sufficient conditions for the compactness of the following weighted Green operators with weight $W \geq 0$ in certain weighted L^p spaces, where $1 \leq p \leq \infty$. Let

$$\mathcal{G}f(x) := \int_M G_P^M(x,y)W(y)f(y)\mathrm{d}m(y), \tag{2.2a}$$

$$\mathcal{G}^{\odot}f(y) := \int_M G_P^M(x,y)W(x)f(x)\mathrm{d}m(x). \tag{2.2b}$$

Let ϕ and $\tilde{\phi}$ be a pair of two positive continuous functions on M, and set

$$L^p(\phi_p) := L^p(M, (\phi_p)^p\mathrm{d}m), \quad L^p(\tilde{\phi}_p) := L^p(M, (\tilde{\phi}_p)^p\mathrm{d}m),$$

where

$$\phi_p := \phi^{-1}(\phi W\tilde{\phi})^{1/p}, \qquad \tilde{\phi}_p := \tilde{\phi}^{-1}(\phi W\tilde{\phi})^{1/p}. \tag{2.3}$$

We have

Theorem 2.11 ([28])**.** *Let P be a subcritical operator in M. Assume that $W>0$ is a semismall perturbation of $P^\star$ and P in M, and let $\lambda_0 := \lambda_0(P,W,M)$. Then*

(1) *The operator $P-\lambda_0 W$ is positive-critical with respect to W, that is,*

$$\int_M \tilde{\phi}(x)W(x)\phi(x)\,\mathrm{d}m(x) < \infty, \tag{2.4}$$

where ϕ and $\tilde{\phi}$ denote the ground states of $P-\lambda_0 W$ and $P^\star-\lambda_0 W$, respectively. Moreover, $\lambda_0 = \|\mathcal{G}\|^{-1}_{L^p(\phi_p)} > 0$ for any $1\le p\le\infty$.

(2) *for any $1\le p\le\infty$, the integral operators $\mathcal{G}$ and $\mathcal{G}^\odot$ defined in* (2.2) *are compact on $L^p(\phi_p)$ and $L^p(\tilde{\phi}_p)$, respectively.*

(3) *For $1\le p\le\infty$, the spectrum of $\mathcal{G}\restriction_{L^p(\phi_p)}$ contains 0, and besides, consists of at most a sequence of eigenvalues of finite multiplicity which has no point of accumulation except 0.*

(4) *For any $1\le p\le\infty$, ϕ (respectively, $\tilde{\phi}$) is the unique nonnegative eigenfunction of the operator $\mathcal{G}\restriction_{L^p(\phi_p)}$ (respectively, $\mathcal{G}^\odot\restriction_{L^p(\tilde{\phi}_p)}$). The corresponding eigenvalue $\nu=(\lambda_0)^{-1}$ is simple.*

(5) *The spectrum of $\mathcal{G}\restriction_{L^p(\phi_p)}$ is p-independent for all $1\le p\le\infty$, and we have*

$$0\in\sigma\left(\mathcal{G}\restriction_{L^p(\phi_p)}\right) = \sigma\left(\mathcal{G}^\odot\restriction_{L^p(\tilde{\phi}_p)}\right) \subset \overline{B\Big(0,(\lambda_0)^{-1}\Big)}.$$

(6) *Suppose further that P is symmetric. Let ϕ_k be the k-th (weighted) eigenfunction in $L^2(M,W\mathrm{d}m)$ (counting multiplicity). Then for each $k\ge1$, the quotient of the eigenfunctions ϕ_k/ϕ is bounded in M and has a continuous extension up to the Martin boundary of the pair (M,P).*

Remark 2.12. We would like to point out that criticality theory, and in particular the results of this paper, are also valid for the class of *classical solutions* of locally uniformly elliptic operators of the form

$$Lu := -\sum_{i,j=1}^{N} a^{ij}(x)\partial_i\partial_j u + b(x)\cdot\nabla u + c(x)u, \tag{2.5}$$

with real and locally Hölder continuous coefficients, and for the class of *strong solutions* of locally uniformly elliptic operators of the form (2.5) with locally bounded coefficients (provided that the formal adjoint operator also satisfies the same assumptions), see [22, 23, 24, 26, 30] and references therein. Nevertheless, for the sake of clarity, we prefer to present our results only for operators in divergence form (1.1) and weak solutions.

3. Aims and objectives

In this section we present the problems that we study in our paper.

3.1. **Maximal interval of equivalence.**

The following problem was posed in [24, Conjecture 3.7], see also [26, Example 8.6] for a counterexample.

Problem 3.1. Suppose that P is subcritical in M of the form (1.1), and assume that $W\ge0$ is a G-bounded perturbation of P in M. Is it true that

$$E_+ = S_+\setminus\{\lambda_0\}?$$

In Section 4 we provide a positive answer to the above question if P is *symmetric* and its positive minimal Green function satisfies the *quasimetric property*. See also Lemma 6.2, where we prove that $SE_+ = S_+ \setminus \{\lambda_0\}$ for a certain family of nonnegative G-semibounded perturbations of a subcritical operator P in M.

3.2. **h-big perturbation.** Next, we discuss a class of perturbations known as *h-big perturbations*. This notion was introduced by A. Grigor'yan and W. Hansen in [14] for the case when $P = -\Delta$, and later it was generalized by M. Murata (see [20, 21]) for elliptic operators of the form (1.1).

Definition 3.2. Suppose that P of the form (1.1) is subcritical in M. Let h be a positive supersolution of the equation

$$P\,u = 0 \quad \text{in } M.$$

We say that a nonnegative potential W is a *h-big in M* if there is no function satisfying

$$(P + W)v = 0 \quad \text{in } M \text{ and } 0 < v \leq h \quad \text{in a neighborohood of infinity in } M.$$

Otherwise, W is said to be *non-h-big*.

Remark 3.3. It is evident from the definition of h-big perturbation that it generalizes the following Liouville property for Schrödinger equation [13]:

Let M be a smooth, noncompact Riemannian manifold M and let $W \neq 0$ be a smooth nonnegative potential on M. We say that the operator $-\Delta + W$ satisfies the *Liouville property* if

$$(-\Delta + W)u = 0 \quad \text{in } M, \text{ and } 0 \leq u \in L^\infty(M), \tag{3.1}$$

implies $u = 0$.

Clearly (see for example [13]), if $W \gneqq 0$ has a compact support the above Liouville property holds true if and only if $P := -\Delta$ is critical in M (in other word, M is parabolic). On the other hand, if $P = -\Delta$ is subcritical in M and

$$\int_M G_P^M(x,y)W(y)\,\mathrm{d}m(y) < \infty,$$

then the Liouville property does not hold [13, 14]. Moreover, it follows from [26, Proposition 3.4] that if P is a subcritical operator in M of the form (1.1), and $h \in \mathcal{C}_P(M)$, then $W \gneqq 0$ is non-h-big if

$$\int_M G_P^M(x,y)W(y)h(y)\,\mathrm{d}m(y) < \infty.$$

For a given subcritical operator P of the form (1.1) there is a natural class of weights satisfying $\lambda_0(P, W, M) > 0$, which are 'big' in a certain sense.

Definition 3.4 ([9])**.** We say that $W \gneqq 0$ is an *optimal-Hardy* weight for P in M if the following three properties hold:

- **Criticality:** $P - W$ is critical in M, and let φ and φ^* be the corresponding ground states of $P - W$ and $P^* - W$.
- **Optimality at infinity:** for any $\lambda > 1$ and $K \Subset M$, $P - \lambda W \ngeq 0$ in $M \setminus K$.
- **Null-criticality with respect to W:** $\varphi\varphi^* \notin L^1(M, W\mathrm{d}m)$.

Remark 3.5. It follows from Theorem 3.4 of the recent paper [16] that if the operator $P - W$ is null-critical in M with respect to W, then W is also optimal at infinity.

The following theorem is a version of [9, Theorem 4.12] (cf. the discussion therein).

Theorem 3.6. *Let P be a subcritical operator in M and let $G_P^M(x,y)$ be its minimal positive Green function. Let $u \in \mathcal{C}_P(M)$ satisfy*

$$\lim_{x\to\infty} \frac{G_P^M(x,y)}{u(x)} = 0, \tag{3.2}$$

where ∞ is the ideal point in the one-point compactification of M.

Let $\phi \gneqq 0$ be a compactly supported smooth function, and consider its Green potential

$$G_\phi(x) := \int_M G_P^M(x,y)\phi(y)\,\mathrm{d}m(y).$$

Then

$$W := \frac{P(\sqrt{G_\phi u})}{\sqrt{G_\phi u}} \tag{3.3}$$

is an optimal Hardy-weight for P in M. Moreover,

$$W(x) = \frac{1}{4}\left|\nabla \log\left(\frac{G_\phi(x)}{u(x)}\right)\right|^2_{A(x)} \qquad \text{in } M \setminus \operatorname{supp}\phi.$$

We omit the proof of Theorem 3.6 since it can be obtained by a slight modification of the proof of [9, Theorem 4.12].

In Section 5, we discuss the following problem.

Problem 3.7. Study the h-bigness property of optimal Hardy-weights W given by Theorem 3.6.

3.3. **Critical Hardy-weights.** An important feature of classical Hardy-weights W is the knowledge of the best Hardy constant. In other words, for such Hardy-weights the value of $\lambda_0(P,W,M)$ is known (in contrary to the case of a general weight). We note that the problem of finding a critical potential for a given subcritical operator was studied in [29, Section 5]. The answer obtained there relies on solving a nontrivial auxiliary variational problem. Moreover, this variational approach is obviously restricted to *symmetric* subcritical operators.

In Section 6 we prove for any subcritical operator P of the form (1.1), the existence of a large family of critical Hardy-weights which are given by a simple explicit formula. More precisely, we present a family of 'small' Hardy-weights W_μ such that each W_μ is a semismall perturbation of P in M, and $P - W_\mu$ is positive-critical with respect to W_μ. In particular, $\lambda_0(P, W_\mu, M) = 1$. Recall that *optimal* Hardy-weights W given by Theorem 3.6 are h-big and $P - W$ is null-critical with respect to W.

3.4. **Liouville comparison principle.** Next, we recall a Liouville comparison principle for nonnegative Schrödinger-type operators.

Theorem 3.8 ([27, Theorem 1.7]). *Let $N \geq 1$ and M be a noncompact connected N-dimensional Riemannian manifold. Consider two Schrödinger operators defined on M of the form* (1.4)*, that is,*

$$P_j := -\operatorname{div}(A_j \nabla) + V_j \qquad j = 0, 1,$$

such that A_j satisfy (1.2)*, and $V_j \in L^q_{\text{loc}}(M; \mathbb{R})$ for some $q > N/2$, where $j = 0, 1$.*

Suppose that the following assumptions hold true:

1. *The operator P_1 is critical in M. Denote by Φ be its ground state.*
2. *P_0 is nonnegative in M, and there exists a real function $\Psi \in H^1_{\text{loc}}(M)$ such that $\Psi_+ \neq 0$, and $P_0 \Psi \leq 0$ in M, where $u_+(x) := \max\{0, u(x)\}$.*
3. *The following inequality holds:*

$$(\Psi_+)^2(x) A_0(x) \leq C \Phi^2(x) A_1(x) \qquad \text{a.e. in } M,$$

where $C > 0$ is a positive constant, and the matrix inequality $A \leq B$ means that $B - A$ is a positive semi-definite matrix.

Then the operator P_0 is critical in M and Ψ is its ground state.

We note that in Theorem 3.8 there is no assumption on the difference of the given potentials V_j. In [27, Problem 5] the author proposed to generalize Theorem 3.8 to the case of *nonsymmetric* elliptic operators of the form (1.1) with the same (or even with comparable) principal parts. In a recent paper [5], the authors gave a partial answer to the above problem using a probabilistic approach along with criticality theory under some assumptions on the difference of the given potentials.

In Section 8, we prove another version of Liouville comparison principle for nonsymmetric nonnegative operators. In particular, we provide a quantitative bound on the difference of the given potentials in terms of a certain Hardy-weight to guarantee the validity of a Liouville comparison principle. Moreover, in contrast to [5, Theorem 2.3] which holds in $\mathbb{R}^N$, our result holds in any noncompact Riemannian manifold. We refer to Theorem 8.1 for more details.

4. Maximal interval of equivalence of Green functions

In the present section we provide a partial answer to Problem 3.1 concerning G-bounded perturbations under the quasimetric assumption. This property of Green functions has been considered previously by several authors, for example in [10, 15, 26].

Definition 4.1. A *quasimetric kernel* K on a measure space (M, μ) is a measurable function from $M \times M \to (0, \infty]$ such that the following conditions hold.

1. The kernel K is symmetric : $K(x, y) = K(y, x)$ for all $x, y \in M$.
2. The function $d := 1/K$ satisfies the quasi-triangle inequality

$$d(x, y) \leq C(d(x, z) + d(z, y)) \qquad \forall x, y, z \in M, \tag{4.1}$$

for some $C > 0$, called the *quasimetric constant* for K.

Remark 4.2. Using Ptolemy inequality [10, Lemma 2.2], it follows that if G_P^M is a quasimetric kernel in the sense of Definition 4.1, then it satisfies the quasimetric inequality of [26, Lemma 7.1]. Therefore, in this case and in light of [26, Lemma 7.1], if W is G-semibounded perturbation, then W is in fact, G-bounded perturbation.

We are now in a position to state the main result of the present section. We have

Theorem 4.3. *Let P be a second-order, symmetric, subcritical elliptic operator of the form* (1.3) *defined on noncompact Riemannian manifold M, and let $0 \lneqq W \in L^q_{\mathrm{loc}}(M)$, with $q > N/2$ be a G-semibounded perturbation of P in M.*

Assume further that G_P^M is a quasimetric kernel. Then

$$G_P^M \asymp G_{P-\varepsilon W}^M \qquad \text{on } M \times M$$

for all $\varepsilon < \lambda_0 = \lambda_0(P, W, M)$. Moreover,

$$E_+ = S_+ \setminus \{\lambda_0\}.$$

Before proving Theorem 4.3, we recall some general results concerning the equivalence of Green functions. We start with the following lemma.

Lemma 4.4 ([19, 22, 23]). *Let P be a second-order, subcritical elliptic operator of the form* (1.1) *defined on noncompact Riemannian manifold M, and let $V \in L^q_{\mathrm{loc}}(M; \mathbb{R})$ with $q > N/2$ be a G-bounded perturbation (that is, the $3G$-inequality* (2.1) *holds true).*

Then $P - \varepsilon V$ is subcritical and

$$G_P^M \asymp G_{P-\varepsilon V}^M \qquad \text{on } M \times M \tag{4.2}$$

for all $|\varepsilon| < (2C_0)^{-1}$. In particular, $\lambda_0 := \lambda_0(P, V, M) > 0$.

Proof. Consider the iterated Green kernel

$$G_P^{(i)}(x,y) := \begin{cases} G_P^M(x,y) & i = 0, \\ \\ \int_M G(x,z)V(z)G_P^{(i-1)}(z,y)\,\mathrm{d}m(z) & i \geq 1. \end{cases} \tag{4.3}$$

Then it follows from the hypothesis and an induction argument that

$$|G_P^{(i)}(x,y)| \leq (C_0)^i G_P^M(x,y),$$

where C_0 is given by (2.1). Hence,

$$\sum_{i=0}^{\infty} |\varepsilon|^i \left| G_P^{(i)}(x,y) \right| \leq \frac{1}{1 - C_0|\varepsilon|} G_P^M(x,y),$$

provided $|\varepsilon| < C_0^{-1}$. Fix $|\varepsilon| < C_0^{-1}$. Using a standard elliptic argument, it follow that the Neumann series

$$H_P^{\varepsilon}(x,y) := \sum_{i=0}^{\infty} \varepsilon^i G_P^{(i)}(x,y)$$

converges locally uniformly in $M \setminus \{y\}$ to a Green function of $(P - \varepsilon V)u = 0$. Moreover, for $|\varepsilon| < C_0^{-1}$, the positive minimal Green function $G_{P-\varepsilon|V|}^M$ exists, and by the minimality of the Green function it satisfies

$$0 \le G^M_{P-|\varepsilon||V|}(x,y) \le \frac{1}{1-|\varepsilon|C_0} G^M_P(x,y).$$

Hence, $G^M_{P-\varepsilon V}$ exists, and by the generalized maximum principle we obtain

$$0 \le G^M_{P-\varepsilon V}(x,y) \le G^M_{P-|\varepsilon||V|}(x,y) \le \frac{1}{1-|\varepsilon|C_0} G^M_P(x,y). \tag{4.4}$$

Using resolvent equation [23, Lemma 2.4]

$$G^M_{P-\varepsilon V}(x,y) = G^M_P(x,y) + \varepsilon \int_M G_{P-\varepsilon V}(x,z)V(z)G^M_P(z,y)\,\mathrm{d}m(z),$$

we obtain

$$G^M_P(x,y) \le G^M_{P-\varepsilon V}(x,y) + \frac{|\varepsilon|C_0}{1-|\varepsilon|C_0} G^M_P(x,y).$$

Hence, for $|\varepsilon| < (2C_0)^{-1}$ we have

$$\frac{1-2|\varepsilon|C_0}{1-|\varepsilon|C_0} G^M_P(x,y) \le G^M_{P-\varepsilon V}(x,y).$$

Hence, the lemma follows. □

We recall a lemma regarding the convergence of the Neumann series of the iterated Green functions in the case of a perturbation by a potential W with a definite sign.

Lemma 4.5 (Lemma 3.1, [26])**.** *Let P be a second-order, subcritical elliptic operator of the form* (1.1) *defined on noncompact Riemannian manifold M, and let $W \in L^q_{\mathrm{loc}}(M;\mathbb{R})$, with $q > N/2$ be a nonzero, nonnegative potential such that $\lambda_0 := \lambda_0(P,V,M) > 0$. Then*

$$\int_M G^M_P(x,z)W(z)G^M_P(z,y)\,\mathrm{d}m(z) < \infty, \tag{4.5}$$

and for every $0 < \varepsilon < \lambda_0$, the Neumann series $\sum_{i=0}^{\infty} \varepsilon^i G^{(i)}_P(x,y)$ converges to $G^M_{P-\varepsilon W}(x,y)$ in the compact-open topology.

Proof of Theorem 4.3. In light of Remark 4.2 we may assume that W is a G-bounded perturbation.

Clearly, E_+ is an open set. Indeed, if $\lambda \in E_+$, then W is G-bounded perturbation of $P - \lambda W$, and by Lemma 4.4, there exists $\varepsilon_0 > 0$ such that $(\lambda - \varepsilon_0, \lambda + \varepsilon_0) \subset E_+$ (see also [24, Corollary 3.6]). In particular, $\lambda_0 \notin E_+$.

Next, We claim that $G^M_P \asymp G^M_{P-\varepsilon W}$ for all $\varepsilon < C_0^{-1}$.

It follows from Lemma 4.4 that $G^M_P \asymp G^M_{P-\varepsilon W}$ for all $|\varepsilon| < (2C_0)^{-1}$. Moreover, by the generalized maximum principle, if $\varepsilon_1 < \varepsilon_2 < \lambda_0$, then

$$G^M_{P-\varepsilon_1 W} \le G^M_{P-\varepsilon_2 W}. \tag{4.6}$$

Therefore, $G_P^M \le G_{P-\varepsilon W}^M$ for all $0 \le \varepsilon < \lambda_0$. On the other hand, for $0 < \varepsilon < C_0^{-1}$, we have by (4.4) that

$$G_P^M \le G_{P-\varepsilon W}^M \le \frac{1}{1-\varepsilon C_0} G_P^M. \tag{4.7}$$

Fix $\varepsilon > 0$, and let

$$G_0 := G_{P+\varepsilon W}^M, \qquad G_1 := G_{P-\frac{W}{2C_0}}^M, \qquad \alpha := \frac{\varepsilon}{\varepsilon + 1/(2C_0)}.$$

In light of [24, Theorem 3.4] and (4.7), we obtain

$$G_0 = G_{P+\varepsilon W}^M \le G_P^M \le (G_1)^\alpha (G_0)^{1-\alpha} \le 2^\alpha (G_P^M)^\alpha G_0^{1-\alpha}.$$

Therefore,

$$G_{P+\varepsilon W} \le G_P^M \le 2^{2C_0\varepsilon} G_{P+\varepsilon W}.$$

Hence, $G_{P-\varepsilon W}^M \asymp G_P^M$ for all $\varepsilon < C_0^{-1}$.

Let $E_0 := \sup E_+$. Thus, $0 < C_0^{-1} \le E_0 \le \lambda_0$. We claim that $E_0 = \lambda_0$. Suppose to the contrary, that there exists $\delta > 0$ such that $E_0 + \delta < \lambda_0$, i.e., $(E_0+\delta)/\lambda_0 < 1$.

Set $\mathrm{d}W := W(x)\mathrm{d}m(x)$, and define the iterated kernel

$$K^{(i)}(x,y) := \begin{cases} (E_0+\delta)\, G_P^M(x,y) & i = 0, \\ \int_M G_P^M(x,z) K^{(i-1)}(z,y)\, \mathrm{d}W(z) & i \ge 1, \end{cases}$$

and an operator $T : L^2(M, \mathrm{d}W) \to L^2(M, \mathrm{d}W)$ by

$$Tf(x) := (E_0+\delta) \int_M G_P^M(x,y) f(y)\, \mathrm{d}W(y).$$

We claim that T is well defined and $||T||_{L^2(M,\, \mathrm{d}W)} < 1$.

Let u be a positive supersolution of $(P - \lambda_0 W)u = 0$. Then it follows from [24] that

$$(E_0+\delta) \int_M G_P^M(x,y) u(y)\, \mathrm{d}W(y) \le \frac{(E_0+\delta)\, u(x)}{\lambda_0},$$

and

$$(E_0+\delta) \int_M u(x) G_P^M(x,y)\, \mathrm{d}W(x) \le \frac{(E_0+\delta)\, u(y)}{\lambda_0}.$$

Therefore, by Schur's test we obtain

$$||T||_{L^2(M,\, \mathrm{d}W)} \le \frac{E_0+\delta}{\lambda_0} < 1.$$

Define

$$H(x,y) := \sum_{i=0}^{\infty} (E_0+\delta)^i\, K^{(i)}(x,y) = (E_0+\delta)\, G_{P-(E_0+\delta)W}^M(x,y), \tag{4.8}$$

which is well defined by Lemma 4.5.

Hence, T is a bounded linear integral operator on $L^2(M, \mathrm{d}W)$, with a quasimetric kernel K and with a norm strictly less than 1. Consequently, [10, Theorem 1.1] implies that

$$\mathrm{e}^{\frac{C_1 K^{(1)}(x,y)}{K^{(0)}(x,y)}} K^{(0)}(x,y) \le H(x,y) \le \mathrm{e}^{\frac{C_2 K^{(1)}(x,y)}{K^{(0)}(x,y)}} K^{(0)}(x,y), \tag{4.9}$$

for some positive constants C_1 and C_2.

Therefore, (4.9) and (4.8) immediately imply

$$(E_0+\delta)\, G^M_{P-(E_0+\delta)W}(x,y) \le K^{(0)}(x,y)\, \mathrm{e}^{\frac{C_2 K^{(1)}(x,y)}{K^{(0)}(x,y)}}. \tag{4.10}$$

Now, observe that

$$\frac{K^{(1)}(x,y)}{K^{(0)}(x,y)} = \frac{1}{G^M_P(x,y)} \int_M G^M_P(x,z)W(z)G^M_P(z,y)\, \mathrm{d}m(z) \le C_0.$$

Hence, (4.10) yields

$$G^M_P(x,y) \le G^M_{P-(E_0+\delta)W}(x,y) \le C G^M_P(x,y),$$

where C is a positive constant. This contradicts the maximality of E_0. Hence, $E_0 = \lambda_0$. $\square$

Remark 4.6. The validity of the conjecture $E_+ = S_+ \setminus \{\lambda_0\}$, for a general nonnegative G-bounded perturbation W of operator P of the form (1.1) remains open (cf. [24, Conjecture 3.7] and [26, Example 8.6]).

5. Optimal Hardy-weights and h-bigness

In the present section we study the h-bigness of *optimal Hardy-weights* $W \ge 0$ given by Theorem 3.6. Recall that G-bounded perturbations are non-h-big [19]. We note that under the conditions of Theorem 3.6, the operator $P_\lambda := P - \lambda W$ is subcritical in M for all $\lambda < 1$. We have

Theorem 5.1. *Consider the operator $P_\lambda := P - \lambda W$, and assume that*

- *The operator P is subcritical, and let G_ϕ be a Green potential with respect to P, with a compactly supported smooth density ϕ.*
- *There exists a positive solution u of the equation $Pv = 0$ in M satisfying* (3.2)*.*
- *W is the corresponding optimal Hardy-weight given by* (3.3)*.*
- $0 < \lambda < 1$, *and set* $\alpha_\pm := (1 \pm \sqrt{1-\lambda})/2$.

Then λW is $h_\pm$-big perturbations for the positive P_λ-supersolutions

$$h_\pm := u^{(1-\alpha_\pm)}(G_\phi)^{\alpha_\pm}.$$

Proof. Let $K := \operatorname{supp} \phi$. Since $\lambda = 4\alpha_\pm(1-\alpha_\pm)$, it follows that $h_\pm$ are indeed positive P_λ-supersolutions in M, which are positive solutions of the equation $P_\lambda v = 0$ in $M \setminus K$ (see [24, Theorem 3.1]).

Let $v_\pm$ be nonnegative solutions of $Pw = (P_\lambda + \lambda W)w = 0$ in M satisfying $0 \le v_\pm \le h_\pm$. Suppose that $v_\pm > 0$. So,

$$\frac{v_\pm(x)}{u(x)} \le \left(\frac{G_\phi(x)}{u(x)}\right)^{\alpha_\pm}.$$

By our assumption, $\lim_{x\to\infty}\frac{G(x)}{u(x)}=0$, therefore, $\lim_{x\to\infty}\frac{G_\phi(x)}{u(x)}=0$. Consequently,

$$\lim_{x\to\infty}\frac{v_\pm(x)}{u(x)}=0.$$

In light of [9, Proposition 6.1], we conclude $v_\pm$ are positive solutions of the equation $Pw=0$ in M of minimal growth in a neighborhood of infinity in M. Hence $v_\pm$ are ground states, and P is critical in M, a contradiction. Hence, we conclude $v_\pm\equiv 0$. □

Remark 5.2. 1. Since near infinity in M we have

$$\left(\frac{G_\phi(x)}{u(x)}\right)^{\alpha_+}\leq\left(\frac{G_\phi(x)}{u(x)}\right)^{\alpha_-},$$

it is enough to prove that λW is h_--big perturbation.

2. Fix $x_0\in M$. We may consider the punctured manifold $M^*:=M\setminus\{x_0\}$, and let u is a positive solution of the equation $Pw=0$ in M, and $G(x):=G_P^M(x,x_0)$ satisfying (3.2). Let

$$W(x):=\frac{1}{4}\left|\nabla\log\left(\frac{G(x)}{u(x)}\right)\right|^2_{A(x)}\qquad\text{in } M\setminus\{x_0\}.$$

As in the proof of Theorem 5.1, it follows that for $0<\lambda<1$, the potential λW is h_--big perturbations for $h_-:=u^{(1-\alpha_-)}(G)^{\alpha_-}$.

6. Critical Hardy-weights

Throughout the present section we assume that P is a subcritical operator in M of the form (1.1). We fix a positive Radon measure μ on M with a 'nice' nonnegative density $\mu(x)$. We denote $\mathrm{d}\mu=\mu(x)\,\mathrm{d}m$, and we assume that the corresponding *Green potential* G_μ is finite. That is, we assume that for some $x\in M$ (and therefore, for any $x\in M$)

$$G_\mu(x):=\int_M G_P^M(x,y)\mathrm{d}\mu(y)<\infty. \tag{6.1}$$

A sufficient condition for (6.1) to hold is obviously, the existence of $k\geq 1$, and a positive (super)solution $\varphi^\star$ of the equation $P^\star u=0$ in $M_k^\star$ such that $\varphi^\star\in L^1(M_k^\star,\mathrm{d}\mu)$.

Set

$$W_\mu(x):=\frac{\mu(x)}{G_\mu(x)}.$$

Since $PG_\mu=\mu$, it follows that the Green potential G_μ is a positive solution of the equation $(P-W_\mu)u=0$ in M, so, $\lambda_0:=\lambda_0(P,W_\mu,M)\geq 1$. Moreover, since

$$\int_M G_P^M(x,y)W_\mu(y)G_\mu(y)\,\mathrm{d}m(y)=G_\mu(x)\quad\forall x\in M, \tag{6.2}$$

it follows that G_μ is a positive *invariant solution* of the equation $(P-W_\mu)u=0$ in M (see [24, 28] and references therein).

Without loss of generality, we assume that $0\in M$, and we denote $G(x):=G_P^M(x,0)$. Since $PG=0$ in $M\setminus\{0\}$, and G has minimal growth at infinity in M, it follows that for a given Green potential G_μ and for $\varepsilon > 0$ small enough, there exists a positive constant C such that

$$G(x) \le CG_\mu(x) \qquad \forall x \in M \setminus B(0, \varepsilon).$$

On the other hand, let $V_\mu(x) := \mu(x)/G(x)$ in M. The following lemma characterizes Green potentials that are comparable (near infinity in M) to G (see [26, Corollary 4.7]).

Lemma 6.1. *There exists a positive constant $C > 0$ such that*

$$C^{-1}G_\mu(x) \le G(x) \qquad \forall x \in M \tag{6.3}$$

if and only if V_μ is a G-semibounded perturbation of $P^\star$ in M.

Moreover, in this case, we have $V_\mu \asymp W_\mu$ near infinity in M, and in particular, W_μ is a G-semibounded perturbation of $P^\star$ in M.

In addition, the convex set of all positive solutions v of the equation $P^\star u = 0$ in M satisfying $v(0) = 1$ is a bounded set in $L^1(M, \mathrm{d}\mu)$.

Proof. Assume first that V_μ is a G-semibounded perturbation of $P^\star$ in M. Then

$$G_\mu(x) = \int_M G_P^M(x,y)\frac{\mu(y)}{G(y)}G(y)\mathrm{d}m(y) = \int_M G_P^M(x,y)V_\mu(y)G(y)\,\mathrm{d}m(y) \le CG(x) \qquad \forall x \in M,$$

and (6.3) holds.

On the other hand, suppose that (6.3) holds. Consequently,

$$\int_M G_P^M(x,y)V_\mu(y)G(y)\,\mathrm{d}m(y) = G_\mu(x) \le CG(x) \quad \forall x \in M. \tag{6.4}$$

Therefore,, V_μ is a G-semibounded perturbation of $P^\star$ in M. In particular, in this case we have $G_\mu \asymp G$ near infinity. This in turn, obviously implies that $V_\mu \asymp W_\mu$ near infinity.

In addition, by (6.4) we have

$$\int_M \frac{G_P^M(x,y)}{G_P^M(x,0)}\,\mathrm{d}\mu(y) = \int_M \frac{G_P^M(x,y)V_\mu(y)G(y)}{G(x)}\,\mathrm{d}m(y) \le C \qquad \forall x \in M.$$

Therefore, the last assertion of the lemma follows from Fatou's lemma and the Martin representation theorem. □

The following lemma gives, in particular, a positive answer to Problem 3.1 for the class of nonnegative G-semibounded perturbations of the form W_μ.

Lemma 6.2. *Suppose that* (6.3) *holds true, then $P - W_\mu$ is positive-critical in M with respect to W_μ, and G_μ is its ground state. Moreover,*

$$SE_+(P, W_\mu, M) = S_+(P, W_\mu, M) = (-\infty, \lambda_0(P, W_\mu, M)) = (-\infty, 1).$$

Proof. Recall that G_μ is a positive solution of the equation $(P - W_\mu)u = 0$ in M. On the other hand, by our assumption $G_\mu \asymp G$ near infinity in M. Note that any positive supersolution v of the equation $(P - W_\mu)u = 0$ near infinity in M is a positive supersolution of the equation $Pu = 0$ in this neighborhood, while G is a positive solution of $Pu = 0$ of minimal growth near infinity.

Consequently,

$$G_\mu \leq CG \leq C_1 v \qquad \text{near infinity in } M.$$

Therefore, G_μ is a ground state of the equation $(P - W_\mu)u = 0$ in M, and $P - W_\mu$ is critical in M. Consequently, for any $0 < \alpha < 1$ and $\varepsilon > 0$ sufficiently small, we have

$$G \asymp G^M_{P-\alpha W_\mu}(\cdot, 0) \asymp G_\mu \qquad \text{in } M \setminus B(0, \varepsilon).$$

Furthermore, in light of [24, Corollary 3.6], $G \asymp G^M_{P-\alpha W_\mu}(\cdot, 0)$ also for any $\alpha < 0$. So, $SE_+(P, W_\mu, M) = S_+(P, W_\mu, M) = (-\infty, 1)$.

Moreover, since $P - W_\mu$ is critical in M, we have that $P^\star - W_\mu$ is also critical in M. Denote by $u^\star_\mu$ its ground state. In particular, $u^\star_\mu$ is a positive invariant solution of the corresponding equation [24, Theorem 2.1]. Therefore,

$$\int_M G_\mu(x) W_\mu(x) u^\star_\mu(x) \mathrm{d}m(x) \asymp \int_M G(x) W_\mu(x) u^\star_\mu(x) \mathrm{d}m(x) = u^\star_\mu(0) < \infty.$$

Hence, $P - W_\mu$ is positive-critical in M with respect to W_μ. $\square$

Lemma 6.3. *For $k \geq 2$, let χ_k be a smooth function on M such that*

$$0 \leq \chi_k(x) \leq 1, \text{ in } M \qquad \chi_k{\restriction_{M_{k-1}}} = 0, \qquad \chi_k{\restriction_{M^\star_k}} = 1,$$

where $\{M_k\}$ is an exhaustion of M (see Section 2). Denote by $\mu_k(x) := \chi_k(x)\mu(x)$. Assume further that

$$\lim_{k\to\infty} \left\| \frac{G_{\mu_k}}{G} \right\|_{\infty; M^\star_k} = 0. \tag{6.5}$$

Then W_μ is a semismall perturbation of the operator $P^\star$ in M, and for any $1 \leq p \leq \infty$ the integral operator

$$\mathcal{G}_\mu f(x) := \int_M G^M_P(x, y) W_\mu(y) f(y)\, \mathrm{d}m(y)$$

is compact on $L^p(\phi_p)$, where

$$\phi_p := G_\mu^{-1} (G_\mu W_\mu u^\star_\mu)^{1/p}. \tag{6.6}$$

Suppose in addition that P is a symmetric operator on $L^2(M, W_\mu(x)\, \mathrm{d}m)$ with a core $C_0^\infty(M)$, Let $\{(\varphi_k, \lambda_k)\}_{k=0}^\infty$ be the set of the corresponding pairs of eigenfunctions and eigenvalues (counting multiplicity), where $\varphi_0 := G_\mu$ and $\lambda_0 = 1$. Then for every $k \geq 1$ there exists a positive constant C_k such that

$$|\varphi_k(x)| \leq C_k \varphi_0(x) \qquad \text{in } M. \tag{6.7}$$

Furthermore, the function φ_k/φ_0 has a continuous extension ψ_k up to the Martin boundary $\partial^M_P M$ of P in M.

Proof. Assumption (6.5) implies

$$\int_{M_k^\star} G_P^M(x,y)W_\mu(y)G(y)\,\mathrm{d}m(y) = \int_{M_k^\star} G_P^M(x,y)\frac{\mu(y)}{G_\mu(y)}G(y)\,\mathrm{d}m(y)$$
$$\leq C\int_{M_k^\star} G_P^M(x,y)\frac{\mu(y)}{G_\mu(y)}G_\mu(y)\,\mathrm{d}m(y) = CG_{\mu_k}(x) < \varepsilon G(x) \qquad \forall x\in M_k^\star,$$

Consequently, W_μ is a semismall perturbation of the operator $P^\star$ in M. Therefore, Theorem 2.11 implies that for any $1\leq p\leq\infty$ the integral operator $\mathcal{G}_\mu f(x)$ is compact on $L^p(\phi_p)$, and its spectrum is p-independent and contained in the closed unit disk. More precisely, the spectrum contains 0, and besides, consists of at most a sequence of eigenvalues of finite multiplicity which has no point of accumulation except 0. Moreover, $\varphi_0 = G_\mu$ is the unique nonnegative eigenfunction of the operator $\mathcal{G}_\mu\restriction_{L^p(\phi_p)}$. Furthermore, the corresponding eigenvalue $\lambda_0 = 1$ is simple.

The statement concerning the symmetric case follows from Theorem 2.11. We note that by [28], the continuous extension ψ_k of φ_k/φ_0 satisfies for $k\geq 1$

$$\begin{aligned}\psi_k(\xi) &= (\psi_0(\xi))^{-1}\lambda_k\int_M K_P^M(z,\xi)W_\mu(z)\varphi_k(z)\,\mathrm{d}m(z) = \\ &\qquad \frac{\lambda_k\int_M K_P^M(z,\xi)W_\mu(z)\varphi_k(z)\,\mathrm{d}m(z)}{\int_M K_P^M(z,\xi)W_\mu(z)\varphi_0(z)\,\mathrm{d}m(z)} \qquad \forall\xi\in\partial_P^M M,\end{aligned} \tag{6.8}$$

where $K_P^M(\cdot,\xi)$ is the Martin kernel of P in M with a pole at $\xi\in\partial_P^M M$, and ψ_0 is the corresponding continuous extension of G_μ/G. □

Remark 6.4. If $\mu = 1$ and (6.1) is satisfied, then G_1 is called the *torsion function* (see for example, [7] and references therein). In a recent paper [6], D. N. Arnold, G. David, M. Filoche, D. Jerison and S. Mayboroda, considered the potential $W_1 = 1/G_1$ (which they called the *effective potential*) associated with a Schrödinger operator L in a bounded Lipschitz domain $M\subset\mathbb{R}^N$. They showed a remarkable connection between the Neumann eigenfunctions of L and the torsion function G_1 (which they call the *landscape function*) by proving that W_1 acts as an effective potential that governs the exponential decay of these eigenfunctions and delivers information on the distribution of eigenvalues near the bottom of the spectrum.

7. Finite torsional rigidity

Throughout the present section we assume that P is subcritical, symmetric operator on $L^2(M,\mathrm{d}m)$ of the form (1.3). Without loss of generality, we assume that $0\in M$, and we denote $G(x) := G_P^M(x,0)$. In addition, we assume that Green potential G_1 is finite and satisfies $G_1\in L^1(M,\mathrm{d}m)$. In other words,

$$G_1(x) = \int_M G_P^M(x,y)\,\mathrm{d}m(y) < \infty, \quad \text{and}\quad T(M) := \int_M G_1(x)\,\mathrm{d}m(x) < \infty.$$

We recall that G_1 (respectively, $T(M)$) is called the *torsion function* (respectively, *torsional rigidity*) with respect to the operator P and the measure $\mathrm{d}m$. Note that if $G_1\asymp G$, then the finiteness of the torsion function G_1 is clearly equivalent to the finiteness of torsional rigidity $T(M)$.

Following [7], we have

Lemma 7.1. *Let P be a symmetric subcritical operator in M with finite torsional rigidity. Assume further that there exists a function*

$$c:(0,\infty)\to(0,\infty)$$

such that $k_P^M(x,y,t)$, the positive minimal heat kernel of P in $(M,\mathrm{d}m)$, satisfies

$$k_P^M(x,y,t)\le c(t)\qquad \forall t>0, x,y\in M. \tag{7.1}$$

Then the spectrum of P on $L^2(M,\mathrm{d}m)$ is discrete.

Suppose further that there exists $\beta\ge 0$ and $\tilde{c}>0$ such that

$$c(t)\le\tilde{c}\min\{t^{-N/2},t^{-\beta/2}\}\qquad \forall t>0.$$

Then there exists a positive function $C:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\lambda_j\ge\min\Big\{C(\beta)T(M)^{-2/(\beta+2)}j^{2/(\beta+2)},C(N)T(M)^{-2/(N+2)}j^{2/(N+2)}\Big\}, \tag{7.2}$$

where $\{\lambda_j\}_{j=0}^\infty$ is the increasing sequence of the eigenvalues of P (counting multiplicity).

Proof. Since

$$G_1(x)=\int_M\int_0^\infty k_P^M(x,y,t)\mathrm{d}t\ \mathrm{d}m,$$

by Tonelli's theorem, it follows that for any $0<\alpha<1$, we have

$$T(M)=(1-\alpha)\int_0^\infty\mathrm{d}t\int_{M\times M}k_P^M(x,y,(1-\alpha)t)\,\mathrm{d}m(y)\,\mathrm{d}m(x).$$

In light of (7.1) and the semigroup property, we have

$$\begin{aligned}T(M)&\ge(1-\alpha)\int_0^\infty\big(c(\alpha t)\big)^{-1}\mathrm{d}t\int_{M\times M}k_P^M(x,y,(1-\alpha)t)k_P^M(x,y,\alpha t)\mathrm{d}m(y)\mathrm{d}m(x)\\ &=(1-\alpha)\int_0^\infty\big(c(\alpha t)\big)^{-1}\mathrm{d}t\int_M k_P^M(x,x,t)\,\mathrm{d}m(x).\end{aligned} \tag{7.3}$$

It follows that the heat operator k_P^M is trace class. So, for each $t>0$ we have

$$\int_M k_P^M(x,x,t)\,\mathrm{d}m(x)=\sum_{j=0}^\infty\exp(-\lambda_j t)<\infty,$$

where $\{\lambda_j\}$ is the nonincreasing sequence of all the eigenvalues of P (counting multiplicity). In particular, P has a discrete $L^2(M,\,\mathrm{d}m)$-spectrum.

Estimate (7.2) is obtained as in [7, Theorem 2]. Indeed, by (7.3) we have

$$T(M)\ge(1-\alpha)(\tilde{c})^{-1}\int_0^\infty(\alpha t)^{\beta/2}\sum_{j=0}^\infty\mathrm{e}^{-\lambda_j t}\mathrm{d}t\ge(1-\alpha)(\tilde{c})^{-1}j\int_0^\infty(\alpha t)^{\beta/2}\mathrm{e}^{-\lambda_j t}\mathrm{d}t.$$

Recall that

$$\int_0^\infty t^\gamma\mathrm{e}^{-\ell t}\,\mathrm{d}t=\frac{\Gamma(\gamma+1)}{\ell^{\gamma+1}}\,.$$

Hence, for $\alpha := \beta/(\beta+2)$, we obtain (7.2) with $C(\beta)$ given by

$$C(\beta) := \frac{\beta^{\beta/(\beta+2)}}{\beta+2}\left(\frac{2\Gamma((\beta+2)/2)}{\tilde{c}}\right)^{2/(\beta+2)}. \qquad \square$$

8. Liouville comparison principle

The present section is devoted to the study of Liouville comparison principle for *nonsymmetric* elliptic operators. The following theorem should be compared with Theorem 3.8 and [5, Theorem 2.3].

Theorem 8.1. *Let M be a smooth, noncompact, connected manifold of dimension N. Consider two operators*

$$P_k := \mathcal{L}_k - V_k \qquad k = 1, 2,$$

where each $\mathcal{L}_k$ is of the form (1.1), and $V_k \in L^p_{\mathrm{loc}}(M;\mathbb{R})$, where $p > N/2$. Let $\overline{V}(x) = \max\{V_1(x), V_2(x)\}$. Suppose that there exists $K_1 \Subset K \Subset M$ such that $\mathcal{L}_1 = \mathcal{L}_2$ in $M \setminus K_1$, and $P_k \geq 0$ in $M \setminus K_1$, for $k = 1, 2$.

Let G_k be a positive supersolution of the equation $P_k u = 0$ in $M \setminus K_1$, such that G_k is a positive solution of the equation $P_k u = 0$ in $M \setminus K$ of minimal growth at infinity in M, where $k = 1, 2$. Suppose that

$$\frac{|V_1 - V_2|}{2} \leq W := \frac{1}{4}\left|\nabla \log\left(\frac{G_1}{G_2}\right)\right|_A^2 \qquad \text{in } M \setminus K. \tag{8.1}$$

Then

(a) $\mathcal{L}_1 - \overline{V} \geq 0$ in $M \setminus K$.

(b) Assume further the that the following assumptions hold true:

(1) *The operator P_1 is critical in M, and let $\Phi \in \mathcal{C}_{P_1}(M)$ be its ground state.*

(2) *$P_2 \geq 0$ in M, and there exists a real function $\Psi \in W^{1,2}_{\mathrm{loc}}(M)$ such that $\Psi_+ \neq 0$ and $P_2\Psi \leq 0$ in M.*

(3) *The following inequality holds:*

$$\Psi_+ \leq C\Phi \qquad \text{in } M.$$

Then the operator P_2 is critical in M and Ψ is its ground state. In particular, the equation $P_2 v = 0$ admits a unique positive supersolution in M. Moreover, $\Psi \asymp \Phi$ in M.

Proof. The proof relies on criticality theory, the supersolution construction [9], and on the well known "maximal ε-trick". We denote the restriction of the operators $\mathcal{L}_k$ on $M \setminus K_1$ by $\mathcal{L}$.

(a) We note that $U := (G_1G_2)^{1/2}$ is a positive solution of the equation

$$\left(\mathcal{L} - \left(\frac{V_1 + V_2}{2}\right) - W\right) v = 0 \qquad \text{in } M \setminus K, \tag{8.2}$$

where W is given in (8.1). Since

$$\overline{V} = \max\{V_1(x), V_2(x)\} = \frac{V_1 + V_2}{2} + \frac{|V_1 - V_2|}{2},$$

assumption (8.1) implies that U is a positive supersolution of the equation $(\mathcal{L} - \overline{V})u \geq 0$ in $M \setminus K_1$. Hence, $\mathcal{L} - \overline{V} \geq 0$ in $M \setminus K_1$.

(b) Let $\overline{G}$ be a positive solution of the equation $(\mathcal{L}-\overline{V})u = 0$ in $M \setminus K$ of minimal growth at infinity in M. Then by the generalized maximum principle and the fact that G_1 has minimal growth at infinity in M we have that

$$G_1 \leq C_1\overline{G} \leq C_2 U = C_2(G_1G_2)^{1/2} \qquad \text{in } M \setminus K. \tag{8.3}$$

Hence, $G_1 \leq C_3G_2$ in $M \setminus K$.

Since $\Phi \leq \tilde{C}G_1$ in $M \setminus K$, and G_2 has minimal growth at infinity in M for P_2, we have that for any positive supersolution f of the equation $P_2u = 0$ in M we have

$$\Psi_+ \leq C\Phi \leq C\tilde{C}G_1 \leq C\tilde{C}C_3G_2 \leq C_4 f \qquad \text{in } M \setminus K. \tag{8.4}$$

Define

$$\varepsilon_0 = \max\{\varepsilon : \varepsilon\Psi(x) \leq f(x) \quad \forall x \in M\}.$$

In light of (8.4), it follows that $\varepsilon_0 > 0$ is well defined, and hence, $w(x) := f(x) - \varepsilon_0\Psi(x)$ is a nonnegative supersolution of the equation $P_2v = 0$ in M.

By the strong maximum principle, either $w > 0$ or $w = 0$ in M. Let us assume that $w > 0$. Then by replacing f with w and repeating the above argument, we conclude that there exists $\delta > 0$ such that $f - (\varepsilon_0 + \delta)\Psi > 0$, which contradicts the maximality of ε_0. Hence, $w = 0$ in M, which in turns implies that

$$\Psi(x) = \Psi_+ = \varepsilon_0 f(x) > 0 \qquad \forall x \in M.$$

Since f is an arbitrary positive supersolution of $P_2u = 0$ in M, it follows that P_2 is critical in M and Ψ is its ground state. The assertion $\Psi \asymp \Phi$ in M follows now from (8.4) since $\Psi(x) = \Psi_+ > 0$ in M and G_2 is a positive solution of the equation $P_2u = 0$ in $M \setminus K$ of minimal growth at infinity in M. □

Remark 8.2. Under the assumptions of Theorem 8.1, it follows that the positive minimal Green functions of P_k in $M \setminus K$, where $k = 1, 2$, are semiequivalent. Moreover, (8.3) implies that these Green functions are also semiequivalent to the positive minimal Green function of $\mathcal{L} - \overline{V}$ in $M \setminus K$. We note that using [25, Theorem 4.3] it follows that under the assumptions of Theorem 8.1, the operators $\mathcal{L}_k - \overline{V}$ might be supercritical in M.

The following example demonstrates that inequality (8.1) might not hold and still the Liouville comparison principle holds true.

Example 8.3. Let $P_1 = -\Delta$, $V_1 = 0$ in $\mathbb{R}^2$. Then it is well known that P_1 is critical and 1 is the corresponding ground state. Let $P_2 = -\Delta - V_2$ be nonnegative in $\mathbb{R}^2$, where $V_2 \in L^\infty(\mathbb{R}^2)$ is a radially symmetric potential that satisfies

$$V_2(x) = \frac{\lambda}{|x|^2} \quad \text{in } \mathbb{R}^2 \setminus B(0, 1), \tag{8.5}$$

where $\lambda < 0$ be any real number. A straightforward computation yields $G_2(x) := |x|^{-\sqrt{-\lambda}}$ is positive solution in $\mathbb{R}^2 \setminus B(0, 1)$ of minimal growth at infinity in $\mathbb{R}^2$ for

P_2. Also $G_1(x) = 1$ is a positive solution of minimal growth at infinity in $\mathbb{R}^2$ for P_1, so, $G_1 \not\asymp G_2$ near infinity. Note that

$$\frac{|V_1 - V_2|}{2} = \frac{|\lambda|}{2|x|^2} > \frac{|\lambda|}{4|x|^2} = \frac{1}{4}\left|\nabla \log\left(\frac{G_1}{G_2}\right)\right|^2.$$

On the other hand, the Liouville comparison principle (Theorem 3.8) applies for the above P_1 and P_2, since these operators are symmetric. In particular, if the equation $P_2 u = 0$ in M admits a nonzero, nonnegative, bounded subsolution, then P_2 is critical in M.

Next, we slightly modify the above example by adding a drift term to the Laplacian.

Example 8.4. Consider the operator

$$P_1 = -\Delta - b\,\frac{\chi_{B(0,1)^*}}{r}\partial_r \quad \text{in } \mathbb{R}^2,$$

and $V_1 = 0$, where $r := |x|$, b is a negative constant, and $\chi_{B(0,1)^*}$ is the indicator function of $B(0,1)^* := \mathbb{R}^2 \setminus B(0,1)$. Then P_1 is critical in $\mathbb{R}^2$, with a ground state equals 1. Let

$$P_2 := -\Delta - b\,\frac{\chi_{B(0,1)^*}}{r}\partial_r - V_2,$$

where $V_2 \in L^\infty(\mathbb{R}^2)$ satisfies (8.5), such that $P_2 \geq 0$ in $\mathbb{R}^2$. Then as before we easily find that $G_2(x) := |x|^{(-b-\sqrt{b^2-4\lambda})/2}$ is a positive solution in $B(0,1)^*$ of minimal growth at infinity in $\mathbb{R}^2$ for P_2. Also, $G_1(x) = 1$ is a positive solution of minimal growth at infinity in $\mathbb{R}^2$ for P_1, so, $G_1 \not\asymp G_2$ near infinity. We note that for $|x| > 1$ we have

$$\frac{1}{4}\left|\nabla \log\left(\frac{G_1}{G_2}\right)\right|^2 = \frac{|\lambda|}{4|x|^2} - \frac{b^2}{8|x|^2}\left[\sqrt{1 + \frac{4|\lambda|}{b^2}} - 1\right].$$

This immediately yields as before

$$\frac{|V_1 - V_2|}{2} = \frac{|\lambda|}{2|x|^2} > \frac{1}{4}\left|\nabla \log\left(\frac{G_1}{G_2}\right)\right|^2.$$

On the other hand, Theorem 2.14 applies for the above P_1 and P_2, since the operator P_1 is symmetric in $L^2(\mathbb{R}^2, \mathrm{d}m)$, where

$$\mathrm{d}m = m(x)\mathrm{d}x := \begin{cases} \mathrm{d}x & \text{if } x \in B(0,1)\,, \\ |x|^b \mathrm{d}x & \text{if } x \in \mathbb{R}^2 \setminus B(0,1)\,. \end{cases}$$

In particular, if the equation $P_2 u = 0$ in M admits a nonzero, nonnegative, bounded subsolution, then P_2 is critical in M.

9. Green function estimate on the hyperbolic space

As an application of our results, we study the behaviour of the positive minimal Green function of the shifted Laplacian on $\mathbb{H}^N$, the real hyperbolic space. For a different approach, see [12, Proposition 7.2]. It is well known that a Cartan-Hadamard manifold M whose sectional curvatures is bounded above by a strictly negative constant satisfies the Poincaré inequality, or in other words, the bottom of the L^2-spectrum of the Laplace-Beltrami on M is strictly positive. The most important example of such a manifold is $\mathbb{H}^N$. Let $\Delta_{\mathbb{H}^N}$ denote the Laplace-Beltrami operator on the hyperbolic space, then the *generalized principal eigenvalue* of $-\Delta_{\mathbb{H}^N}$ is given by

$$\lambda_0(-\Delta_{\mathbb{H}^N}, \mathbf{1}, \mathbb{H}^N) = \frac{(N-1)^2}{4}\ .$$

Moreover, by using explicit bounds for the heat kernel on $\mathbb{H}^N$ (see e.g. [8]) one can show that the nonnegative operator

$$P := -\Delta_{\mathbb{H}^N} - \frac{(N-1)^2}{4}$$

admits a positive minimal Green function (for $N \geq 2$). In other words, P is subcritical in $\mathbb{H}^N$.

Fix $x_0 \in \mathbb{H}^N$, and let $G(x) := G^{\mathbb{H}^N}_{-\Delta_{\mathbb{H}^N}}(x, x_0)$. For $0 < \lambda < 1$, let

$$0 < \alpha_- < 1/2 < \alpha_+ < 1$$

be the roots of the equation $\lambda = 4\alpha(1-\alpha)$. Using the supersolution construction [9], it follows that $G^{\alpha_\pm}$ are solutions of the equation

$$(-\Delta_{\mathbb{H}^N} - \lambda W)G^{\alpha_\pm} = 0 \quad \text{in } \mathbb{H}^N \setminus \{x_0\}, \quad \text{where } W := \frac{1}{4}\frac{|\nabla G|^2}{|G|^2}\ .$$

The asymptotic of W is given by the following lemma.

Lemma 9.1. Let $N \geq 2$ and $r := \mathrm{d}(x, x_0)$. Then $W(r)$ satisfies

$$W(r) = \frac{(N-1)^2}{4} + \frac{(N-1)^3}{N+1}\mathrm{e}^{-2r} + o(\mathrm{e}^{-2r}) \quad \text{as } r \to \infty.$$

Proof. For the hyperbolic space $\mathbb{H}^N$, the Green function of the Laplace-Beltrami operator is given by

$$G(x) = \tilde{G}(r) := \int_r^\infty (\sinh s)^{-(N-1)}\,\mathrm{d}s.$$

We have

$$(\sinh s)^{-(N-1)} = 2^{N-1}\mathrm{e}^{-(N-1)s}(1-\mathrm{e}^{-2s})^{-(N-1)}.$$

Therefore, $r \to \infty$ yields

$$(\sinh r)^{-(N-1)} = 2^{N-1}\left(\mathrm{e}^{-(N-1)r} + (N-1)\mathrm{e}^{-(N+1)r} + o\big(\mathrm{e}^{-(N+1)r}\big)\right).$$

Furthermore, as $r \to \infty$ we have

$$\int_r^\infty (\sinh s)^{-(N-1)}\mathrm{d}s = 2^{N-1}\left[\frac{1}{N-1}\mathrm{e}^{-(N-1)r} + \frac{N-1}{N+1}\mathrm{e}^{-(N+1)r} + o\big(\mathrm{e}^{-(N+1)r}\big)\right].$$

Hence, as $r \to \infty$ we have

$$W(r) = \frac{1}{4}\left[\frac{(\sinh r)^{-2(N-1)}}{\left(\int_r^\infty (\sinh s)^{-(N-1)}\mathrm{d}s\right)^2}\right] = \frac{(N-1)^2}{4} + \frac{(N-1)^3}{N+1}\mathrm{e}^{-2r} + o(\mathrm{e}^{-2r}).$$

$\square$

Now we state the following perturbative result.

Theorem 9.2. *Let $N \geq 2$ and $0 < \lambda < 1$. Then there holds*

$$G^{\mathbb{H}^N}_{-\Delta_{\mathbb{H}^N} - \lambda\frac{(N-1)^2}{4}}(x, x_0) \asymp G^{\mathbb{H}^N}_{-\Delta_{\mathbb{H}^N} - \lambda W}(x, x_0) \asymp G^{\alpha_+}(x) \tag{9.1}$$

in $\mathbb{H}^N \setminus B(x_0, 1)$, where $\lambda = 4\alpha_+(1 - \alpha_+)$ and $1/2 < \alpha+ < 1$.

Proof. Recall that $G^{\mathbb{H}^N}_{-\Delta_{\mathbb{H}^N} - \lambda W}(x, x_0)$ is a positive solution of minimal growth at infinity of the equation $(-\Delta_{\mathbb{H}^N} - \lambda W)v = 0$ in $\mathbb{H}^N$. On the other hand,

$$\lim_{r\to\infty} \frac{G^{\alpha_+}(r)}{G^{\alpha_-}(r)} = 0.$$

Therefore, [9, Proposition 6.1] implies that G^{α_+} is also a positive solution of minimal growth at infinity of the equation $(-\Delta_{\mathbb{H}^N} - \lambda W)v = 0$ in $\mathbb{H}^N$.

Thus,

$$G^{\mathbb{H}^N}_{-\Delta_{\mathbb{H}^N} - \lambda W}(x, x_0) \asymp G^{\alpha_+}(x) \qquad \text{in } \mathbb{H}^N \setminus B(x_0, 1).$$

Hence, it remains to prove that

$$G^{\mathbb{H}^N}_{-\Delta_{\mathbb{H}^N} - \lambda\frac{(N-1)^2}{4}}(x, x_0) \asymp G^{\mathbb{H}^N}_{-\Delta_{\mathbb{H}^N} - \lambda W}(x, x_0) \qquad \text{in } \mathbb{H}^N \setminus B(x_0, 1).$$

Note that for $r \to \infty$, we have

$$\lambda W(r) - \lambda\frac{(N-1)^2}{4} = \lambda\frac{(N-1)^3}{N+1}\mathrm{e}^{-2r} + o(\mathrm{e}^{-2r}).$$

Consequently, Remark 2.10 implies that it suffices to show that $\tilde{W}(r) := \mathrm{e}^{-2r} + o(\mathrm{e}^{-2r})$ is a small perturbation of the operator

$$P_\lambda := -\Delta_{\mathbb{H}^N} - \lambda\frac{(N-1)^2}{4} \qquad \text{in } \mathbb{H}^N.$$

We follow the approach of Ancona [3, corollary 6.1]. Let us choose $\Phi(r) := \mathrm{e}^{-(2-\varepsilon)r}$ with $0 < \varepsilon < 1$. Then it follows

$$\lim_{r\to\infty} \frac{\Phi(r)}{\tilde{W}(r)} = +\infty. \tag{9.2}$$

Moreover, Φ is nonnegative, nonincreasing and $\int_0^\infty \Phi(r)\mathrm{d}r < \infty$. Therefore, by [3, Theorem 1], we conclude

$$G^{\mathbb{H}^N}_{P_\lambda} \asymp G^{\mathbb{H}^N}_{P_\lambda + \Phi(r)\mathbf{1}_{\mathbb{H}^N\setminus B(x_0,R)}} \quad \text{in } \mathbb{H}^N \times \mathbb{H}^N \tag{9.3}$$

for large R. Consequently, (9.3) and arguments given in [22, 23] implies that Φ is a G-bounded perturbation of P_λ in $\mathbb{H}^N$.

Hence, it follows from (9.2) that $\tilde{W}$ is a small perturbation for P_λ. In particular, by Remark 2.10 we have

$$G^{\mathbb{H}^N}_{P_\lambda} \asymp G^{\mathbb{H}^N}_{-\Delta_{\mathbb{H}^N} - \lambda W} \quad \text{in } \mathbb{H}^N \times \mathbb{H}^N \setminus \{(x, x) \mid x \in \mathbb{H}^N\}.$$

Thus, (9.1) follows. ∎

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D. Ganguly
Department of Mathematics, Indian Institute of Technology Delhi, IIT Campus, Hauz Khas, New Delhi, Delhi 110016, India
E-mail address: debdipmath@gmail.com

Y. Pinchover
Department of Mathematics, Technion - Israel Institute of Technology, Haifa 3200003, Israel
E-mail address: pincho@technion.ac.il