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ON LIOUVILLE'S SYSTEMS CORRESPONDING TO SELF SIMILAR SOLUTIONS OF THE KELLER-SEGEL SYSTEMS OF SEVERAL POPULATIONS

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Abstract. We study a modified version of the Liouville's system on $\mathbb{R}^2$. One of the motivation for this system is the Keller-Segel system of several interacting populations, under the existence of an additional drift for each component which decays in time at the rate $O(1/\sqrt{t})$. We show that self-similar solutions always exist in the sub-critical case, while the existence of such self-similar solution in the critical case depends on the gap between the decaying drifts for each of the components. For this, we study the conditions for existence/non-existence of solutions for the corresponding Liouville's systems, which, in turn, are related to the existence/non-existence of minimizers to a corresponding Free Energy functional.

1. Introduction

In this paper we study the modified Liouville system:

$$\Delta u_i(x) + \frac{\beta_i e^{\sum_j a_{ij} u_j(x) - \alpha_i |x-v_i|^2/2}}{\int_{\mathbb{R}^2} e^{\sum_j a_{ij} u_j(z) - \alpha_i |z-v_i|^2/2} d^2 z} = 0 \quad , \; i = 1, \dots, n$$

on $\mathbb{R}^2$ where (a_{ij}) is a symmetric $n \times n$ matrix of nonnegative entries, $\beta_i > 0$, $\alpha_i > 0$ and $v_i \in \mathbb{R}^2$. As it turns out, the solvability of this system depends on some conditions on the matrix (a_{ij}), on β_i and (to some extent) on v_i's, but not on α_i (as long as these are positive). Thus, we will assume $\alpha_i = 1$.

Before discussing the analysis of this system we describe a possible motivation for studying it, which is originated from the celebrated Keller-Segel system.

The Keller-Segel system represents the evolution of living cells under self-attraction and diffusive forces [15], [18]. Its general form is given by

$$\frac{\partial \rho}{\partial t} = \Delta \rho - \nabla \cdot \rho \left(a \nabla_x u \right) \; ; \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+ \tag{1.1}$$

where $a > 0$, $\rho = \rho(x,t)$ stands for the distribution of living cells and $u = u(x,t)$ is a self-induced potential describing the concentration of the chemical substance attracting the cells. In the parabolic/elliptic limit this concentration is given by the

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Newtonian potential

$$u(x,t) := -\frac{1}{2\pi}\int_{\mathbb{R}^2} \rho(y,t)\ln|x-y|d^2y \ , \quad i.e \quad -\Delta u = \rho \ . \tag{1.2}$$

Since (1.1) is a parabolic equation of divergence type it follows that the *total population number* $\int \rho d^2x := \beta > 0$ is conserved in time under suitable boundary conditions at infinity. The steady states of (1.1,1.2) takes the form of *Liouville's Equation*

$$\Delta u(x) + \frac{\beta e^{au(x)}}{\int_{\mathbb{R}^2} e^{au(z)}d^2z} = 0 \ . \tag{1.3}$$

The spacial dimension 2 which we discuss here was studied by many authors [3, 4, 5, 6]. The two dimensional case is special in the sense that there is a critical mass $\beta_c = 8\pi/a$. If $\beta < \beta_c$ then, under some natural assumptions on the initial data $\rho(x,0) := \rho_0$, the solutions exists globally in time and, moreover, $\lim_{t\to\infty}\rho(x,t) = 0$ locally uniformly on $\mathbb{R}^2$ [3]. In particular, there is no solution of (1.3). If $\beta > \beta_c$ then there is no global in time solution of (1.1, 1.2) [13] and, again, no solution of (1.3) exists. In the case $\beta = \beta_c$ there is a family of solutions of (1.3) and the (free-energy) solutions of (1.1, 1.2) exist globally in time. Moreover, if the initial data has finite second moment then any such solution converges asymptotically to the Dirac measure $\beta_c\delta_0$ [6], otherwise, any radial solution to (1.1, 1.2) converges asymptotically to one of the solutions of (1.3) [5].

In the sub-critical case $\beta \le \beta_c$ it is natural to ask whether there exists self similar solutions of (1.1,1.2) of the form

$$\rho(x,t) := (2t)^{-1}\bar{\rho}\left(\frac{x}{\sqrt{2t}}, \frac{1}{2}\ln 2t\right) \ , \quad u(x,t) = \bar{u}\left(\frac{x}{\sqrt{2t}}, \frac{1}{2}\ln 2t\right) \ . \tag{1.4}$$

where $t > 0$.

It follows that

$$\bar{u}(y,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\bar{\rho}(x,t)\ln|x-y|d^2x - \frac{\beta}{2\pi}t$$

in particular $\nabla_x u(x,t) = (2t)^{-1/2}\nabla_y\bar{u}(x/\sqrt{2t}, \frac{1}{2}\ln 2t)$. Substituting in the KS equation we get under the change of variables $x \to \frac{x}{\sqrt{2t}}$, $t \to \frac{1}{2}\ln 2t$,

$$\partial_t\bar{\rho} = \Delta\bar{\rho} - \nabla\cdot\bar{\rho}\left(a\nabla\bar{u} - x\right). \tag{1.5}$$

The corresponding steady state of (1.5) is

$$\Delta_x\bar{u} + \frac{\beta e^{a\bar{u}-|x|^2/2}}{\int_{\mathbb{R}^2} e^{a\bar{u}(z)-|z|^2/2}d^2z} = 0 \tag{1.6}$$

The existence and uniqueness (up to a constant) of the solutions to (1.6) in the sub-critical case $\beta < \beta_c$ was given in [10, 7]. In [2] the authors considered the existence of such self-similar solution of (1.4) for sub-critical data. Non existence of solutions of (1.6) in the critical case was also proved in [7].

In this paper we are motivated by a generalization of (1.1, 1.2) to the case of a system of n populations whose densities are given by $\rho_1, \dots, \rho_n$, and assume the presence of $O(t^{-1/2})$ decaying drift forces:

$$\frac{\partial \rho_i}{\partial t} - t^{-1/2} v_i \cdot \nabla \rho_i = \Delta \rho_i - \nabla \cdot \rho_i \left(\sum_{j=1}^{n} a_{ij} \nabla_x u_j \right) \quad ; \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+ \tag{1.7}$$

where $A := (a_{ij})_{n\times n}$ is a symmetric and nonnegative (i.e., $a_{ij} \geq 0$ for all i, j) matrix,

$$u_i(x,t) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \rho_i(y,t) \ln |x-y| d^2 y \tag{1.8}$$

and $v_i \in \mathbb{R}^2$ are constant vectors.

In the case $v_i = 0$ the stationary solution of such systems, subjected to the initial data satisfying $\int \rho_i(x,0) d^2 x = \beta_i$ solves the *Liouville's systems:*

$$\Delta u_i + \frac{\beta_i e^{\sum_j a_{ij} u_j}}{\int_{\mathbb{R}^2} e^{\sum_j a_{ij} u_j(z)} d^2 z} = 0 \ . \tag{1.9}$$

Again, such Liouville's systems have been studied intensively in [11, 22, 16], and the cases where a_{ij} are not necessarily nonnegative (in connection with the chemotactic system known as the conflict case) have also been explored in [12, 24].

The solvability of such systems was considered in [11, 22] and [23]. The criticality condition is determined, in that case, by the functions

$$\Lambda_J(\boldsymbol{\beta}) = \sum_{i \in J} \beta_i \left(8\pi - \sum_{j \in J} a_{ij} \beta_j \right) .$$

where $\phi \neq J \subseteq I := \{1, \ldots, n\}$. The criticality condition $\beta_c = 8\pi/a$ in the case of single composition is replaced by

$$\Lambda_I(\boldsymbol{\beta}) = 0 \ .$$

In particular it was proved in [11] that an entire solution of (1.9) exists *only* in the critical case iff, in addition, $\Lambda_J(\boldsymbol{\beta}) > 0$ for all $\phi \neq J \subsetneq I$ hold.

Under the scaling (1.4) we recover the modified KS system from (1.7)

$$\frac{\partial \bar{\rho}_i}{\partial t} = \Delta \bar{\rho}_i - \nabla \cdot \bar{\rho}_i \left(\sum_{j=1}^{n} a_{ij} \nabla_x \bar{u}_j - (x - v_i) \right) \quad ; \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+ \tag{1.10}$$

where

$$\bar{u}_i(x,t) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \bar{\rho}_i(y,t) \ln |x-y| d^2 y - \frac{\beta_i}{2\pi} t \ . \tag{1.11}$$

The steady states of (1.10, 1.11) are given by the modified Liouville's system

$$\Delta_x \bar{u}_i + \frac{\beta_i e^{\sum_j a_{ij} \bar{u}_j - |x - v_i|^2/2}}{\int_{\mathbb{R}^2} e^{\sum_j a_{ij} \bar{u}_j(z) - |z - v_i|^2/2} d^2 z} = 0. \tag{1.12}$$

Note that if $n = 1$, $(a_{ij}) \equiv a$ and $v \equiv v_i$ then the system (1.10) is reduced, under the shift $x \to x - v$ to the modified Liouville's *equation*

$$\Delta_x \bar{u} + \frac{\beta e^{a\bar{u} - |x|^2/2}}{\int_{\mathbb{R}^2} e^{a\bar{u}(z) - |z|^2/2} d^2 z} = 0 \tag{1.13}$$

which is independent of v. The same holds for the system (1.12) only when $v_1 = v_2 = \cdots = v_n$. The modified KS system (1.10, 1.11) and the modified Liouville's system (1.12) are closely related to the *Free energy functional*

(1.14)
$$\mathcal{F}_{\boldsymbol{v}}(\bar{\boldsymbol{\rho}}) := \sum_{i=1}^{n} \int_{\mathbb{R}^2} \bar{\rho}_i(x) \ln \bar{\rho}_i(x) d^2x + \frac{1}{4\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{\rho}_i(x) \ln |x-y| \bar{\rho}_j(y) d^2x d^2y + \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^2} |x - v_i|^2 \bar{\rho}_i(x) d^2x,$$

defined over the set
$$\Gamma^{\boldsymbol{\beta}} := \left\{ \bar{\boldsymbol{\rho}} = (\bar{\rho}_1, \ldots, \bar{\rho}_n) |\ \bar{\rho}_i \geq 0, \int_{\mathbb{R}^2} \bar{\rho}_i \ln \bar{\rho}_i < \infty, \int_{\mathbb{R}^2} |x|^2 \bar{\rho}_i < \infty, \int_{\mathbb{R}^2} \bar{\rho}_i = \beta_i, \forall i \right\}.$$

Indeed, we observe formally that (1.10, 1.11) can be written as a gradient descend system in the Wasserstein sense [1]

$$\frac{\partial \bar{\rho}_i}{\partial t} = \nabla \cdot \left(\bar{\rho}_i \nabla \left(\frac{\delta \mathcal{F}_{\boldsymbol{v}}}{\delta \bar{\rho}_i} \right) \right) , i = 1, \ldots, n, \tag{1.15}$$

and, in particular

$$\frac{d}{dt} \mathcal{F}_{\boldsymbol{v}}(\bar{\boldsymbol{\rho}}) = - \sum_i \int_{\mathbb{R}^2} \rho_i \left| \nabla \frac{\delta \mathcal{F}_{\boldsymbol{v}}}{\delta \bar{\rho}_i} \right|^2 . \tag{1.16}$$

Every critical point of $\mathcal{F}_{\boldsymbol{v}}$ on $\Gamma^{\boldsymbol{\beta}}$ induces a solution of (1.12) [11], [21]. In particular, any minimizer is such a solution. Moreover, we expect such minimizers to be a stable stationary solutions of (1.10, 1.11) and thus to represent stable self similar limit of (1.7, 1.8).

Unless otherwise stated, in this article we assume the matrix $A = (a_{ij})_{n \times n}$ satisfies

$$(H) \qquad \textit{A is symmetric and nonnegative},$$

and $\boldsymbol{\beta}$ satisfies

$$\begin{cases} \Lambda_J(\boldsymbol{\beta}) \geq 0, \text{ for all } \emptyset \neq J \subseteq I, \\ \text{if, for some } J \neq \emptyset \ , \ \Lambda_J(\boldsymbol{\beta}) = 0, \text{ then } a_{ii} + \Lambda_{J \setminus \{i\}} > 0, \forall i \in J. \end{cases} \tag{1.17}$$

Let
$$Var(v_1, \ldots, v_n) := \min_{x \in \mathbb{R}^2} \sum_{1}^{n} |x - v_i|^2 \ .$$

The main result of this article is:

Theorem 1.1. *Suppose A satisfies (H) and $\boldsymbol{\beta}$ satisfies* (1.17). *Then*
- (a) (1.17) *is necessary and sufficient condition for the boundedness from below of $\mathcal{F}_{\boldsymbol{v}}$ on $\Gamma^{\boldsymbol{\beta}}$.*
- (b) *If $\Lambda_J(\boldsymbol{\beta}) > 0$ for all $\emptyset \neq J \subseteq I$, then there exists a minimizer of $\mathcal{F}_{\boldsymbol{v}}$ on $\Gamma^{\boldsymbol{\beta}}$, for all $(v_1, \ldots, v_n) \in (\mathbb{R}^2)^n$.*

(c) *If* $\Lambda_I(\boldsymbol{\beta}) = 0$ *and* $Var(v_1, \ldots, v_n) = 0$ *then there is no minimizer of* $\mathcal{F}_{\boldsymbol{v}}$ *in* $\Gamma^{\boldsymbol{\beta}}$.

(d) *If* $n = 2$ *and* $\Lambda_{\{1,2\}}(\boldsymbol{\beta}) = 0$, $\Lambda_{\{1\}}(\boldsymbol{\beta}), \Lambda_{\{2\}}(\boldsymbol{\beta}) > 0$ *and* $|v_1 - v_2|$ *is large enough then there exists a minimizer of* $\mathcal{F}_{\boldsymbol{v}}$ *on* $\Gamma^{\boldsymbol{\beta}}$.

For a given such matrix A, we define

Definition 1.2.
- $\boldsymbol{\beta}$ is sub-critical if $\Lambda_J(\boldsymbol{\beta}) > 0$ for any $\emptyset \neq J \subseteq I$.
- $\boldsymbol{\beta}$ is critical if $\Lambda_I(\boldsymbol{\beta}) = 0$ and $\Lambda_J(\boldsymbol{\beta}) > 0$ for any $\emptyset \neq J \subset I$.

Theorem 1.3.

(a) *There exists a solution of* (1.12) *for any sub-critical* $\boldsymbol{\beta}$ *and any* $v_1, \ldots, v_n \in \mathbb{R}^2$.

(b) *If* $\boldsymbol{\beta}$ *is critical,* $Var(v_1, \ldots, v_n) = 0$, *and* A *is invertible and irreducible, then there is no solution to* (1.12).

(c) *There exists a solution of* (1.12) *for* $n = 2$ *in the critical case provided* $|v_1 - v_2|$ *is large enough.*

Remark 1.
- Theorem 1.3-a,c follows immediately from Theorem 1.1-a,b,d.
- Theorem 1.1-c implies the non-existence of minimizers in the critical case. The non-existence of solutions in the critical case (Theorem 1.3-c) follows from a different argument.
- The results of Theorem 1.1-d and Theorem 1.3-c can be easily extended to the case $n > 2$, provided $Var(v_1, \ldots, v_n)$ is large enough. It is not known whether $Var(v_1, \ldots, v_n) \neq 0$ is sufficient for existence of solutions of (1.12) in the critical case for any $n \geq 2$.

Our organization of the article is as follows: in Section 2 we discuss the boundedness from below of the functional $\mathcal{F}_{\boldsymbol{v}}$ over $\Gamma^{\boldsymbol{\beta}}$. Section 3 is devoted to the basic lemmas required for the proof of our main theorem. In Section 4 we proved the existence of minimizers for sub critical $\boldsymbol{\beta}$. The critical case has been analyzed in Sections 5 and 6 and we established a sufficient criterion (Proposition 6.1) for the existence of minimizers. More precisely, we proved that a minimizer exists if strict inequality holds in (5.3). At the end of this article we exhibited certain examples (when $Var(v_1, v_2)$ large) for which the minimum is actually attained and proved the nonexistence result (Theorem 2(b)) when $Var(v_1, \ldots, v_n) = 0$.

2. Boundedness from below

Since we can shift $(v_1, \ldots, v_n)$ by any constant vector we can set $v_1 = v_2 = \cdots = v_n = 0$ if $Var(v_1, \ldots, v_n) = 0$. The functional $\mathcal{F}_{\boldsymbol{v}}$ will be denoted by $\mathcal{F}_0$ in that case. Also, we omit the bars from $\bar{\rho}_i$ from now on.

We will actually prove the boundedness from below of a little more general functional. For $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}_+)^n$, (where $\mathbb{R}_+$ is the set of all positive real numbers) define

$$\begin{aligned}\mathcal{F}_{\boldsymbol{v},\boldsymbol{\alpha}}(\boldsymbol{\rho}) :=& \sum_{i=1}^{n} \int_{\mathbb{R}^2} \rho_i(x) \ln \rho_i(x) d^2x \\ &+ \frac{1}{4\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_i(x) \ln |x-y| \rho_j(y) d^2x d^2y \end{aligned} \tag{2.1}$$

$$+\sum_{i=1}^{n}\alpha_i\int_{\mathbb{R}^2}|x-v_i|^2\rho_i(x)d^2x.$$

When $v_i=0$ for all i, it will be denoted by $\mathcal{F}_{0,\boldsymbol{\alpha}}$.

Theorem 2.1. *Condition* (1.17) *is necessary and sufficient condition for the boundedness from below of* $\mathcal{F}_{\boldsymbol{v},\boldsymbol{\alpha}}$ *on* $\Gamma^{\boldsymbol{\beta}}$.

Proof. First we recall [11, 22] that if $\boldsymbol{\rho}$ is supported in a given bounded set then $\mathcal{F}_{\boldsymbol{v},\mathbf{0}}$ is bounded from below iff (1.17) is satisfied. This implies the necessary part. For the sufficient part we know from the same references that (1.17) together with the condition $\Lambda_I(\boldsymbol{\beta})=0$ imply that $\mathcal{F}_{\boldsymbol{v},\mathbf{0}}$ is bounded from below. We only need to show that for any positive $\boldsymbol{\alpha}$ we still obtain the bound from below in the case $\Lambda_I(\boldsymbol{\beta})>0$. Note also that since $|x-v|^2>|x|^2/2-C$ for any $x\in\mathbb{R}^2$ and C depending on $|v|$ it is enough to prove the sufficient condition for $\boldsymbol{v}=0$.

The proof is a straight forward adaptation of the corresponding proof in [22] without the potential $|x|^2$. For $\boldsymbol{\rho}=(\rho_1,\dots,\rho_n)\in\Gamma^{\boldsymbol{\beta}}$ let ρ_i^* be the symmetric decreasing rearrangement of ρ_i. Then clearly we have

$$\int_{\mathbb{R}^2}\rho_i\ln\rho_i=\int_{\mathbb{R}^2}\rho_i^*\ln\rho_i^*,\ \int_{\mathbb{R}^2}\rho_i|\ln\rho_i|=\int_{\mathbb{R}^2}\rho_i^*|\ln\rho_i^*|,\ \int_{\mathbb{R}^2}|x|^2\rho_i^*\le\int_{\mathbb{R}^2}|x|^2\rho_i.$$

Thus if we define $\boldsymbol{\rho}^*=(\rho_1^*,\dots,\rho_n^*)$ then $\boldsymbol{\rho}^*\in\Gamma^{\boldsymbol{\beta}}$. Furthermore, we have (see [8, 22])

$$\int_{\mathbb{R}^2}\rho_i^*(x)\ln|x-y|\rho_j^*(y)\le\int_{\mathbb{R}^2}\rho_i(x)\ln|x-y|\rho_j(y),\ \forall i,j.$$

and hence $\mathcal{F}_{0,\boldsymbol{\alpha}}(\boldsymbol{\rho}^*)\le\mathcal{F}_{0,\boldsymbol{\alpha}}(\boldsymbol{\rho})$. Therefore it is enough to prove the theorem for radially symmetric decreasing function of $|x|$. Let $\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}$ be a radially symmetric decreasing function of $r=|x|$. As in [11, 22] we define

$$m_i(r)=2\pi\int_0^r\tau\rho_i(\tau)\ d\tau,\ r\in(0,\infty),$$
$$u_i(x)=-\frac{1}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|\rho_i(y)\ d^2y.$$

Then we get using $\int_{\mathbb{R}^2}|x|^2\rho_i<\infty$ and [22, equation (5.6)]

$$\begin{cases}\lim_{R\to\infty}\left[u_i(R)+\frac{\beta_i}{2\pi}\ln R\right]=0,\\ \lim_{R\to\infty}(\beta_i-m_i(R))R^2=0.\end{cases}\tag{2.2}$$

Furthermore, by density we can assume the support of $\boldsymbol{\rho}$ lies within the ball $B(0,\tilde{R})$. Therefore, for any $R>\tilde{R}$

$$\mathcal{F}_{0,\boldsymbol{\alpha}}(\boldsymbol{\rho}):=\sum_{i=1}^{n}\int_{B(0,R)}\rho_i\ln\rho_i-\frac{1}{2}\sum_i\sum_j a_{ij}\int_{B(0,R)}\rho_iu_j+\sum_{i=1}^{n}\alpha_i\int_{B(0,R)}|x|^2\rho_i(x),$$

Again following [22], we define $w_i(s)=m_i(e^s)$. Then

$$\sum_{i=1}^{n}\alpha_i\int_{B(0,R)}|x|^2\rho_i(x)\ d^2x=\sum_{i=1}^{n}2\pi\alpha_i\int_0^R r^3\rho_i(r)\ dr$$

$$
\begin{aligned}
&= \sum_{i=1}^n \alpha_i \int_0^R r^2 m_i'(r)\ dr \\
&= -2\sum_{i=1}^n \alpha_i \int_0^R r m_i(r)\ dr + \sum_{i=1}^n \alpha_i m_i(R) R^2 \\
&= -2\sum_{i=1}^n \alpha_i \int_{-\infty}^{\ln R} e^{2s} w_i(s)\ ds + \sum_{i=1}^n \alpha_i m_i(R) R^2
\end{aligned}
$$

and therefore we can write $\mathcal{F}_{0,\boldsymbol{\alpha}}(\boldsymbol{\rho}) = G_R(w) - (\ln 2\pi)\sum_{i=1}^n m_i(R)$, where

$$
\begin{aligned}
G_R(w) = &\int_{-\infty}^{\ln R} \sum_{i=1}^n w_i' \ln w_i'\ ds + \int_{-\infty}^{\ln R}\left[2\sum_{i=1}^n w_i - \frac{1}{4\pi}\sum_{i,j=1}^n a_{ij} w_i w_j\right] ds \\
&- 2\sum_{i=1}^n \alpha_i \int_{-\infty}^{\ln R} e^{2s} w_i\ ds - \sum_{i=1}^n m_i(R)\left(2\ln R + \frac{1}{2}\sum_{j=1}^n a_{ij} u_j(R) - \alpha_i R^2\right).
\end{aligned}
$$

Now define $\nu_i = 2 - \frac{1}{4\pi}\sum_{j=1}^n a_{ij}\beta_j$. Using the identity $\frac{\Lambda_I(\boldsymbol{\beta})}{4\pi} = \sum_{i=1}^n \nu_i\beta_i$ and (2.2) we get

(2.3)

$$
-\sum_{i=1}^n m_i(R)\left[2\ln R + \frac{1}{2}\sum_{j=1}^n a_{ij}u_j(R)\right] + \sum_{i=1}^n 2\nu_i\beta_i \ln R = \frac{\Lambda_I(\boldsymbol{\beta})}{4\pi}\ln R + o_R(1),
$$

where $o_R(1)$ stands for a quantity going to zero as $R \to \infty$. Utilizing (2.3), we can decompose $G_R(w)$ as follows

$$
G_R(w) = J_{-\infty}(w) + J_\infty(w) + E_R(w) + o_R(1),
$$

where

$$
\begin{aligned}
J_{-\infty}(w) = &\int_{-\infty}^0 \sum_{i=1}^n w_i' \ln w_i'\ ds + \int_{-\infty}^0 \left[2\sum_{i=1}^n w_i - \frac{1}{4\pi}\sum_{i,j=1}^n a_{ij} w_i w_j\right] ds \\
&- 2\sum_{i=1}^n \alpha_i \int_{-\infty}^0 e^{2s} w_i\ ds, \\
J_\infty(w) = &\int_0^{\ln R} \sum_{i=1}^n w_i' \ln w_i'\ ds \\
&+ \int_0^{\ln R}\left[\sum_{i=1}^n 2(1-\nu_i) w_i - \frac{1}{4\pi}\sum_{i,j=1}^n a_{ij} w_i w_j + \frac{\Lambda_I(\boldsymbol{\beta})}{4\pi}\right] ds \\
E_R(w) = &-2\sum_{i=1}^n \alpha_i \int_0^{\ln R} e^{2s} w_i\ ds + \sum_{i=1}^n 2\nu_i \int_0^{\ln R} w_i\ ds - 2\left(\sum_{i=1}^n \nu_i\beta_i\right)\ln R \\
&+ \sum_{i=1}^n \alpha_i m_i(R) R^2.
\end{aligned}
$$

By [22] we have $J_{-\infty}$ and J_∞ are bounded from below on $\Gamma^{\boldsymbol{\beta}}$, once we observe that

$$\int_{-\infty}^{0} e^{2s} w_i \leq \beta_i \int_{-\infty}^{0} e^{2s} = \frac{\beta_i}{2}.$$

Therefore, we only need to show that $E_R(w)$ is bounded from below. We can rewrite $E_R(w)$ in the following way

$$\begin{aligned} E_R(w) = & \int_0^{\ln R} \left[\sum_{i=1}^{n} 2\left(\nu_i - \alpha_i e^{2s}\right) w_i - 2\sum_{i=1}^{n} \nu_i\beta_i + 2\sum_{i=1}^{n} \alpha_i\beta_i e^{2s} \right] \, ds \\ & + \sum_{i=1}^{n} \alpha_i \beta_i + o(1) \\ = & \int_0^{\ln R} \left[2\sum_{i=1}^{n} (\beta_i - w_i(s))(\alpha_i e^{2s} - \nu_i) \right] \, ds + \sum_{i=1}^{n} \alpha_i\beta_i + o(1). \end{aligned}$$

Now $w_i(s) \leq \beta_i$ for all s and $\alpha_i > 0, \nu_i$ are being fixed numbers, we can find a $R_0 > 0$, independent of w_i such that $(\beta_i - w_i(s))(\alpha_i e^{2s} - \nu_i) \geq 0$ for all $s \geq \ln R_0$. Again since

$$\left| \int_0^{\ln R_0} \left[2\sum_{i=1}^{n} (\beta_i - w_i(s))(\alpha_i e^{2s} - \nu_i) \right] \, ds \right| \leq \sum_{i=1}^{n} 4\beta_i \left(\frac{\alpha_i}{2} R_0^2 - \nu_i \ln R_0 - \frac{\alpha_i}{2} \right).$$

we have $E_R(w) \geq -|E_{R_0}(w)| \geq -C$. This proves the sufficiency of the condition (1.17). □

3. Basic Lemmas

In this section we will recall a few definitions and lemmas and also prove some basic ingredients required for the proof of our main results. We define the space $\mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ as the Orlicz space determined by the N-function $N(t) = (1+t)\ln(1+t) - t, t \geq 0$:

$$\mathbb{L}\ln\mathbb{L}(\mathbb{R}^2) := \left\{ \rho : \mathbb{R}^2 \to \mathbb{R} \text{ measurable } : \int_{\mathbb{R}^2} [(1+|\rho|)\ln(1+|\rho|) - |\rho|] d^2x < \infty \right\}.$$

Then $\mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ is a Banach space with respect to the Luxemberg norm (because $N(t)$ satisfies the Δ_2 condition: $N(2t) \leq 2N(t)$ for all $t \geq 0$).

The dual space of $\mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ is the Orlicz space determined by the N-function $M(t) = (e^t - t - 1), t \geq 0$. It is important to remark that $\mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ is not reflexive (because $M(t)$ does not satisfy the Δ_2 condition). However, there is a notion of weak convergence which is slightly weaker than the usual weak convergence in Banach spaces. A sequence $\rho_m \in \mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ is said to converge L_M-weakly to ρ if

$$\int_{\mathbb{R}^2} \rho_m \phi \to \int_{\mathbb{R}^2} \rho\phi, \text{ for all bounded measurable functions } \phi \text{ with bounded support.}$$

It is well known from the general Orlicz space theory [14] that $\mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ is L_M-weakly compact. To simplify our notations we will denote the weak convergence (in the above sense) by $\rho_m \rightharpoonup \rho$.

We begin with the following elementary lemma whose proof can be found in [3]:

Lemma 3.1. *For $1 \le i \le n$ let $\rho_i \in L^1(\mathbb{R}^2)$ be such that $\rho_i \ge 0$ and satisfies*

$$\int_{\mathbb{R}^2} \rho_i \ln \rho_i \le C_0, \int_{\mathbb{R}^2} |x|^2 \rho_i \le C_0.$$

Then

$$\sum_{i=1}^n \int_{\mathbb{R}^2} \rho_i |\ln \rho_i| \le \sum_{i=1}^n \int_{\mathbb{R}^2} \rho_i \ln \rho_i + 2 \ln 2\pi \left(\sum_{i=1}^n \int_{\mathbb{R}^2} \rho_i \right) + 2 \sum_{i=1}^n \int_{\mathbb{R}^2} |x|^2 \rho_i + 2ne^{-1}.$$

Lemma 3.2. *Let $\{\rho_m\}$ be a sequence in $\mathbb{L} \ln \mathbb{L}(\mathbb{R}^2)$ such that*

$$\int_{\mathbb{R}^2} \rho_m \ln \rho_m \le C_0, \ \int_{\mathbb{R}^2} \rho_m = \beta, \ \int_{\mathbb{R}^2} |x|^2 \rho_m \le C_0.$$

Then there exists $\rho \in \mathbb{L} \ln \mathbb{L}(\mathbb{R}^2)$ such that up to a subsequence $\rho_m \rightharpoonup \rho$ in the weak topology of $\mathbb{L} \ln \mathbb{L}(\mathbb{R}^2)$ and satisfies

$$\int_{\mathbb{R}^2} \rho \ln \rho \le \liminf_{n \to \infty} \int_{\mathbb{R}^2} \rho_m \ln \rho_m. \tag{3.1}$$

Remark 2. The conclusion of the lemma is false without the assumption on the uniform boundedness of $\int_{\mathbb{R}^2} |x|^2 \rho_m$. As a counter example, let $\phi \in C_c^\infty(\mathbb{R}^2)$ be a smooth cutoff function such that $0 \le \phi \le 1 - \delta$, for some $\delta \in (0, 1)$. Let x_m be a sequence in $\mathbb{R}^2$ such that $|x_m| \nearrow \infty$ and define the sequence

$$\rho_m(x) = \phi(x + x_m).$$

Then it is easy to check that $\int_{\mathbb{R}^2} |x|^2 \rho_m \to \infty$, and

$$\int_{\mathbb{R}^2} \rho_m \ln \rho_m = \int_{\mathbb{R}^2} \phi \ln \phi < 0, \text{ for all } m.$$

But $\rho_m \rightharpoonup \rho \equiv 0$ in $\mathbb{L} \ln \mathbb{L}(\mathbb{R}^2)$ and hence $\int_{\mathbb{R}^2} \rho \ln \rho = 0$. Therefore the assumption $\int_{\mathbb{R}^2} |x|^2 \rho_m$ bounded is a necessary condition for the Fatou's type estimate (3.1) to hold true.

We need some supplementary lemmas to prove Lemma 3.2.

Lemma 3.3. *Let $\{\rho_m\}$ be a sequence in $\mathbb{L} \ln \mathbb{L}(\mathbb{R}^2)$ such that*

$$\int_{\mathbb{R}^2} \rho_m \ln \rho_m \le C_0, \ \int_{\mathbb{R}^2} \rho_m = \beta, \ \int_{\mathbb{R}^2} |x|^2 \rho_m \le C_0. \tag{3.2}$$

Then there exists a $\rho \in L^1(\mathbb{R}^2, (1+|x|^2)d^2x)$ such that (up to a subsequence) $\rho_m \rightharpoonup \rho$ weakly in $L^1(\mathbb{R}^2)$, i.e.,

$$\int_{\mathbb{R}^2} \rho_m g \to \int_{\mathbb{R}^2} \rho g, \ \textit{for all } g \in L^\infty(\mathbb{R}^2).$$

Proof. By Lemma 3.1, the assumption (3.2) implies that

$$\int_{\mathbb{R}^2} |\rho_m| |\ln \rho_m| \le C.$$

for some constant C, and hence $\int_{\mathbb{R}^2}[(1+\rho_m)\ln(1+\rho_m)-\rho_m]$ is uniformly bounded. Since $\int_{\mathbb{R}^2}\rho_m=\beta$ by weak* compactness in L^1 there exists a finite measure μ on $\mathbb{R}^2$ such that

$$\int_{\mathbb{R}^2}\rho_m\phi\to\int_{\mathbb{R}^2}\phi\,d\mu,\ \text{for all } \phi\in C_0(\mathbb{R}^2).$$

Furthermore, the uniform boundedness of $\int_{\mathbb{R}^2}[(1+\rho_m)\ln(1+\rho_m)-\rho_m]$ implies μ has a density $\rho\in L^1_{loc}(\mathbb{R}^2)$. Now we claim that $\int_{\mathbb{R}^2}|x|^2\rho<+\infty$. To prove it we let $\phi\in C_0(\mathbb{R}^2)$ be such that $\phi(x)=|x|^2$ in $B(0,R), 0\le\phi\le|x|^2$ in $\mathbb{R}^2$. Then by (3.2) and L^1 weak* convergence we get

$$\int_{\{|x|<R\}}|x|^2\rho\le\int_{\mathbb{R}^2}\rho\phi=\lim_{m\to\infty}\int_{\mathbb{R}^2}\rho_m\phi\le C_0.$$

Letting $R\to\infty$ we reach at the desired claim. Moreover, the assumption $\int_{\mathbb{R}^2}|x|^2\rho_m\le C_0$ gives $\int_{\mathbb{R}^2}\rho=\beta$. Therefore, by Portmanteau's theorem

$$\int_{\mathbb{R}^2}\rho_m\phi\to\int_{\mathbb{R}^2}\rho\phi, \tag{3.3}$$

for all bounded continuous functions ϕ on $\mathbb{R}^2$. Using Lusin's theorem and Tietz's extension theorem we can extend this result to $\phi\in L^\infty(\mathbb{R}^2)$. □

Lemma 3.4. *The set*

$$\mathcal{S}:=\left\{\rho\in L^1(\mathbb{R}^2):\rho\ge0,\int_{\mathbb{R}^2}\rho\ln\rho\le\alpha,\int_{\mathbb{R}^2}\rho=\beta,\int_{\mathbb{R}^2}|x|^2\rho\le C_0\right\}$$

is a weakly closed subset in $L^1(\mathbb{R}^2)$.

Proof. We will show that the set $\mathcal{S}$ is a convex and strongly closed subset of $L^1(\mathbb{R}^2)$. Then by Mazur's lemma it will imply the weak closeness of $\mathcal{S}$. Again by the convexity of $t\ln t$ we only need to show that $\mathcal{S}$ is strongly closed in $L^1(\mathbb{R}^2)$. Let $\{\rho_m\}_m$ be a sequence in $L^1(\mathbb{R}^2)$ such that $\rho_m\to\rho$ in $L^1(\mathbb{R}^2)$. Let ρ_m^*,ρ^* be the symmetric decreasing rearrangement of ρ_m and ρ respectively. Then $\rho_m^*\to\rho^*$ in $L^1(\mathbb{R}^2)$ and up to a subsequence ρ_m (respectively ρ_m^*) converges pointwise a.e. in $\mathbb{R}^2$. By strong convergence and Fatou's lemma we have

$$\int_{\mathbb{R}^2}\rho=\beta,\ \int_{\mathbb{R}^2}|x|^2\rho\le C_0.$$

Furthermore, by Lemma 3.1 and the pointwise convergence we obtain

$$\int_{\mathbb{R}^2}\rho|\ln\rho|<+\infty.$$

To conclude the proof of the lemma we will show that $\int_{\mathbb{R}^2}\rho^*\ln\rho^*\le\alpha$. Using Fatou's lemma we get

$$\int_{B(0,R)}\rho^*\ln\rho^*\le\liminf\int_{B(0,R)}\rho_m^*\ln\rho_m^*,$$

for any $R > 0$. Now to estimate for $|x| > R$ we will use the bound $0 \leq \rho^*(|x|) \leq \frac{\beta}{\pi|x|^2}$. The bound follows from

$$\beta = \int_{\mathbb{R}^2} \rho = \int_{\mathbb{R}^2} \rho^* = 2\pi \int_0^\infty s\rho^*(s)ds \geq 2\pi \int_0^r s\rho^*(s)ds \geq \pi r^2 \rho^*(r).$$

Choosing $\epsilon \in (0, \frac{1}{2})$ and using $\ln(1/t) \leq 1/t$ for $t < 1$ we get, after multiplying by ϵ and using $\rho^*(x) < 1$ for sufficiently large R

$$\begin{aligned}
\int_{\{|x|>R\}} \rho_m^* |\ln \rho_m^*| &\leq \frac{1}{\epsilon} \int_{\{|x|>R\}} \rho_m^* \frac{1}{(\rho_m^*)^\epsilon} \\
&= \frac{1}{\epsilon} \int_{\{|x|>R\}} (\rho_m^*)^{1-\epsilon}, \\
&= \frac{1}{\epsilon} \int_{\{|x|>R\}} \frac{\left(|x|^2 \rho_m^*\right)^{1-\epsilon}}{|x|^{2(1-\epsilon)}}, \\
&\leq \frac{1}{\epsilon} \left(\int_{\{|x|>R\}} |x|^2 \rho_m^* \right)^{1-\epsilon} \left(\int_{\{|x|>R\}} |x|^{2(1-\frac{1}{\epsilon})} \right)^{\epsilon}, \\
&= O\left(\frac{1}{R^{2(\frac{1}{\epsilon}-2)}} \right)
\end{aligned}$$

Thus we obtain

$$\int_{B(0,R)} \rho_m^* \ln \rho_m^* \leq \int_{\mathbb{R}^2} \rho_m^* \ln \rho_m^* + O\left(\frac{1}{R^{2(\frac{1}{\epsilon}-2)}} \right),$$

and hence

$$\int_{B(0,R)} \rho^* \ln \rho^* \leq \liminf \int_{\mathbb{R}^2} \rho_m^* \ln \rho_m^* + O\left(\frac{1}{R^{2(\frac{1}{\epsilon}-2)}} \right).$$

Letting $R \to \infty$ we get the desired result. □

Proof of Lemma 3.2:

Proof. Define $\alpha = \liminf \int_{\mathbb{R}^2} \rho_m \ln \rho_m + \epsilon$, where $\epsilon > 0$ is a small fixed number. Let ρ_{m_k} be a subsequence such that $\lim \int_{\mathbb{R}^2} \rho_{m_k} \ln \rho_{m_k} = \liminf \int_{\mathbb{R}^2} \rho_m \ln \rho_m$. By Lemma 3.3, up to a subsequence ρ_{m_k} converges to some ρ weakly in $L^1(\mathbb{R}^2)$. Since for sufficiently large k, $\rho_{m_k} \in \mathcal{S}$, which is weak L^1-closed by Lemma 3.4, we conclude that $\rho \in \mathcal{S}$ and hence

$$\int_{\mathbb{R}^2} \rho \ln \rho \leq \liminf \int_{\mathbb{R}^2} \rho_m \ln \rho_m + \epsilon.$$

Since $\epsilon > 0$ is arbitrary the proof of the lemma is completed. □

Lemma 3.5. *Let $\rho \in L^1(\mathbb{R}^2)$ satisfies*

$$\int_{\mathbb{R}^2} \rho \ln \rho \leq C_0, \int_{\mathbb{R}^2} \rho = \beta, \int_{\mathbb{R}^2} |x|^2 \rho \leq C_0.$$

Define

$$u(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y| \rho(y) \, d^2y, \text{ for } x \in \mathbb{R}^2. \tag{3.4}$$

Then there exists a constants C, R depending only on C_0 and β such that

$$\left| u(x) + \frac{\beta}{2\pi} \ln|x| \right| \le C, \ \textit{for all } |x| > R.$$

Proof. The proof goes in the same line as in Chen and Li [9] with slight modifications. As in [9] we write

$$\frac{u(x)}{\ln|x|} + \frac{\beta}{2\pi} = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\ln|x-y| - \ln|x|}{\ln|x|} \rho(y)\ d^2y = I_1 + I_2 + I_3,$$

where the integral I_1 is over the domain $\{|x-y| < 1\}, I_2$ is over the domain $\{|x-y| > 1, |y| \le \frac{|x|}{2}\}$ and I_3 is over the domain $\{|x-y| > 1, |y| > \frac{|x|}{2}\}$. We want to show that each I_j is bounded by $C(\beta, C_0)/\ln|x|$. Now

$$|I_1| \le \int_{\{|x-y|<1\}} \rho(y)\ d^2y + \frac{1}{\ln|x|} \int_{\{|x-y|<1\}} |\ln|x-y|| \rho(y)\ d^2y \tag{3.5}$$

Since $\{|x-y| < 1\} \subset \{|y| > |x| - 1\}$, and $\int_{\mathbb{R}^2} |x|^2 \rho \le C_0$ the first integral in (3.5) is bounded by $C(\beta, C_0)/(|x|-1)^2$. To estimate the second integral in (3.5) we divide it into two parts $\{|x-y| > 1, \rho \le 1\}$ and $\{|x-y| > 1, \rho > 1\}$. Clearly,

$$\frac{1}{\ln|x|} \int_{\{|x-y|<1, \rho \le 1\}} |\ln|x-y|| \rho(y)\ d^2y \le \frac{C(\beta, C_0)}{\ln|x|}.$$

Choose $\epsilon \in (0,1)$. Then

$$\begin{aligned}
\int_{\{|x-y|<1, \rho>1\}} |\ln|x-y|| \rho(y)\ d^2y &\le \int_{\{|x-y|<1, \ln\rho < \epsilon \ln \frac{1}{|x-y|}\}} |\ln|x-y|| \rho(y)\ d^2y \\
&+ \int_{\{|x-y|<1, \ln\rho > \epsilon \ln \frac{1}{|x-y|}\}} |\ln|x-y|| \rho(y)\ d^2y \\
&\le \int_{\{|x-y|<1\}} \left(\ln \frac{1}{|x-y|} \right) e^{\epsilon \ln \frac{1}{|x-y|}}\ d^2y \\
&+ \frac{1}{\epsilon} \int_{\{|x-y|<1\}} \rho(y) \ln\rho(y)\ d^2y \\
&\le \int_{\{|x-y|<1\}} \left(\ln \frac{1}{|x-y|} \right) \frac{1}{|x-y|^\epsilon}\ d^2y \\
&+ \frac{1}{\epsilon} \int_{\{|x-y|<1\}} \rho(y) \ln\rho(y)\ d^2y \\
&\le C(\beta, C_0, \epsilon).
\end{aligned}$$

Combining all we get the estimate

$$|I_1| \le C(\beta, C_0) \left[\frac{1}{\ln|x|} + \frac{1}{(|x|-1)^2} \right].$$

To estimate I_2 we see that on the domain $\{|x-y| > 1, |y| \le \frac{|x|}{2}\}, |\ln|x-y| - \ln|x|| \le 1$. Thus

$$|I_2| \le \frac{1}{\ln|x|} \int_{\{|y| \le \frac{|x|}{2}\}} \rho(y)\ d^2y \le \frac{C(\beta)}{\ln|x|}.$$

Now on $I_3, \ln|x-y| \geq 0, |x-y| \leq 3|y|$ and hence $|\ln|x-y| - \ln|x|| \leq \ln 3|y| + \ln|x|$. Therefore

$$\begin{aligned} |I_3| &\leq \frac{1}{\ln|x|}\int_{\{|y|>\frac{|x|}{2}\}} \rho(y)\ln(3|y|)\ d^2y + \int_{\{|y|>\frac{|x|}{2}\}} \rho(y)\ d^2y \\ &\leq C(\beta, C_0)\left[\frac{1}{|x|\ln|x|} + \frac{1}{|x|^2}\right]. \end{aligned}$$

□

We end this section with the following compactness lemma whose proof can be found in [20].

Theorem A. Suppose we have a sequence $\{u_m\} \subset H^1(B(0,2R))$ of weak solutions to

$$-\Delta u_m = f_m, \text{ in } B(0,2R), \tag{3.6}$$

and $\{f_m\} \subset \mathbb{L}\ln\mathbb{L}(B(0,2R))$. Suppose there exists a constant $C < +\infty$ such that

$$||u_m||_{L^1(B(0,2R))} + ||f_m||_{\mathbb{L}\ln\mathbb{L}(B(0,2R))} \leq C. \tag{3.7}$$

Then there exists $u \in H^1_{\text{loc}}(B(0,2R))$ such that

$$||u_m - u||_{H^1(B(0,R))} \to 0, \text{ as } m \to \infty.$$

In [20], the authors actually proved the above compactness theorem for $R = \frac{1}{2}$ but for more general inhomogeneity $\Omega_m \cdot \nabla u_m + f_m$ under some smallness condition on Ω_m. For our purpose we can take $\Omega_m \equiv 0$, and the general R can be dealt with through a simple scaling argument. To be meticulous, define $\tilde{u}_m(x) = u_m(2Rx)$ and $\tilde{f}_m(x) = (2R)^2 f_m(2Rx)$. Then one can easily verify that (3.6),(3.7) holds with u_m, f_m replaced by $\tilde{u}_m, \tilde{f}_m$ in the domain $B(0,1)$. Hence by compactness theorem there exists $\tilde{u} \in H^1_{loc}(B(0,1))$ such that $\tilde{u}_m \to \tilde{u}$ in $H^1(B(0,\frac{1}{2}))$. Scaling back to the original variable we see that $u_m(\cdot) \to u(\cdot) := \tilde{u}(\frac{\cdot}{2R})$ in $H^1(B(0,R))$. We refer the reader to [20] for more details.

4. Existence of minimizers: sub-critical case

In this section we assume $\boldsymbol{\beta}$ is sub-critical (Definition 1.2).

Theorem 4.1. *If $\boldsymbol{\beta}$ is sub-critical then for all $\boldsymbol{v} = (v_1, \ldots, v_n) \in (\mathbb{R}^2)^n$ there exists a minimizer of $\mathcal{F}_{\boldsymbol{v}}$ on $\Gamma^{\boldsymbol{\beta}}$.*

Proof. Let $\boldsymbol{\rho}^m = (\rho_1^m, \ldots, \rho_n^m)$ be a minimizing sequence for $\mathcal{F}_{\boldsymbol{v}}$ on $\Gamma^{\boldsymbol{\beta}}$.

Step 1: $\int_{\mathbb{R}^2} |x - v_i|^2 \rho_i^m$ is uniformly bounded by some constant C_0.

Choose $\delta \in (0, \frac{1}{2})$. By Theorem 2.1

$$\sum_{i=1}^n \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m + \frac{1}{4\pi}\sum_{i=1}^n\sum_{j=1}^n a_{ij}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho_i^m(x)\ln|x-y|\rho_j^m(y)$$
$$+ \sum_{i=1}^n(\frac{1}{2} - \delta)\int_{\mathbb{R}^2}|x - v_i|^2\rho_i^m \geq -C$$

which implies $\mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m) - \delta \sum_{i=1}^n \int_{\mathbb{R}^2} |x - v_i|^2 \rho_i^m \geq -C$. Since along a minimizing sequence $\mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m)$ is bounded above, the conclusion of Step 1 is proved.

Step 2: ρ_i^m are uniformly bounded in $\mathbb{L}\ln\mathbb{L}$.

Since $\boldsymbol{\beta}$ is sub-critical we can choose $\epsilon > 0$, small such that

$$\sum_{i\in J} \beta_i \left(8\pi - \sum_{j\in J}(a_{ij}+\epsilon)\beta_j\right) > 0, \text{ for all } \emptyset \neq J \subset I. \tag{4.1}$$

Define

$$I_{ij}^m := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho_i^m(x) \ln|x-y| \rho_j^m(y).$$

Using Step 1 and the following inequality

$$\ln|x-y| \leq \frac{1}{2}\ln(1+|x|^2) + \frac{1}{2}\ln(1+|y|^2) \quad , \quad |x|^2 > \ln(1+|x|^2)$$

we see that $I_{ij}^m \leq \frac{C_1}{2}(\beta_i + \beta_j)$. Since $\boldsymbol{\beta}$ satisfies (4.1) we obtain by Theorem 2.1

$$\sum_{i=1}^n \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m + \frac{1}{4\pi}\sum_{i=1}^n\sum_{j=1}^n (a_{ij}+\epsilon)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho_i^m(x)\ln|x-y|\rho_j^m(y)$$
$$+ \sum_{i=1}^n \frac{1}{2}\int_{\mathbb{R}^2} |x-v_i|^2 \rho_i^m \geq -C.$$

Therefore we have

$$\mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m) + \frac{\epsilon}{4\pi}\sum_{I_{ij}^m>0} I_{ij}^m - \frac{\epsilon}{4\pi}\sum_{I_{ij}^m<0} |I_{ij}^m| \geq -C.$$

Since along a minimizing sequence $\mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m)$ is bounded we obtain $\sum_{I_{ij}^m<0} |I_{ij}^m|$ is uniformly bounded. Hence $\sum_{i=1}^n \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m$ is bounded above and by Lemma 3.1 we get the uniform bound of $\boldsymbol{\rho}^m$ in $\mathbb{L}\ln\mathbb{L}$.

Step 3: Existence of a limit.

By Lemma 3.2 there exists $\rho_i \in \mathbb{L}\ln\mathbb{L}(\mathbb{R}^2)$ such that up to a subsequence $\rho_i^m \rightharpoonup \rho_i$ in the topology of $\mathbb{L}\ln\mathbb{L}$ and satisfies the inequality

$$\int_{\mathbb{R}^2} \rho_i \ln\rho_i \leq \liminf \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m, \text{ for all } i. \tag{4.2}$$

Furthermore, it also follows from the proof of Lemma 3.2 that

$$\sum_{i=1}^n \frac{1}{2}\int_{\mathbb{R}^2} |x-v_i|^2 \rho_i \leq \liminf \sum_{i=1}^n \frac{1}{2}\int_{\mathbb{R}^2} |x-v_i|^2\rho_i^m, \quad \int_{\mathbb{R}^2} \rho_i = \beta_i, \tag{4.3}$$

and hence $\boldsymbol{\rho} := (\rho_1, \ldots, \rho_n) \in \Gamma^{\boldsymbol{\beta}}$. To complete the proof of the theorem we need to show that

$$\int_{\mathbb{R}^2} \rho_i^m u_j^m \to \int_{\mathbb{R}^2} \rho_i u_j \text{ for all } 1 \leq i,j \leq n,$$

where u_j, u_j^m are defined by (3.4) via ρ_j, ρ_j^m respectively.

By Lemma 3.5 we have for R large

$$\left|\int_{\{|x|>R\}} \rho_i^m u_j^m\right| \le C\left[\int_{\{|x|>R\}} \rho_i^m \ln|x| + \int_{\{|x|>R\}} \rho_i^m\right] \tag{4.4}$$

$$\le \frac{C}{R}. \tag{4.5}$$

For $\{|x| \le R\}$ we will use Theorem A to prove the convergence. For that we need to show that $u_i^m \in H^1_{loc}(\mathbb{R}^2)$ and $||u_i^m||_{L^1(B(0,2R))}$ is uniformly bounded for all $i = 1, \dots, n$:

$$\begin{aligned}\int_{\{|x|<2R\}} |u_i^m| &\le \frac{1}{2\pi}\int_{\{|x|<2R\}}\int_{\mathbb{R}^2} |\ln|x-y||\rho_i^m(y)\ d^2y d^2x \\ &\le \frac{1}{2\pi}\int_{\{|x|<2R\}}\int_{\{|y|<4R\}} |\ln|x-y||\rho_i^m(y)\ d^2y d^2x \\ &+ \frac{1}{2\pi}\int_{\{|x|<2R\}}\int_{\{|y|>4R\}} |\ln|x-y||\rho_i^m(y)\ d^2y d^2x \\ &\le \frac{1}{2\pi}\int_{\{|y|<4R\}} \rho_i^m(y)\int_{\{|x|<2R\}} |\ln|x-y||\ d^2x d^2y \\ &+ C(R)\int_{\mathbb{R}^2} |y|^2\rho_i^m(y)\ d^2y \\ &\le C(R).\end{aligned}$$

By compactness result of Theorem A, there exists $u_i \in H^1(B(0,R))$ such that u_i^m converges to u_i in $H^1(B(0,R))$. Therefore u_i^m converges to u_i in the strong topology of Orlicz space determined by the N-function $(e^t - t - 1)$. By duality

$$\int_{B(0,R)} \rho_i^m u_j^m \to \int_{B(0,R)} \rho_i u_j. \tag{4.6}$$

Hence by (4.4) and (4.6) we see that

$$\int_{\mathbb{R}^2} \rho_i^m u_j^m \to \int_{\mathbb{R}^2} \rho_i u_j, \text{ for all } i, j. \tag{4.7}$$

Therefore by (4.2), (4.3) and (4.7) we have $\boldsymbol{\rho} \in \Gamma^{\boldsymbol{\beta}}$ and

$$\mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}) \le \liminf \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m) = \inf_{\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_{\boldsymbol{v}}.$$

This completes the proof of the theorem. □

Remark 3. It follows from the proof of Theorem 4.1 that if a minimizing sequence is bounded in the $\mathbb{L}\ln\mathbb{L}$ topology and has bounded second moment then the minimizing sequence converges and the limit is a minimizer. More precisely, if $\boldsymbol{\rho}^m$ is minimizing sequence that satisfies

$$\sum_{i=1}^n \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m \le C_0, \text{ and } \sum_{i=1}^n \int_{\mathbb{R}^2} |x|^2 \rho_i^m \le C_0$$

for some constant C_0 independent of m then there exists $\boldsymbol{\rho}_0 \in \Gamma^{\boldsymbol{\beta}}$ such that $\boldsymbol{\rho}^m \rightharpoonup \boldsymbol{\rho}_0$ in the topology of $\mathbb{L}\ln\mathbb{L}$ and $\mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}_0) = \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho})$.

5. The Critical case

Recall the definition of the functional $\mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho})$ (1.14). In this section we assume the critical case

$$\Lambda_I(\boldsymbol{\beta}) = 0 \quad , \Lambda_J(\boldsymbol{\beta}) > 0 \; \forall J \subset I, J \neq I, \emptyset \; . \tag{5.1}$$

Lemma 5.1. *Assume $\boldsymbol{\beta}$ satisfies* (5.1). *Then any minimizing sequence $\{\boldsymbol{\rho}^m\}$ for $\mathcal{F}_0$ concentrates at the origin, i.e.,*

$$\lim_{m\to\infty} \int_{\mathbb{R}^2} |x|^2 \rho_i^m(x) d^2x = 0, \quad for\ all\ i = 1, \ldots, n.$$

In particular, $\mathcal{F}_0$ does not attain its infimum on $\Gamma^{\boldsymbol{\beta}}$.

Proof. Let $\boldsymbol{\rho}^m$ be a minimizing sequence. Define

$$\tilde{\rho}_i^m(x) = R^2 \rho_i^m(Rx), \;\; x \in \mathbb{R}^2, R > 0 \; .$$

Direct computation gives

$$\mathcal{F}_0(\tilde{\boldsymbol{\rho}}_m) = \mathcal{F}_0(\boldsymbol{\rho}^m) + \left(\frac{1}{R^2} - 1\right) \sum_{i=1}^n \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho_i^m.$$

Thus we have (using $\liminf(a_m + b_m) = \lim a_m + \liminf b_m$, if a_m converges)

$$\inf_{\boldsymbol{\rho} \in \Gamma^{\boldsymbol{\beta}}} \mathcal{F}_0(\boldsymbol{\rho}) \leq \lim \mathcal{F}_0(\boldsymbol{\rho}^m) + \liminf \left(\frac{1}{R^2} - 1\right) \sum_{i=1}^n \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho_i^m.$$

which gives

$$\liminf \left(\frac{1}{R^2} - 1\right) \sum_{i=1}^n \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho_i^m \geq 0. \tag{5.2}$$

Choosing $R > 1$ in (5.2) gives $\limsup \left(\sum_{i=1}^n \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho_i^m\right) \leq 0$. On the other hand ρ_i^m being non-negative $\liminf \sum_{i=1}^n \left(\frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho_i^m\right) \geq 0$ and hence

$$\lim \left(\sum_{i=1}^n \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho_i^m\right) = 0.$$

Therefore all the components of $\boldsymbol{\rho}^m$ concentrates at the origin and hence there does not exists a minimizer of $\mathcal{F}_0$ on $\Gamma^{\boldsymbol{\beta}}$. □

5.1. **A Functional inequality:**

Lemma 5.2. *The following inequality holds true*

$$\inf_{\boldsymbol{\rho} \in \Gamma^{\boldsymbol{\beta}}} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}) \leq \inf_{\boldsymbol{\rho} \in \Gamma^{\boldsymbol{\beta}}} \mathcal{F}_0(\boldsymbol{\rho}) + \min_{x_0 \in \mathbb{R}^2} \sum_{i=1}^n \frac{1}{2} \beta_i |x_0 - v_i|^2. \tag{5.3}$$

Proof. Let $\boldsymbol{\rho}_{\boldsymbol{m}}$ be a minimizing sequence for $\inf_{\boldsymbol{\rho} \in \Gamma^{\boldsymbol{\beta}}} \mathcal{F}_0(\boldsymbol{\rho})$. Define for $x_0 \in \mathbb{R}^2$,

$$\tilde{\boldsymbol{\rho}}_m(x) = \boldsymbol{\rho}^m(x - x_0), \;\; x \in \mathbb{R}^2.$$

Then a direct computation gives

$$\inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}) \leq \mathcal{F}_{\boldsymbol{v}}(\tilde{\boldsymbol{\rho}}_m) = \mathcal{F}_0(\boldsymbol{\rho}^m) + \sum_{i=1}^n \frac{1}{2}\int_{\mathbb{R}^2} \left(|x+x_0-v_i| - |x|^2\right)\rho_i^m$$

$$= \mathcal{F}_0(\boldsymbol{\rho}^m) + \sum_{i=1}^n \int_{\mathbb{R}^2} \langle x, x_0 - v_i\rangle \rho_i^m + \sum_{i=1}^n \frac{1}{2}\beta_i |x_0 - v_i|^2. \tag{5.4}$$

Since by Lemma 5.1 $\lim\left(\sum_{i=1}^n \frac{1}{2}\int_{\mathbb{R}^2}|x|^2\rho_i^m\right) = 0$ we get

$$\left|\sum_{i=1}^n \int_{\mathbb{R}^2} \langle x, x_0 - v_i\rangle \rho_i^m\right| \leq \sum_{i=1}^n \beta_i^{\frac{1}{2}} |x_0 - v_i| \left(\int_{\mathbb{R}^2}|x|^2\rho_i^m\right)^{\frac{1}{2}} \to 0,$$

as $m\to\infty$. Therefore letting $m\to\infty$ in (5.4) we get

$$\inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}) \leq \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_0(\boldsymbol{\rho}) + \sum_{i=1}^n \frac{1}{2}\beta_i|x_0 - v_i|^2.$$

Since $x_0 \in \mathbb{R}^2$ is arbitrary the proof of the lemma is completed. □

Remark 4. If the equality occurs in (5.3) then there exists a minimizing sequence $\boldsymbol{\rho}^m$ for $\mathcal{F}_{\boldsymbol{v}}$ such that the sequence $\tilde{\boldsymbol{\rho}}^m := \boldsymbol{\rho}^m(\cdot + x_0)$ is a minimizing sequence for $\mathcal{F}_0$, where x_0 is the unique minimizer of $\min_{x\in\mathbb{R}^2}\sum_{i=1}^n \frac{1}{2}\beta_i|x-v_i|^2$. Hence, for any such minimizing sequence we get $\sum_{i=1}^n\int_{\mathbb{R}^2}\rho_i^m \ln\rho_i^m \to \infty$. Otherwise, as in Theorem 4.1 (Remark 3) we can prove the existence of a minimizer of $\mathcal{F}_0$ on $\Gamma^{\boldsymbol{\beta}}$, which contradicts Lemma 5.1.

6. Blow up analysis: Brezis Merle type argument

We pose the following:

Proposition 6.1. *Suppose $\boldsymbol{\beta}$ satisfies* (5.1), *then either*

(a) *there exists a minimizer of $\mathcal{F}_{\boldsymbol{v}}$ over $\Gamma^{\boldsymbol{\beta}}$, or*
(b) *equality holds in the functional inequality* (5.3).

In particular, if strict inequality holds in (5.3) *then there exists a minimizer of $\mathcal{F}_{\boldsymbol{v}}$ over $\Gamma^{\boldsymbol{\beta}}$.*

For the proof of this Proposition will need the two Lemmas below:
Let $\boldsymbol{\beta}_m$ be a sequence such that $\boldsymbol{\beta}_m \nearrow \boldsymbol{\beta}$ and satisfies

$$\Lambda_J(\boldsymbol{\beta}_m) > 0, \text{ for all } \phi \neq J \subset I.$$

One can indeed choose such sequence $\boldsymbol{\beta}_m$, see for example [11, Lemma 5.1 and equation (5.4)]. By Theorem 4.1 the infimum $\inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}_m}}\mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho})$ is attained. Let us denote the minimizer by $\boldsymbol{\rho}^m \in \Gamma^{\boldsymbol{\beta}_m}$.

Lemma 6.2. *The following holds*

$$\sup_m \int_{\mathbb{R}^2}|x|^2\rho_i^m < +\infty, \textit{ for all } i\ .$$

Proof. For each fixed m and $R > 0$ define

$$\tilde{\rho}_i^m(x) = R^2 \rho_i^m(Rx).$$

A direct computations gives

$$\mathcal{F}_{\boldsymbol{v}}(\tilde{\boldsymbol{\rho}}_m) = \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m) + f_m(R),$$

where $f_m : (0, \infty) \to \mathbb{R}$ is defined by

$$f_m(t) = a_m \ln t + \frac{b_m}{2t^2} + \frac{2c_m}{t} + d_m,$$

and a_m, b_m, c_m, d_m are defined as follows:

$$a_m = \frac{1}{4\pi}\Lambda_I(\boldsymbol{\beta}_m) \to 0, \quad 2c_m = -\sum_{i=1}^n \int_{\mathbb{R}^2} \langle x, v_i\rangle \rho_i^m$$

$$b_m = \sum_{i=1}^n \int_{\mathbb{R}^2} |x|^2 \rho_i^m, \quad d_m = -\frac{1}{2}\sum_{i=1}^n \int_{\mathbb{R}^2} |x - v_i|^2 \rho_i^m + \frac{1}{2}\sum_{i=1}^n |v_i|^2 \beta_i^m.$$

One can easily verify that the following inequality holds:

$$|c_m| \le \left(\sup_i \frac{\sqrt{n}}{2} |v_i| \beta_i^{\frac{1}{2}} \right) b_m^{\frac{1}{2}}, \tag{6.1}$$

Since $\boldsymbol{\rho}^m$ minimizes $\mathcal{F}_{\boldsymbol{v}}$ over $\Gamma^{\boldsymbol{\beta}_m}$ we have

$$\inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}_m}} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}) \le \mathcal{F}_{\boldsymbol{v}}(\tilde{\boldsymbol{\rho}}_m) = \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m) + f_m(R) = \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}_m}} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}) + f_m(R),$$

and therefore $f_m(R) \ge 0$. $R > 0$ being arbitrary we obtain $\inf_{t\in(0,\infty)} f_m(t) \ge 0$. Since for each $m, f_m(1) = 0$ and $f_m(t) \to \infty$ as $t \to 0+$ and $t \to \infty$ we have that $f_m'(1) = 0$ for all m. Which gives

$$a_m - b_m - 2c_m = 0 \quad \textit{for all } m. \tag{6.2}$$

Now the desired conclusion follows from the estimate (6.1) and (6.2) and hence the proof of the lemma is completed. □

Lemma 6.3. *The followings hold true:*

(a)

$$\lim_{m\to\infty} \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}_m}} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}) \le \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}) \tag{6.3}$$

(b)

$$\lim_{m\to\infty} \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}_m}} \mathcal{F}_0(\boldsymbol{\rho}) = \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_0(\boldsymbol{\rho}) \tag{6.4}$$

Proof. We first prove inequality (6.3). Let $\boldsymbol{\rho} \in \Gamma^{\boldsymbol{\beta}}$ be a fixed element. Choose $R_i^m > 0$ such that $\int_{B(0,R_i^m)} \rho_i = \beta_i^m$ and define $\rho_i^m = \rho_i \chi_{B(0,R_i^m)}$. Then $\boldsymbol{\rho}^m \in \Gamma^{\boldsymbol{\beta}_m}$ and by dominated convergence theorem

$$\lim_{m\to\infty} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m) = \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}).$$

Thus we have

$$\lim_{m\to\infty} \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}_m}} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}) \le \lim_{m\to\infty} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m) = \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}).$$

Since $\boldsymbol{\rho} \in \Gamma^{\boldsymbol{\beta}}$ is arbitrary, we have proved the inequality (6.3). Next we prove (6.4). Thanks to (6.3), we only need to show $\lim_{m\to\infty} \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}_m}} \mathcal{F}_0(\boldsymbol{\rho}) \geq \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_0(\boldsymbol{\rho})$. This step is a little bit technical and therefore we divide the proof into several parts.

(1) By Theorem 4.1, there exists $\boldsymbol{\rho}^m \in \Gamma^{\boldsymbol{\beta}_m}$ such that

$$\mathcal{F}_0(\boldsymbol{\rho}^m) = \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}_m}} \mathcal{F}_0(\boldsymbol{\rho}).$$

Furthermore, we may assume that ρ_i^m are radially symmetric and decreasing function of $r = |x|$. By abuse of notation, we will also denote the radial function by $\rho_i^m(r)$.

(2) A simple adoption of the proof of Lemma 5.1 gives $\int_{\mathbb{R}^2} |x|^2\rho_i^m(x) \to 0$ as $m \to \infty$. Therefore for any $r \in (0,\infty)$

$$o_m(1) = \int_{\mathbb{R}^2} |x|^2\rho_i^m(x) = 2\pi\int_0^\infty s^3\rho_i^m(s)\ ds \geq 2\pi\int_0^r s^3\rho_i^m(s)\ ds \geq \frac{\pi}{2}r^4\rho_i^m(r),$$

where $o_m(1)$ denotes a quantity going to 0 as $m \to \infty$. Thus we have $\sup_{r\in(0,\infty)} r^4\rho_i^m(r) = o_m(1)$ as $m \to \infty$.

A similar argument using $\int_{\mathbb{R}^2} \rho_i^m = \beta_i^m$ gives $\sup_{r\in(0,\infty)} r^2\rho_i^m(r) \leq \frac{\beta_i^m}{\pi}$.

(3) Let ϕ be a smooth, nonnegative, radial, compactly supported function such that $\int_{\mathbb{R}^2} \phi = 1$. Define $\epsilon_i^{(m)} = \beta_i - \beta_i^m > 0$ and

$$\tilde{\rho}_i^m(x) = \rho_i^m(x) + \epsilon_i^{(m)}\phi(x),\ x \in \mathbb{R}^2.$$

Then $\tilde{\boldsymbol{\rho}}_m \in \Gamma^{\boldsymbol{\beta}}$ for all m and hence $\inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_0(\boldsymbol{\rho}) \leq \mathcal{F}_0(\tilde{\boldsymbol{\rho}}_m)$. Now we will estimate each term of $\mathcal{F}_0(\tilde{\boldsymbol{\rho}}_m)$ and show that

$$\mathcal{F}_0(\tilde{\boldsymbol{\rho}}_m) = \mathcal{F}_0(\boldsymbol{\rho}^m) + o_m(1).$$

(4)

$$\int_{\mathbb{R}^2} \tilde{\rho}_i^m \ln\tilde{\rho}_i^m - \int_{\mathbb{R}^2} \rho_i^m \ln\rho_i^m = o_m(1). \tag{6.5}$$

Let us denote by $k_m := \max_{1\leq i\leq n}\max\{\sup_{r\in(0,\infty)} r^4\rho_i^m(r), \sup_{r\in(0,\infty)} r^4\tilde{\rho}_i^m(r)\}$, then using (2) and $\epsilon_i^{(m)} \to 0$, we see that $k_m \to 0$. Let δ_m be a sequence such that $\delta_m \to 0$ and $k_m \ln k_m/\delta_m^3 \to 0$. Clearly we have

$$\int_{B(0,\delta_m)} (\tilde{\rho}_i^m \ln\tilde{\rho}_i^m - \rho_i^m\ln\rho_i^m)\chi_{\{\rho_i^m\leq 2\}} = o_m(1),$$

because $t\ln t$ is bounded on any compact subset of $[0,\infty)$. Now using mean value theorem we get

$$\begin{aligned}
&\int_{B(0,\delta_m)} (\tilde{\rho}_i^m \ln\tilde{\rho}_i^m - \rho_i^m\ln\rho_i^m)\chi_{\{\rho_i^m> 2\}}\\
=&2\pi\epsilon_i^{(m)}\int_0^{\delta_m}\int_0^1 r\left[1+\ln(\rho_i^m(r)+t\epsilon_i^{(m)}\phi(r))\right]\phi(r)\chi_{\{\rho_i^m>2\}}\ dtdr\\
=&o_m(1) + 2\pi\epsilon_i^{(m)}\int_0^1\int_0^{\delta_m} r\ln(\rho_i^m(r)+t\epsilon_i^{(m)}\phi(r))\phi(r)\chi_{\{\rho_i^m>2\}}\ drdt
\end{aligned}$$

On the set $\{\rho_i^m > 2\}$, we have $\rho_i^m(r)+t\epsilon_i^{(m)}\phi(r) > 1$. Moreover, using the estimate of (2) we see that $\rho_i^m(r)+t\epsilon_i^{(m)}\phi(r) \leq \frac{C}{r^2}$ where C is some positive constant. Therefore $0 \leq r\ln(\rho_i^m(r) + t\epsilon_i^{(m)}\phi(r)) \leq r\ln\frac{C}{r^2}$ and hence

$$2\pi\epsilon_i^{(m)}\int_0^1\int_0^{\delta_m} r\ln(\rho_i^m(r) + t\epsilon_i^{(m)}\phi(r))\phi(r)\chi_{\{\rho_i^m>2\}}\,drdt = o_m(1),$$

which gives

$$\int_{B(0,\delta_m)}(\tilde{\rho}_i^m\ln\tilde{\rho}_i^m - \rho_i^m\ln\rho_i^m)\chi_{\{\rho_i^m>2\}} = o_m(1).$$

Now let us estimate $\int_{B(0,\delta_m)^c}\rho_i^m\ln\rho_i^m$.

$$\begin{aligned}\left|\int_{B(0,\delta_m)^c}\rho_i^m\ln\rho_i^m\right| &= \left|2\pi\int_{\delta_m}^\infty r\rho_i^m(r)\ln\rho_i^m(r)\;dr\right|\\
&\leq 2\pi\int_{\delta_m}^\infty \frac{|r^4\rho_i^m\ln(r^4\rho_i^m)|}{r^3}\;dr + 8\pi\int_{\delta_m}^\infty\frac{r^4\rho_i^m|\ln r|}{r^3}\;dr\\
&\leq 2\pi|k_m\ln k_m|\int_{\delta_m}^\infty\frac{dr}{r^3} + 8\pi k_m\int_{\delta_m}^\infty\frac{|\ln r|}{r^3}dr\\
&\leq \frac{2\pi|k_m\ln k_m|}{\delta_m^2} + C\frac{k_m}{\delta_m^{2-\epsilon}},\;\text{ for some }\epsilon>0\\
&= o_m(1).\end{aligned}$$

In an entirely similar way we can verify that $\left|\int_{B(0,\delta_m)^c}\tilde{\rho}_i^m\ln\tilde{\rho}_i^m\right| = o_m(1)$, and hence we have proved (6.5).

(5) Next we estimate

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\tilde{\rho}_i^m(x)\ln|x-y|\tilde{\rho}_j^m(y)\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho_i^m(x)\ln|x-y|\rho_j^m(y) + \epsilon_i^m\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\phi(x)\ln|x-y|\rho_j^m(y)\\
&\quad+\epsilon_j^m\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho_i^m(x)\ln|x-y|\phi(y) + \epsilon_i^m\epsilon_j^m\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\phi(x)\ln|x-y|\tilde{\phi}(y)\end{aligned}$$

$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho_i^m(x)\ln|x-y|\rho_j^m(y) + o_m(1). \tag{6.6}$$

Where we have used the fact that $|\int_{\mathbb{R}^2}\ln|x-y|\phi(y)\;d^2y| \leq C(1+\ln(1+|x|))$ for all $x\in\mathbb{R}^2$.

(6) Finally we have

$$\int_{\mathbb{R}^2}|x|^2\tilde{\rho}_i^m(x) = \int_{\mathbb{R}^2}|x|^2\rho_i^m(x) + o_m(1). \tag{6.7}$$

Combining (6.5), (6.6) and (6.7) we get

$$\inf_{\boldsymbol{\rho}\in\Gamma^\beta}\mathcal{F}_0(\boldsymbol{\rho}) \leq \mathcal{F}_0(\tilde{\boldsymbol{\rho}}_m) = \mathcal{F}_0(\boldsymbol{\rho}^m) + o_m(1) = \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}_m}}\mathcal{F}_0(\boldsymbol{\rho}) + o_m(1).$$

Letting $m \to \infty$, we reach at the desired conclusion. This completes the proof of the lemma. □

6.1. **Proof of Proposition 6.1.** Recall that $\boldsymbol{\rho}^m$ is a minimizer of $\mathcal{F}_{\boldsymbol{v}}$ over $\Gamma^{\boldsymbol{\beta}_m}$, where $\boldsymbol{\beta}_m \nearrow \boldsymbol{\beta}$. Define the Newtonian potentials

$$u_i^m(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|\rho_i^m(y)\ d^2y,\ x \in \mathbb{R}^2.$$

By variational principle and Lemma 6.2, u_i^m satisfies the following equation:

$$\begin{cases} -\Delta u_i^m(x) = \mu_i^m e^{\sum_{j=1}^n a_{ij}u_j^m(x) - \frac{1}{2}|x-v_i|^2},\ \text{in } \mathbb{R}^2, \\ \mu_i^m \int_{\mathbb{R}^2} e^{\sum_{j=1}^n a_{ij}u_j^m - \frac{1}{2}|x-v_i|^2} = \beta_i^m, \\ \mu_i^m \int_{\mathbb{R}^2} |x|^2 e^{\sum_{j=1}^n a_{ij}u_j^m - \frac{1}{2}|x-v_i|^2} \le C_0, \end{cases}$$

where C_0 is a constant independent of m. Define

$$v_i^m(x) = \ln \mu_i^m + \sum_{j=1}^n a_{ij}u_j^m(x),\ x \in \mathbb{R}^2.$$

Let us consider the two cases:
Case (A): Suppose there exists $R > 0$ such that

$$\max_{1\le i\le n}\ \sup_{x\in B(0,R)}\ v_i^m(x) \to \infty,\ \text{as } m \to \infty. \tag{6.8}$$

Case (B): For any $R > 0$ there exists a constant $C(R)$ such that

$$\max_{1\le i\le n}\ \sup_{x\in B(0,R)}\ v_i^m(x) \le C(R).$$

We first prove:

Lemma 6.4. *Under the assumption of Case (A), the following equality holds:*

$$\inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}) = \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_0(\boldsymbol{\rho}) + \min_{x_0\in\mathbb{R}^2} \sum_{i=1}^n \frac{1}{2}\beta_i|x_0 - v_i|^2. \tag{6.9}$$

Proof. By definition $v_i^m, 1 \le i \le n$ satisfies the equation

$$\begin{cases} -\Delta v_i^m(x) = \sum_{j=1}^n a_{ij} e^{v_j^m(x) - \frac{1}{2}|x-v_j|^2},\ \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{v_i^m - \frac{1}{2}|x-v_i|^2} = \beta_i^m, \\ \int_{\mathbb{R}^2} |x|^2 e^{v_i^m - \frac{1}{2}|x-v_i|^2} \le C_0. \end{cases}$$

Furthermore, the following relation holds:

$$\rho_i^m(x) = \mu_i^m e^{\sum_{j=1}^n a_{ij}u_j^m(x) - \frac{1}{2}|x-v_i|^2} = e^{v_i^m(x) - \frac{1}{2}|x-v_i|^2},\ x \in \mathbb{R}^2. \tag{6.10}$$

After passing to a subsequence if necessary we may assume the supremum in (6.8) is attained by v_1^m for all m. That is, there exists $x_m \in \overline{B(0,R)}$ such that

$$v_1^m(x_m) = \max_i\ \sup_{x\in B(0,R)}\ v_i^m(x) \to \infty,\ \text{as } m \to \infty.$$

Let $x_m \to x_0$ for some $x_0 \in \overline{B(0,R)}$, and choose a $\tilde{R} > 0$ large enough so that $\overline{B(0,R)} \subset B(x_0, \tilde{R})$. Since $v_1^m(x_m) \to \infty$ we have

$$\sup\{v_i^m(x) + 2\ln(\tilde{R} - |x - x_0|) \ : \ x \in B(x_0, \tilde{R}), 1 \le i \le n\} \to \infty, \text{ as } m \to \infty. \tag{6.11}$$

Again after passing to a subsequence me may assume $y_m \in B(x_0, \tilde{R})$ be the point and i_0 be the index such that the supremum in (6.11) is attained for all m. Since $2\ln(\tilde{R} - |x - x_0|)$ is bounded above on $B(x_0, \tilde{R})$ we have $v_{i_0}^m(y_m) \to \infty$.

Define $\delta_m = e^{-\frac{v_{i_0}^m(y_m)}{2}}$, then $\delta_m \to 0$ and it follows from (6.11) that

$$\left(\frac{\tilde{R} - |y_m - x_0|}{\delta_m}\right) \to \infty, \text{ as } m \to \infty. \tag{6.12}$$

Now define

$$\tilde{v}_i^m(x) = v_i^m(y_m + \delta_m(x - x_0)) + 2\ln\delta_m.$$

We note that $\tilde{v}_{i_0}^m(x_0) = 0$ for all m. Furthermore, it follows from (6.12) that for any $M > 0$ fixed and $x \in B(x_0, M), y_m + \delta_m(x - x_0) \in B(x_0, \tilde{R})$ for large m. Now $\tilde{v}_i^m(x)$ satisfies the equation

$$\begin{cases} -\Delta\tilde{v}_i^m(x) = \sum_{j=1}^n a_{ij} e^{\tilde{v}_j^m(x) - \frac{1}{2}|y_m + \delta_m(x - x_0) - v_j|^2} \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{\tilde{v}_i^m(x) - \frac{1}{2}|y_m + \delta_m(x - x_0) - v_i|^2} = \beta_i^m. \end{cases} \tag{6.13}$$

Let $y_m \to y_0 \in \overline{B(x_0, \tilde{R})}$. Since $\tilde{v}_{i_0}^m(x_0) = 0$ either $\tilde{v}_i^m$ converges to some $\tilde{v}_i$ in $C_{loc}^0(\mathbb{R}^2)$ for all i or $\tilde{v}_i^m$ converges to $-\infty$ uniformly on compact subsets of $\mathbb{R}^2$ for some $i \ne i_0$.

Let $I' \subset I$ is the set of indices such that $\tilde{v}_i \ne -\infty$ iff $i \in I'$. Then $\tilde{v}_i^m$ converges to $\tilde{v}_i$ in $C_{loc}^0(\mathbb{R}^2)$ for $i \in I'$ and, by (6.13)

$$\begin{cases} -\Delta\tilde{v}_i = \sum_{j \in I'} a_{ij} e^{\tilde{v}_j - \frac{1}{2}|y_0 - v_j|^2} \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{\tilde{v}_i - \frac{1}{2}|y_0 - v_i|^2} = \tilde{\beta}_i, \end{cases} \tag{6.14}$$

Letting $z_i(x) = \tilde{v}_i(x) - \frac{1}{2}|y_0 - v_i|^2$ we obtain

$$\begin{cases} -\Delta z_i = \sum_{j \in I'} a_{ij} e^{z_j} \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{z_i} = \tilde{\beta}_i \ . \end{cases} \tag{6.15}$$

holds for $i \in I'$ for some $\tilde{\beta}_i \le \beta_i$.

A necessary condition for the existence of solution to (6.15) is $\Lambda_{I'}(\tilde{\boldsymbol{\beta}}) = 0$ ([11], see also [17, 19]). Since we assumed $\Lambda_I(\boldsymbol{\beta}) = 0$ this implies $I' = I$ and $\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}$. (see [11]).

It follows that, in Case (A), ρ_i^m concentrates at some point $y_0 \in \mathbb{R}^2$. In particular

$$\lim_{m\to\infty} \int_{\mathbb{R}^2} |x - v_i|^2 \rho_i^m(x) d^2x \ge \beta_i |y_0 - v_i|^2 \text{ for all } 1 \le i \le n \ . \tag{6.16}$$

We want to show that y_0 is the global minima of $\sum_{i=1}^n \frac{1}{2}\beta_i|x-v_i|^2$ on $\mathbb{R}^2$. Let us define $\tilde{\boldsymbol{\rho}}_m$ as

$$\tilde{\boldsymbol{\rho}}_m(x) = \frac{1}{\delta^2}\boldsymbol{\rho}^m\left(\frac{x}{\delta} - y_0\right).$$

Then

$$\begin{aligned}\mathcal{F}_0(\tilde{\boldsymbol{\rho}}_m) = \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m) - \frac{\Lambda_I(\boldsymbol{\beta}_m)}{4\pi}\ln\delta + \delta^2\sum_{i=1}^n \frac{1}{2}\int_{\mathbb{R}^2}|x+y_0|^2\rho_i^m \\ - \sum_{i=1}^n \frac{1}{2}\int_{\mathbb{R}^2}|x-v_i|^2\rho_i^m.\end{aligned}$$

Therefore we obtain

$$\inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}_m}} \mathcal{F}_0(\boldsymbol{\rho}) \le \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m) - \frac{\Lambda_I(\boldsymbol{\beta}_m)}{4\pi}\ln\delta + \delta^2 O(1) - \sum_{i=1}^n \frac{1}{2}\int_{\mathbb{R}^2}|x-v_i|^2\rho_i^m.$$

Letting $m\to\infty$ and using (6.16) and Lemma 6.3(b) we get

$$\inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_0(\boldsymbol{\rho}) \le \inf_{\boldsymbol{\rho}\in\Gamma^{\boldsymbol{\beta}}} \mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}) + \delta^2 O(1) - \sum_{i=1}^n \frac{1}{2}\beta_i|y_0-v_i|^2.$$

Since $\delta > 0$ is arbitrary, by (5.3) we get y_0 is the global minima of $\sum_{i=1}^n \frac{1}{2}\beta_i|x-v_i|^2$ on $\mathbb{R}^2$ and (6.9) holds true. ☐

Lemma 6.5. *Under the assumption of Case (B) there exists a minimizer of $\mathcal{F}_{\boldsymbol{v}}$ in $\Gamma^{\boldsymbol{\beta}}$.*

Proof. Under this assumption, we have from (6.10) that $||\rho_i^m||_{L^\infty(B(0,R))} \le C_0$, for some constant C_0 independent of m. In the proof C_0 will stand for some universal constant independent of m but may depend on R. Then

$$\left|\sum_{i=1}^n \int_{B(0,R)} \rho_i^m(x)\ln\rho_i^m(x)\ d^2x\right| \le C_0. \tag{6.17}$$

Now let

$$\tilde{u}_i^m(x) := -\frac{1}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|\rho_i^m(y)\chi_{B(0,R)}(y)\ d^2y,$$

then it follows from Lemma 3.5 (using the fact $||\rho_i^m||_{L^\infty(B(0,R))} \le C_0$) that

$$|\tilde{u}_i^m(x)| \le \begin{cases} C_0, & \text{if } |x|\le 1,\\ C_0(1+\ln|x|), & \text{if } |x|>1.\end{cases}$$

Thus we have

$$\begin{aligned}&\left|\int_{B(0,R)^c}\int_{B(0,R)}\rho_i^m(x)\ln|x-y|\rho_j^m(y)\ d^2y d^2x\right| \\ &\le \int_{\mathbb{R}^2}\rho_i^m(x)|\tilde{u}_i^m(x)|\ d^2x\end{aligned}$$

$$\leq C_0 \left[\int_{\mathbb{R}^2} \rho_i^m \; d^2x + \int_{\{|x|>1\}} \ln|x| \rho_i^m \; d^2x \right]$$

$$\leq C_0 \left[\beta_i^m + \int_{\mathbb{R}^2} |x|^2 \rho_i^m \; d^2x \right] \leq C_0. \tag{6.18}$$

Let us define $\hat{\boldsymbol{\rho}}_m^R(x) = \boldsymbol{\rho}^m(x)\chi_{B(0,R)^c}(x)$. Let

(6.19)

$$\mathcal{F}_{\boldsymbol{v},R}(\boldsymbol{\rho}) := \sum_{i=1}^n \int_{B(0,R)} \rho_i \ln \rho_i + \frac{1}{4\pi} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \int_{B(0,R)} \int_{B(0,R)} \rho_i^m(x) \ln|x-y| \rho_j^m(y) + \frac{1}{2} \sum_{i=1}^n \int_{B(0,R)} |x - v_i|^2 \rho_i \; .$$

We can write $\mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m)$ as

$$\mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m) = \mathcal{F}_{\boldsymbol{v},R}(\boldsymbol{\rho}^m) + \mathcal{F}_{\boldsymbol{v}}(\hat{\boldsymbol{\rho}}_m^R) + \frac{1}{2\pi} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \int_{B(0,R)} \int_{B(0,R)^c} \rho_i^m(x) \ln|x-y| \rho_j^m(y) d^2x d^2y \; . \tag{6.20}$$

Since $\|\boldsymbol{\rho}^m\|_{L^\infty(B(0,R))} \leq C_0$ we obtain that $\mathcal{F}_{\boldsymbol{v},R}(\boldsymbol{\rho}^m) = O(1)$. Also, (6.18) implies that the second line in (6.20) is $O(1)$ as well. Since $\mathcal{F}_{\boldsymbol{v}}(\boldsymbol{\rho}^m)$ is a bounded sequence (as $\boldsymbol{\rho}^m$ is a minimizer of $\inf_{\Gamma^{\boldsymbol{\beta}_m}} \mathcal{F}_{\boldsymbol{v}}$, see Lemma 6.3(a)) this implies that

$$\mathcal{F}_{\boldsymbol{v}}(\hat{\boldsymbol{\rho}}_m^R) = O(1) \tag{6.21}$$

uniformly in m.

Next, observe that we can choose R large enough for which $\int_{\mathbb{R}^2} \hat{\boldsymbol{\rho}}_m^R < \boldsymbol{\beta}/2$. Indeed, since $\int_{\mathbb{R}^2} |x|^2 \rho_i^m \leq C$ then $\int_{\{|x|>R\}} \rho_i^m \leq R^{-2} \int_{\{|x|>R\}} |x|^2 \rho_i^m \leq C/R^2$. For such R, $\hat{\boldsymbol{\rho}}_m$ is sub-critical, uniformly in m, thus

$$\mathcal{F}_{\boldsymbol{v}}(\hat{\boldsymbol{\rho}}_m^R) \geq C \sum_{i=1}^n \int_{\mathbb{R}^2} \hat{\rho}_i^m \ln \hat{\rho}_i^m \; . \tag{6.22}$$

From (6.21) and (6.22) we acquire that $\hat{\boldsymbol{\rho}}_m^R$ has a uniform bound in $\mathbb{L}\ln\mathbb{L}$. Since by assumption $\|\boldsymbol{\rho}^m\|_{L^\infty(B(0,R))} = O(1)$ we obtain that $\boldsymbol{\rho}^m$ is bounded in $\mathbb{L}\ln\mathbb{L}$ as well.

Proceeding as in the sub critical case (Theorem 4.1, see Remark 3) we can prove the existence of a minimizer of $\mathcal{F}_{\boldsymbol{v}}$ over $\Gamma^{\boldsymbol{\beta}}$. □

6.2. **Case of** $Var(v_1, \dots, v_n)$ **large: Proof of Theorem 1.1-d.** According to Proposition 6.1 we only have to exclude case A.

Lemma 6.6. *Suppose* $\boldsymbol{\beta}$ *satisfies* (5.1). *Then there exists a constant* $\kappa(\boldsymbol{\beta})$ *such that whenever* $|v_1 - v_2| > \kappa$, *then strict inequality holds in* (5.3).

Proof. Let $\bar{\rho}$ be any non-negative, bounded function of compact support (say $\bar{\rho}(x) = 0$ if $|x| > 1$) such that $\int_{\mathbb{R}^2} \bar{\rho} = 1$. Define $\rho_i(x) := \beta_i \bar{\rho}(x - v_i)$ so that $\boldsymbol{\rho} \in \Gamma^{\boldsymbol{\beta}}$.

Then we immediately see that

$$\left|\int_{\mathbb{R}^2} \rho_i \ln \rho_i\right| = O(1), \quad \left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho_i(x)\ln|x-y|\rho_i(y)\right| = O(1),$$
$$\int_{\mathbb{R}^2} |x-v_i|^2\rho_i = \beta_i\int_{\mathbb{R}^2}|x|^2\bar\rho = O(1),$$

for all $i = 1, 2$, where $O(1)$ denotes a quantity independent of v_i. Now

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho_1(x)\ln|x-y|\rho_2(y) = \beta_1\beta_2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\bar\rho(x)\ln|x-y+(v_1-v_2)|\bar\rho(y). \tag{6.23}$$

One can easily estimate that $|\ln|x-y+(v_1-v_2)| - \ln|v_1-v_2|| \le \frac{2}{|v_1-v_2|-2}$, for all $x, y \in (0,1)$ provided $|v_1-v_2| > 2$ (this condition on $|v_1-v_2|$ is unnecessary, because we can choose the support of $\bar\rho$ accordingly). Since $\bar\rho$ has support in $B(0,1)$ we get

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\bar\rho(x)\ln|x-y+(v_1-v_2)|\bar\rho(y) - \ln|v_1-v_2|\\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\bar\rho(x)\left(\ln|x-y+(v_1-v_2)| - \ln|v_1-v_2|\right)\bar\rho(y) = O(1).\end{aligned} \tag{6.24}$$

Thus we obtain from (6.23) and (6.24),

$$\inf_{\boldsymbol\rho\in\Gamma^{\boldsymbol\beta}} \mathcal{F}_{\boldsymbol v}(\boldsymbol\rho) \le \mathcal{F}_{\boldsymbol v}(\boldsymbol\rho) = O(1) + \frac{a_{12}}{2\pi}\beta_1\beta_2\ln|v_1-v_2|. \tag{6.25}$$

While the right hand side of (5.3) becomes

$$\inf_{\boldsymbol\rho\in\Gamma^{\boldsymbol\beta}} \mathcal{F}_0(\boldsymbol\rho) + \min_{x_0\in\mathbb{R}^2}\sum_{i=1}^n \frac{\beta_i}{2}|x_0-v_i|^2 = O(1) + \frac{\beta_1\beta_2}{2(\beta_1+\beta_2)}|v_1-v_2|^2. \tag{6.26}$$

We see from (6.25) and (6.26) that the equality can not occur in (5.3) provided $|v_1-v_2|$ is very large. Hence by Proposition 6.1, there exists a minimizer of $\mathcal{F}_{\boldsymbol v}$ on $\Gamma^{\boldsymbol\beta}$. This completes the proof of the lemma. □

Proof of Theorem 1.3:

Proof. The proof of (*a*) and (*c*) follows from Theorem 1.1 (*b*) and (*d*) respectively. We only need to prove (*b*). Since A is invertible and all the v_i are equal by translating and adding constants to the solution we can assume $u_i, 1 \le i \le n$ satisfies

$$\begin{cases} -\Delta u_i = e^{\sum_{j=1}^n a_{ij}u_j - \frac12|x|^2}, \text{ in } \mathbb{R}^2,\\ \int_{\mathbb{R}^2} e^{\sum_{j=1}^n a_{ij}u_j - \frac12|x|^2} = \beta_i. \end{cases} \tag{6.27}$$

Again using the invertibility and irreducibility of A we get by [11, Proposition 4.1] with $V_i(x) = e^{-\frac{|x|^2}{2}}$ that u_i in (6.27) are radially symmetric with respect to the origin. By abuse of notation we still denote the radial function by $u_i(r), r = |x|$. Then u_i satisfies

$$-\frac1r\frac{d}{dr}\left(r\frac{du_i}{dr}\right) = e^{\sum_{j=1}^n a_{ij}u_j(r) - \frac{r^2}{2}}, r \in (0,\infty). \tag{6.28}$$

Define

$$m_i(r) = 2\pi \int_0^r s e^{\sum_{j=1}^n a_{ij} u_j(s) - \frac{s^2}{2}} ds = -2\pi r \frac{du_i}{dr}, \ r \in (0,\infty), i = 1, \dots, n.$$

Then m_i satisfies

$$\lim_{r\to 0+} m_i(r) = 0, \lim_{r\to\infty} m_i(r) = \beta_i, \text{ and } m_i \text{ are non decreasing.} \tag{6.29}$$

Furthermore, since u_i has log decay at infinity i.e., $|u_i(r) + \frac{\beta_i}{2\pi} \ln r| = O(1)$ as $r \to \infty$ (see [11, Proposition 3.1]) we see that

$$\lim_{r\to\infty} r^2 m_i'(r) = 0. \tag{6.30}$$

Now define $w_i(s) = m_i(e^s), s \in (-\infty,\infty)$ then it follows from (6.29), (6.30) that w_i is non decreasing and satisfies

$$\lim_{s\to-\infty} w_i(s) = 0, \lim_{s\to\infty} w_i(s) = \beta_i, \lim_{s\to-\infty} e^{-s} w_i'(s) = 0, \int_{-\infty}^{\infty} e^s w_i'(s) ds < \infty.$$

Therefore using the equation (6.28) we see that w_i satisfies

$$w_i''(s) = w_i'(s) \left[2 - \frac{1}{2\pi} \sum_{j=1}^n a_{ij} w_j(s) - e^s \right]. \tag{6.31}$$

Summing over all i we can rewrite (6.31) as

$$\left(\sum_{i=1}^n w_i'(s) \right)' = \left[2 \sum_{i=1}^n w_i(s) - \frac{1}{4\pi} \sum_{i=1}^n \sum_{j=1}^n a_{ij} w_i(s) w_j(s) \right]' - \sum_{i=1}^n e^s w_i'(s). \tag{6.32}$$

Since $\lim_{s\to\infty} \sum_{i=1}^n w_i(s) = \sum_{i=1}^n \beta_i, w_i$ are non decreasing we can find a sequence s_m converging to ∞ such that $\sum_{i=1}^n w_i'(s_m) \to 0$ as $m \to \infty$. Therefore integrating (6.32) from $-\infty$ to s_m and letting $m \to \infty$ we obtain

$$2 \sum_{i=1}^n \beta_i - \frac{1}{4\pi} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \beta_i \beta_j = \sum_{i=1}^n \int_{-\infty}^{\infty} e^s w_i'(s) ds$$

which implies $\Lambda_I(\boldsymbol{\beta}) > 0$, contradicting our assumption. This completes the proof of the corollary. □

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