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SHARP POINTWISE ESTIMATES FOR SOLUTIONS OF THE MODIFIED HELMHOLTZ EQUATION

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ABSTRACT. Modified Helmholtz equation $(\Delta - c^2)u = 0$, $c > 0$, in the half-space $\mathbb{R}^n_+ = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}$ is considered. It is assumed that the boundary data of the Dirichlet and Neumann problems in $\mathbb{R}^n_+$ belong to the space L^p. Representations for the sharp coefficients in pointwise estimates involving the gradient of solution to this equation in $\mathbb{R}^n_+$ are obtained. Each of these representations includes an extremal problem with respect to a vector parameter inside of an integral over the unit sphere in $\mathbb{R}^n$. The extremal problems are solved for $p \in [2, \infty]$ and $p \in [2, (n+2)/2]$ in the cases of Dirichlet and Neumann boundary data, respectively. Besides, the explicit formula for the sharp coefficient in the pointwise estimate for the modulus of the gradient of solution to the equation $\left(c^2 - \Delta\right)^{\alpha/2} u = f$ with $\alpha > 1$ and $f \in L^\infty(\mathbb{R}^n)$ is found.

1. Introduction

In the present paper we find, mainly, the sharp coefficients in certain pointwise estimates for solutions to the modified Helmholtz equation $(\Delta - c^2)u = 0$, $c > 0$, in the half-space $\mathbb{R}^n_+ = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}$. Henceforth we use the term *sharp estimate* if the coefficient in front of a function characteristic in the majorant part of an inequality can't be diminished. This best coefficient we call also *sharp*. It is assumed that the boundary data of the Dirichlet and Neumann problems in $\mathbb{R}^n_+$ for the modified Helmholtz equation belong to the space $L^p(\mathbb{R}^{n-1})$. Previous results of similar nature were obtained in the works [3]-[8], where solutions of the Laplace, Lamé, Stokes and heat equations in $\mathbb{R}^n_+$ were considered.

In particular, in [7] the explicit formula for the sharp coefficient $\mathcal{A}_{n,p}(x)$ in the inequality

$$\left| \nabla \left\{ \frac{u(x)}{x_n} \right\} \right| \leq \mathcal{A}_{n,p}(x) \left\| u(\cdot, 0) \right\|_p \tag{1.1}$$

was derived, where x is an arbitrary point $\mathbb{R}^n_+$, u is a harmonic function in $\mathbb{R}^n_+$, represented by the Poisson integral with boundary values in $L^p(\mathbb{R}^{n-1})$, $||\cdot||_p$ is the norm in $L^p(\mathbb{R}^{n-1})$, $1 \leq p \leq \infty$. It was shown that

$$\mathcal{A}_{n,p}(x) = \frac{A_{n,p}}{x_n^{2+(n-1)/p}},$$

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where

$$A_{n,p} = \frac{2n}{\omega_n} \left\{ \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{3p+n-1}{2(p-1)}\right)}{\Gamma\left(\frac{(n+2)p}{2(p-1)}\right)} \right\}^{1-\frac{1}{p}}$$

for $1 < p < \infty$, and $A_{n,1} = 2n/\omega_n$, $A_{n,\infty} = 1$. Here and henceforth, we denote by $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

Another sharp estimate for the modulus of the gradient of harmonic function u in $\mathbb{R}^n_+$ was obtained in [4]:

$$|\nabla u(x)| \leq \mathcal{N}_{n,p}(x) \left\| \frac{\partial u}{\partial \boldsymbol{\nu}}(\cdot, 0) \right\|_p , \tag{1.2}$$

where $\boldsymbol{\nu}$ is the unit normal vector to $\partial\mathbb{R}^n_+$, $p \in [1, n]$, $x \in \mathbb{R}^n_+$. The best value of the coefficient in (1.2) is given by

$$\mathcal{N}_{n,p}(x) = \frac{N_{n,p}}{x_n^{(n-1)/p}} ,$$

where

$$N_{n,p} = \frac{2^{1/p}}{\omega_n} \left\{ \frac{2\pi^{(n-1)/2} \Gamma\left(\frac{n+p-1}{2p-2}\right)}{\Gamma\left(\frac{np}{2p-2}\right)} \right\}^{1-\frac{1}{p}}$$

for $1 < p \leq n$, and $N_{n,1} = 2/\omega_n$.

The plan of the present paper is as follows. Introduction is followed by four sections. Section 2 is auxiliary. It is devoted to a certain optimization problem with respect to vector parameter inside of an integral over the unit sphere of $\mathbb{R}^n$. In sections 3 and 4 we study solutions of the modified Helmholtz equation in the half-space $\mathbb{R}^n_+$ with Dirichlet and Neumann boundary data, respectively. We note that in these sections we apply the result of section 2 to solution of some n-dimensional extremal problems. In section 5 we deal with solutions in $\mathbb{R}^n$ of non-homogeneous equation containing a power of the modified Helmholtz operator $c^2 - \Delta$.

In what follows, K_ν denotes the modified Bessel function of the third kind, or the Macdonald function.

Now we describe the results of this paper in more detail.

The Dirichlet boundary value problem

$$(\Delta - c^2)u = 0 \;\text{ in }\; \mathbb{R}^n_+, \quad u\big|_{x_n=0} = f(x') \tag{1.3}$$

is considered in Section 3, where $f \in L^p\big(\mathbb{R}^{n-1}\big)$, $1 \leq p \leq \infty$. In this section we obtain the inequality

$$\left| \nabla \left\{ \frac{u(x)}{x_n} \right\} \right| \leq \mathcal{C}_p(x) ||f||_p$$

with the best coefficient

$$\mathcal{C}_1(x) = \left(\frac{c^{n+2}}{2^{n-2}\pi^n} \right)^{1/2} \frac{K_{(n+2)/2}(cx_n)}{x_n^{n/2}}$$

for $p=1$ and

$$\mathcal{C}_p(x)=\frac{2^{1/p}}{\pi^{\frac{n}{2}}x_n^{2+\frac{n-1}{p}}}\max_{|\boldsymbol{z}|=1}\left\{\int_{\mathbb{S}^{n-1}}\rho_n^q\big((\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big)|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)|^{\frac{n+p}{p-1}}|(\boldsymbol{e}_\sigma,\boldsymbol{z})|^q d\sigma\right\}^{\frac{1}{q}} \tag{1.4}$$

for $p\in(1,\infty]$, where $p^{-1}+q^{-1}=1$, u is solution of problem (1.3), x is an arbitrary point in $\mathbb{R}^n_+$ and

$$\rho_m(t)=\int_0^\infty \xi^{m/2}e^{-\xi-\frac{c^2x_n^2}{4\xi t^2}}d\xi.$$

The extremal problem in (1.4) is solved for the case $p\in[2,\infty]$. Namely, it is shown the maximum in (1.4) is attained at $\boldsymbol{z}=\boldsymbol{e}_n$. As a consequence, the explicit formula

$$\mathcal{C}_p(x)=\frac{\omega_{n-1}^{1-\frac{1}{p}}c^{\frac{n+2}{2}}x_n^{\frac{n-2}{2}-\frac{n-1}{p}}}{\pi^{\frac{n}{2}}2^{\frac{n-2}{2}}}\left\{\int_0^{\pi/2}K^q_{\frac{n+2}{2}}\left(\frac{cx_n}{\cos\vartheta}\right)\cos^{\frac{2p-n(p-2)}{2(p-1)}}\vartheta\sin^{n-2}\vartheta d\vartheta\right\}^{\frac{1}{q}}$$

is derived. In particular,

$$\mathcal{C}_\infty(x)=\frac{c^{(n+2)/2}x_n^{(n-2)/2}}{2^{\frac{n-4}{2}}\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}\int_0^{\pi/2}K_{\frac{n+2}{2}}\left(\frac{cx_n}{\cos\vartheta}\right)\frac{\sin^{n-2}\vartheta}{\cos^{(n-2)/2}\vartheta}d\vartheta\ .$$

In Section 4 we obtain an analog of (1.2) for solutions of the Neumann problem

$$(\Delta-c^2)u=0\ \text{ in }\ \mathbb{R}^n_+,\qquad \left.\frac{\partial u}{\partial x_n}\right|_{x_n=0}=g(x')$$

with $g\in L^p\big(\mathbb{R}^{n-1}\big)$, $1\le p\le\infty$. It is shown that for an arbitrary point $x\in\mathbb{R}^n_+$, the sharp coefficient $\mathcal{K}_p(x)$ in the inequality

$$|\nabla u(x)|\le\mathcal{K}_p(x)||g||_p$$

is given by

$$\mathcal{K}_1(x)=\frac{c}{2^{(n-2)/2}\pi^{n/2}}\frac{K_{n/2}(cx_n)}{x_n^{(n-2)/2}}$$

for $p=1$ and

$$\mathcal{K}_p(x)=\frac{2^{\frac{1}{p}-1}}{\pi^{\frac{n}{2}}c^{\frac{n-2}{2}}x_n^{\frac{n-1}{p-1}}}\max_{|\boldsymbol{z}|=1}\left\{\int_{\mathbb{S}^{n-1}}\rho_{n-2}^q\big((\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big)|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)|^{\frac{n-p}{p-1}}|(\boldsymbol{e}_\sigma,\boldsymbol{z})|^q d\sigma\right\}^{\frac{1}{q}}$$

for $p\in(1,\infty]$. It is proven that the maximum in the last equality for the case $p\in[2,(n+2)/2]$ is attained at $\boldsymbol{z}=\boldsymbol{e}_n$. As a corollary, the explicit formula

$$\mathcal{K}_p(x)=\frac{c\,\omega_{n-1}^{1-\frac{1}{p}}x_n^{\frac{n}{2}-\frac{n-1}{p-1}}}{\pi^{\frac{n}{2}}2^{\frac{n-2}{2}}}\left\{\int_0^{\pi/2}K_{n/2}^{p/(p-1)}\left(\frac{cx_n}{\cos\vartheta}\right)\cos^{\frac{(2-p)n}{2(p-1)}}\vartheta\sin^{n-2}\vartheta d\vartheta\right\}^{\frac{p-1}{p}}$$

is obtained, where $p\in[2,(n+2)/2]$. In particular,

$$\mathcal{K}_2(x)=\frac{cx_n^{1/2}}{\pi^{(n+1)/4}2^{(n-3)/2}\sqrt{\Gamma\left(\frac{n-1}{2}\right)}}\left\{\int_0^{\pi/2}K^2_{n/2}\left(\frac{cx_n}{\cos\vartheta}\right)\sin^{n-2}\vartheta\ d\vartheta\right\}^{1/2}.$$

In Section 5 we consider solution of the equation $\left(c^2 - \Delta\right)^{\alpha/2} u = f$ in the whole space with $\alpha > 1$ and $f \in L^{\infty}(\mathbb{R}^n)$. It is shown that for an arbitrary point $x \in \mathbb{R}^n$, the sharp coefficient $\mathcal{B}_\alpha$ in the inequality

$$|\nabla u(x)| \le \mathcal{B}_\alpha ||f||_\infty$$

is given by

$$\mathcal{B}_\alpha = \frac{\Gamma\left(\frac{\alpha-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\alpha}{2}\right) c^{\alpha-1}} \ .$$

In particular,

$$\mathcal{B}_{2m} = \frac{(2m-3)!!}{(2m-2)!! c^{2m-1}}$$

for a natural number m. As a special case of the last formula, for the non-homogeneous modified Helmholtz equation one has

$$\mathcal{B}_2 = \frac{1}{c} \ .$$

2. Extremal problem for integral over $\mathbb{S}^{n-1}$ with vector parameter

Let assume X is the space with σ-finite measure μ defined on the σ-algebra $\mathfrak{S}$ of measurable sets, parameters y and y_0 are elements of a set Y, $\rho(x; y)$ and $f(x; y)$ are $[0, +\infty]$-valued $\mathfrak{S}$-measurable functions on X for any fixed $y \in Y$.

The following assertion was proved in [8].

Proposition 2.1. *Let y_0 be a fixed point of Y, and let $\rho(x; y)$ and $f(x; y)$ be non-negative measurable functions on the space X for any fixed point $y \in Y$. Let $\gamma > 0$ and let the integral*

$$\int_X \rho(x; y_0) f^\gamma(x; y) d\mu \tag{2.1}$$

attains its supremum (the case of $+\infty$ is not excluded) on $y \in Y$ at the point $y_0 \in Y$. Further on, let

$$\mathcal{I}(y, y_0) = \int_X \rho(x; y_0) f^\alpha(x; y) f^\beta(x; y_0) d\mu \ , \tag{2.2}$$

where $\alpha > 0, \beta \ge 0$ and $\alpha + \beta = \gamma$.

Then the equality holds

$$\sup_{y \in Y} \mathcal{I}(y, y_0) = \mathcal{I}(y_0, y_0) = \int_X \rho(x; y_0) f^\gamma(x; y_0) d\mu. \tag{2.3}$$

In particular, the supremum of $\mathcal{I}(y, y_0)$ over $y \in Y$ is independent of y_0 if the value of integral

$$\int_X \rho(x; y) f^\gamma(x; y) d\mu$$

doesn't depend on y.

A particular case of Proposition 2.1 with $\rho \equiv 1$ and somewhat weaker assumption was proved in [9].

Let $\boldsymbol{e}_\sigma$ stand for the n-dimensional unit vector joining the origin to a point σ on the sphere $\mathbb{S}^{n-1}$. In what follows by $\boldsymbol{e}_i$ we mean the unit vector of the i-th coordinate axis. We denote by $\boldsymbol{e}$ and $\boldsymbol{z}$ the n-dimensional unit vectors and assume that $\boldsymbol{e}$ is a fixed vector. Let ρ and f be non-negative Lebesgue measurable functions in $[-1, 1]$.

The next assertion is an immediate consequence of Proposition 2.1.

Corollary 2.2. *Let $\gamma > 0$ and let the integral*

$$\int_{\mathbb{S}^{n-1}} \rho\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big) f^\gamma\big((\boldsymbol{e}_\sigma, \boldsymbol{z})\big) d\sigma \tag{2.4}$$

attains its supremum on $\boldsymbol{z} \in \mathbb{R}^n$, $|\boldsymbol{z}| = 1$ at the vector $\boldsymbol{e}$. Further, let $\alpha \geq 0, \beta > 0$ and $\alpha + \beta = \gamma$. Then

$$\begin{aligned} &\sup_{|\boldsymbol{z}|=1} \int_{\mathbb{S}^{n-1}} \rho\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big) f^\alpha\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big) f^\beta\big((\boldsymbol{e}_\sigma, \boldsymbol{z})\big) d\sigma \\ &= \int_{\mathbb{S}^{n-1}} \rho\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big) f^\gamma\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big) d\sigma. \end{aligned} \tag{2.5}$$

Remark 2.3. The equality

$$\int_{\mathbb{S}^{n-1}} F\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big) d\sigma = \omega_{n-1} \int_0^\pi F\big(\cos\vartheta\big) \sin^{n-2}\vartheta d\vartheta \tag{2.6}$$

shows that value of the integral on the right-hand side of (2.5) is independent of $\boldsymbol{e}$.

In the case of the even function F, the last equality can be written as

$$\int_{\mathbb{S}^{n-1}} F\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big) d\sigma = 2\omega_{n-1} \int_0^{\pi/2} F\big(\cos\vartheta\big) \sin^{n-2}\vartheta d\vartheta. \tag{2.7}$$

Indeed, by (2.6)

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} F\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big) d\sigma \; &= \; \omega_{n-1} \int_0^{\pi/2} F\big(\cos\vartheta\big) \sin^{n-2}\vartheta d\vartheta \\ &+ \; \omega_{n-1} \int_{\pi/2}^{\pi} F\big(\cos\vartheta\big) \sin^{n-2}\vartheta d\vartheta. \end{aligned} \tag{2.8}$$

By the change of variable $\vartheta = \pi - \varphi$ in the second integral on the right-hand side of the last equality, we obtain

$$\int_{\pi/2}^{\pi} F\big(\cos\vartheta\big) \sin^{n-2}\vartheta d\vartheta = -\int_{\pi/2}^{0} F\big(-\cos\varphi\big) \sin^{n-2}\varphi d\varphi,$$

which together with (2.8) and the evenness of function F leads to (2.7).

The following assertion plays an important role in two next sections.

Lemma 2.4. *Let*

$$G_\nu(\boldsymbol{z}) = \int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big) \big|(\boldsymbol{e}_\sigma, \boldsymbol{e})|^\nu \big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|^{2-\nu} d\sigma, \tag{2.9}$$

where ω is a continuous non-negative even function on $[-1,1]$ with continuous positive derivative on $(0,1)$. Then for any $\nu \in [0,2)$, the equality

$$\max_{|\boldsymbol{z}|=1} G_\nu(\boldsymbol{z}) = G_\nu(\boldsymbol{e}) = \int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big) (\boldsymbol{e}_\sigma, \boldsymbol{e})^2 d\sigma \tag{2.10}$$

holds.

Proof. (i) *The case $\nu = 0$.* Let $\boldsymbol{z}' = \boldsymbol{z} - (\boldsymbol{z}, \boldsymbol{e})\boldsymbol{e}$. Then $(\boldsymbol{z}', \boldsymbol{e}) = 0$. We choose the Cartesian coordinates with origin $\mathcal{O}$ at the center of the sphere $\mathbb{S}^{n-1}$ such that $\boldsymbol{e}_1 = \boldsymbol{e}$ and $\boldsymbol{e}_n$ is collinear to $\boldsymbol{z}'$. Then $\boldsymbol{z} = \alpha \boldsymbol{e}_1 + \beta \boldsymbol{e}_n$, where

$$\alpha^2 + \beta^2 = 1. \tag{2.11}$$

Now, we rewrite (2.9) for the case $\nu = 0$ in the form

$$\begin{aligned} G_0(\boldsymbol{z}) &= \int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_1)\big) \big(\boldsymbol{e}_\sigma, \alpha \boldsymbol{e}_1 + \beta \boldsymbol{e}_n\big)^2 d\sigma \\ &= \int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_1)\big) \big[\alpha^2 (\boldsymbol{e}_\sigma, \boldsymbol{e}_1)^2 + 2\alpha\beta (\boldsymbol{e}_\sigma, \boldsymbol{e}_1)(\boldsymbol{e}_\sigma, \boldsymbol{e}_n) + \beta^2 (\boldsymbol{e}_\sigma, \boldsymbol{e}_n)^2\big] d\sigma. \end{aligned} \tag{2.12}$$

Let us show that

$$\int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_1)\big) (\boldsymbol{e}_\sigma, \boldsymbol{e}_1)(\boldsymbol{e}_\sigma, \boldsymbol{e}_n) d\sigma = 0\ . \tag{2.13}$$

The last equality is obvious for the case $n = 2$. We suppose that $n \geq 3$. Let us denote by $\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}$ the spherical coordinates in $\mathbb{R}^n$ with the center at $\mathcal{O}$, where $\vartheta_i \in [0, \pi]$ for $1 \leq i \leq n-2$, and $\vartheta_{n-1} \in [0, 2\pi]$. Then for any $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in \mathbb{S}^{n-1}$ we have

$$\begin{aligned} &\sigma_1 = \cos \vartheta_1, \\ &\sigma_2 = \sin \vartheta_1 \cos \vartheta_2, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\sigma_{n-1} = \sin \vartheta_1 \dots \sin \vartheta_{n-2} \cos \vartheta_{n-1}, \\ &\sigma_n = \sin \vartheta_1 \dots \sin \vartheta_{n-2} \sin \vartheta_{n-1}. \end{aligned}$$

Using the equalities

$$(\boldsymbol{e}_\sigma, \boldsymbol{e}_1) = \sigma_1 = \cos \vartheta_1, \qquad (\boldsymbol{e}_\sigma, \boldsymbol{e}_n) = \sigma_n = \sin \vartheta_1 \dots \sin \vartheta_{n-2} \sin \vartheta_{n-1}$$

in view of

$$d\sigma = \sin^{n-2} \vartheta_1 \sin^{n-3} \vartheta_2 \dots \sin \vartheta_{n-2}\, d\vartheta_1 d\vartheta_2 \dots d\vartheta_{n-1},$$

we calculate the integral on the left-hand side of (2.13):

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_1)\big) (\boldsymbol{e}_\sigma, \boldsymbol{e}_1)(\boldsymbol{e}_\sigma, \boldsymbol{e}_n) d\sigma \\ &= \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \omega\big(\cos \vartheta_1\big) \cos \vartheta_1 \left(\prod_{i=1}^{n-2} \sin^{n-i} \vartheta_i \right) \sin \vartheta_{n-1} d\vartheta_1 \dots d\vartheta_{n-2} d\vartheta_{n-1} \end{aligned}$$

$$= I \int_0^{2\pi} \sin\vartheta_{n-1} d\vartheta_{n-1} = 0 \ , \tag{2.14}$$

where

$$I = \int_0^{\pi} ... \int_0^{\pi} \omega\big(\cos\vartheta_1\big)\cos\vartheta_1 \left(\prod_{i=1}^{n-2} \sin^{n-i}\vartheta_i\right) d\vartheta_1 ... d\vartheta_{n-2} \ .$$

Equality (2.14) proves (2.13).

Hence, by (2.11), (2.12) and (2.13) we obtain

$$G_0(\boldsymbol{z}) = \int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_1)\big)\big[\alpha^2(\boldsymbol{e}_\sigma, \boldsymbol{e}_1)^2 + \beta^2(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)^2\big] d\sigma \le \max\{L, M\}, \tag{2.15}$$

where

$$L = \int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_1)\big)(\boldsymbol{e}_\sigma, \boldsymbol{e}_1)^2 d\sigma \tag{2.16}$$

and

$$M = \int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_1)\big)(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)^2 d\sigma. \tag{2.17}$$

In view of the evenness of the function $\omega(t)$, by virtue of (2.7) we can write (2.16) as

$$L = 2\omega_{n-1} \int_0^{\pi/2} \omega(\cos\vartheta_1)\cos^2\vartheta_1 \sin^{n-2}\vartheta_1 d\vartheta_1.$$

By the change of variable $\vartheta_1 = \frac{\pi}{2} - \varphi$ in the integral on the right-hand side of the last equality, we obtain

$$L = 2\omega_{n-1} \int_0^{\pi/2} \omega(\sin\varphi)\sin^2\varphi \cos^{n-2}\varphi d\varphi \ . \tag{2.18}$$

Now, we calculate the integral on the right-hand side of (2.17):

$$\begin{aligned} M &= \int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_1)\big)(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)^2 d\sigma \\ &= \int_0^{\pi} \!\!\cdots\! \int_0^{\pi} \int_0^{2\pi} \omega(\cos\vartheta_1) \left(\prod_{i=1}^{n-1} \sin^{n+1-i}\vartheta_i\right) d\vartheta_1 \ldots d\vartheta_{n-2} d\vartheta_{n-1} \\ &= \left\{ \int_0^{\pi} \omega(\cos\vartheta_1)\sin^n\vartheta_1 d\vartheta_1 \right\} \left\{ 2\int_0^{\pi} \!\!...\! \int_0^{\pi} \left(\prod_{i=2}^{n-1} \sin^{n+1-i}\vartheta_i\right) d\vartheta_2 ... d\vartheta_{n-1} \right\}. \end{aligned} \tag{2.19}$$

Changing the variable $\vartheta_1 = \varphi + \frac{\pi}{2}$ in the first integral on the right-hand side of (2.19) and using the evenness of $\omega(t)$, we arrive at equality

$$\int_0^{\pi} \omega(\cos\vartheta_1)\sin^n\vartheta_1 d\vartheta_1 = 2\int_0^{\pi/2} \omega(\sin\varphi)\varphi \cos^n\varphi d\varphi \ . \tag{2.20}$$

Calculating the multiple integral on the right-hand side of (2.19), we obtain

$$2\int_0^{\pi} ... \int_0^{\pi} \left(\prod_{i=2}^{n-1} \sin^{n+1-i}\vartheta_i\right) d\vartheta_2 ... d\vartheta_{n-1} = 2 \cdot 2^{n-2} \prod_{k=2}^{n-1} \int_0^{\pi/2} \sin^k\vartheta d\vartheta$$

$$= \frac{2^{n-1}}{2^{n-2}} \prod_{k=2}^{n-1} \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)} = \frac{2\pi^{(n-1)/2}}{(n-1)\Gamma\left(\frac{n-1}{2}\right)} = \frac{\omega_{n-1}}{n-1} \ ,$$

which together with (2.19) and (2.20) leads to

$$M = \frac{2\omega_{n-1}}{n-1} \int_0^{\pi/2} \omega(\sin\varphi)\cos^n\varphi d\varphi \ . \tag{2.21}$$

Let us show that $L \geq M$. Choosing two positive numbers ϵ and δ such that $\epsilon < \frac{\pi}{2} - \delta$, we transform the integral

$$2\omega_{n-1} \int_\epsilon^{\frac{\pi}{2}-\delta} \omega(\sin\varphi)\sin^2\varphi\cos^{n-2}\varphi d\varphi$$

$$= -\frac{2\omega_{n-1}}{n-1} \int_\epsilon^{\frac{\pi}{2}-\delta} \omega(\sin\varphi)\sin\varphi \, d\big(\cos^{n-1}\varphi\big)$$

$$= -\frac{2\omega_{n-1}}{n-1} \left\{ \omega(\sin\varphi)\sin\varphi\cos^{n-1}\varphi \Big|_\epsilon^{\frac{\pi}{2}-\delta} - \int_\epsilon^{\frac{\pi}{2}-\delta} \cos^{n-1}\varphi \, d\left(\omega(\sin\varphi)\sin\varphi\right) \right\}$$

$$= \frac{2\omega_{n-1}}{n-1} \left\{ -\omega(\sin\varphi)\sin\varphi\cos^{n-1}\varphi \Big|_\epsilon^{\frac{\pi}{2}-\delta} + \int_\epsilon^{\frac{\pi}{2}-\delta} \omega(\sin\varphi)\cos^n\varphi d\varphi \right\}$$

$$+ \frac{2\omega_{n-1}}{n-1} \int_\epsilon^{\frac{\pi}{2}-\delta} \omega\,'(\sin\varphi)\cos^n\varphi\sin\varphi \, d\varphi.$$

Since $\omega'(t) > 0$ in the interval $t \in (0,1)$ by assumption of the Lemma, it follows from the last equality that

$$2\omega_{n-1} \int_\epsilon^{\frac{\pi}{2}-\delta} \omega(\sin\varphi)\sin^2\varphi\cos^{n-2}\varphi d\varphi$$

$$> \frac{2\omega_{n-1}}{n-1} \left\{ -\omega(\sin\varphi)\sin\varphi\cos^{n-1}\varphi \Big|_\epsilon^{\frac{\pi}{2}-\delta} + \int_\epsilon^{\frac{\pi}{2}-\delta} \omega(\sin\varphi)\cos^n\varphi d\varphi \right\}.$$

Passing to the limits as $\delta \to 0, \epsilon \to 0$ in the last inequality and taking into account (2.18) and (2.21), we arrive at

$$L \geq M.$$

This, by (2.15) and (2.16), leads to the inequality

$$\max_{|\boldsymbol{z}|=1} G_0(\boldsymbol{z}) \leq \int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_1)\big)(\boldsymbol{e}_\sigma, \boldsymbol{e}_1)^2 d\sigma \ . \tag{2.22}$$

By (2.7), the value of the integral

$$\int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big)(\boldsymbol{e}_\sigma, \boldsymbol{e})^2 d\sigma$$

is independent of $\boldsymbol{e}$. Hence, by (2.22),

$$\max_{|\boldsymbol{z}|=1} G_0(\boldsymbol{z}) \leq \int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big)(\boldsymbol{e}_\sigma, \boldsymbol{e})^2 d\sigma \tag{2.23}$$

for an arbitrary n-dimensional unit vector $\boldsymbol{e}$.

The lower estimate

$$\max_{|\boldsymbol{z}|=1} G_0(\boldsymbol{z}) \geq G_0(\boldsymbol{e}) = \int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e})\big)(\boldsymbol{e}_\sigma, \boldsymbol{e})^2 d\sigma$$

follows from (2.9) with $\nu = 0$, which together with (2.23) proves equality (2.10) for the case $\nu = 0$.

(ii) *The case* $\nu \in (0, 2)$. Equality (2.10) is an immediate consequence of part (i) of the Lemma and Corollary 2.2. □

3. Sharp weighted estimate for the gradient of solution to the Dirichlet problem in the half-space

We denote by $||\cdot||_p$ the norm in the space $L^p(\mathbb{R}^{n-1})$, that is

$$||f||_p = \left\{ \int_{\mathbb{R}^{n-1}} |f(x')|^p \, dx' \right\}^{1/p},$$

if $1 \leq p < \infty$, and $||f||_\infty = \text{ess sup}\{|f(x')| : x' \in \mathbb{R}^{n-1}\}$.

Solution of the Dirichlet problem in $\mathbb{R}^n_+$ for the modified Helmholtz equation,

$$(\Delta - c^2)u = 0 \ \text{ in } \ \mathbb{R}^n_+, \quad u\big|_{x_n=0} = f(x') \tag{3.1}$$

with continuous and bounded function f on $\mathbb{R}^{n-1}$, is given by (e.g. [13]):

$$u(x) = \frac{c^n x_n}{2^{(n-2)/2}\pi^{n/2}} \int_{\mathbb{R}^{n-1}} \frac{K_{n/2}\big(c|y-x|\big)}{\big(c|y-x|\big)^{n/2}} f(y')dy' \,, \tag{3.2}$$

where $y = (y', 0), y' \in \mathbb{R}^{n-1}$.

Let us consider solution of problem (3.1) with $f \in L^p(\mathbb{R}^{n-1})$ represented by (3.2), where $p \in [1, \infty]$. A related theory of harmonic functions in $\mathbb{R}^n_+$ with boundary values from $L^p(\mathbb{R}^{n-1})$ is described, for instance, in [14] (Ch. 2, Sect. 2).

In this section we prove the following assertion.

Theorem 3.1. *Let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_p(x)$ in the inequality*

$$\left| \nabla \left\{ \frac{u(x)}{x_n} \right\} \right| \leq \mathcal{C}_p(x)||f||_p \tag{3.3}$$

is given by

$$\mathcal{C}_1(x) = \left(\frac{c^{n+2}}{2^{n-2}\pi^n} \right)^{1/2} \frac{K_{(n+2)/2}(cx_n)}{x_n^{n/2}} \tag{3.4}$$

for $p = 1$ and

$$\mathcal{C}_p(x) = \frac{2^{1/p}}{\pi^{\frac{n}{2}} x_n^{2+\frac{n-1}{p}}} \max_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}} \rho_n^q\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big) |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{n+p}{p-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{z})|^q d\sigma \right\}^{\frac{1}{q}} \tag{3.5}$$

for $p \in (1, \infty]$, where $p^{-1} + q^{-1} = 1$ and

$$\rho_m(t) = \int_0^\infty \xi^{m/2} e^{-\xi - \frac{c^2 x_n^2}{4\xi t^2}} d\xi. \tag{3.6}$$

In particular,

$$\mathcal{C}_p(x) = \frac{2^{1/p}}{\pi^{\frac{n}{2}} x_n^{2+\frac{n-1}{p}}} \left\{ \int_{\mathbb{S}^{n-1}} \rho_n^q \big((\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big) |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{n+2p}{p-1}} d\sigma \right\}^{\frac{1}{q}} \tag{3.7}$$

$$= \frac{\omega_{n-1}^{1-\frac{1}{p}} c^{\frac{n+2}{2}} x_n^{\frac{n-2}{2}-\frac{n-1}{p}}}{\pi^{\frac{n}{2}} 2^{\frac{n-2}{2}}} \left\{ \int_0^{\pi/2} K_{\frac{n+2}{2}}^q \left(\frac{c x_n}{\cos \vartheta} \right) \cos^{\frac{2p-n(p-2)}{2(p-1)}} \vartheta \sin^{n-2} \vartheta d\vartheta \right\}^{\frac{1}{q}} \tag{3.8}$$

for $2 \le p \le \infty$.

As a special case of (3.8) *one has*

$$\mathcal{C}_\infty(x) = \frac{c^{(n+2)/2} x_n^{(n-2)/2}}{2^{\frac{n-4}{2}} \sqrt{\pi}\, \Gamma\left(\frac{n-1}{2}\right)} \int_0^{\pi/2} K_{(n+2)/2} \left(\frac{c x_n}{\cos \vartheta} \right) \frac{\sin^{n-2} \vartheta}{\cos^{(n-2)/2} \vartheta} d\vartheta \,. \tag{3.9}$$

Proof. By (3.2),

$$\frac{u(x)}{x_n} = 2 \left(\frac{c^2}{2\pi} \right)^{n/2} \int_{\mathbb{R}^{n-1}} \frac{K_{n/2}\big(c|y-x|\big)}{\big(c|y-x|\big)^{n/2}} f(y') dy' \,.$$

Differentiating the last equality with respect to x_j, $j = 1, \dots, n$, in view of (see e.g. [10])

$$\frac{d}{dt} \left(\frac{K_\nu(t)}{t^\nu} \right) = -\frac{K_{\nu+1}(t)}{t^\nu} \,, \tag{3.10}$$

we obtain

$$\nabla \left\{ \frac{u(x)}{x_n} \right\} = -2 \left(\frac{c^2}{2\pi} \right)^{n/2} \int_{\mathbb{R}^{n-1}} \frac{K_{(n+2)/2}\big(c|y-x|\big)}{\big(c|y-x|\big)^{n/2}} \nabla \big(c|y-x|\big) f(y') dy'$$

$$= 2 \frac{c}{c^{n/2}} \left(\frac{c^2}{2\pi} \right)^{n/2} \int_{\mathbb{R}^{n-1}} \frac{K_{(n+2)/2}\big(c|y-x|\big)}{|y-x|^{n/2}} \frac{y-x}{|y-x|} f(y') dy'.$$

Denoting $\boldsymbol{e}_{xy} = (y-x)/|y-x|$, we rewrite the last equality as

$$\nabla \left\{ \frac{u(x)}{x_n} \right\} = \left(\frac{c^{n+2}}{2^{n-2} \pi^n} \right)^{1/2} \int_{\mathbb{R}^{n-1}} \frac{K_{(n+2)/2}\big(c|y-x|\big)}{|y-x|^{n/2}} \boldsymbol{e}_{xy} f(y') dy',$$

which leads to

$$\left(\nabla \left\{ \frac{u(x)}{x_n} \right\}, \boldsymbol{z} \right) = \left(\frac{c^{n+2}}{2^{n-2} \pi^n} \right)^{1/2} \int_{\mathbb{R}^{n-1}} \frac{K_{(n+2)/2}\big(c|y-x|\big)}{|y-x|^{n/2}} \big(\boldsymbol{e}_{xy}, \boldsymbol{z}\big) f(y') dy', \tag{3.11}$$

where $\boldsymbol{z}$ is a unit n-dimensional vector.

The known integral representation (see e.g. [10])

$$K_\nu(t) = \frac{1}{2} \left(\frac{t}{2} \right)^\nu \int_0^\infty \xi^{-\nu-1} e^{-\xi - \frac{t^2}{4\xi}} d\xi$$

in view of the property $K_\nu(t) = K_{-\nu}(t)$ of the Macdonald function, can be written in the form

$$K_\nu(t) = \frac{1}{2} \left(\frac{2}{t} \right)^\nu \int_0^\infty \xi^{\nu-1} e^{-\xi - \frac{t^2}{4\xi}} d\xi \,, \tag{3.12}$$

which together with (3.11) implies

$$\left(\nabla \left\{ \frac{u(x)}{x_n} \right\}, \boldsymbol{z}\right) = \frac{2}{\pi^{n/2}} \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \xi^{n/2} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi \right) \frac{(\boldsymbol{e}_{xy}, \boldsymbol{z})}{|y-x|^{n+1}} f(y') dy'.$$

The last equality, by the property of the inner product in $\mathbb{R}^n$, leads to

$$\left| \nabla \left\{ \frac{u(x)}{x_n} \right\} \right| = \frac{2}{\pi^{n/2}} \max_{|\boldsymbol{z}|=1} \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \xi^{n/2} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi \right) \frac{(\boldsymbol{e}_{xy}, \boldsymbol{z})}{|y-x|^{n+1}} f(y') dy'.$$

Therefore, the sharp coefficient $\mathcal{C}_p(x)$ in inequality (3.3) is given by

$$\mathcal{C}_p(x) = \frac{2}{\pi^{n/2}} \sup_{||f||_p=1} \max_{|\boldsymbol{z}|=1} \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \xi^{n/2} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi \right) \frac{(\boldsymbol{e}_{xy}, \boldsymbol{z})}{|y-x|^{n+1}} f(y') dy',$$

which after permutation of suprema becomes

$$\mathcal{C}_p(x) = \frac{2}{\pi^{\frac{n}{2}}} \max_{|\boldsymbol{z}|=1} \sup_{||f||_p=1} \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \xi^{\frac{n}{2}} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi \right) \frac{(\boldsymbol{e}_{xy}, \boldsymbol{z})}{|y-x|^{n+1}} f(y') dy'. \tag{3.13}$$

(i) *The case* $p = 1$. In view of (3.13) with $p = 1$ and the theorem on the norm of a linear functional in the space of summable functions, we obtain

$$\mathcal{C}_1(x) = \frac{2}{\pi^{n/2}} \max_{|\boldsymbol{z}|=1} \sup_{y \in \partial \mathbb{R}^n_+} \frac{|(\boldsymbol{e}_{xy}, \boldsymbol{z})|}{|y-x|^{n+1}} \int_0^\infty \xi^{n/2} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi.$$

Using the permutation of suprema in the last equality, we arrive at

$$\begin{aligned} \mathcal{C}_1(x) &= \frac{2}{\pi^{n/2}} \sup_{y \in \partial \mathbb{R}^n_+} \frac{1}{|y-x|^{n+1}} \int_0^\infty \xi^{n/2} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi \\ &= \frac{2}{\pi^{n/2} x_n^{n+1}} \int_0^\infty \xi^{n/2} e^{-\xi - \frac{c^2 x_n^2}{4\xi}} d\xi\ , \end{aligned}$$

which together with (3.12) proves (3.4).

(ii) *Representation of the sharp coefficient* $\mathcal{C}_p(x)$ *in inequality* (3.3) *in the case* $p \in (1, \infty]$. By (3.13) and the theorem on the norm of a linear functional in L^p, we have

$$\mathcal{C}_p(x) = \frac{2}{\pi^{\frac{n}{2}}} \max_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \xi^{n/2} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi \right)^q \frac{|(\boldsymbol{e}_{xy}, \boldsymbol{z})|^q}{|y-x|^{(n+1)q}} dy' \right\}^{\frac{1}{q}}. \tag{3.14}$$

Using the equality

$$|y-x| |(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)| = x_n\ , \tag{3.15}$$

we represent the inner integral on the right-hand side of (3.14) as

$$\int_0^\infty \xi^{n/2} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi = \rho_n\big((\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\big), \tag{3.16}$$

where the function $\rho_m(t)$ is defined by (3.6).

In view of (3.15), we have

$$\frac{1}{|y-x|^{(n+1)q}} = \frac{1}{x_n |y-x|^{n(q-1)+q}} \frac{x_n}{|y-x|^n}$$

$$= \frac{1}{x_n}\left(\frac{|(\boldsymbol{e}_{xy},\boldsymbol{e}_n)|}{x_n}\right)^{n(q-1)+q}\frac{x_n}{|y-x|^n}$$

$$= \frac{1}{x_n^{n(q-1)+q+1}}|(\boldsymbol{e}_{xy},\boldsymbol{e}_n)|^{n(q-1)+q}\frac{x_n}{|y-x|^n}\ ,$$

which together with (3.16) allows us to represent (3.14) in the form

$$\mathcal{C}_p(x)=\frac{2}{\pi^{\frac{n}{2}}x_n^{2+\frac{n-p}{p}}}\max_{|\boldsymbol{z}|=1}\left\{\int_{\mathbb{S}_-^{n-1}}\rho_n^q\big((\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big)|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)|^{\frac{n+p}{p-1}}|(\boldsymbol{e}_\sigma,\boldsymbol{z})|^q d\sigma\right\}^{\frac{1}{q}}, \tag{3.17}$$

where $\mathbb{S}_-^{n-1}=\{\sigma\in\mathbb{S}^{n-1}:(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)<0\}$.

Using the evenness of the function $\rho_m(t)$ defined by (3.6), we rewrite (3.17) as

$$\mathcal{C}_p(x)=\frac{2^{1/p}}{\pi^{\frac{n}{2}}x_n^{2+\frac{n-1}{p}}}\max_{|\boldsymbol{z}|=1}\left\{\int_{\mathbb{S}^{n-1}}\rho_n^q\big((\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big)|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)|^{\frac{n+p}{p-1}}|(\boldsymbol{e}_\sigma,\boldsymbol{z})|^q d\sigma\right\}^{\frac{1}{q}}, \tag{3.18}$$

which proves (3.5).

(iii) *Solution of extremal problem in* (3.5) *for the case* $p\in[2,\infty]$. We introduce the function

$$\omega(t)=|t|^{\frac{n+2}{p-1}}\rho_n^{\frac{p}{p-1}}=|t|^{\frac{n+2}{p-1}}\left(\int_0^\infty \xi^{n/2}e^{-\xi-\frac{c^2x_n^2}{4\xi t^2}}d\xi\right)^{\frac{p}{p-1}} \tag{3.19}$$

for $t\neq 0$, $\omega(0)=0$, and rewrite (3.18) as

$$\mathcal{C}_p(x)=\frac{2^{1/p}}{\pi^{\frac{n}{2}}x_n^{2+\frac{n-1}{p}}}\max_{|\boldsymbol{z}|=1}\left\{\int_{\mathbb{S}^{n-1}}\omega\big((\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big)|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)|^{\frac{p-2}{p-1}}|(\boldsymbol{e}_\sigma,\boldsymbol{z})|^{2-\frac{p-2}{p-1}}d\sigma\right\}^{\frac{1}{q}}. \tag{3.20}$$

Since the quantity $\gamma=(p-2)/(p-1)$ for $p\in[2,\infty]$ satisfies inequality $0\le\gamma\le 1$ and the function (3.19) obeys the assumptions of Lemma 2.4, we can apply Lemma 2.4 to (3.20). As the result, we obtain

$$\mathcal{C}_p(x)=\frac{2^{1/p}}{\pi^{n/2}x_n^{2+\frac{n-1}{p}}}\left\{\int_{\mathbb{S}^{n-1}}\omega\big((\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big)(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)^2 d\sigma\right\}^{\frac{p-1}{p}}. \tag{3.21}$$

Substituting $\omega(t)$ from (3.19) into (3.21), we arrive at (3.7).

(iv) *Representation of* $\mathcal{C}_p(x)$ *as definite integral for the case* $p\in[2,\infty]$. Using (2.7), we rewrite (3.7) in the form

$$\mathcal{C}_p(x)=\frac{2\omega_{n-1}^{1/q}}{\pi^{\frac{n}{2}}x_n^{2+\frac{n-1}{p}}}\left\{\int_0^{\pi/2}\rho_n^{\frac{p}{p-1}}\big(\cos\vartheta\big)\cos^{\frac{n+2p}{p-1}}\vartheta\sin^{n-2}\vartheta d\vartheta\right\}^{\frac{p-1}{p}}. \tag{3.22}$$

In view of (3.6) and (3.12),

$$\rho_n\big(\cos\vartheta\big)=2\left(\frac{cx_n}{2\cos\vartheta}\right)^{\frac{n+2}{2}}K_{\frac{n+2}{2}}\left(\frac{cx_n}{\cos\vartheta}\right)\ ,$$

which together with (3.22) leads to (3.8). □

4. Sharp estimate for the gradient of solution to the Neumann problem in the half-space

Solution of the Neumann problem in $\mathbb{R}^n_+$ for the modified Helmholtz equation,

$$(\Delta - c^2)u = 0 \text{ in } \mathbb{R}^n_+, \qquad \left.\frac{\partial u}{\partial x_n}\right|_{x_n=0} = g(x') \tag{4.1}$$

with continuous and bounded function g on $\mathbb{R}^{n-1}$, is given by (e.g. [12], sect. 7.3, 8.3)

$$u(x) = -\frac{2c^{(n-2)/2}}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \frac{K_{(n-2)/2}\big(c|y-x|\big)}{\big(c|y-x|\big)^{(n-2)/2}} g(y')dy'. \tag{4.2}$$

Here, as before, $y = (y', 0), y' \in \mathbb{R}^{n-1}$.

In this section we consider solution of problem (4.1) with $g \in L^p(\mathbb{R}^{n-1})$ represented by (4.2), where $p \in [1, \infty]$.

Now, we prove

Theorem 4.1. *Let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{K}_p(x)$ in the inequality*

$$|\nabla u(x)| \le \mathcal{K}_p(x)||g||_p \tag{4.3}$$

is given by

$$\mathcal{K}_1(x) = \frac{c}{2^{(n-2)/2}\pi^{n/2}} \frac{K_{n/2}(cx_n)}{x_n^{(n-2)/2}} \tag{4.4}$$

for $p = 1$ and

$$\mathcal{K}_p(x) = \frac{2^{(1-p)/p}}{\pi^{n/2} c^{\frac{n-2}{2}} x_n^{\frac{n-1}{p-1}}} \max_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}} \rho^q_{n-2}\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big) |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{n-p}{p-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{z})|^q d\sigma \right\}^{\frac{1}{q}} \tag{4.5}$$

for $p \in (1, \infty]$, where $p^{-1} + q^{-1} = 1$ and the function $\rho_m(t)$ is defined by (3.6)*.*

In particular,

$$\mathcal{K}_p(x) = \frac{2^{(1-p)/p}}{\pi^{\frac{n}{2}} c^{\frac{n-2}{2}} x_n^{\frac{n-1}{p-1}}} \left\{ \int_{\mathbb{S}^{n-1}} \rho^q_{n-2}\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big) |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{n}{p-1}} d\sigma \right\}^{\frac{1}{q}} \tag{4.6}$$

$$= \frac{c\, \omega^{1/q}_{n-1} x_n^{\frac{n}{2} - \frac{n-1}{p-1}}}{\pi^{\frac{n}{2}} 2^{\frac{n-2}{2}}} \left\{ \int_0^{\pi/2} K^q_{n/2}\left(\frac{cx_n}{\cos\vartheta}\right) \cos^{\frac{(2-p)n}{2(p-1)}} \vartheta \sin^{n-2}\vartheta d\vartheta \right\}^{\frac{1}{q}} \tag{4.7}$$

for $p \in [2, (n+2)/2]$.

As a special case of (4.7) *one has*

$$\mathcal{K}_2(x) = \frac{c x_n^{1/2}}{\pi^{\frac{n+1}{4}} 2^{\frac{n-3}{2}} \sqrt{\Gamma\left(\frac{n-1}{2}\right)}} \left\{ \int_0^{\pi/2} K^2_{n/2}\left(\frac{cx_n}{\cos\vartheta}\right) \sin^{n-2}\vartheta \, d\vartheta \right\}^{1/2}. \tag{4.8}$$

Proof. Differentiating in (4.2) with respect to x_j, $j = 1, \dots, n$, and using (3.10), we obtain

$$\begin{aligned}\nabla u(x) &= \frac{2c^{(n-2)/2}}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \frac{K_{n/2}\big(c|y-x|\big)}{\big(c|y-x|\big)^{(n-2)/2}} \nabla\big(c|y-x|\big) g(y')dy' \\ &= -\frac{2c}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \frac{K_{n/2}\big(c|y-x|\big)}{|y-x|^{(n-2)/2}} \frac{y-x}{|y-x|} g(y')dy',\end{aligned}$$

which can be written as

$$\nabla u(x) = -\frac{2c}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \frac{K_{n/2}\big(c|y-x|\big)}{|y-x|^{(n-2)/2}} \boldsymbol{e}_{xy} g(y')dy',$$

where $\boldsymbol{e}_{xy} = (y-x)/|y-x|$. It follows

$$\big(\nabla u(x), \boldsymbol{z}\big) = -\frac{2c}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \frac{K_{n/2}\big(c|y-x|\big)}{|y-x|^{(n-2)/2}} (\boldsymbol{e}_{xy}, \boldsymbol{z}) g(y')dy',$$

where $\boldsymbol{z}$ is a unit n-dimensional vector. Therefore,

$$\big|\nabla u(x)\big| = \frac{2c}{(2\pi)^{n/2}} \max_{|\boldsymbol{z}|=1} \int_{\mathbb{R}^{n-1}} -\frac{K_{n/2}\big(c|y-x|\big)}{|y-x|^{(n-2)/2}} (\boldsymbol{e}_{xy}, \boldsymbol{z}) g(y')dy'.$$

From the last equality, by permutation of suprema, we obtain the representation of the sharp coefficient $\mathcal{K}_p(x)$ in inequality (4.3)

$$\mathcal{K}_p(x) = \frac{2c}{(2\pi)^{n/2}} \max_{|\boldsymbol{z}|=1} \sup_{||g||_p=1} \int_{\mathbb{R}^{n-1}} -\frac{K_{n/2}\big(c|y-x|\big)}{|y-x|^{(n-2)/2}} (\boldsymbol{e}_{xy}, \boldsymbol{z}) g(y')dy',$$

which in view of (3.12), can be written as

$$\mathcal{K}_p(x) = k_n \max_{|\boldsymbol{z}|=1} \sup_{||g||_p=1} \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \xi^{\frac{n-2}{2}} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi \right) \frac{(\boldsymbol{e}_{xy}, \boldsymbol{z})}{|y-x|^{n-1}} g(y')dy', \tag{4.9}$$

where

$$k_n = \frac{1}{\pi^{n/2} c^{(n-2)/2}}.$$

(i) *The case* $p = 1$. In view of (4.9) with $p = 1$ and the theorem on the norm of a linear functional in the space of summable functions, we obtain

$$\mathcal{K}_1(x) = \frac{1}{\pi^{n/2} c^{(n-2)/2}} \max_{|\boldsymbol{z}|=1} \sup_{y \in \partial\mathbb{R}^n_+} \frac{|(\boldsymbol{e}_{xy}, \boldsymbol{z})|}{|y-x|^{n-1}} \int_0^\infty \xi^{(n-2)/2} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi.$$

Using the permutation of suprema in the last equality, we arrive at

$$\begin{aligned}\mathcal{K}_1(x) &= \frac{1}{\pi^{n/2} c^{(n-2)/2}} \sup_{y \in \partial\mathbb{R}^n_+} \frac{1}{|y-x|^{n-1}} \int_0^\infty \xi^{(n-2)/2} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi \\ &= \frac{1}{\pi^{n/2} c^{(n-2)/2} x_n^{n-1}} \int_0^\infty \xi^{(n-2)/2} e^{-\xi - \frac{c^2 x_n^2}{4\xi}} d\xi\ ,\end{aligned}$$

which together with (3.12) proves (4.4).

(ii) *Representation of the sharp coefficient* $\mathcal{K}_p(x)$ *in inequality* (4.3) *in the case* $p \in (1, \infty]$. By (4.9) and the theorem on the norm of a linear functional in L^p, we have

$$\mathcal{K}_p(x) = k_n \max_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \xi^{\frac{n-2}{2}} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi \right)^q \frac{|(\boldsymbol{e}_{xy}, \boldsymbol{z})|^q}{|y-x|^{(n-1)q}} dy' \right\}^{\frac{1}{q}}. \tag{4.10}$$

Using equality (3.15), we represent the inner integral on the right-hand side of (4.10) as

$$\int_0^\infty \xi^{(n-2)/2} e^{-\xi - \frac{c^2|y-x|^2}{4\xi}} d\xi = \rho_{n-2}\big((\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\big), \tag{4.11}$$

where the function $\rho_m(t)$ is defined by (3.6).

In view of (3.15), we have

$$\begin{aligned} \frac{1}{|y-x|^{(n-1)q}} &= \frac{1}{x_n |y-x|^{n(q-1)-q}} \frac{x_n}{|y-x|^n} \\ &= \frac{1}{x_n} \left(\frac{|(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)|}{x_n} \right)^{n(q-1)-q} \frac{x_n}{|y-x|^n} \\ &= \frac{1}{x_n^{n(q-1)-q+1}} |(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)|^{n(q-1)-q} \frac{x_n}{|y-x|^n}, \end{aligned}$$

which together with (4.11) allows us to represent (4.10) as

$$\mathcal{K}_p(x) = \frac{k_n}{x_n^{\frac{n-1}{p-1}}} \max_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}_-} \rho^q_{n-2}\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big) |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{n-p}{p-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{z})|^q d\sigma \right\}^{\frac{1}{q}}, \tag{4.12}$$

where $\mathbb{S}^{n-1}_- = \{\sigma \in \mathbb{S}^{n-1} : (\boldsymbol{e}_\sigma, \boldsymbol{e}_n) < 0\}$.

Using the evenness of the function $\rho_m(t)$ defined by (3.6), we rewrite (4.12) as

$$\mathcal{K}_p(x) = \frac{k_n 2^{-\frac{1}{q}}}{x_n^{\frac{n-1}{p-1}}} \max_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}} \rho^q_{n-2}\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big) |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{n-p}{p-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{z})|^q d\sigma \right\}^{\frac{1}{q}}, \tag{4.13}$$

which proves (4.5).

(iii) *Solution of extremal problem in* (4.5) *for the case* $p \in [2, (n+2)/2]$. We introduce the function

$$\omega(t) = |t|^{\frac{n+2-2p}{p-1}} \rho^{\frac{p}{p-1}}_{n-2} = |t|^{\frac{n+2-2p}{p-1}} \left(\int_0^\infty \xi^{(n-2)/2} e^{-\xi - \frac{c^2 x_n^2}{4\xi t^2}} d\xi \right)^{\frac{p}{p-1}} \tag{4.14}$$

for $t \neq 0$, which is defined at $t = 0$ by continuity, and rewrite (4.13) as

$$\mathcal{K}_p(x) = \frac{k_n 2^{-\frac{1}{q}}}{x_n^{\frac{n-1}{p-1}}} \max_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big) |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{p-2}{p-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{z})|^{2-\frac{p-2}{p-1}} d\sigma \right\}^{\frac{1}{q}}. \tag{4.15}$$

Since $p \in [2, (n+2)/2]$, the quantity $\gamma = (p-2)/(p-1)$ satisfies inequality $0 \leq \gamma < 1$ and the function (4.14) obeys the assumptions of Lemma 2.4. Applying Lemma 2.4

to (4.15), we arrive at

$$\mathcal{K}_p(x) = \frac{2^{\frac{1}{p}-1}}{\pi^{n/2}c^{(n-2)/2}x_n^{\frac{n-1}{p-1}}}\left\{\int_{\mathbb{S}^{n-1}} \omega\big((\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big)(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)^2 d\sigma\right\}^{\frac{p-1}{p}}. \tag{4.16}$$

Substituting $\omega(t)$ from (4.14) into (4.16), we get (4.6).

(iv) *Representation of $\mathcal{K}_p(x)$ as definite integral for the case $p \in [2, (n+2)/2]$.* Using (2.7), we rewrite (4.6) in the form

$$\mathcal{K}_p(x) = \frac{\omega_{n-1}^{1/q}}{\pi^{\frac{n}{2}}c^{\frac{n-2}{2}}x_n^{\frac{n-1}{p-1}}}\left\{\int_0^{\pi/2} \rho_{n-2}^q\big(\cos\vartheta\big)\cos^{\frac{n}{p-1}}\vartheta \sin^{n-2}\vartheta d\vartheta\right\}^{\frac{1}{q}}. \tag{4.17}$$

In view of (3.12) and (3.6),

$$\rho_{n-2}\big(\cos\vartheta\big) = 2\left(\frac{cx_n}{2\cos\vartheta}\right)^{\frac{n}{2}} K_{\frac{n}{2}}\left(\frac{cx_n}{\cos\vartheta}\right),$$

which together with (4.17) leads to (4.7). □

5. Sharp estimate for the gradient of solution to non-homogeneous equation in $\mathbb{R}^n$ containing a power of the operator $c^2 - \Delta$

First, we describe the notions of the positive power of the modified Helmholtz operator, the Bessel kernel and Bessel potential with a parameter $c > 0$.

Let $\alpha > 0$. The positive power of the modified Helmholtz operator $c^2 - \Delta$ is defined as

$$(c^2 - \Delta)^{\alpha/2}u(x) = \mathcal{F}^{-1}\big((c^2 + |\xi|^2)^{\alpha/2}\mathcal{F}u(\xi)\big)(x),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier and inverse Fourier transforms, respectively, and u belongs to the Schwartz class $\mathcal{S}$ of rapidly decreasing C^∞-functions on $\mathbb{R}^n$.

The parametric Bessel potential

$$u(x) = G_{\alpha,c} * f = \int_{\mathbb{R}^n} G_{\alpha,c}(x-y)f(y)dy, \tag{5.1}$$

where

$$\begin{aligned} G_{\alpha,c}(x) &= \mathcal{F}^{-1}\big((c^2 + |\xi|^2)^{-\alpha/2}\big)(x) \\ &= \frac{c^{n-\alpha}}{\pi^{n/2}2^{(n+\alpha-2)/2}\Gamma\left(\frac{\alpha}{2}\right)}\frac{K_{(n-\alpha)/2}\big(c|x|\big)}{\big(c|x|\big)^{(n-\alpha)/2}} \end{aligned} \tag{5.2}$$

is the parametric Bessel kernel and $f \in L^\infty(\mathbb{R}^n)$, represents continuous and bounded in $\mathbb{R}^n$ solution of the equation

$$\big(c^2 - \Delta\big)^{\alpha/2}u = f. \tag{5.3}$$

The definitions and facts given above for any positive parameter c are completely analogous (including the proofs) to those discussed in the bibliography for the case $c = 1$ (e.g. [1], Ch. 1, [11], Ch. 10).

Various estimates, including pointwise ones, for the Bessel potential are known (e.g. [1], Ch. 3). In the statement below we give a simple sharp pointwise estimate for the modulus of the gradient of the parametric Bessel potential with respect to the norm of its density in the space $L^\infty(\mathbb{R}^n)$.

Theorem 5.1. *Let u be solution of* (5.3) *with $\alpha > 1$ and $f \in L^\infty(\mathbb{R}^n)$, and let x be an arbitrary point in $\mathbb{R}^n$. The sharp coefficient $\mathcal{B}_\alpha$ in the inequality*

$$|\nabla u(x)| \le \mathcal{B}_\alpha ||f||_\infty \tag{5.4}$$

is given by

$$\mathcal{B}_\alpha = \frac{\Gamma\left(\frac{\alpha-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\alpha}{2}\right) c^{\alpha-1}}\ . \tag{5.5}$$

In particular,

$$\mathcal{B}_{2m} = \frac{(2m-3)!!}{(2m-2)!!c^{2m-1}}\ . \tag{5.6}$$

As a special case of (5.6) *one has*

$$\mathcal{B}_2 = \frac{1}{c}\ . \tag{5.7}$$

Proof. By (5.1) and (5.2),

$$u(x) = \frac{c^{n-\alpha}}{\pi^{n/2} 2^{(n+\alpha-2)/2}\Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \frac{K_{(n-\alpha)/2}\big(c|x-y|\big)}{\big(c|x-y|\big)^{(n-\alpha)/2}} f(y)dy\ . \tag{5.8}$$

Differentiating in (5.8) with respect to x_j, $j = 1, \dots, n$, in view of (3.10) we obtain

$$\begin{aligned}\nabla u(x) &= -\frac{c^{n-\alpha}}{\pi^{n/2} 2^{(n+\alpha-2)/2}\Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \frac{K_{(n-\alpha+2)/2}\big(c|y-x|\big)}{\big(c|y-x|\big)^{(n-\alpha)/2}} \nabla\big(c|y-x|\big) f(y)dy \\ &= \frac{c^{n-\alpha+1}}{\pi^{n/2} 2^{(n+\alpha-2)/2}\Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \frac{K_{(n-\alpha+2)/2}\big(c|y-x|\big)}{\big(c|y-x|\big)^{(n-\alpha)/2}} \frac{y-x}{|y-x|} f(y)dy.\end{aligned}$$

Denoting $\boldsymbol{e}_{xy} = (y-x)/|y-x|$, we rewrite the last equality as

$$\nabla u(x) = k_{n,\alpha} \int_{\mathbb{R}^n} \frac{K_{(n-\alpha+2)/2}\big(c|y-x|\big)}{|y-x|^{(n-\alpha)/2}} \boldsymbol{e}_{xy} f(y)dy,$$

which leads to

$$(\nabla u(x), \boldsymbol{z}) = k_{n,\alpha} \int_{\mathbb{R}^n} \frac{K_{(n-\alpha+2)/2}\big(c|y-x|\big)}{|y-x|^{(n-\alpha)/2}} \big(\boldsymbol{e}_{xy}, \boldsymbol{z}\big) f(y)dy, \tag{5.9}$$

where $\boldsymbol{z}$ is a unit n-dimensional vector and

$$k_{n,\alpha} = \frac{c^{(n-\alpha+2)/2}}{\pi^{n/2} 2^{(n+\alpha-2)/2}\Gamma\left(\frac{\alpha}{2}\right)}\ . \tag{5.10}$$

By (5.9) and the property of the inner product in $\mathbb{R}^n$, we arrive at

$$|\nabla u(x)| = k_{n,\alpha} \max_{|\boldsymbol{z}|=1} \int_{\mathbb{R}^n} \frac{K_{(n-\alpha+2)/2}\big(c|y-x|\big)}{|y-x|^{(n-\alpha)/2}} \big(\boldsymbol{e}_{xy}, \boldsymbol{z}\big) f(y)dy. \tag{5.11}$$

Using permutation of suprema in (5.11), we obtain the representation for the sharp coefficient in inequality (5.4),

$$\mathcal{B}_\alpha = k_{n,\alpha} \max_{|\boldsymbol{z}|=1} \sup_{||f||_\infty=1} \int_{\mathbb{R}^n} \frac{K_{(n-\alpha+2)/2}\big(c|y-x|\big)}{|y-x|^{(n-\alpha)/2}} \big(\boldsymbol{e}_{xy}, \boldsymbol{z}\big) f(y) dy,$$

that is

$$\mathcal{B}_\alpha = k_{n,\alpha} \max_{|\boldsymbol{z}|=1} \int_{\mathbb{R}^n} \frac{K_{(n-\alpha+2)/2}\big(c|y-x|\big)}{|y-x|^{(n-\alpha)/2}} |(\boldsymbol{e}_{xy}, \boldsymbol{z})| dy. \tag{5.12}$$

Now, we write the integral in (5.12) as

$$\begin{aligned}\int_{\mathbb{R}^n} &\frac{K_{(n-\alpha+2)/2}\big(c|y-x|\big)}{|y-x|^{(n-\alpha)/2}} |(\boldsymbol{e}_{xy}, \boldsymbol{z})| dy \\ &= \int_0^\infty \frac{K_{(n-\alpha+2)/2}\big(c\rho\big)}{\rho^{(n-\alpha)/2}} \rho^{n-1} d\rho \int_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{z})| d\sigma\ .\end{aligned} \tag{5.13}$$

Using the known formula (see, e.g. [2], item 6.561/16)

$$\int_0^\infty x^\mu K_\nu(ax)dx = 2^{\mu-1} a^{-(\mu+1)} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right)\ ,$$

we calculate the first integral on the right-hand side of (5.13)

$$\int_0^\infty K_{\frac{n-\alpha+2}{2}}\big(c\rho\big) \rho^{\frac{n+\alpha-2}{2}} d\rho = \frac{2^{(n+\alpha-4)/2}}{c^{(n+\alpha)/2}} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{\alpha-1}{2}\right)\ . \tag{5.14}$$

Further, by (2.7),

$$\int_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{z})| d\sigma = 2\omega_{n-1} \int_0^{\pi/2} \cos\vartheta \sin^{n-2}\vartheta d\vartheta = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}\ . \tag{5.15}$$

Substituting (5.14) and (5.15) into (5.13), we obtain

$$\int_{\mathbb{R}^n} \frac{K_{(n-\alpha+2)/2}\big(c|y-x|\big)}{|y-x|^{(n-\alpha)/2}} |(\boldsymbol{e}_{xy}, \boldsymbol{z})| dy = \frac{2^{(n+\alpha-2)/2} \pi^{(n-1)/2}}{c^{(n+\alpha)/2}} \Gamma\left(\frac{\alpha-1}{2}\right)\ ,$$

which together with (5.12) and (5.10) leads to (5.5).

Applying formula

$$\Gamma\left(m - \frac{1}{2}\right) = \frac{\sqrt{\pi}(2m-2)!}{2^{2(m-1)}(m-1)!}$$

to transform of (5.5) in the case $\alpha = 2m$, we arrive at (5.6). □

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