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GLOBAL SOLUTION OF THE INITIAL VALUE PROBLEM FOR THE FOCUSING DAVEY-STEWARTSON II SYSTEM

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Abstract. We consider the two dimensional focusing Davey-Stewartson II system and construct the global solution of the Cauchy problem for a dense in $L^2(\mathbb{C})$ set of initial data. We do not assume that the initial data is small. So, the solutions may have singularities. We show that the blow-up may occur only on a real analytic variety and the variety is bounded in each strip $t \leq T$.

1. Introduction

Let $q_0(z),\ z=x+iy,\ (x,y)\in\mathbb{R}^2$, be a compactly supported (or fast decaying) sufficiently smooth function. Consider the two dimensional focusing Davey-Stewartson II (DSII) system of equations for unknown functions $q=q(z,t),\ \phi=\phi(z,t),\ (x,y)\in\mathbb{R}^2,\ t\geq 0:$

$$\begin{aligned} q_t &= 2iq_{xy} - 4q(\overline{\varphi}-\varphi),\\ \partial\varphi &= \overline{\partial}|q|^2,\\ q(z,0) &= q_0(z). \end{aligned} \tag{1.1}$$

A smooth, decaying in z at infinity solution of (1.1) exists for all $t>0$ if q_0 is small enough, [1, 3, 20–22]. We will call this solution classical. If q_0 is not small, the solution was constructed locally in our previous work [14] via the IST (inverse scattering transform) using the $\overline{\partial}$-method that has been generalized in [16], [12], [13] to the case when Faddeev type exceptional points may be present. The solution was obtained in a neighbourhood of any point (z_0,t_0) for generic initial data q_0 that depend on the point. The main objectives of this article are to obtain the solution for an arbitrary initial data from a specific set and to get the global solution defined in the whole space, including a description of the set where the solution blows up. It will be shown that the latter set is a real analytic variety that is bounded in every strip $0\leq t\leq T$.

Let us recall that the focusing DSII equation may have a finite time blow-up (e.g., [17]). While the uniqueness is known for smooth (in some sense) solutions, see [8], [9], one has to be careful with the definition of the solution that has singularities. We understand these solutions in the following sense. Let us multiply the initial data q_0 by a positive parameter $a\in(0,1]$. We will show that the classical solution

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that exists when $a \ll 1$ allows an analytic continuation into a complex neighborhood of $(0,1]$, and this analytic continuation will be used to single out the solution with singularities when $a=1$. The main statement of the present paper is given in Theorem 2.2 of Section 2.

Let us mention some recent articles on DSII: [11], [18], [19].

2. The solution of the Cauchy problem, main results

Let $q_0(z) \in L^2(\mathbb{C})$. Denote

$$Q_0(z) = \begin{pmatrix} 0 & q_0(z) \\ -q_0(z) & 0 \end{pmatrix}, \quad z \in \mathbb{C}. \tag{2.1}$$

Let $\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, and let the 2×2 matrix $\psi(\cdot,k)$, $k \in \mathbb{C}$, be a solution of the following problem for the Dirac equation in $\mathbb{C}$:

$$\frac{\partial \psi}{\partial \overline{z}} = Q_0\overline{\psi}, \quad \psi(z,k)e^{-i\overline{k}z/2} \to I, \quad z \to \infty. \tag{2.2}$$

The corresponding generalized Lippmann-Schwinger equation has the following form:

$$\psi(z,k) = e^{i\overline{k}z/2}I + \int_{\mathbb{C}} G(z-z',k)Q_0(z')\overline{\psi}(z',k)d\sigma_{z'}, \tag{2.3}$$

where $G(z,k) = \frac{1}{\pi}\frac{e^{i\overline{k}z/2}}{z}$, $d\sigma_{z'} = dx'dy'$. Here and below we use the same notation for functional spaces, irrespectively of whether those are the spaces of matrix-valued or scalar-valued functions. After the substitution,

$$\mu(z,k) = \psi(z,k)e^{-i\overline{k}z/2}, \qquad \mu(z,k) - I \to 0, \quad z \to \infty, \tag{2.4}$$

equation (2.3) takes the form

$$\mu(z,k) = I + \frac{1}{\pi}\int_{\mathbb{C}} \frac{e^{i\Re(\overline{k}z)}}{z-z'}Q_0(z')\overline{\mu}(z',k)d\sigma_{z'}, \tag{2.5}$$

and becomes Fredholm in $L^q(\mathbb{C})$, $q>2$, after the additional substitution $\nu = \mu - I$ (see, e.g., [15, lemma 5.3]).

Solutions ψ of (2.3) are called the generalized *scattering solutions*, and the values of k such that the homogeneous equation (2.5) has a non-trivial solution are called *exceptional points*. The set of exceptional points will be denoted by $\mathcal{E}$. Thus the scattering solution may not exist if $k \in \mathcal{E}$. Note that the operator in equation (2.5) is not analytic in k, and $\mathcal{E} \subset \mathbb{C}$ may contain one-dimensional components. There are no exceptional points in a neighborhood of infinity (e.g., [20, lemma 2.8], [2, lemma C]). Let us choose $A \gg 1$ and $k_0 \in \mathbb{C}$ such that all the exceptional points are contained in the disk

$$D = \{k \in \mathbb{C} : 0 \le |k| < A\}, \tag{2.6}$$

and k_0 belongs to the same disc $\overline{D}$ and is not exceptional.

The generalized scattering data (an analogue of the scattering amplitude in the standard scattering problem) are defined by the following integral (when the integral converges)

$$h_0(\varsigma,k)=\frac{1}{(2\pi)^2}\int_{\mathbb{C}} e^{-i\bar{\varsigma}z/2}Q_0(z)\overline{\psi}(z,k)d\sigma_z,\ \varsigma\in\mathbb{C},\ k\in\mathbb{C}\backslash\mathcal{E}. \tag{2.7}$$

In fact, from the Green formula, it follows that h_0 can be determined without using the potential Q_0 or the solution ψ of the Dirac equation (2.2) if the Dirichlet data at $\partial\Omega$ are known for the solution of (2.2) in a bounded region Ω containing the support of Q_0:

$$h_0(\varsigma,k)=\frac{-i}{8\pi^2}\int_{\partial\mathcal{O}} e^{-i\bar{\varsigma}z/2}\overline{\psi}(z,k)dz,\ \varsigma\in\mathbb{C},\ k\notin\mathcal{E}. \tag{2.8}$$

Note that h_0 is continuous when $k\notin D$ under minimal assumptions on Q [20], [21], and moreover,

$$h_0=h_0(\varsigma,k)\in C^\infty \quad \text{when } |k|\geq A \tag{2.9}$$

if Q is bounded and decays faster then any power at infinity. This follows from the fact that (2.5) admits differentiation in z and k when $k\notin D$.

The inverse problem (recovery of Q when h_0 is given) was solved using $\overline{\partial}-$method in [1], [20–22] when the potential is small enough to guarantee the absence of exceptional points. When $\mathcal{E}\neq\varnothing$, the inverse problem was solved in a generic sense in [13]. The latter results were applied in [14] to construct solutions of the focusing DSII system. Let us recall some results obtained in [14].

Consider the space

$$\mathcal{B}^s=\left\{u\in L^s(\mathbb{C}\backslash D)\bigcap C(D)\right\},\quad s>2, \tag{2.10}$$

where $C(D)$ is the space of analytic functions in D with the norm $\|u\|=\sup_D|u|$. Here and below, we use the same space notation for matrices as for their entries.

Let operator $T_z:\mathcal{B}^s\to\mathcal{B}^s,\ s>2$, be defined as follows:

$$\begin{aligned}T_z\phi(k)=&\frac{1}{\pi}\int_{\mathbb{C}\backslash D} e^{i(\bar{\varsigma}z+\bar{z}\varsigma)/2}\overline{\phi}(\varsigma)\Pi^o h(\varsigma,\varsigma)\frac{d\sigma_\varsigma}{\varsigma-k}\\ &+\frac{1}{2\pi i}\int_{\partial D}\frac{d\varsigma}{\varsigma-k}\int_{\partial D}[e^{i(\varsigma\bar{z}+\overline{\varsigma'}z)/2}\overline{\phi^-(\varsigma')}\Pi^o\\ &+e^{i(\varsigma-\varsigma')\bar{z}/2}\phi^-(\varsigma')\Pi^d\mathbf{C}]\left[\mathrm{Ln}\frac{\overline{\varsigma'}-\bar{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}\right]h(\varsigma',\varsigma)\overline{d\varsigma'},\end{aligned} \tag{2.11}$$

where $d\sigma_\varsigma=d\varsigma_Rd\varsigma_I,\ z\in\mathbb{C},\ \phi\in\mathcal{B}^s$, ϕ^- is the boundary trace of ϕ from the interior of D, $\mathbf{C}$ is the operator of complex conjugation, Π^oM is the off-diagonal part of a matrix M, Π^dM is the diagonal part. Let us specify the logarithmic function in (2.12). Let us shift the coordinates in $\mathbb{C}$ and move the origin to the point $\overline{\varsigma'}\in\partial D$. Then we rotate the plane in such a way that the direction of the x-axis is defined by the vector from $\overline{\varsigma'}$ to $-\overline{\varsigma'}$. Then $|\arg(\overline{\varsigma'}-\bar{\varsigma})|<\pi/2,\ \varsigma'\neq\varsigma$, and $|\arg(\overline{\varsigma'}-\overline{k_0})|\leq\pi/2$, i.e.,

$$\left|\arg\frac{\overline{\varsigma'}-\bar{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}\right|<\pi,\quad \varsigma',\varsigma\in\partial D,\ \varsigma'\neq\varsigma.$$

This defines the values of the logarithmic function uniquely.

It turns out that, after the substitution $w = v - I \in \mathcal{B}^s,\ s > 2$, the equation

$$(I + T_z)v = I$$

becomes Fredholm in $\mathcal{B}^s$, and the potential q_0 can be expressed explicitly in terms of v (see [12, 13]).

In order to solve the DSII problem (1.1), we apply this reconstruction procedure to a specially chosen scattering data. We start with the scattering data h_0 defined by q_0 and extend it in time as follows:

$$h(\varsigma, k, t) := e^{-t(k^2-\bar{\varsigma}^2)/2}\Pi^o h_0(\varsigma, k) + e^{-t(\bar{k}^2-\bar{\varsigma}^2)/2}\Pi^d h_0(\varsigma, k), \tag{2.12}$$

where $\varsigma \in \mathbb{C},\ k \in \mathbb{C}\backslash\mathcal{E},\ t \geq 0$. For $t \geq 0$, we define the operator

$$\begin{aligned} T_{z,t}\phi(k) = \frac{1}{\pi}\int_{\mathbb{C}\backslash D} e^{i(\bar{\varsigma}z+\bar{z}\varsigma)/2}\overline{\phi}(\varsigma)\Pi^o h(\varsigma,\varsigma,t)\frac{d\sigma_\varsigma}{\varsigma - k} \\ +\frac{1}{2\pi i}\int_{\partial D}\frac{d\varsigma}{\varsigma - k}\int_{\partial D}[e^{i(\varsigma\bar{z}+\overline{\varsigma'}z)/2}\overline{\phi^-(\varsigma')}\Pi^o \\ +e^{i(\varsigma-\varsigma')\bar{z}/2}\phi^-(\varsigma')\Pi^d\mathbf{C}]\left[\mathrm{Ln}\frac{\overline{\varsigma'}-\bar{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}\right]h(\varsigma',\varsigma,t)\overline{d\varsigma'}. \end{aligned} \tag{2.13}$$

Theorem 2.1 ([14]). *Let $q_0(\cdot)$ be a function with compact support. Assume that it is 6 times differentiable in x and y. Alternatively, this condition can be replaced*[1] *by the superexponential decay of q_0:*

$$\lim_{z\to\infty} e^{\widetilde{A}|z|}\partial_x^i\partial_y^j q_0(z) = 0 \ for\ each\ \widetilde{A} > 0,\ i + j \leq 6. \tag{2.14}$$

Then, for each $s > 2$, the following statements are valid.

1) The operator $T_{z,t}$ is compact in $\mathcal{B}^s$ for all $z \in \mathbb{C},\ t \geq 0$, and depends continuously on z and $t \geq 0$. The same property holds for its first derivative in time and all the derivatives in x, y up to the third order, where the derivatives are defined in the norm convergence. The function $T_{z,t}I$ belongs to $\mathcal{B}^s$ for all $t \geq 0$.

2) Let the kernel of $I + T_{z,t}$ in the space $\mathcal{B}^s$ be trivial for (z, t) in an open or half open[2] *set $\omega \subset \mathbb{R}^3$. Let $v_{z,t} = w_{z,t} + I$, where $w_{z,t} \in \mathcal{B}^s$ is the solution of the equation*

$$(I + T_{z,t})w_{z,t} = -T_{z,t}I. \tag{2.15}$$

Then functions q, φ defined by

$$\begin{aligned} \begin{pmatrix} \varphi(z,t) & q(z,t) \\ -q(z,t) & \varphi(z,t) \end{pmatrix} := \frac{-i}{2\pi}(\Pi^o + \bar{\partial}\Pi^d)\Bigg(\int_{\mathbb{C}\backslash D} e^{i(\bar{\varsigma}z+\bar{z}\varsigma)/2}\overline{v_{z,t}}(\varsigma)\Pi^o h(\varsigma,\varsigma,t)d\sigma_\varsigma \\ -\frac{1}{2i}\int_{\partial D}d\varsigma\int_{\partial D}[e^{i(\varsigma\bar{z}+\overline{\varsigma'}z)/2}\overline{v^-_{z,t}(\varsigma')}\Pi^o \\ -e^{i(\varsigma-\varsigma')\bar{z}/2}v^-_{z,t}(\varsigma')\Pi^d\mathbf{C}]\left[\mathrm{Ln}\frac{\overline{\varsigma'}-\bar{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}\right]h(\varsigma',\varsigma,t)\overline{d\varsigma'}\Bigg), \end{aligned} \tag{2.16}$$

[1]This fact was not mentioned in the paper, but it can be easily checked

[2]We will say that a set ω of points (z, t) in $\mathbb{R}^3_+ = \mathbb{R}^3 \bigcap\{t \geq 0\}$ is half-open if ω contains points where $t = 0$ and, for each point $(z_0, 0) \in \omega$, there is a ball B_0 centered at this point such that $B_0 \bigcap\{t \geq 0\} \subset \omega$.

satisfy all the relations (1.1) in the classical sense when $(z,t) \in \omega$. *In particular,* $q(z,0) = q_0(z)$.

3) Consider a set of initial data $aq_0(z)$ *that depend on* $a \in (0,1]$. *Then equation (2.15) with* Q^0 *replaced by* aQ^0 (Q^0 *is fixed) is uniquely solvable for almost every* $(z,t,a) \in \mathbb{R}^2 \times \mathbb{R}^+ \times (0,1]$. *Moreover,*[3] *for each* (z,t), *the solution of (2.15) is meromorphic in* $a \in [0,1]$ *and has at most a finite set of poles* $a = a_j(z,t)$.

Remark. All the exceptional points are located in a disk whose radius depends only on the norm of aq_0. Hence D and k_0 can be chosen independently of $a \in [0,1]$ (see [13, Lemma 5.1]). From now on, we assume that the disk D is fixed and contains the exceptional points for all the potentials $aq_0,\ a \in [0,1]$.

In order to state the main results of the present paper, we need to recall the construction (e.g., [20]) of the global solution of (1.1) when q_0 is small. The latter expression ("q_0 is small") will be used below only for problem (1.1) with initial data aq_0 where q_0 is infinitely smooth and satisfies (2.14), and $0 < a \ll 1$. Let us recall that the scattering problem (2.2) and the Lippmann-Schwinger equation (2.3) are uniquely solvable for all k when q_0 is small, i.e., there are no exceptional points in this case and $h_0(k,k)$ is defined for all the values of k. Operator $T_{z,t}$ is needed only with $D = \emptyset$ if q_0 is small. Hence, only the first term is present in the right-hand side of (2.13). Moreover, $\|T_{z,t}\| < 1$ when q_0 is small, and therefore equation (2.15) is uniquely solvable for all $z \in \mathbb{C},\ t \geq 0$. Then (q,ϕ) given by (2.16) with $D = \emptyset$ is a smooth global solution of problem (1.1) with the small initial data. We will call this solution *classical*. It exists under a weaker assumption on the decay of q_0 than in Theorem 2.1.

We will consider analytic continuations of functions h_0, q_0, and we need some notation. Let $\gamma = (\gamma_1,\gamma_2) \in \mathbb{C}^2$ and let $A_\gamma : \mathbb{C} \to \mathbb{C}^2$ be the map defined by $A_\gamma z = A_\gamma(x+iy) = (x+\gamma_1, y+\gamma_2) \in \mathbb{C}^2$, i.e., the map A_γ shifts real points x,y into complex planes. If a function $f = f(z)$ is analytic in (x,y), then $B_\gamma f(z) := f(A_\gamma z)$ is the value of the analytic continuation of f at point $A_\gamma z$. We will use notation $A'_\sigma, B'_\sigma,\ \sigma \in \mathbb{C}^2$, for the same operations applied to a function of $k \in \mathbb{C}$, and $A''_\eta, B''_\eta,\ \eta \in \mathbb{C}^2$, if they are applied to a function of $\varsigma \in \mathbb{C}$.

The main result of the paper is obtained under the following condition on the initial data that must hold for large enough R:

Condition $Q(R)$. The initial data q_0 admits analytic continuation in (x,y) and, for a given $R > 0$, there exist a $C = C(R)$ such that

$$|B_\gamma q_0(z)| \leq Ce^{-R|z|}, \quad z \in \mathbb{C}, \text{ when } |\gamma| \leq R.$$

Remark. Clearly, linear combinations of Gaussian functions satisfy Condition $Q(R)$ for all $R > 0$, and it was shown in [4,5] that these combinations form a dense set in $L^2(\mathbb{C})$.

We will show that Condition $Q(R)$ implies a similar behavior of the scattering data, i.e., the validity of the following assumption.

Condition $H(R)$. For a given $R > 0$, the estimate

$$|h_0(\varsigma,\varsigma)| \leq e^{-R|\varsigma|} \quad \text{as} \quad |\varsigma| \to \infty$$

[3] that statement can be found in [13, lemma 5.1]

holds, and there exists $C = C(R)$ and $a_0(R)$ such that the scattering data $h_0(\varsigma, \varsigma)$ for the potential aq_0 with $a \in (0, a_0)$ admits analytic continuation $B''_\eta h_0(\varsigma, \varsigma)$, $|\eta| \le R$, with respect to variables ς_i, and

$$|B''_\eta h_0(\varsigma, \varsigma)| \le C(R)e^{-R|\varsigma|}, \quad \varsigma \in \mathbb{C}, \text{ when } |\eta| \le R.$$

We will show that there is a duality between these two conditions. To be more exact, the validity of $Q(R)$ implies the validity of $H(R-\varepsilon)$. Conversely, if the initial data is small, Condition $H(R)$ holds, and h_0 is extended in time according to (2.12), then the potential $q(z, t)$ that corresponds to the extended data $h(\varsigma, \varsigma, t)$ satisfies Condition $Q(R)$ with a smaller R that depends on t. These results will be obtained in the next section. Note that they are an analogue of similar results of L. Sung ([21, Cor. 4.16]) who established a duality of the non-linear Fourier transform in the Schwartz class. We need a refined result to study the more complicated form (2.13) of operator $T_{z,t}$ that appears in the presence of exceptional points.

Below is the main statement of the present paper.

Theorem 2.2. *Let us fix an arbitrary disk D containing all the exceptional points for the potentials $aq_0(z)$, $a \in [0, 1]$. Let Condition $Q(R)$ hold with $R > (1 + 2T)A$, where A is the radius of the disk D. Then*

1) for each point (z, t), $0 \le t \le T$, the classical solution (q, ϕ) of problem (1.1) with the initial data aq_0 and small enough $a > 0$ admits a meromorphic continuation with respect to a in a neighborhood of the segment $[0, 1]$. This meromorphic continuation is given by (2.16) with an arbitrary choice of the disk D and an arbitrary choice of point $k_0 \in \partial D$[4].

2) when $a = 1$, the analytic continuation of (q, ϕ) is infinitely smooth and satisfies (1.1) everywhere, except possibly a set S that is bounded in the strip $0 \le t \le T$, $(x, y) \in R^2$, and is such that $S_t = S \bigcap \{t = const\}$ is a bounded 1D real analytic variety.

Remark. The theorem implies that the local solutions found in Theorem 2.1 are analytic continuations in a of the global classical solutions (under the assumption that condition $Q(R)$ holds). At the same time, the theorem does not prohibit the solution from blowing up at an arbitrarily small time $t > 0$ (see the recent paper [10] and citations there on instantaneous blow-ups). We can't say anything about relation between our global solution and local solutions found in [8].

Two important technical improvements of the previous results will be used in the proof of Theorem 2.2. First, we will show that the Hilbert space $\mathcal{B}^2$ can be used in Theorem 2.1 instead of the Banach space $\mathcal{B}^s$, $s > 2$. The space $\mathcal{B}^2$ is defined as follows:

$$\mathcal{B}^2 = \left\{ u \in \left(L^2(\mathbb{C}\backslash D) \oplus \mathbb{C}^1\right) \bigcap L^2_+(\partial D) \right\}. \tag{2.17}$$

Here $\mathbb{C}^1$ is the one-dimensional space of functions of the form $\frac{c\beta(k)}{k}$, where c is a complex constant, $\beta \in C^\infty$ is a fixed function that vanishes in a neighbourhood of the disk D and equals one in a neighbourhood of infinity. By $L^2_+(\partial D)$ we denote

[4]all the exceptional points are inside D, i.e., all the points on ∂D are non-exceptional.

the space of analytic functions $u=\sum_{n\geq 0}c_nz^n$ in D with the boundary values in $L^2(\partial D)$ and the norm

$$\|u\|_{L^2_+(\partial D)}=\left(\sum_{n\geq 0}A^{2n}|c_n|^2\right)^{1/2},$$

where A is the radius of the disk D.

Secondly, we will simplify the form of the operator $T_{z,t}$ by writing the second term in (2.12) and (2.13) without the logarithmic factor. We also will allow k_0 to be on ∂D, and not necessarily in D, and show that formula (2.13) in the latter case can be written as

$$T_{z,t}\phi(k)=\frac{1}{\pi}\int_{\mathbb{C}\backslash D}e^{i(\overline{\varsigma}z+\overline{z}\varsigma)/2}\overline{\phi}(\varsigma)\Pi^o h(\varsigma,\varsigma,t)\frac{d\sigma_\varsigma}{\varsigma-k}- \tag{2.18}$$
$$i\int_{\partial D}\frac{d\varsigma}{\varsigma-k}\int_{\widehat{k_0,\varsigma}}[e^{i(\varsigma\overline{z}+\overline{\varsigma'}z)/2}\overline{\phi^-(\varsigma')}\Pi^o+e^{i(\varsigma-\varsigma')\overline{z}/2}\phi^-(\varsigma')\Pi^d\mathbf{C}]\left[h(\varsigma',\varsigma,t)\overline{d\varsigma'}\right],$$

where $\widehat{k_0,\varsigma}$ is the arc on ∂D between points k_0 and ς with the counter clock-wise direction on it.

The following two difficulties were resolved in the paper. We show that if one starts with a small potential q_0 and its scattering data $h_0(\varsigma,\varsigma)$, and extends $h_0(\varsigma,\varsigma)$ in time according to (2.12), then the solution $q(z,t)$ of the inverse scattering problem with the scattering data (2.12) decays exponentially at infinity, and the scattering data (2.7) for this potential $q(z,t)$ coincides with the scattering data $h(\varsigma,\varsigma,t)$ from which the potential was obtained (this will be done in the next section). Another difficulty concerns the proof of the invertibility of operator $I+T_{z,t}$ for large $|z|$ in spite of the exponential growth of the integrands in the second terms of (2.13) and (2.18) as $|z|\to\infty$ (see section 5).

3. Exponential decay of the scattering data and of $q(z,t)$

Lemma 3.1. *Let*

$$I(z)=\int_{\mathbb{C}}\frac{f(z_1)}{z-z_1}d\sigma_{z_1},\quad J(z)=\int_{\mathbb{C}}\frac{f(z_1)}{\overline{z}-\overline{z}_1}d\sigma_{z_1},\quad z\in\mathbb{C},$$

where $f(z)$ is analytic in (x,y), and

$$|f(A_\gamma z)|,|\nabla_\gamma f(A_\gamma z)|\leq\frac{C(\gamma)}{1+x^2+y^2}.$$

Then $I(z),J(z)$ admit analytic continuation in (x,y), and

$$B_\gamma I(z)=\int_{\mathbb{C}}\frac{f(A_\gamma z_1)}{z-z_1}d\sigma_{z_1},\quad B_\gamma J(z)=\int_{\mathbb{C}}\frac{f(A_\gamma z_1)}{\overline{z}-\overline{z}_1}d\sigma_{z_1}.$$

Proof. Let us rewrite $I(z)$ in the form

$$I(z)=-\int_{\mathbb{C}}\frac{f(z+z_1)}{z_1}d\sigma_{z_1}.$$

This immediately implies that $I(z)$ is analytic in (x, y), and

$$B_\gamma I(z) = \int_{\mathbb{C}} \frac{B_\gamma f(z+z_1)}{-z_1} d\sigma_{z_1} = \int_{\mathbb{C}} \frac{f(x+\gamma_1+x_1, y+\gamma_2+y_2)}{-z_1} d\sigma_{z_1}$$
$$= \int_{\mathbb{C}} \frac{f(A_\gamma z_1)}{z-z_1} d\sigma_{z_1}.$$

The statement for J can be proved absolutely similarly. □

Let us provide some examples of analytic continuations of functions from $\mathbb{C}$ into $\mathbb{C}^2$: (1) If $f(z) = z = x+iy$, then $B_\gamma f(z) = x+\gamma_1+i(y+\gamma_2) = z+\gamma'$, $\gamma' = \gamma_1+i\gamma_2 \in \mathbb{C}$. (2) If $f(z) = \overline{z} = x-iy$, then $B_\gamma f(z) = x+\gamma_1-i(y+\gamma_2) = \overline{z}+\gamma''$, $\gamma'' = \gamma_1-i\gamma_2 \in \mathbb{C}$ (note that $\gamma' \neq \overline{\gamma''}$ since γ_i are complex.) (3) if $f(z) = \Re(k\overline{z}) = k_1x + k_2y$, then $B_\gamma f(z) = \Re(k\overline{z}) + k_1\gamma_1 + k_2\gamma_2$ and $B_\sigma \Re(k\overline{z}) = \Re(k\overline{z}) + \sigma_1 x + \sigma_2 y$.

Lemma 3.2. *Let the potential be $aq_0(z)$ where q_0 satisfies Condition $Q(R)$ for some $R > 0$. Then there exists $a_0 = a_0(R)$ such that function $\overline{\mu} = \overline{\mu}(z, k)$ defined by (2.3) via the solution of the Lippmann-Schwinger equation with the potential aq_0, $a \in (0, a_0)$, admits analytic continuation to $\mathbb{C}^4$ with respect to variables x, y, k_1, k_2, and*

$$|B'_\sigma B_\gamma \overline{\mu}(z, k)| < C(R, \varepsilon) \quad \text{when } z, k \in \mathbb{C},\ |\sigma|, |\gamma| \leq R-\varepsilon,\ a < a_0. \tag{3.1}$$

The statement remains valid if $a = 1$, but $|k| \geq \rho(R)$ with large enough ρ.

Proof. We will prove the statement of the lemma for the component μ_{11} of the matrix μ. Other components can be treated similarly. Let us iterate equation (2.5). The following equation is valid for the first component:

$$\overline{\mu_{11}} = 1 + \frac{1}{\pi^2} \int_{\mathbb{C}} d\sigma_{z_1} \int_{\mathbb{C}} d\sigma_{z_2} \frac{e^{i\Re(k\overline{z}_1)}}{\overline{z}-\overline{z_1}} \overline{Q}_{12}(z_1) \frac{e^{-i\Re(k\overline{z}_2)}}{z_1-z_2} Q_{21}(z_2) \overline{\mu_{11}}(z_2, k), \tag{3.2}$$

where Q_{21} and Q_{12} are entries of the matrix Q_0. Denote $Q = Q_{12} = -Q_{21}$. Assume that the analytic continuation $B_\gamma \overline{\mu}_{11}$ exists. Then from Lemma 3.1, formula (3.2) and the relation

$$B_\gamma e^{\pm i\Re(k\overline{z})} = e^{\pm i\Re(k\overline{z}) \pm i<k,\gamma>}$$

it follows that $B_\gamma \overline{\mu}_{11}$ is equal to

$$1 - \frac{1}{\pi^2} \int_{\mathbb{C}} d\sigma_{z_1} \frac{B_\gamma e^{i\Re(k\overline{z}_1)}}{\overline{z}-\overline{z_1}} B_\gamma \overline{Q}(z_1) B_\gamma \int_{\mathbb{C}} d\sigma_{z_2} \frac{e^{-i\Re(k\overline{z}_2)}}{z_1-z_2} Q(z_2) \overline{\mu_{11}}(z_2, k)$$
$$= 1 - \frac{1}{\pi^2} \int_{\mathbb{C}} d\sigma_{z_1} \frac{e^{i\Re(k\overline{z}_1)}}{\overline{z}-\overline{z_1}} B_\gamma \overline{Q}(z_1) \int_{\mathbb{C}} \frac{e^{-i\Re(k\overline{z}_2)}}{z_1-z_2} B_\gamma Q(z_2) B_\gamma \overline{\mu_{11}}(z_2, k) d\sigma_{z_2}.$$

Hence, if the analytic continuation $\Psi := B'_\sigma B_\gamma \overline{\mu}_{11}$ exists, then it satisfies the equation

$$\Psi(z, k) = 1 - \frac{1}{\pi^2} \int_{\mathbb{C}} d\sigma_{z_1} \frac{e^{i\Re(k\overline{z}_1)+i<\sigma,z_1>}}{\overline{z}-\overline{z_1}} B_\gamma \overline{Q}(z_1) \cdot$$
$$\cdot \int_{\mathbb{C}} \frac{e^{-i\Re(k\overline{z}_2)-i<\sigma,z_2>}}{z_1-z_2} B_\gamma Q(z_2) \Psi(z_2, k) d\sigma_{z_2}.$$

Denote by $K^{\pm} = K^{\pm}_{k,\sigma,\gamma}$ the integral operators given by the exterior and interior integrals above, respectively. Their norms in the space $L^\infty(\mathbb{C})$ can be estimated from above by the norms of the potential (see [20]):

$$\|K^-\| < C(\|e^{-i<\sigma,z_2>}B_\gamma Q(z_2)\|_{L^p(\mathbb{C})} + \|e^{-i<\sigma,z_2>}B_\gamma Q(z_2)\|_{L^q(\mathbb{C})}),$$

where $1 < p < 2 < q < \infty$. A similar estimate is valid for K^+. Thus the assumption $a_0 \ll 1$ and Condition Q imply that $\|K^{\pm}\| < 1$, Ψ exists, and

$$|\Psi| < C(R) \quad \text{when } |\Im\sigma|, |\gamma| \le R,\ a < a_0.$$

Moreover, the derivatives of $K^{\pm}$ with respect to complex variables σ_i, γ_j also have small norms, i.e., $\Psi = \Psi(z,k,\sigma,\gamma)$ is analytic in $(\sigma_1,\sigma_2,\gamma_1,\gamma_2)$. One can easily see that $\Psi = \Psi(z+\gamma, k+\sigma)$. Hence Ψ is the analytic continuation of $\overline{\mu}$. The proof of (3.1) is complete.

In order to prove the statement of Lemma 3.2 concerning $a = 1$, one needs only to show that operator $K := K^+K^-$ and its derivatives in σ_i, γ_j are small (less than one) as $|k| \to \infty$. This can be done by a standard procedure: one splits K into two terms $K = K_1 + K_2$, where K_1 is obtained by adding the factor $\alpha(\frac{z_1-z}{\varepsilon})\alpha(\frac{z_1-z_2}{\varepsilon})$ in the integral kernel of K. Here $\alpha = \alpha(z)$ is a cut-off function that is equal to one when $|z| < 1$ and vanishes when $|z| > 2$. Then $\|K_1\| \to 0$ as $\varepsilon \to 0$, and $\|K_2\| = O(|k|^{-1})$ as $|k| \to \infty$. The latter can be shown by appropriate integration by parts in x_1, y_1. □

Theorem 3.3. *If Condition $Q(R)$ holds for some $R > 0$, then Condition $H(R-\varepsilon)$ holds for each $\varepsilon > 0$.*

Proof. Recall that

$$h_0(\varsigma,\varsigma) = \frac{1}{(2\pi)^2}\int_{\mathbb{C}} e^{-i\Re(\overline{\varsigma}z)}Q_0(z)\overline{\mu}(z,\varsigma)d\sigma_z,\ k,\varsigma \in \mathbb{C}.$$

We shift the complex plane $\mathbb{C}$ in the integral above by vector $\gamma = -i\frac{(\varsigma_1,\varsigma_2)}{|\varsigma|}(R-\varepsilon)$, and then apply operator B_η. This leads to

$$|B''_\eta h_0| < \frac{1}{(2\pi)^2}\int_{\mathbb{C}} \left| e^{-i(<\eta,z>+<\varsigma,\gamma>+<\eta,\gamma>)}Q_0(A_\gamma z)B_\gamma B''_\eta \overline{\mu}(z,\varsigma)\right| d\sigma_z.$$

It remains to use Lemma 3.2 and Condition $Q(R)$. □

Theorem 3.4. *Let Condition $Q(R)$ hold for some $R > 0$ and let the scattering data h_0 be defined by the potential aq_0, $0 < a < a_0(R)$, where $a_0(R)$ is defined in Lemma 3.2. Then the time dependent scattering data $h(\varsigma,\varsigma,t)$, $0 \le t \le T$, given by (2.12), admits an analytic continuation in $(\varsigma_1,\varsigma_2)$, and*

$$|B''_\eta h(\varsigma,\varsigma,t)| \le C(R,\varepsilon)e^{(-\frac{R}{1+2T}+\varepsilon)|\varsigma|}, \quad |\eta| \le \frac{R}{1+2T}.$$

The statement remains valid if $a = 1$, but $|\varsigma| > \rho$, where $\rho = \rho(R)$ is large enough.

Proof. The statement follows immediately from Theorem 3.3 and formula (2.12). One needs only to combine the upper bound $Ce^{(-R+\varepsilon)|\varsigma|}$ for the analytic continuation of h_0 obtained in Theorem 3.3 with the upper bound $Ce^{\frac{2TR}{1+2T}|\varsigma|}$ for the time-dependent factor in (2.12). □

Let us recall again the procedure to obtain the classical solution of the focusing DSII equation with initial data aq_0 and a very small a such that there are no exceptional points. As the first step, one needs to solve the equation $(I+T_{z,t})v = I$, where $T_{z,t}$ is given by (2.13) with $D=\emptyset$, i.e., the equation for $v = v_{z,t}$ has the form

$$v_{z,t}(k) + \frac{1}{\pi}\int_{\mathbb{C}} e^{i(\bar\varsigma z+\bar z\varsigma)/2}\overline{v_{z,t}}(\varsigma)\Pi^o h(\varsigma,\varsigma,t)\frac{d\sigma_\varsigma}{\varsigma-k} = I, \tag{3.3}$$

where $w_{z,t}(\cdot) = v_{z,t}(\cdot) - I \in \mathcal{B}^s$. Then the solution of the focusing DSII equation with initial data aq_0 is given by (2.16). In particular,

$$q(z,t) = \frac{1}{2\pi i}\int_{\mathbb{C}} e^{i(\bar\varsigma z+\bar z\varsigma)/2}(\overline{v_{z,t}})_{11}(\varsigma)h_{12}(\varsigma,\varsigma,t)d\sigma_\varsigma. \tag{3.4}$$

Theorem 3.5. *Let Condition $Q(R)$ hold for q_0, and let the potential $q(z,t)$, $0\le t\le T$, in (3.4) be constructed from the initial data $aq_0(z)$ with $0<a<a_1\ll 1$. Then there exists $a_1 = a_1(R,T)$ such that Condition $Q(\frac{R}{1+2T}-\varepsilon)$ holds for the potential (3.4) for all $t\in[0,T]$.*

Proof. There is a complete duality (e.g. [21, Th. 4.15]) between the nonlinear Fourier transform given by (2.5), (2.7) and the inverse transform (3.3), (3.4). Function h in (3.3) plays the role of the potential Q_0 in (2.5). Theorem 3.4 implies that the Condition $Q(R')$ holds for h with $R' = \frac{R}{1+2T}-\frac{\varepsilon}{2}$. From Lemma 3.2 applied to (3.3) instead of (2.5), it follows that v has the same properties as the properties of μ established in Lemma 3.2. One needs only to take a small enough to guarantee that the analogues of operators $K^\pm$ have norms that do not exceed one. Then

$$|B'_\sigma B_\gamma \bar v(z,k)| < C(R',\varepsilon) \quad \text{when } z,k\in\mathbb{C},\ |\sigma|,|\gamma|\le R'-\frac{\varepsilon}{2},\ a\ll 1.$$

Then the statement of the theorem can be obtained similarly to the proof of Theorem 3.3, i.e., by using the shift of the complex plane $\mathcal{C}$ in (3.4) by the vector $\eta = i\frac{(x,y)}{|z|}(R'-\frac{\varepsilon}{2})$. □

4. Proof of the first statement of Theorem 2.2

Consider problem (1.1) with q_0 replaced by aq_0, $a\in(0,1]$. Let D be a disk containing all the exceptional points for problems (2.2), (2.3) for all $a\in(0,1]$. Let $k_0\in\partial D$ be a non-exceptional point for all $a\in(0,1]$. We will use notation v^1 for the solution of (2.15) and (q^1,φ^1) for the pair defined by (2.16) when the operator $T_{z,t}$ is defined using the disk D. We preserve the notations v, (q,φ) for the same objects when there are no exceptional points and $D=\emptyset$. Since q^1,φ^1 are meromorphic in a in a neighbourhood of $(0,1]$ (see Theorem 2.1), the first statement of Theorem 2.2 will be proved if we show that $(q^1,\varphi^1) = (q,\varphi)$ when $a>0$ is small and $t>0$.

From (2.4), (2.7) and Condition $Q(R)$ with $R>(1+2T)A>A$, it follows that the scattering data $h_0 = h_0(\varsigma,k)$ is defined for all the potentials aq_0 when $|\varsigma|,|k|\le A$ (i.e., $\varsigma,k\in\overline{D}$) and also for all $\varsigma=k$. We define $h(\varsigma,k,t)$ (extension of h_0 in t) according to (2.12). Let $v=v_{z,t}=w_{z,t}+I$, where $w_{z,t}\in\mathcal{B}^s$, $s>2$, is the solution of (2.15) with $T_{z,t}$ given by (2.13) with $D=\emptyset$ (i.e., the right-hand side in (2.13) contains only the first term, see equation (3.3)). Then (q,ϕ) given by (2.16) with $D=\emptyset$ solves the DSII equation (1.1) (see [7]), and

$$\psi=\psi(z,k,t):=\Pi^d\overline{v}e^{i\overline{k}z/2}+e^{-i\overline{z}k/2}\Pi^o v,\ \varsigma,k\in\mathbb{C},\ t\geq 0, \tag{4.1}$$

is the solution of the scattering problem (2.2) (and the Lippmann-Schwinger equation (2.3)) with the potential $Q_t(z)=\begin{pmatrix} 0 & q(z,t)\\ -q(z,t) & 0\end{pmatrix}$ instead of Q_0.

Consider now the scattering data

$$\widehat{h}(\varsigma,k,t):=\frac{1}{(2\pi)^2}\int_{\mathbb{C}}e^{-i\overline{\varsigma}z/2}Q_t(z)\overline{\psi}(z,k,t)d\sigma_z \tag{4.2}$$

defined by the solution ψ of the Lippmann-Schwinger equation (2.3) with the potential $Q_t(z)$. If $0\leq t\leq T$, then from Theorem 3.5 (it is assumed there that $R>(1+2T)A$) it follows that integral (4.2) converges when $|\varsigma|,|k|\leq A$ (i.e., $\varsigma,k\in\overline{D}$) and when $\varsigma=k$. Moreover, $\widehat{h}(\varsigma,k,t)=\widehat{h}(k+\alpha,k,t)$ is an anti-analytic continuation of $\widehat{h}(k,k,t)$ in α. We will prove that $\widehat{h}$ coincides with the scattering data $h(\varsigma,k,t)$ defined in (2.12). We also will prove that there exists an analytic in k function $\widehat{v}_1^+=\widehat{v}_1^+(k,t),\ k\in D$, such that

$$(v-\widehat{v}_1^+)|_{\varsigma\in\partial D}= \tag{4.3}$$

$$=\int_{\partial D}[e^{i/2(\varsigma\overline{z}+\overline{\varsigma'}z)}\overline{\widehat{v}_1^+(\varsigma')}\Pi^o-e^{i/2(\varsigma-\varsigma')\overline{z}}\widehat{v}_1^+(\varsigma')\Pi^d\mathbf{C}]\mathrm{Ln}\frac{\overline{\varsigma'}-\overline{\varsigma}}{\overline{\varsigma'}-\overline{k_0^1}}\widehat{h}_t(\varsigma',\varsigma)d\varsigma'.$$

From these two facts and the $\overline{\partial}$-equation (see [20])

$$\frac{\partial}{\partial\overline{k}}v(z,k,t)=e^{i(\overline{k}z+\overline{z}k)/2}\overline{v}(z,k,t)\Pi^o h(k,k,t),\quad k\in\mathbb{C}\backslash D, \tag{4.4}$$

it follows (see [13, Lemma 3.3]) that the function

$$v'(z,k):=\begin{cases} v(z,k), & k\in\mathbb{C}\backslash D,\\ \widehat{v}_1^+(z,k), & k\in D,\end{cases} \tag{4.5}$$

satisfies the integral equation (2.15), where operator $T_{z,t}$ is constructed using the scattering data $\widehat{h}$. Equation (2.15) has a unique solution when a is small enough. Under the assumption that $\widehat{h}=h$, we have $v^1\equiv v'$. Therefore $v^1(z,k)=v(z,k)$ when $k\in\mathbb{C}\backslash D$. Solution (q,ϕ) of the DSII equation can be determined via the asymptotics of v at large values of k (e.g., [20, (1.17)], [14, Lemma 3.3]). Hence $(q^1,\phi^1)=(q,\phi)$ for small a. Thus the first statement of the theorem will be proved as soon as we show that $\widehat{h}=h,\ t>0$, and that $\widehat{v}^+$ exists.

Justification of the equality $\widehat{h}=h,\ t>0$. Everywhere below, till the end of the section, we omit mentioning the parameter a and assume that the initial data q_0 is small. Let us recall (see [14, Lemmas 4.1, 4.2]) that the symmetry of the matrix Q_0 (see (2.1)) implies that $h_{11}=h_{22},h_{12}=-h_{21}$, and the same relations hold for matrix v determined from the integral equation (2.15) and related to ψ by (4.1).

Let us introduce functions

$$\begin{pmatrix} a & b\\ -b & a\end{pmatrix}=h_0(k+\alpha,k),$$

and note that

$$b(\alpha,k)=\frac{1}{(2\pi)^2}\int_{\mathbb{C}}e^{-\overline{\alpha}z/2}e^{-i(k\overline{z}+\overline{k}z)/2}q_0(z)v_{11}(z,k)d\sigma_z,$$

$$a(\alpha,k)=\frac{1}{(2\pi)^2}\int_{\mathbb{C}}e^{-\overline{\alpha}z/2}q_0(z)\overline{v}_{12}(z,k)d\sigma_z.$$

Now define

$$\begin{pmatrix} a(\alpha,k,t) & b(\alpha,k,t)\\ -b(\alpha,k,t) & a(\alpha,k,t)\end{pmatrix}=h(k+\alpha,k,t),$$

where h is given by (2.12). Similar quantities $\widehat{a},\widehat{b}$ are defined via the solutions $v(\cdot,k,t)$:

$$\widehat{b}(\alpha,k,t):=\frac{1}{(2\pi)^2}\int_{\mathbb{C}}e^{-\overline{\alpha}z/2}e^{-i(k\overline{z}+\overline{k}z)/2}q(z,t)v_{11}(z,k,t)d\sigma_z,$$

$$\widehat{a}(\alpha,k,t):=\frac{1}{(2\pi)^2}\int_{\mathbb{C}}e^{-\overline{\alpha}z/2}q(z,t)\overline{v}_{12}(z,k,t)d\sigma_z.$$

These quantities are well defined due to Theorem 3.5. Let

$$\widehat{h}=\begin{pmatrix} \widehat{a}(\alpha,k,t) & \widehat{b}(\alpha,k,t)\\ -\widehat{b}(\alpha,k,t) & \widehat{a}(\alpha,k,t)\end{pmatrix}.$$

Consider solution $\psi(z,k,t)$ of (2.2) with potential Q_0 replaced by Q_t, and let v be defined by (4.1). From (4.1) it follows that $v\to I$ uniformly on each compact with respect to the variable z when $k\to\infty$. Therefore, from Theorem 3.5 it follows that

$$\widehat{a}(\alpha,k,t)\to 0,\ \ k\to\infty. \tag{4.6}$$

Obviously (see (2.12)), the same relation holds for $a(\alpha,k,t)$.

The $\overline{\partial}$-equation (4.4) implies that the following rules are valid when $t=0$:

$$\frac{\partial b}{\partial\overline{k}}=\frac{\partial b}{\partial\overline{\alpha}}+ab_0,\quad \frac{\partial a}{\partial k}=b\overline{b_0},\quad \text{where}\quad b_0=b(0,k),\ |\alpha|\le A,\ k\in\mathbb{C}. \tag{4.7}$$

Due to Theorem 3.5, the same relations are valid for $\widehat{a}(\alpha,k,t),\widehat{b}(\alpha,k,t)$:

$$\frac{\partial\widehat{b}}{\partial\overline{k}}=\frac{\partial\widehat{b}}{\partial\overline{\alpha}}+\widehat{a}\widehat{b}_0,\quad \frac{\partial\widehat{a}}{\partial k}=\widehat{b}\overline{\widehat{b}_0},\ \widehat{b}_0=\widehat{b}(0,k,t),\quad |\alpha|\le A,\ k\in\mathbb{C},\ 0\le t\le T. \tag{4.8}$$

From (2.12), (4.7), and the obvious relations

$$e^{-t(\overline{k}^2-(\overline{k+\alpha})^2)/2}=e^{-t(k^2-(\overline{k+\alpha})^2)/2}\overline{e^{-i\Im k^2}},$$

$$\frac{\partial}{\partial\overline{\alpha}}e^{-t(k^2-(\overline{k+\alpha})^2)/2}=\frac{\partial}{\partial\overline{k}}e^{-t(k^2-(\overline{k+\alpha})^2)/2},$$

it follows that (4.8) holds for $a(\alpha,k,t),b(\alpha,k,t)$:

$$\frac{\partial b}{\partial\overline{k}}=\frac{\partial b}{\partial\overline{\alpha}}+ab_0,\quad \frac{\partial a}{\partial k}=b\overline{b_0},\quad b_0=b(0,k,t), \tag{4.9}$$

when $|\alpha|\le A,\ k\in\mathbb{C},\ 0\le t\le T$.

Now we note that $\widehat{b}(0,k,t)=b(0,k,t)$ (see [22, Theorem 5.3]). The second relations in (4.8), (4.9) with $\alpha=0$ imply that $(\widehat{a}-a)|_{\alpha=0}$ is anti-analytic in k.

Then the maximum principle, together with (4.6) for both $\widehat{a}$ and a, imply that $\widehat{a}|_{\alpha=0} = a|_{\alpha=0}$. Now from the first relations in (4.8), (4.9), with $\alpha = 0$, it follows that $\frac{\partial \widehat{b}}{\partial \overline{\alpha}}|_{\alpha=0} = \frac{\partial b}{\partial \overline{\alpha}}|_{\alpha=0}$. Then we differentiate the second relations in (4.8), (4.9) in $\overline{\alpha}$ and put $\alpha = 0$ there. This leads to the anti-analyticity in k of $\frac{\partial \widehat{a}}{\partial \overline{\alpha}}|_{\alpha=0} - \frac{\partial a}{\partial \overline{\alpha}}|_{\alpha=0}$. The maximum principle with (4.6) imply that $\frac{\partial \widehat{a}}{\partial \overline{\alpha}}|_{\alpha=0} = \frac{\partial a}{\partial \overline{\alpha}}|_{\alpha=0}$. After the differentiation in $\overline{\alpha}$ of the first relations in (4.8), (4.9), we obtain that $\frac{\partial^2 \widehat{b}}{\partial \overline{\alpha}^2}|_{\alpha=0} = \frac{\partial b^2}{\partial \overline{\alpha}^2}|_{\alpha=0}$, and so on. Hence all the derivatives in $\overline{\alpha}$ of the vectors $(\widehat{a}, \widehat{b})$ and (a, b) coincide when $\alpha = 0$. Since both vectors are anti-analytic in α, they are identical, i.e., $\widehat{h} = h,\ t > 0$.

The *existence of* $\widehat{v}^+$ can be shown similarly to the proof of same statement in [13], where the potential was assumed to be compactly supported. Namely, consider the following analogue of the Lippmann-Schwinger equation with different values $k_0, k \in \overline{D}$ of the spectral parameter in the operator and in the free term of the equation:

$$\psi^+(z,k) = e^{i\frac{\overline{k}z}{2}} I + \int_{z\in\mathbb{C}} G(z - z', k_0) Q_t(z') \overline{\psi^+}(z', k) d\sigma_{z'}, \tag{4.10}$$

where $G(z,k) = \frac{1}{\pi}\frac{e^{i\overline{k}z/2}}{z}$. We substitute here $\psi^+ = \mu^+ e^{i\overline{k_0}z/2}$ and rewrite the equation in terms of

$$w^+ = \mu^+(z,k) - e^{i(\overline{k-k_0})z/2} I \in L^\infty_z(L^p_k),\ p > 1. \tag{4.11}$$

The equation takes the form

$$w^+(z,k) - \int_{z\in\mathbb{C}} \frac{e^{-i\Re(\overline{k_0}z')}}{z - z'} Q_t(z') \overline{w^+}(z',k) d\sigma_{z'} =$$

$$= \int_{z\in\mathbb{C}} \frac{e^{-i\Re(\overline{k_0}z')}}{z - z'} \left[Q_t(z') e^{-i\overline{z'}(k-k_0)/2} \right] d\sigma_{z'}. \tag{4.12}$$

Theorem 3.5 implies that function $\left[Q_t(z') e^{-i\overline{z'}(k-k_0)/2} \right]$ decays exponentially as $z \to \infty$, and $|k|, |k_0| \le A$. The unique solvability of the problem (4.12) is obvious since the potential is small.

Function $\widehat{v}^+$ is defined by ψ^+ in the same way as v is defined by ψ in (4.1). The analyticity of $\widehat{v}^+$ and (4.3) are proved in Lemmas 3.1 and 3.5 of [13]. ☐

5. Proof of statement 2 of the Theorem 2.2.

Reduction to Theorem 5.2 and Lemma 5.3. Theorem 3.5 immediately implies that the operator $T_{z,t} : \mathcal{B}^s \to \mathcal{B}^s,\ s > 2$, is analytic in x and y in a complex neighborhood of $\mathbb{R}^2$. In order to use the multidimensional analytic Fredholm theory ([24, Th. 4.11, 4.12] or [23]) and obtain a decay of operator norm $\|T^2_{z,t}\|$ as $|z| \to \infty$, we would like to consider this operator in the Hilbert space $\mathcal{B}^2$ instead of the Banach space $\mathcal{B}^s,\ s > 2$. All the previous and new results mentioned in this paper remain valid if $s > 2$ is replaced by $s = 2$ (with the appropriate definition of the space $\mathcal{B}^2$ given in (2.17)). In order to justify the latter statement, one needs to show that the properties of the operator $T_{z,t}$ are preserved when $s > 2$ is replaced by $s = 2$. This will be done in Theorem 5.2 below (we will not discuss the properties that obviously are s-independent), but we will show that operator $T_{z,t} : \mathcal{B}^2 \to \mathcal{B}^2, 0 \le t \le T$, is compact, continuous in (z,t), and analytic in (x,y) in a complex neighborhood of $\mathbb{R}^2$. After that, we will show (Lemma 5.3) the invertibility of $I+T_{z,t}$ at large values of $|z|$. Then the second statement of the theorem will be a simple consequence of the first statement and the analytic Fredholm theory. Note that the invertibility of operator $I+T_{z,t}$ will be proved for z on each ray $\arg z=\psi=const, |z|\geq Z_0$, with ψ-independent Z_0 and with $T_{z,t}$ defined (see (2.13)) using a special value of $k_0=k_0(\psi)$. Since the solution (q,ϕ) of problem (1.1) does not depend on the choice of k_0 (see Theorem 2.2), it remains only to prove Theorem 5.2 and Lemma 5.3.

5.1. **Compactness of operator T.** We will need the following lemma.

Lemma 5.1. *Let operator $M:\mathcal{B}^2\to\mathcal{B}^2$ have the form*

$$(Mf)(k)=\int_{\mathbb{C}\backslash D}\frac{g(\varsigma)}{\varsigma-k}f(\varsigma)d\sigma_\varsigma,\quad k\in\mathbb{C},$$

where function $g_\delta=g(\varsigma)(1+|\varsigma|)^\delta$ has the following properties

$$|g_\delta|<a_1<\infty,\ g_\delta\to 0 \text{ as } \varsigma\to\infty, \text{ and } \|g_\delta\|_{L^2(\mathbb{C}\backslash D)}=a_2<\infty$$

for some $\delta>0$. Then M is compact and $\|M\|\leq C(a_1+a_2)$.

Proof. Let P be the following operator in $\mathcal{B}^2$ of rank one:

$$Pf=-\frac{\beta(k)}{k}\int_{\mathbb{C}\backslash D}g(\varsigma)f(\varsigma)d\sigma_\varsigma, \tag{5.1}$$

where β is the function introduced in the definition of the space $\mathcal{B}^2$. Since $Pf=0$ in a neighborhood of D, and

$$\int_{\mathbb{C}\backslash D}g(\varsigma)f(\varsigma)d\sigma_\varsigma\leq a_2\int_{\mathbb{C}\backslash D}|\frac{f(\varsigma)}{(1+|\varsigma|)^\delta}|^2d\sigma_\varsigma\leq Ca_2\|f\|_{\mathcal{B}^2},$$

it is enough to prove the statement of the lemma for operator $M-P=M_1+M_2$, where

$$M_if=\int_{\mathbb{C}\backslash D}K_i(k,\varsigma)f(\varsigma)d\sigma_\varsigma,$$

$$K_1(k,\varsigma)=\frac{\alpha(\varsigma-k)}{\varsigma-k}g(\varsigma),\ K_2(k,\varsigma)=[\frac{\beta(\varsigma-k)}{\varsigma-k}+\frac{\beta(k)}{k}]g(\varsigma),$$

and $\alpha:=1-\beta$ is a cut-off function which is equal to one in a neighborhood of D.

Let M_i' be the operator defined by the same formulas as operators M_i, but considered as operators in $L^2(\mathbb{C})$. Let us show that operators M_i' are compact and their norms do not exceed $C(a_1+a_2)$.

Since $|g|\leq a_1$, we have

$$\sup_{k\in\mathbb{C}}\int_{\mathbb{C}}|K_1(k,\varsigma)|d\sigma_\varsigma+\sup_{\varsigma\in\mathbb{C}}\int_{\mathbb{C}}|K_1(k,\varsigma)|d\sigma_k\leq Ca_1.$$

Hence, from the Young theorem, it follows that $\|M_i'\|\leq Ca_1$. Similarly, using the decay of g_1 at infinity, we obtain that $M_1'=\lim_{R\to\infty}M_{1,R}'$, where $M_{1,R}'$ are operators in $L^2(\mathbb{C})$ with the integral kernels $K_1(k,\varsigma)\alpha(\varsigma/R)$. Operators $M_{1,R}$ are pseudo-differential operators of order -1 (they increase the smoothness of functions by one) defined in a bounded domain. Hence operators $M'_{1,R}$ and their limit M'_1 are compact operators in $L^2(\mathbb{C})$.

The boundedness (with the upper bound Ca_2) and compactness of the operator M_2 will be proved if we show that

$$\int_{\mathbb{C}}\int_{\mathbb{C}} |K_2(k,\varsigma)|^2 d\sigma_k d\sigma_\varsigma \leq Ca_2.$$

We split the interior integral in two parts: over region $|k| < 2|\varsigma|$ and over region $|k| > 2|\varsigma|$, and estimate each of them separately. We have

$$\int_{|k|<2|\varsigma|} |K_2(k,\varsigma)|^2 d\sigma_k \leq 2|g(\varsigma)|^2 \int_{|k|<2|\varsigma|} [\frac{\beta^2(\varsigma-k)}{|\varsigma-k|^2} + \frac{\beta^2(k)}{|k|^2}] d\sigma_k$$
$$\leq C|g(\varsigma)|^2(1+|\varsigma|)^\delta.$$

A better estimate with a logarithmic factor is valid, but we do not need this accuracy. Next,

$$\int_{|k|>2|\varsigma|} |K_2(k,\varsigma)|^2 d\sigma_k = |g(\varsigma)|^2 \int_{|k|>2|\varsigma|} \frac{|k\beta(\varsigma-k)+(\varsigma-k)\beta(k)|^2}{|(\varsigma-k)k|^2} d\sigma_k.$$

The denominator of the integrand can be estimated from below by $\frac{1}{4}|k|^4$. The numerator, denoted by n, has the following properties. If $|\varsigma|$ is large enough, than both beta functions in n are equal to one, and $n = |\varsigma|^2$. The same is true if $|\varsigma|$ is bounded and $|k|$ is large. If both variables are bounded, than $|n|$ is bounded. Thus $|n| < (C+|\varsigma|)^2$, and the integrand above does not exceed $C\frac{1+|\varsigma|^2}{|k|^4}$. Obviously, the integrand vanishes when $|k|$ is small enough. Thus there is a constant $c > 0$ such that

$$\int_{|k|>2|\varsigma|} |K_2(k,\varsigma)|^2 d\sigma_k \leq C|g(\varsigma)|^2 \int_{|k|>\max(c,2|\varsigma|)} \frac{1+|\varsigma|^2}{|k|^4} d\sigma_k$$
$$\leq C|g(\varsigma)|^2 \int_{|k|>c} \frac{1}{|k|^4} d\sigma_k + C|g(\varsigma)|^2 \int_{|k|>2|\varsigma|} \frac{|\varsigma|^2}{|k|^4} d\sigma_k = C_1|g(\varsigma)|^2.$$

Hence

$$\int_{\mathbb{C}}\int_{\mathbb{C}} |K_2(k,\varsigma)|^2 d\sigma_k d\sigma_\varsigma \leq C\int_{\mathbb{C}} |g(\varsigma)|^2(1+|\varsigma|)^\delta d\sigma_\varsigma \leq C'a_2.$$

Thus, operators $M'_i : L^2(\mathbb{C}) \to L^2(\mathbb{C})$ are compact and $\|M'_i\| \leq C(a_1+a_2)$.

Denote by $M''_i : \mathcal{B}^2 \to L^2(\mathbb{C})$ operators with the same integral kernels K_i as for operators M'_i, but with the domain $\mathcal{B}^2$ instead of $L^2(\mathbb{C})$. Compactness of these operators will be proved if we show the boundedness of M'_i on the one-dimensional space of functions of the form $f_c(\varsigma) = c\frac{\beta(\varsigma)}{\varsigma}$, $c =$const. The upper estimate on $\|M''_i f_c\|$ can be obtained by repeating the arguments above used to estimate $\|M'_i\|$. One needs only to replace f_c by the function $f = f_c/|\varsigma|^{\delta/2} \in L^2(\mathbb{C})$ and replace the kernel K_i by $K_i|\varsigma|^{\delta/2}$. Hence, operators M''_i are compact and $\|M'_i\| \leq C(a_1+a_2)$.

Obviously, for each $f \in \mathcal{B}^2$, the function $(M_1+M_2)f$ is analytic in D. Consider its trace on ∂D. Let $M_D : \mathcal{B}^2 \to L^2(\partial D)$ be the operator that maps each $f \in \mathcal{B}^2$ into the trace of $(M_1+M_2)f$ on ∂D. In order to complete the proof of the lemma, it remains to show that operator M_D is well defined, compact, and $\|M_D\| \leq C(a_1+a_2)$. To prove these properties of M_D, we split the operator into two terms $M_D =$

$M_D\phi + M_D(1-\phi)$, where ϕ is the operator of multiplication by the indicator function of a disk D_1 of a larger radius than the radius of D. Then $M(1-\phi)f$ is analytic in D_1, and

$$\|M(1-\phi)f\|_{L^2(D_1)} \leq \|Mf\|_{L^2(\mathbb{C})} \leq C(a_1+a_2)\|f\|_{\mathcal{B}^2}.$$

From a priori estimates for elliptic operators, it follows that

$$\|M(1-\phi)f\|_{H^s(D)} \leq C_s\|M(1-\phi)f\|_{L^2(D_1)} \leq C_s(a_1+a_2)\|f\|_{\mathcal{B}^2},$$

where H^s is the Sobolev space and s is arbitrary. Hence

$$\|M(1-\phi)f\|_{H^{s-1/2}(\partial D)} \leq C(a_1+a_2)\|f\|_{\mathcal{B}^2}.$$

This implies that operator $M_D(1-\phi)$ is compact and its norm does not exceed $C(a_1+a_2)$. We will take D_1 not very large, so that function β vanishes on D_1. Then $M\phi f$ is the convolution of $1/k$ and $\phi g f$, i.e., $M\phi f = \frac{1}{k} * (\phi g f)$. The latter expression is a pseudo differential operator of order -1 applied to the function $\phi g f$ with a compact support. Thus,

$$\|M\phi f\|_{H^1(D)} \leq C\|\phi g f\|_{L^2(D_1)} \leq Ca_1\|f\|_{\mathcal{B}^2},$$

and therefore $\|M_D\phi f\|_{H^{1/2}(D)} \leq Ca_1\|f\|_{\mathcal{B}^2}$. Hence, operator $M_D\phi$ is compact and its norm does not exceed Ca_1. $\square$

Theorem 5.2. *Let conditions of Theorem 2.2 hold. Then operator $T_{z,t} : \mathcal{B}^2 \to \mathcal{B}^2, 0 \leq t \leq T$, is compact, continuous in (z,t), and analytic in (x,y) in a complex neighborhood of $\mathbb{R}^2$. The same properties are valid for derivatives of $T_{z,t}$ of any order in t,x,y.*

Remark. $T_{z,t}$ is analytic in x,y in the region $|\Im x|^2 + |\Im y|^2 \leq R^2$.

Proof. The operator $T_{z,t}$ can be naturally split into two terms: $T_{z,t} = \mathcal{M} + \mathcal{D}$, where $\mathcal{M}$ involves integration over $\mathbb{C}\backslash D$ and $\mathcal{D}$ involves integration over ∂D. In particular,

$$\mathcal{M}\phi = \frac{1}{\pi}\int_{\mathbb{C}\backslash D} \frac{e^{i\Re(\varsigma\overline{z})}\overline{\phi}(\varsigma)\Pi^o h(\varsigma,\varsigma,t)}{\varsigma - k} d\sigma_\varsigma.$$

The statements of the theorem are valid for operator $\mathcal{M}$ due to (2.9), Lemma 5.1 and Theorem 3.4. Indeed, the compactness and continuity of M in (z,t) is proved in Lemma 5.1. The analyticity in (x,y) follows from the fast decay of h at infinity which is established in Theorem 3.4.

Let us show that the same properties are valid for $\mathcal{D}$. We write $\mathcal{D}$ in the form $D = I_1 I_2$, where operator $I_2 : L^2(\partial D) \to C^\alpha(\partial D)$ is defined by the interior integral in the expression for $\mathcal{D}$ in (2.13), and operator $I_1 : C^\alpha(\partial D) \to \mathcal{H}^s$ is defined by the exterior integral in the same expression. Here $C^\alpha(\partial D)$ is the Holder space and α is an arbitrary number in $(0,1/2)$. The integral kernel of operator I_2 has a logarithmic singularity at $\varsigma = \varsigma'$, i.e., I_2 is a pseudo differential operator of order -1, and therefore I_2 is a bounded operator from $L^2(\partial D)$ into the Sobolev space $H^1(\partial D)$. Thus it is compact as operator from $C(\partial D)$ to $C^\alpha(\partial D), \alpha \in (0,1/2)$, due to the Sobolev embedding theorem. Thus the compactness of $\mathcal{D}$ will be proved as soon as we show that I_1 is bounded.

For each $\phi \in C^\alpha(\partial D)$, function $I_1\phi$ is analytic outside of ∂D and vanishes at infinity. Due to the Sokhotski–Plemelj theorem, the limiting values $(I_1\phi)_\pm$ of $(I_1\phi)$

on ∂D from inside and outside of D, respectively, are equal to $\frac{\pm\phi}{2}+P.V.\frac{1}{2\pi i}\int_{\partial D}\frac{\phi(\varsigma)d\varsigma}{\varsigma-\lambda}$. Thus

$$\max_{\partial D}|(I_1\phi)_\pm|\leq C\|\phi\|_{C^\alpha(\partial D)}.$$

From the maximum principle for analytic functions, it follows that the same estimate is valid for function $I_1\phi$ on the whole plane. Taking also into account that $I_2\phi$ has the following behavior at infinity $I_2\phi\sim c/k+O(|k|^2)$, we obtain that operator I_1 is bounded. Hence operator $\mathcal{D}$ is compact. Since h decays superexponentially at infinity, the arguments above allow one to obtain not only the compactness of $\mathcal{D}$, but also its smoothness in t,x,y and analyticity in (x,y). □

5.2. **The invertibility of $I+T_{z,t}$ at large values of z.** We will prove the following lemma.

Lemma 5.3. *The following relation is valid for operator norm of $T^2_{z,t}$ in $\mathcal{B}^2$:*

$$\max_{0\leq t\leq T}\|T^2_{z,t}\|\to 0,\quad z\in\mathbb{C},\ z\to\infty.$$

Hence the operator $I+T_{z,t}$ is invertible when $z\in\mathbb{C},\ |z|\gg 1$.

We split operator $T_{z,t}$ into two terms $T_{z,t}=\mathcal{M}+\mathcal{D}$ that correspond to the integration over $\mathcal{C}\backslash D$ and D, respectively, in (2.13). The entries $M^{ij},\ D^{ij},\ i,j=1,2$, of the matrix operators $\mathcal{M}$ and $\mathcal{D}$ are

$$M^{11}=M^{22}=0,$$

$$M^{12}\phi=-M^{21}\phi=\frac{1}{\pi}\int_{\mathbb{C}\backslash D}\frac{e^{i\Re(\varsigma\overline{z})-t(\varsigma^2-\overline{\varsigma}^2)/2}\overline{\phi}(\varsigma)h_{12}(\varsigma,\varsigma)}{\varsigma-k}d\sigma_\varsigma,$$

$$D^{11}\phi=D^{22}\phi=$$

$$=\frac{1}{2\pi i}\int_{\partial D}\frac{d\zeta}{\zeta-k}\int_{\partial D}\overline{\mathrm{Ln}\frac{\overline{\varsigma'}-\overline{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}h_{11}(\varsigma',\varsigma)e^{\frac{i}{2}(\varsigma-\varsigma')\overline{z}+\frac{t}{2}(\varsigma'^2-\varsigma^2)}}\phi(\varsigma')d\zeta',$$

$$D^{12}\phi=-D^{21}\phi=$$

$$=\frac{1}{2\pi i}\int_{\partial D}\frac{d\zeta}{\zeta-k}\int_{\partial D}\mathrm{Ln}\frac{\overline{\varsigma'}-\overline{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}h_{12}(\varsigma',\varsigma)e^{\frac{i}{2}(\varsigma\overline{z}+\overline{\varsigma'}z)+\frac{t}{2}(\overline{\varsigma'}^2-\varsigma^2)}\overline{\phi}(\varsigma')\overline{d\varsigma'}.$$

We used here the relations $h_{12}=-h_{12},h_{11}=h_{22}$ for the entries of h_0 that were established, for example, in [14, Lemma 4.1].

Lemma 5.1 implies the uniform boundedness of M^{21},M^{12} when $0\leq t\leq T,\ z\in\mathbb{C}$. Thus Lemma 5.3 will be proved if we show that operator norms of $M^{21}\overline{M^{12}}$ and $D^{ij},i,j=1,2$, vanish as $z\to\infty$. Let us prove the statement about D^{ij}.

Lemma 5.4. *For each $T>0$, there exists a constant C_T such that*

$$\|D\varphi\|_{\mathcal{B}^2}\leq\frac{C_{\alpha,T}}{1+|z|^{1/4}}\|\varphi\|_{\mathcal{B}^2},\quad z\in\mathbb{C},\ 0\leq t\leq T,$$

if k_0 in the definition of operator $\mathcal{D}$ is chosen to belong to ∂D and equal to $k_0=-iAe^{i\psi}$, where $\psi=\arg z$ and A is the radius of the disk D.

Proof. We will prove the estimate for the component D^{12} of the matrix D. Other components of D can be estimated similarly. Consider the interior integral in D^{12}:

$$R^{12}\phi = \int_{\partial D} \mathrm{Ln}\frac{\overline{\varsigma'} - \overline{\varsigma}}{\overline{\varsigma'} - \overline{k_0}} h_{12}(\varsigma', \varsigma) e^{\frac{i}{2}(\varsigma\overline{z} + \overline{\varsigma'}z) + \frac{t}{2}(\overline{\varsigma'}^2 - \varsigma^2)} \overline{\phi}(\varsigma') d\overline{\varsigma'}, \tag{5.2}$$

where $\varsigma \in \partial D,\ \phi \in \mathcal{B}^2$. Our goal is to show that

$$\|R^{12}\phi\|_{L^\infty(\partial D)} \leq \frac{C_T}{1 + |z|^{1/4}} \|\phi\|_{L^2(\partial D)},\ \phi \in \mathcal{B}^2. \tag{5.3}$$

The integrand in (5.2) is anti-holomorphic in $\varsigma' \in D$ with logarithmic branching points at k_0 and ς. If k_0 is strictly inside D, then the integration over ∂D in (5.2) can be replaced by the integration over two sides of the segment $[k_0, \varsigma]$, which are passed in the counter clock-wise direction. The values of the logarithm on these sides differ by the constant 2π. This leads to an alternative form of the operator $\mathcal{D}$:

$$D^{12}\phi = -D^{21}\phi = i \int_{\partial D} \frac{d\zeta}{\zeta - k} \int_{[k_0, \varsigma]} h_{12}(\varsigma', \varsigma) e^{\frac{i}{2}(\varsigma\overline{z} + \overline{\varsigma'}z) + \frac{t}{2}(\overline{\varsigma'}^2 - \varsigma^2)} \overline{\phi}(\varsigma') \overline{d\varsigma'}.$$

If $k_0 \in \partial D$, the contour of integration above can be replaced by arc$[k_0, \varsigma]$. Thus

$$R^{12}\phi = i \int_{\widehat{k_0, \varsigma}} h_{12}(\varsigma', \varsigma) e^{\frac{i}{2}(\varsigma\overline{z} + \overline{\varsigma'}z) + \frac{t}{2}(\overline{\varsigma'}^2 - \varsigma^2)} \phi(\varsigma') d\overline{\zeta'}, \quad \varsigma \in \partial D,\ \phi \in L^2(\partial D).$$

Consider the following function (from the exponent in the integrand above): $\Phi = \Re\left[\frac{i}{2}\varsigma\overline{z}\right]$. This function is linear in ς, and for each fixed $z = |z|e^{i\psi}, \psi \in [0, 2\pi)$, it has the unique global maximum on D. The maximum occurs on the boundary at the point $\varsigma_0 = -iAe^{i\psi}$, which depends only on the argument of z. Due to Theorem 2.2, point $k_0 \in \partial D$ can be chosen arbitrarily. We choose $k_0 = \varsigma_0 \in \partial D$, and we get that

$$|R^{12}\phi| \leq C \left(\int_{\widehat{\varsigma_0, \varsigma}} \exp 2\left(\Phi(\varsigma) - \Phi(\varsigma')\right) |d\varsigma'| \right)^{1/2} \|\phi\|_{L^2}.$$

Let us estimate the integral above. Let $\varsigma = -iAe^{i(\psi+\varphi)},\ |\varphi| \leq \pi$. For $\varsigma' \in \widehat{\varsigma_0, \varsigma}$, we have

$$\Phi(\varsigma') = A|z|(\cos\varphi')/2, \quad \Phi(\varsigma) = A|z|(\cos\varphi)/2,$$

and the integral is equal to

$$\int_0^\varphi e^{A|z|(\cos\varphi - \cos\varphi')/2} d\varphi' = O(\frac{1}{\sqrt{|z|}}),\ z \to \infty.$$

This justifies (5.3). □

Let us show now that the following statement holds.

Lemma 5.5.

$$\max_{0 \leq t \leq T} \|M^{21}\overline{M^{12}}\|_{\mathcal{B}^2} \to 0, \quad z \in \mathbb{C},\ z \to \infty. \tag{5.4}$$

Proof. Kernels of M^{12}, M^{21} are smooth, see (2.9). From Theorem 3.4, it follows that the kernels and rapidly decaying functions in $\mathbb{C}$. Therefore, Lemma 5.1 implies that operators M^{12}, M^{21} can be approximated in $\mathcal{B}^2$ by operators with function h_{12}

replaced by a compactly supported one. Therefore, without loss of the generality, we will assume below that the supports of h_{12}, h_{21} belong to a bounded domain $\mathcal{O}$.

We will use the notation P for the one-dimensional operator defined in (5.1) with the density $g = e^{i\Re(\varsigma\overline{z})-t(\varsigma-\overline{\varsigma}^2)/2}h_{12}(\varsigma,\varsigma)$. Let $\widehat{M} := (M^{12}-P)\overline{(M^{21}-P)}$. We will prove that

$$\max_{0\le t\le T}\|\widehat{M}\|_{\mathcal{B}^2}\to 0,\quad z\in\mathbb{C},\ z\to\infty. \tag{5.5}$$

The other three terms $M^{12}\overline{(M^{21}-P)}$, $(M^{12}-P)\overline{M^{21}}$, and $P\overline{P}$ can be treated in the same way. We have

$$\widehat{M}\varphi = \frac{1}{\pi^2}\int_{\mathcal{O}\backslash D} A(z,\varsigma,\varsigma_2)\overline{h_{21}}(\varsigma_2,\varsigma_2)e^{-i\Re(\varsigma_2\overline{z})+t(\varsigma_2-\overline{\varsigma_2}^2)/2}\varphi(\varsigma_2)d\sigma_{\varsigma_2},$$

where $A(z,\varsigma,\varsigma_2)$ is given by the following integral

$$\int_{\mathcal{O}\backslash D} e^{i\Re(\varsigma_1\overline{z})-t(\varsigma_1-\overline{\varsigma_1}^2)/2}h_{12}(\varsigma_1,\varsigma_1)\left(\frac{1}{\varsigma_1-\varsigma}+\frac{\beta(\varsigma)}{\varsigma}\right)\overline{\left(\frac{1}{\varsigma_2-\varsigma_1}+\frac{\beta(\varsigma_1)}{\varsigma_1}\right)}d\sigma_{\varsigma_1}. \tag{5.6}$$

The Minkovsky inequality in the integral form implies the following two estimates, that are valid when $f\in\mathcal{B}^2$:

$$\|\widehat{M}f\|_{L^2(\mathbb{C}\backslash D)} \le \int_{\mathcal{O}\backslash D}\left[\int_{\mathcal{O}\backslash D}|A(z,\varsigma,\varsigma_2)|^2 d\sigma_\varsigma\right]^{1/2}|h_{21}(\varsigma_2,\varsigma_2)f(\varsigma_2)|d\sigma_{\varsigma_2},$$

$$\|\widehat{M}f\|_{L^2(\partial D)} \le \int_{\mathcal{O}\backslash D}\left[\int_{\partial D}|A(z,\varsigma,\varsigma_2)|^2 |d\varsigma|\right]^{1/2}|h_{21}(\varsigma_2,\varsigma_2)f(\varsigma_2)|d\sigma_{\varsigma_2}.$$

Since the norm of the operator $L^2(\mathbb{C}\backslash D)\to L^1(\mathbb{C}\backslash D)$ of multiplication by h_{21} can be estimated by a constant, the validity of (5.5) will follow from the estimates above if we show that the following relations hold as $z\to\infty$:

$$\sup_{\varsigma_2\in\mathbb{C}\backslash D}\int_{\mathcal{O}\backslash D}|A(z,\varsigma,\varsigma_2)|^2 d\sigma_\varsigma\to 0,\qquad \sup_{\varsigma_2\in\mathbb{C}\backslash D}\int_{\partial D}|A(z,\varsigma,\varsigma_2)|^2|d\varsigma|\to 0.$$

We will prove only the first of them, since the second one can be proved similarly. Note that, uniformly in $\varsigma_2\in\mathcal{O}$,

$$\int_{\mathcal{O}\backslash D}|A(z,\varsigma,\varsigma_2)|^2 d\sigma_\varsigma \le$$

$$\int_{\mathcal{O}\backslash D}\left|\int_{\mathcal{O}\backslash D}h_{12}(\varsigma_1,\varsigma_1)\left(\frac{1}{\varsigma_1-\varsigma}+\frac{\beta(\varsigma)}{\varsigma}\right)\overline{\left(\frac{1}{\varsigma_2-\varsigma_1}+\frac{\beta(\varsigma_1)}{\varsigma_1}\right)}d\sigma_{\varsigma_1}\right|^2 d\sigma_\varsigma < C.$$

The boundedness follows from the fact that the internal integral is $O(\ln|\varsigma-\varsigma_2|), \varsigma-\varsigma_2\to 0$. Let A^s be given by (5.6) with the extra factor $\eta_s := \eta(s|\varsigma-\varsigma_1|)\eta(s|\varsigma_1-\varsigma_2|)), s>0$, in the integrand, where $\eta\in C^\infty(\mathbb{R})$, $\eta=1$ outside of a neighborhood of the origin, and η vanishes in a smaller neighborhood of the origin.

For each ε, there exists $s=s_0(\varepsilon)$ such that

$$\int_{\mathcal{O}\backslash D}|A-A^{s_0}|^2\, d\sigma_\varsigma < \varepsilon$$

for all the values of $\varsigma_2 \in \mathcal{O}, z \in \mathbb{C}$. Denote by R^{s_0} the function A^{s_0} with the potential h_{12} replaced by its L_1-approximation $\widetilde{h}_{12} \in C_0^\infty(\mathbb{C}\backslash D)$. We can choose this approximation in such a way that

$$\int_{\mathcal{O}\backslash D} |A^{s_0} - R^{s_0}|^2 \, d\sigma_\varsigma < \varepsilon$$

for all the values of ς_2, z. Now it is enough to show that

$$|R^{s_0}(\varsigma, \varsigma_2, z)| \to 0 \quad \text{as} \quad z \to \infty$$

uniformly in $\varsigma, \varsigma_2 \in \mathcal{O}$. The latter can be obtained by integration by parts in $R^{s_0}(\varsigma, \varsigma_2, z)$, defined by integral (5.6) with h_{12} replaced by $(1-\eta_s)\widetilde{h_{12}}(\varsigma_1, \varsigma_1)$ (integrating $e^{i\Re(\varsigma_1 \bar{z})}$ and differentiating the complementary factor). This completes the proof of (5.4). $\square$

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