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AN EXTENSION OF THE BEALE-KATO-MAJDA CRITERION FOR THE 3D NAVIER-STOKES EQUATION WITH HEREDITARY VISCOSITY

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ABSTRACT. In this work, we consider the three dimensional Navier-Stokes equations on the whole space with a hereditary viscous term which depends on the past history. We study a blow-up criterion of smooth solutions to such systems. The existence and uniqueness of smooth solution is proved via a frequency truncation method. We also give the example of Maxwell's fluid flow equations, which is a linear viscoelastic fluid flow model.

1. INTRODUCTION

The Navier-Stokes equation with hereditary viscosity models arise in the dynamics of non-Newtonian fluids and also as viscoelastic models for the dynamics of turbulence statistics in Newtonian fluids (cf. [19, 20, 25]). The celebrated paper [3] established that the maximum norm of the vorticity controls the breakdown of smooth solutions of the 3D Euler equations. Many mathematicians extended the work of Beale-Kato-Majda (see [3]) for the Euler and Navier-Stokes equations (see for example [4, 7–9, 16, 17], references therein). In this paper, we consider the 3D Navier-Stokes equations with hereditary viscosity (see [1]) and establish an extended version of the Beale-Kato-Majda blow-up criterion of smooth solutions using a logarithmic Sobolev inequality. The local smooth solutions to such systems are established via a frequency truncation method.

Using the following logarithmic Sobolev inequality: for $f \in \mathbb{H}^s(\mathbb{R}^3)$ with div $f = 0$ (see [3, 6])

$$\|\nabla f\|_{\mathbb{L}^\infty} \leq C\Big\{1 + \|\nabla \times f\|_{\mathbb{L}^2} + \|\nabla \times f\|_{\mathbb{L}^\infty} \log_e\left(e + \|f\|_{\mathbb{H}^s}\right)\Big\}, \ s > 5/2; \tag{1.1}$$

the authors in [3] showed a regularity criterion for the smooth solution of the Euler equation ($\mathbf{u}(\cdot,\cdot)$ is the velocity field, $p(\cdot,\cdot)$ is the fluid pressure and $\mathbf{f}(\cdot,\cdot)$ is the

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external forcing):

$$
\left.\begin{aligned}
\frac{\partial \mathbf{u}(x,t)}{\partial t}+(\mathbf{u}(x,t)\cdot\nabla)\mathbf{u}(x,t)&=-\nabla p(x,t)+\mathbf{f}(x,t)\ \text{ in }\ \mathbb{R}^3\times(0,\infty),\\
\nabla\cdot\mathbf{u}(x,t)&=0\ \text{ in }\ \mathbb{R}^3\times(0,\infty),\\
\mathbf{u}(x,0)&=\mathbf{u}_0(x)\ \text{ in }\ \mathbb{R}^3,
\end{aligned}\right\}\tag{1.2}
$$

in $\mathbb{R}^3$ on $[0,T)$ in terms of the vorticity $\omega=\operatorname{curl}\mathbf{u}=\nabla\times\mathbf{u}$. That is, a smooth solution $\mathbf{u}$ is regular after $t\geq T$, provided

$$\omega\in \mathrm{L}^1([0,T];\mathbb{L}^\infty).\tag{1.3}$$

Later [8](see Theorem 1) refined the estimate (1.1) to the Bounded Mean Oscillation(BMO) spaces as

$$\|f\|_{\mathbb{L}^\infty}\leq C\Big\{1+\|f\|_{\mathrm{BMO}}\log_e\left(e+\|f\|_{\mathbb{H}^s}\right)\Big\},\ s>3/2.\tag{1.4}$$

In (1.4), BMO is the space defined as a set of locally $\mathbb{L}^1$-functions f such that

$$\|f\|_{\mathrm{BMO}}\equiv\sup_{R>0,x\in\mathbb{R}^3}\frac{1}{|\mathrm{B}_R(x)|}\int_{\mathrm{B}_R(x)}|f(y)-\overline{f}_{\mathrm{B}_R(x)}|\mathrm{d}y<+\infty,$$

where $\mathrm{B}_R(x)$ is the ball of radius R centered at x, $|B_R(x)|$ is its volume and $\overline{f}_{\mathrm{B}_R(x)}$ stands for the average of f over $\mathrm{B}_R(x)$, i.e., $\overline{f}_{B_R}=\dfrac{1}{|\mathrm{B}_R|}\displaystyle\int_{y\in\mathrm{B}_R(x)}f(y)\mathrm{d}y$. Using the fact that the Riesz transforms are bounded in BMO, but not in $\mathbb{L}^\infty$, the authors in [8] extended the result (1.3) to the space

$$\omega\in\mathrm{L}^1([0,T];\mathrm{BMO}).\tag{1.5}$$

The authors in [9] (see Theorem 2.1) obtained the logarithmic Sobolev inequality in Besov spaces as: For $f\in\mathbb{H}^s(\mathbb{R}^3)$:

$$\|\nabla f\|_{\mathbb{L}^\infty}\leq C\Big\{1+\|\nabla\times f\|_{\dot{\mathrm{B}}^0_{\infty,\infty}}\left(1+\log_e\left(e+\|f\|_{\mathbb{H}^s}\right)\right)\Big\},\ s>5/2,\tag{1.6}$$

where $\dot{\mathrm{B}}^0_{\infty,\infty}$ is the homogeneous Besov space. In general, the homogeneous Besov space $\dot{\mathrm{B}}^s_{p,q}$, $1\leq p,q\leq\infty$ is defined as follows: Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions and $\mathcal{S}'$ be its dual, i.e., the space of tempered distributions. By introducing the Littlewood-Paley dyadic partition of unity $\phi_j(x)$, for $1\leq p,q\leq\infty$, $s\in\mathbb{R}$, the homogeneous Besov space $\dot{\mathrm{B}}^s_{p,q}$ is defined as

$$\dot{\mathrm{B}}^s_{p,q}=\left\{f\in\mathcal{Z}':\|f\|_{\dot{\mathrm{B}}^s_{p,q}}\equiv\left(\sum_{j=-\infty}^{\infty}2^{jsq}\|\phi_j*f\|^q_{\mathbb{L}^p}\right)^{1/q}<+\infty\right\},$$

where $\mathcal{Z}'$ can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomial space $\mathcal{P}$ (see [4, 21, 24] for more details). By the embedding of $\mathrm{BMO}\subset\dot{\mathrm{B}}^0_{\infty,\infty}$ and using (1.6), the authors in [9] extended the regularity criterion in (1.5) as

$$\omega\in\mathrm{L}^1([0,T];\dot{\mathrm{B}}^0_{\infty,\infty}).\tag{1.7}$$

In [1], the three dimensional Navier-Stokes equations with hereditary viscosity are rigorously studied and the finite speed propagation property of the vorticity field is also established. With the above motivations, we try to examine a blow-up criterion for the 3D Navier-Stokes equations with hereditary viscosity, which easily covers linear viscoelastic fluid flow models like Maxwell's fluid flow equations. The solvability results and blow-up criterion of such models obtained in this work suggests us that these models are more close to Euler equations rather than Navier-Stokes equations.

1.1. **Main results.** The main results obtained in this paper are summarized as follows:

Theorem 1.1. *1. Suppose the initial velocity* $\mathbf{u}_0 \in \mathbb{V}_s$*, for* $s > 5/2$ *with* $\|\mathbf{u}_0\|_{\mathbb{V}_s} \leq N_0$*, for some* $N_0 > 0$*. Then there exists a time* $\widetilde{T}$ *depending only on* N_0 *such that the system (2.9) (see below) has a* unique smooth solution *in the class*

$$\mathbf{u} \in \mathrm{C}([0,T];\mathbb{V}_s) \cap \mathrm{C}^1(0,T;\mathbb{V}_{s-2}), \tag{1.8}$$

at least for $T = \widetilde{T}(N_0)$*.*

2. For the solution $\mathbf{u}(\cdot)$ *of (2.9), suppose there are constants* L_0 *and* T^* *so that on any interval* $[0,T]$ *of existence of the solution in class (1.8), with* $T < T^*$*, the vorticity satisfies the a-priori estimate:*

$$\int_0^{T^*} \|\omega(t)\|_{\dot{\mathrm{B}}^0_{\infty,\infty}} \mathrm{d}t \leq L_0.$$

Then the solution can be continued in the class

$$\mathrm{C}([0,T];\mathbb{V}_s) \cap \mathrm{C}^1(0,T;\mathbb{V}_{s-2}),$$

to the interval $[0,T^*]$*.*

1.2. **Layout of the paper.** The organization of the paper is as follows: In the next section, we give the mathematical formulation of the 3D Navier-Stokes equations with hereditary viscosity and describe properties of the kernel. The existence and uniqueness of smooth solutions to the system via a frequency truncation method is obtained in section 3. The Beale-Kato-Majda blow-up criterion of smooth solutions in various spaces is established in section 4.

2. Navier-Stokes equations with hereditary viscosity

In this section, we describe the three dimensional Navier-Stokes equations with hereditary viscosity and discuss about the functional spaces needed to obtain the unique local solvability results.

In the theory of viscoelastic fluids, one assumes that the stress tensor is represented as $\mathrm{T} = -p\mathrm{I} + \mathcal{F}(\mathrm{G}(s))$, where $\mathrm{G}(s)$ is the history of the strain tensor, $p : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ is the fluid pressure and I is the identity matrix. We use this constitutive relationship to obtain the momentum equation as

$$\left.\begin{aligned}\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}&=-\nabla p+\nabla\cdot\mathcal{F}(\mathrm{G}(s))+\mathbf{f}\ \text{ in } \mathbb{R}^3\times(0,T),\\ \nabla\cdot\mathbf{u}&=0\ \text{ in } \mathbb{R}^3\times(0,T),\end{aligned}\right\} \tag{2.1}$$

where $\mathbf{u}=(u_1,u_2,u_3):\mathbb{R}^3\times[0,T]\to\mathbb{R}^3$ is the fluid velocity and $\mathbf{f}=(f_1,f_2,f_3)$ is the external forcing. The functional $\mathcal{F}$ may be represented in terms of multiple integrals over polynomials in the histories G (see [1]) and by restricting to the representation to first order linear terms, we obtain

$$\mathcal{F}(\mathrm{G})=2\int_0^\infty a(s)\mathrm{G}(s)\mathrm{d}s=\int_0^\infty a(s)(\nabla_x\mathbf{u}(x,t-s)+(\nabla_x\mathbf{u}(x,t-s))^\top)\mathrm{d}s,$$

where the kernel $a(\cdot)$ has the properties given in (2.10) and $(\cdot)^\top$ denotes the transpose. Thus we consider the system (2.1) in the form:

$$\left.\begin{aligned}\frac{\partial \mathbf{u}(x,t)}{\partial t}+(\mathbf{u}(x,t)\cdot\nabla)\mathbf{u}(x,t)-\int_{-\infty}^t a(t-s)\Delta\mathbf{u}(x,s)\mathrm{d}s&=-\nabla p(x,t)+\mathbf{f}(x,t)\\ &\quad\text{in } \mathbb{R}^3\times\mathbb{R},\\ \nabla\cdot\mathbf{u}(x,t)&=0\ \text{ in } \mathbb{R}^3\times\mathbb{R},\\ \mathbf{u}(x,s)&=\mathbf{u}^0(x,s)\\ &\quad\text{in } \mathbb{R}^3\times(-\infty,0),\\ \mathbf{u}(x,0)&=\mathbf{u}_0(x)\ \text{ in } \mathbb{R}^3.\end{aligned}\right\} \tag{2.2}$$

Without loss of generality, we take $\mathbf{u}^0(x,s)\equiv 0$ for all $(x,s)\in\mathbb{R}^3\times(-\infty,0)$ and $\mathbf{f}(x,t)\equiv 0$, for all $(x,t)\in\mathbb{R}^3\times(0,\infty)$. We define the following function spaces:

$$\mathbb{H}:=\Big\{\mathbf{u}\in\mathbb{L}^2(\mathbb{R}^3):\nabla\cdot\mathbf{u}=0\Big\}, \tag{2.3}$$

with the norm denoted by $\|\cdot\|_{\mathbb{H}}$ and inner product by $(\cdot,\cdot)_{\mathbb{H}}$, and

$$\mathbb{V}_s:=\Big\{\mathbf{u}\in\mathbb{H}^s(\mathbb{R}^3):\nabla\cdot\mathbf{u}=0\Big\},$$

with the Hilbertian norm $\|\mathbf{u}\|_{\mathbb{V}_s}=\|(\mathrm{I}-\Delta)^{s/2}\mathbf{u}\|_{\mathbb{H}}=:\|\mathrm{J}^s\mathbf{u}\|_{\mathbb{H}}$ and the inner product $(\mathbf{u},\mathbf{v})_{\mathbb{V}_s}=(\mathrm{J}^s\mathbf{u},\mathrm{J}^s\mathbf{v})_{\mathbb{H}}$.

Remark 2.1. The following properties of the Sobolev space $\mathbb{V}_s(\mathbb{R}^n)$ are used in the paper frequently.

1. For $s>\dfrac{n}{2}+k$, the space $\mathbb{V}_s(\mathbb{R}^n)\subset\mathrm{C}^k(\mathbb{R}^n)$, and $\|f\|_{\mathrm{C}^k}\leq C\|f\|_{\mathbb{V}_s}$, where $\|f\|_{\mathrm{C}^k}:=\sup\limits_{x\in\mathbb{R}^n}\{|f|,|\nabla f|,\cdots,|\nabla^k f|\}$, $k=0,1,2,\ldots$.
2. For $s>n/2$, $\mathbb{V}_s(\mathbb{R}^n)$ is an algebra, i.e., $\|fg\|_{\mathbb{V}_s}\leq\|f\|_{\mathbb{V}_s}\|g\|_{\mathbb{V}_s}$, for $s>n/2$ and $f,g\in\mathbb{V}_s(\mathbb{R}^n)$.
3. *Sobolev interpolation inequality.* For $0<s'<s$ and $f\in\mathbb{V}_s(\mathbb{R}^n)$, we have

$$\|f\|_{\mathbb{V}_{s'}}\leq\|f\|_{\mathbb{H}}^{1-s'/s}\|f\|_{\mathbb{V}_s}^{s'/s}.$$

2.1. **Linear and nonlinear operators.** Let us define the Stokes operator

$$\mathrm{A}\mathbf{u} = -\mathrm{P}_{\mathbb{H}}\Delta\mathbf{u}, \text{ for all } \mathbf{u} \in \mathrm{D}(\mathrm{A}) = \mathbb{H}^2(\mathbb{R}^3) \cap \mathbb{V}_1, \tag{2.4}$$

where $\mathrm{P}_{\mathbb{H}}$ is the the Helmholtz-Hodge orthogonal projection from $\mathbb{L}^2$ onto $\mathbb{H}$ (see [12] for more details).

We define the bilinear operator $\mathrm{B} : \mathrm{D}(\mathrm{B}) \subset \mathbb{H}\times\mathbb{V}_1 \to \mathbb{H}$ by $\mathrm{B}(\mathbf{u},\mathbf{v}) = \mathrm{P}_{\mathbb{H}}(\mathbf{u}\cdot\nabla)\mathbf{v}$, with $\mathrm{B}(\mathbf{u}) = \mathrm{B}(\mathbf{u},\mathbf{u})$. Moreover, for any $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{V}_1$, we have

$$(\mathrm{B}(\mathbf{u},\mathbf{v}),\mathbf{w})_{\mathbb{H}} = \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i\frac{\partial v_j}{\partial x_i}w_j\mathrm{d}x = -\left(\mathrm{B}(\mathbf{u},\mathbf{w}),\mathbf{v}\right)_{\mathbb{H}} \text{ and } (\mathrm{B}(\mathbf{u},\mathbf{v}),\mathbf{v})_{\mathbb{H}} = 0. \tag{2.5}$$

Using Hölder's inequality, for $\mathbf{u},\mathbf{v} \in \mathbb{V}_s$ and all $\mathbf{w} \in \mathbb{H}$, we have

$$|\left(\mathrm{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\right)_{\mathbb{H}}| \leq \|\mathrm{B}(\mathbf{u},\mathbf{v})\|_{\mathbb{H}}\|\mathbf{w}\|_{\mathbb{H}} \leq \|\mathbf{u}\|_{\mathbb{H}}\|\nabla\mathbf{v}\|_{\mathbb{L}^\infty}\|\mathbf{w}\|_{\mathbb{H}},$$

and hence

$$\|\mathrm{B}(\mathbf{u},\mathbf{v})\|_{\mathbb{H}} \leq \|\mathbf{u}\|_{\mathbb{H}}\|\nabla\mathbf{v}\|_{\mathbb{L}^\infty} \leq C\|\mathbf{u}\|_{\mathbb{V}_s}\|\mathbf{v}\|_{\mathbb{V}_s}, \quad \text{for} \quad s > 5/2.$$

For more details and properties of the operators A and $\mathrm{B}(\cdot)$, we refer the readers to [23].

Let us now recall the Kato-Ponce commutator estimate used in this paper.

Lemma 2.2 (Lemma XI, [6])**.** *If $s \geq 0$ and $1 < p < \infty$, then*

$$\|\mathrm{J}^s(fg) - f(\mathrm{J}^s g)\|_{\mathbb{L}^p} \leq C_p\left(\|\nabla f\|_{\mathbb{L}^\infty}\|\mathrm{J}^{s-1}g\|_{\mathbb{L}^p} + \|\mathrm{J}^s f\|_{\mathbb{L}^p}\|g\|_{\mathbb{L}^\infty}\right). \tag{2.6}$$

For $p = 2$, Lemma 2.2 implies

$$\|\mathrm{J}^s\left[\mathrm{B}(\mathbf{u},\mathbf{v})\right] - \mathrm{B}(\mathbf{u},\mathrm{J}^s\mathbf{v})\|_{\mathbb{H}} \leq C\left(\|\nabla\mathbf{u}\|_{\mathbb{L}^\infty}\|\mathbf{v}\|_{\mathbb{V}_s} + \|\mathbf{u}\|_{\mathbb{V}_s}\|\nabla\mathbf{v}\|_{\mathbb{L}^\infty}\right). \tag{2.7}$$

The divergence free condition yields $\left(\mathrm{B}\left(\mathbf{u},\mathrm{J}^s\mathbf{u}\right),\mathrm{J}^s\mathbf{u}\right)_{\mathbb{H}} = 0$ and hence we have

$$\begin{aligned}(\mathrm{J}^s\mathrm{B}(\mathbf{u},\mathbf{u}),\mathrm{J}^s\mathbf{u})_{\mathbb{H}} &= (\mathrm{J}^s\mathrm{B}(\mathbf{u},\mathbf{u}) - \mathrm{B}(\mathbf{u},\mathrm{J}^s\mathbf{u}),\mathrm{J}^s\mathbf{u})_{\mathbb{H}}\\ &\leq \|\mathrm{J}^s\mathrm{B}(\mathbf{u},\mathbf{u}) - \mathrm{B}(\mathbf{u},\mathrm{J}^s\mathbf{u})\|_{\mathbb{H}}\|\mathrm{J}^s\mathbf{u}\|_{\mathbb{H}}\\ &\leq C\|\nabla\mathbf{u}\|_{\mathbb{L}^\infty}\|\mathbf{u}\|_{\mathbb{V}_s}^2. \end{aligned}\tag{2.8}$$

2.2. **Abstract formulation.** Under the above functional setting, the system (2.2) (after taking the orthogonal projection $\mathrm{P}_{\mathbb{H}}$) can be written in the abstract form as

$$\left.\begin{aligned}\frac{\mathrm{d}\mathbf{u}(t)}{\mathrm{d}t} + \mathrm{B}(\mathbf{u}(t)) + (a*\mathrm{A}\mathbf{u})(t) &= \mathbf{0},\\ \mathbf{u}(0) &= \mathbf{u}_0,\end{aligned}\right\} \tag{2.9}$$

where $(a*\mathrm{A}\mathbf{u})(t) = \displaystyle\int_0^t a(t-s)\mathrm{A}\mathbf{u}(s)\mathrm{d}s$. In this paper, we assume that the non-constant kernel $a(\cdot)$ satisfies the following conditions:

$$\left.\begin{aligned}&a \in \mathrm{C}^2(\mathbb{R}^+) \cap \mathrm{C}[0,\infty),\\ &(-1)^k a^{(k)}(t) \geq 0, \text{ for all } t > 0, k = 0,1,2.\end{aligned}\right\} \tag{2.10}$$

We also take $\mathbf{u}_0 \in \mathbb{V}_s$, for all $s > 5/2$.

2.3. **Properties of the kernel.** Let us define

$$(L\mathbf{u})(t) := (a * \mathbf{u})(t) = \int_0^t a(t-s)\mathbf{u}(s)\mathrm{d}s.$$

A function $a(\cdot)$ is called *positive kernel* if the operator L is positive on $\mathrm{L}^2([0,T];\mathbb{H})$ for all T. That is,

$$\int_0^T (L\mathbf{u}(t),\mathbf{u}(t))\mathrm{d}t = \int_0^T\int_0^t a(t-s)(\mathbf{u}(s),\mathbf{u}(t))\mathrm{d}s\mathrm{d}t \geq 0,$$
$$\text{for all } \mathbf{u}\in\mathbb{H} \text{ and every } T>0.$$

Let $\widehat{a}(\theta)$ be the *Laplace transform* of $a(t)$, i.e.,

$$\widehat{a}(\theta) = \int_0^\infty e^{-\theta r}a(r)\mathrm{d}r,\ \theta\in\mathbb{C}.$$

Then, we have

Lemma 2.3 (Lemma 4.1, [2])**.** *Let $a\in\mathrm{L}^\infty(0,\infty)$ be such that*

$$Re\ \widehat{a}(\theta)>0, \quad \textit{if} \quad Re\ \theta>0.$$

Then, $a(t)$ defines a positive kernel.

Also, $a(\cdot)$ is said to be a *strongly positive kernel* if there exists constants $\varepsilon>0$ and $\alpha>0$ such that $a(t)-\varepsilon e^{-\alpha t}$ is a positive kernel, that is

$$\int_0^T\int_0^t a(t-s)(\mathbf{u}(s),\mathbf{u}(s))\mathrm{d}s\mathrm{d}t \geq \varepsilon\int_0^T\int_0^t e^{-\alpha(t-s)}(\mathbf{u}(s),\mathbf{u}(s))\mathrm{d}s\mathrm{d}t\geq 0, \tag{2.11}$$

for all $\mathbf{u}\in\mathrm{L}^2([0,T];\mathbb{H})$.

Lemma 2.4 (Proposition 4.1, [2])**.** *Let $a(t)$ satisfy the following conditions:*

(i) $a\in\mathrm{C}[0,\infty)\cap\mathrm{C}^2(0,\infty)$,
(ii) $(-1)^k\dfrac{\mathrm{d}^k}{\mathrm{d}t^k}a(t)\geq 0$, *for* $t>0, k=0,1,2$,
(iii) $a(t)\neq$ *constant.*

Then $a(t)$ is a strongly positive kernel.

We recall (see for e.g. Proposition 1.3.3, [18]) that this condition implies the integral operator $y\mapsto a*y$ is positive in the space $\mathrm{L}^2([0,T];\mathbb{X})$, where $\mathbb{X}$ is an arbitrary real Hilbert space with the scalar product $(\cdot,\cdot)_{\mathbb{X}}$ and the norm $\|\cdot\|_{\mathbb{X}}$. More precisely, we have

$$\int_0^T\left(y(t),\int_0^t a(t-s)y(s)\mathrm{d}s\right)_{\mathbb{X}}\mathrm{d}t\geq 0, \text{ for all } y\in\mathrm{L}^2([0,T];\mathbb{X}). \tag{2.12}$$

Remark 2.5. 1. For any $\mathbf{v}\in\mathrm{L}^2([0,T];\mathbb{V}_s)$, using integration by parts and boundary conditions, it can be easily seen that

$$\int_0^T\left(\mathbf{v}(t),\int_0^t a(t-s)\mathrm{A}\mathbf{v}(s)\mathrm{d}s\right)_{\mathbb{H}}\mathrm{d}t = \int_0^T\left(\nabla\mathbf{v}(t),\int_0^t a(t-s)\nabla\mathbf{v}(s)\mathrm{d}s\right)_{\mathbb{H}}\mathrm{d}t\geq 0. \tag{2.13}$$

2. Even though we are assuming that $a(\cdot)$ is strongly positive (see (2.10)), we only exploit (2.13) in the proofs and hence the results obtained in the paper are also true for positive kernels.

2.4. **The vorticity equation.** Taking curl in the first equation in (2.1), we get the vorticity $\omega = \nabla \times \mathbf{u}$ equation as

$$\left.\begin{aligned} \frac{\partial \omega(t)}{\partial t} + (\mathbf{u}(t)\cdot\omega(t)) - (\omega(t)\cdot\nabla)\mathbf{u}(t) = (a * \Delta\omega)(t), \\ \omega(0) = \omega_0, \end{aligned}\right\} \tag{2.14}$$

where $\omega_0 = \text{curl } \mathbf{u}_0 = \nabla\times\mathbf{u}_0$. Note that $\nabla\times\nabla p = 0$ and since $\omega = \nabla\times\mathbf{u}$, $\nabla\cdot\omega = 0$. The derivative of velocity $\mathbf{u}(\cdot)$ is described by the vorticity $\omega(\cdot)$ through the singular integral operator (Biot-Savart law) as

$$\nabla\mathbf{u} = (-\Delta)^{-1}\nabla(\nabla\times\omega) \quad \text{or} \quad \mathbf{u}(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-y)\times\omega(y,t)}{|x-y|^3}\mathrm{d}y. \tag{2.15}$$

Remark 2.6. There exists a constant C independent of $\mathbf{u}$ such that

$$\|\nabla\mathbf{u}\|_{\mathbb{L}^p} \leq C\|\omega\|_{\mathbb{L}^p},\ 1 < p < \infty. \tag{2.16}$$

The inequality (2.16) can be derived from the Biot-Savart law (see [11]) and the bounds of the Riesz transforms on $\mathbb{L}^p(1 < p < \infty)$ (see [22]).

2.5. **Maxwell's fluid flow.** Let us now give an example, which motivated us to consider the system like (2.9). A linear viscoelastic fluid with a finite discretely distributed relaxation times $\{\lambda_l\}$ and retardation times $\{\kappa_m\}$ is a fluid whose defining (or rheological) equation, connecting the deviator of the stress tensor σ and the strain tensor D has the form:

$$\left(1 + \sum_{l=1}^{L}\lambda_l\frac{\partial^l}{\partial t^l}\right)\sigma = 2\nu\left(1 + \sum_{m=1}^{M}\kappa_m\nu^{-1}\frac{\partial^m}{\partial t^m}\right)\mathrm{D}, \quad \text{where } \nu,\ \lambda_L,\ \kappa_M > 0, \tag{2.17}$$

and the numbers L and M are connected by the relation $M = L-1$, $L = 1, 2, \ldots$. We call such fluid flows as the *Maxwell's fluid flow* of order L. We assume that the relaxation times $\{\lambda_l\}$ satisfy the following conditions: the roots $\{\alpha_l\}$ of the polynomial

$$\mathrm{Q}(p) = 1 + \sum_{l=1}^{L}\lambda_l p^l$$

are distinct, i.e., $\mathrm{Q}'(\alpha_l) \neq 0$, for all $l = 1,\ldots,L$, real, negative: $\alpha_l < 0$, for all $l = 1,\ldots,L$ and, in addition, the relaxation times $\{\lambda_l\}$, the viscosity coefficient ν, and the delay times $\{\kappa_m\}$ satisfy the following conditions:

$$a_l = \nu\mathrm{P}(\alpha_l)[\mathrm{Q}'(\alpha_l)]^{-1} > 0,\ l = 1,\ldots,L, \tag{2.18}$$

where for the Maxwell's fluid, we have $\mathrm{P}(p) = 1 + \sum\limits_{m=1}^{M}\kappa_m\nu^{-1}p^m$.

Without loss of generality, we assume that $\sigma(x,t)=0$, for $t<0$ and we apply to (2.17) the Laplace transform $\mathcal{L}$ with respect to the variable t. Then we obtain

$$\sigma(x,t) \equiv \int_0^t \mathrm{G}(t-\tau)\mathrm{D}(x,\tau)\mathrm{d}\tau, \tag{2.19}$$

and moreover

$$\mathrm{G}(t) = 2\nu\mathcal{L}^{-1}\left\{\left(1+\sum_{m=1}^{M}\kappa_m\nu^{-1}p^m\right)\mathrm{Q}^{-1}(p)\right\}. \tag{2.20}$$

From (2.19) and (2.20), under the conditions given in (2.18), we obtain that for Maxwell's fluids of order $L=1,2,\ldots$, we have

$$\sigma(x,t) = \sum_{m=1}^{L} a_m \int_0^t e^{\alpha_m(t-\tau)}\mathrm{D}(x,\tau)\mathrm{d}\tau. \tag{2.21}$$

Introducing (2.21) into the equations of motion of a continuous incompressible medium in the Cauchy form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nabla\cdot\sigma + \mathbf{f}(x,t),\ \text{div } \mathbf{u} = 0,$$

we obtain the integrodifferential equations of the motion of Maxwell's fluids of order L as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \sum_{m=1}^{L} a_m \int_0^t e^{\alpha_m(t-\tau)}\Delta\mathbf{u}(\tau)\mathrm{d}\tau + \nabla p = \mathbf{f},\ \nabla\cdot\mathbf{u} = 0.$$

The Maxwell fluid of order one can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \gamma\int_0^t e^{-\delta(t-\tau)}\Delta\mathbf{u}(\tau)\mathrm{d}\tau + \nabla p = \mathbf{f},\ \nabla\cdot\mathbf{u} = 0, \tag{2.22}$$

where $\gamma = \frac{2\nu}{\lambda} > 0$ and $\delta = \frac{1}{\lambda} > 0$ so that $a(t) = \gamma e^{-\delta t}$. If we define $\mathbf{v}(t) := \int_0^t e^{-\delta(t-s)}\mathbf{u}(s)\mathrm{d}s$, then the system (2.22) can be written as

$$\left\{\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \gamma\Delta\mathbf{v} + \nabla p = \mathbf{f},\\ &\qquad\frac{\partial \mathbf{v}}{\partial t} - \mathbf{u} + \delta\mathbf{v} = \mathbf{0},\ \nabla\cdot\mathbf{u} = 0. \end{aligned}\right. \tag{2.23}$$

For the kernel $a(t) = \gamma e^{-\delta t}$, we know that

$$\int_0^\infty a(t)\mathrm{d}t = \frac{\gamma}{\delta}\ \text{ and }\ \widehat{a}(\theta) = \frac{\gamma}{\theta+\delta} > 0\ \text{ for }\ \mathrm{Re}\,\theta > 0,$$

and by Lemma 2.3, $a(t)$ is a positive kernel. Hence, we have

$$\int_0^T ((a*\mathrm{A})\mathbf{u}(t),\mathbf{u}(t))\mathrm{d}t = \int_0^T ((a*\nabla\mathbf{u})(t),\nabla\mathbf{u}(t))\mathrm{d}t \geq 0.$$

Also, we know that

$$a(t) = \gamma e^{-\delta t} > 0, a'(t) = -\gamma\delta e^{-\delta t} < 0 \text{ and } a''(t) = \gamma\delta^2 e^{-\delta t} > 0,$$

and by thus Lemma 2.4, $a(t)$ is a strongly positive kernel.

3. Local existence and uniqueness

In this section, we establish the local existence and uniqueness of smooth solutions to the system (2.2) using a frequency truncation method. Interested readers are referred to see [5, 12–15], etc for the local solvability of various systems using this method. The local solvability results are already known due to [1], in which the authors used a suitable intermediate m-accretive quantization of the nonlinear term to get the local existence. The stochastic counter part of this problem is considered in [14].

3.1. **The truncated system.**

Let us define the Fourier truncation $\mathcal{S}_R$ (see Fefferman et.al. [5]) as follows:

$$\widehat{\mathcal{S}_R f}(\xi) = \mathbb{1}_{\mathrm{B}_R}(\xi)\widehat{f}(\xi),$$

where B_R, a ball of radius $R>0$ centered at the origin and $\mathbb{1}_{\mathrm{B}_R}(\cdot)$ is the indicator function. For $s \geq 0$, we have

$$\left.\begin{aligned}\|\mathcal{S}_R f\|_{\mathbb{V}_s} &\leq C\|f\|_{\mathbb{V}_s},\\ \|\mathcal{S}_R f - f\|_{\mathbb{V}_s} &\leq C\left(\frac{1}{R}\right)^k \|f\|_{\mathbb{V}_{s+k}},\\ \|(\mathcal{S}_R - \mathcal{S}_{R'})f\|_{\mathbb{V}_s} &\leq C\max\left\{\left(\frac{1}{R}\right)^k, \left(\frac{1}{R'}\right)^k\right\}\|f\|_{\mathbb{V}_{s+k}},\end{aligned}\right\} \tag{3.1}$$

where C is a generic constant independent of R. Let us consider the truncated (in the frequency domain with cut off $\mathcal{S}_R$) 3D Navier-Stokes equation with hereditary viscosity in the whole space $\mathbb{R}^3$ as

$$\left.\begin{aligned}\frac{\partial \mathbf{u}^R(x,t)}{\partial t} + \mathcal{S}_R(\mathbf{u}^R(x,t)\cdot\nabla)\mathbf{u}^R(x,t) - \left(a * \Delta\mathbf{u}^R\right)(x,t) &= -\nabla p^R(x,t),\\ \nabla\cdot\mathbf{u}^R(x,t) &= 0,\\ \mathbf{u}^R(x,0) &= \mathcal{S}_R\mathbf{u}_0(x),\end{aligned}\right\} \tag{3.2}$$

for $(x,t)\in\mathbb{R}^3\times[0,T]$. By taking the truncated initial data, we ensure that $\mathbf{u}^R$ lies in the space

$$\mathscr{H}_R := \left\{f\in\mathbb{V}_s : \widehat{f} \text{ is supporthed in } \mathrm{B}_R\right\}.$$

On the space $\mathscr{H}_R$, we have $\mathcal{S}_R\mathbf{u}^R = \mathbf{u}^R$ and it is easy to show that $\mathcal{S}_R\left(\left(\mathbf{u}^R\cdot\nabla\right)\mathbf{u}^R\right)$ is locally Lipschitz in $\mathbf{u}^R$. Hence, by Picard's theorem for infinite-dimensional ODEs (see Theorem 3.1, [11]), there exists a solution $\mathbf{u}^R$ in $\mathscr{H}_R$ to (3.2) for some time interval $[0,T(R)]$. The solution exists as long as $\|\mathbf{u}^R\|_{\mathbb{V}_s}$ remains finite. We take the orthogonal projection $\mathrm{P}_{\mathbb{H}}$ on (3.2) to find

$$\left.\begin{aligned}\frac{\mathrm{d}\mathbf{u}^R(t)}{\mathrm{d}t} + \mathcal{S}_R\mathrm{B}\left(\mathbf{u}^R(t)\right) + (a*\mathrm{A}\mathbf{u}^R)(t) &= 0,\\ \mathbf{u}^R(0) &= \mathcal{S}_R\mathbf{u}_0,\end{aligned}\right\} \tag{3.3}$$

where we also used the fact that the operators $\mathcal{S}_R$ and $\mathrm{P}_{\mathbb{H}}$ commute. Using the divergence free condition, it can be easily seen that

$$(3.4)\qquad \left(\mathcal{S}_R\mathrm{B}\left(\mathbf{u}^R\right),\mathbf{u}^R\right)_{\mathbb{H}} = \left(\mathrm{B}\left(\mathbf{u}^R\right),\mathcal{S}_R\mathbf{u}^R\right)_{\mathbb{H}} = \left(\mathrm{B}\left(\mathbf{u}^R\right),\mathbf{u}^R\right)_{\mathbb{H}} = 0.$$

We take inner product with $\mathbf{u}^R(\cdot)$ in (3.3) and integrate it from 0 to t to obtain

$$(3.5)\qquad \|\mathbf{u}^R(t)\|_{\mathbb{H}}^2 = \|\mathbf{u}^R(0)\|_{\mathbb{H}}^2 - \int_0^t \left(\left(a*\mathrm{A}\mathbf{u}^R\right)(s),\mathbf{u}^R(s)\right)_{\mathbb{H}}\mathrm{d}s.$$

Using (2.13), (3.4) and (3.1), we get $\|\mathbf{u}^R(t)\|_{\mathbb{H}} \leq \|\mathbf{u}_0\|_{\mathbb{H}}$, for all $t\in[0,T]$. The next proposition gives a higher order a-priori estimate for the system (3.3).

Proposition 3.1. *Given initial data $\mathbf{u}_0\in\mathbb{V}_s$ with $s>5/2$, there exists a time $\widetilde{T}$ such that $\|\mathbf{u}^R(t)\|_{\mathbb{V}_s}$ is bounded uniformly on $[0,\widetilde{T}]$ and the bound is independent of R.*

Proof. Let us apply J^s on (3.3) to get

$$(3.6)\qquad \frac{\mathrm{d}\mathrm{J}^s\mathbf{u}^R(t)}{\mathrm{d}t} + \mathcal{S}_R\mathrm{J}^s\mathrm{B}\left(\mathbf{u}^R(t)\right) + \mathrm{J}^s(a*\mathrm{A}\mathbf{u}^R)(t) = \mathbf{0},$$

since J^s and $\mathcal{S}_R$ commute (see [12]). We now take inner product with $\mathrm{J}^s\mathbf{u}^R(\cdot)$ in (3.6) and integrate it from 0 to t to obtain

$$(3.7)\qquad \begin{aligned}\|\mathbf{u}^R(t)\|_{\mathbb{V}_s}^2 = \|\mathbf{u}^R(0)\|_{\mathbb{V}_s}^2 &- \int_0^t \left(\mathcal{S}_R\mathrm{B}\left(\mathbf{u}^R(s)\right),\mathbf{u}^R(s)\right)_{\mathbb{V}_s}\mathrm{d}s\\ &- \int_0^t \left((a*\mathrm{A}\mathbf{u}^R)(s),\mathbf{u}^R(s)\right)_{\mathbb{V}_s}\mathrm{d}s.\end{aligned}$$

Using the Cauchy-Schwarz inequality and (2.8), we have

$$(3.8)\qquad \left|\left(\mathcal{S}_R\mathrm{B}\left(\mathbf{u}^R\right),\mathbf{u}^R\right)_{\mathbb{V}_s}\right| \leq C\|\nabla\mathbf{u}^R\|_{\mathbb{L}^\infty}\|\mathbf{u}^R\|_{\mathbb{V}_s}^2 \leq C\|\mathbf{u}^R\|_{\mathbb{V}_s}^3,$$

for $s>5/2$. Now we use (2.13), (3.1) and (3.8) in (3.7) to get

$$(3.9)\qquad \|\mathbf{u}^R(t)\|_{\mathbb{V}_s}^2 \leq \|\mathbf{u}_0\|_{\mathbb{V}_s}^2 + C\int_0^t \|\mathbf{u}^R(s)\|_{\mathbb{V}_s}^3\mathrm{d}s.$$

Let us set $\mathrm{Y}(t)=\|\mathbf{u}^R(t)\|_{\mathbb{V}_s}^2$ and $\mathrm{Y}(0)=\|\mathbf{u}_0\|_{\mathbb{V}_s}^2$, so that from (3.9), we obtain

$$(3.10)\qquad \sqrt{\mathrm{Y}(t)} \leq \frac{\sqrt{\mathrm{Y}(0)}}{(1-Ct\sqrt{\mathrm{Y}(0)})},$$

for all $t\in[0,T]$. If we choose $\widetilde{T} < \frac{1}{C\sqrt{\mathrm{Y}(0)}}$, then we have

$$(3.11)\qquad \sup_{t\in[0,\widetilde{T}]}\|\mathbf{u}^R(t)\|_{\mathbb{V}_s} \leq \frac{\|\mathbf{u}_0\|_{\mathbb{V}_s}}{1-C\widetilde{T}\|\mathbf{u}_0\|_{\mathbb{V}_s}} = \|\mathbf{u}_0\|_{\mathbb{V}_s} + \frac{\|\mathbf{u}_0\|_{\mathbb{V}_s}^2 C\widetilde{T}}{1-C\widetilde{T}\|\mathbf{u}_0\|_{\mathbb{V}_s}},$$

and hence $\|\mathbf{u}^R\|_{\mathbb{V}_s}$ is bounded uniformly on $[0,\widetilde{T}]$ and the bound is independent of R. □

Proposition 3.2. *The family $\{\mathbf{u}^R(\cdot)\}$ of solutions of (3.3) are Cauchy (as $R\to\infty$) in $\mathrm{L}^\infty([0,\widetilde{T}];\mathbb{H})$.*

Proof. We consider the equation (3.3) and take the difference between the equations for R and R' to obtain ($R' \geq R$)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}^R - \mathbf{u}^{R'})(t) + (\mathcal{S}_R \mathrm{B}(\mathbf{u}^R(t)) - \mathcal{S}_{R'}\mathrm{B}(\mathbf{u}^{R'}(t))) + (a * \mathrm{A}(\mathbf{u}^R - \mathbf{u}^{R'}))(t) = \mathbf{0}. \tag{3.12}$$

Let us take inner product with $\mathbf{u}^R - \mathbf{u}^{R'}$ in (3.12) and then integrate from 0 to t to obtain

$$\begin{aligned} \|(\mathbf{u}^R - \mathbf{u}^{R'})(t)\|_{\mathbb{H}}^2 &= \|(\mathbf{u}^R - \mathbf{u}^{R'})(0)\|_{\mathbb{H}}^2 \\ &\quad - 2\int_0^t ((\mathcal{S}_R \mathrm{B}(\mathbf{u}^R(s)) - \mathcal{S}_{R'}\mathrm{B}(\mathbf{u}^{R'}(s))), (\mathbf{u}^R - \mathbf{u}^{R'})(s))_{\mathbb{H}} \mathrm{d}s \\ &\quad - 2\int_0^t ((a * \mathrm{A}(\mathbf{u}^R - \mathbf{u}^{R'}))(s), (\mathbf{u}^R - \mathbf{u}^{R'})(s))_{\mathbb{H}} \mathrm{d}s. \end{aligned} \tag{3.13}$$

For $0 < \varepsilon < 1$, we use (3.1) to find

$$\|(\mathbf{u}^R - \mathbf{u}^{R'})(0)\|_{\mathbb{H}} = \|(\mathcal{S}_R - \mathcal{S}_{R'})\mathbf{u}_0\|_{\mathbb{H}} \leq \frac{C}{R^\varepsilon}\|\mathbf{u}_0\|_{\mathbb{V}_\varepsilon} \leq \frac{C}{R^\varepsilon}\|\mathbf{u}_0\|_{\mathbb{V}_s}. \tag{3.14}$$

A rearrangement gives us

$$\begin{aligned} &(\mathcal{S}_R \mathrm{B}(\mathbf{u}^R) - \mathcal{S}_{R'}\mathrm{B}(\mathbf{u}^{R'}), \mathbf{u}^R - \mathbf{u}^{R'})_{\mathbb{H}} \\ &= ((\mathcal{S}_R - \mathcal{S}_{R'})\mathrm{B}(\mathbf{u}^R), \mathbf{u}^R - \mathbf{u}^{R'})_{\mathbb{H}} + (\mathcal{S}_{R'}(\mathrm{B}(\mathbf{u}^R) - \mathrm{B}(\mathbf{u}^{R'})), \mathbf{u}^R - \mathbf{u}^{R'})_{\mathbb{H}} \\ &= ((\mathcal{S}_R - \mathcal{S}_{R'})\mathrm{B}(\mathbf{u}^R), \mathbf{u}^R - \mathbf{u}^{R'})_{\mathbb{H}} + (\mathcal{S}_{R'}\mathrm{B}(\mathbf{u}^R - \mathbf{u}^{R'}, \mathbf{u}^R), \mathbf{u}^R - \mathbf{u}^{R'})_{\mathbb{H}}, \end{aligned} \tag{3.15}$$

since $(\mathcal{S}_{R'}\mathrm{B}(\mathbf{u}^{R'}, \mathbf{u}^R - \mathbf{u}^{R'}), \mathbf{u}^R - \mathbf{u}^{R'})_{\mathbb{H}} = 0$. Let us take the first term from the right hand side of the inequality in (3.15) and use the Cauchy-Schwarz inequality, (3.1) and (2.8) to obtain

$$\begin{aligned} \left|((\mathcal{S}_R - \mathcal{S}_{R'})\mathrm{B}(\mathbf{u}^R), \mathbf{u}^R - \mathbf{u}^{R'})_{\mathbb{H}}\right| &\leq \|(\mathcal{S}_R - \mathcal{S}_{R'})\mathrm{B}(\mathbf{u}^R)\|_{\mathbb{H}}\|\mathbf{u}^R - \mathbf{u}^{R'}\|_{\mathbb{H}} \\ &\leq \frac{C}{R^\varepsilon}\|\mathrm{B}(\mathbf{u}^R)\|_{\mathbb{V}_\varepsilon}\|\mathbf{u}^R - \mathbf{u}^{R'}\|_{\mathbb{H}} \\ &\leq \frac{C}{R^\varepsilon}\|(\mathbf{u}^R \cdot \nabla)\mathbf{u}^R\|_{\mathbb{V}_{s-1}}\|\mathbf{u}^R - \mathbf{u}^{R'}\|_{\mathbb{H}} \\ &\leq \frac{C}{R^\varepsilon}\|\mathbf{u}^R\|_{\mathbb{V}_{s-1}}\|\nabla\mathbf{u}^R\|_{\mathbb{V}_{s-1}}\|\mathbf{u}^R - \mathbf{u}^{R'}\|_{\mathbb{H}} \\ &\leq \frac{C}{R^\varepsilon}\|\mathbf{u}^R\|_{\mathbb{V}_s}^2\|\mathbf{u}^R - \mathbf{u}^{R'}\|_{\mathbb{H}}, \end{aligned} \tag{3.16}$$

for $0 < \varepsilon < 1 < s - 1$ and $s > 5/2$. In (3.16), we have also used the fact that $\mathbb{V}_s$ is an algebra for $s > 3/2$. We estimate the second term in the right hand side of the inequality (3.15) using (3.1), Cauchy-Schwarz and Hölder's inequalities as

$$\begin{aligned} \left|(\mathcal{S}_{R'}\mathrm{B}(\mathbf{u}^R - \mathbf{u}^{R'}, \mathbf{u}^R), \mathbf{u}^R - \mathbf{u}^{R'})_{\mathbb{H}}\right| &\leq \|\mathrm{B}(\mathbf{u}^R - \mathbf{u}^{R'}, \mathbf{u}^R)\|_{\mathbb{H}}\|\mathbf{u}^R - \mathbf{u}^{R'}\|_{\mathbb{H}} \\ &\leq C\|\nabla\mathbf{u}^R\|_{\mathbb{L}^\infty}\|\mathbf{u}^R - \mathbf{u}^{R'}\|_{\mathbb{H}}^2 \\ &\leq C\|\mathbf{u}^R\|_{\mathbb{V}_s}\|\mathbf{u}^R - \mathbf{u}^{R'}\|_{\mathbb{H}}^2, \end{aligned} \tag{3.17}$$

for $s > 5/2$. Combining (3.16) and (3.17), substituting in (3.13), and then using (2.13), Young's inequality and Proposition 3.1, we obtain

$$\begin{aligned}\|(\mathbf{u}^R-\mathbf{u}^{R'})(t)\|_{\mathbb{H}}^2 &\leq \frac{C}{R^{2\varepsilon}}\|\mathbf{u}_0\|_{\mathbb{V}_s}^2+\frac{C}{R^{\varepsilon}}\int_0^t\|\mathbf{u}^R(s)\|_{\mathbb{V}_s}^2\|(\mathbf{u}^R-\mathbf{u}^{R'})(s)\|_{\mathbb{H}}\mathrm{d}s\\ &\quad+C\int_0^t\|\mathbf{u}^R(s)\|_{\mathbb{V}_s}\|(\mathbf{u}^R-\mathbf{u}^{R'})(s)\|_{\mathbb{H}}^2\mathrm{d}s\\ &\leq C\left(\frac{1}{R^{2\varepsilon}}\|\mathbf{u}_0\|_{\mathbb{V}_s}^2+\frac{M^2\widetilde{T}}{R^{\varepsilon}}\right)\\ &\quad+CM\left(1+\frac{M}{R^{\varepsilon}}\right)\int_0^t\|(\mathbf{u}^R-\mathbf{u}^{R'})(s)\|_{\mathbb{H}}^2\mathrm{d}s,\end{aligned}\tag{3.18}$$

for $t\in[0,\widetilde{T}]$, where we used

$$\sup_{t\in[0,\widetilde{T}]}\|\mathbf{u}^R(t)\|_{\mathbb{V}_s}\leq M.\tag{3.19}$$

For $0<\varepsilon<1$, an application of Gronwall's inequality in (3.18) yields

$$\sup_{t\in[0,\widetilde{T}]}\|(\mathbf{u}^R-\mathbf{u}^{R'})(t)\|_{\mathbb{H}}^2\leq C\left(\frac{1}{R^{2\varepsilon}}\|\mathbf{u}_0\|_{\mathbb{V}_s}^2+\frac{M^2\widetilde{T}}{R^{\varepsilon}}\right)\exp\left(CM\left(1+\frac{M}{R^{\varepsilon}}\right)\widetilde{T}\right),\tag{3.20}$$

and the right-hand side tends to zero as $R,R'\to\infty$, as required. □

Proposition 3.3. *The family $\{\mathbf{u}^R(\cdot)\}$ of solutions of (3.3) are Cauchy (as $R\to\infty$) in $\mathrm{L}^\infty([0,\widetilde{T}];\mathbb{V}_{s'})$, for any $0<s'<s$, and $\mathbf{u}^R\to\mathbf{u}$ strongly in $\mathrm{L}^\infty([0,\widetilde{T}];\mathbb{V}_{s'})$.*

Proof. It follows from Proposition 3.2 that $\mathbf{u}^R\to\mathbf{u}$ strongly in $\mathrm{L}^\infty([0,\widetilde{T}];\mathbb{H})$. For $0<s'<s$, by using the Sobolev interpolation inequality, and Propositions 3.1 and 3.2, we also have

$$\begin{aligned}\sup_{t\in[0,\widetilde{T}]}\|(\mathbf{u}^R-\mathbf{u}^{R'})(t)\|_{\mathbb{V}_{s'}} &\leq \sup_{t\in[0,\widetilde{T}]}\|(\mathbf{u}^R-\mathbf{u}^{R'})(t)\|_{\mathbb{H}}^{1-s'/s}\sup_{t\in[0,\widetilde{T}]}\|(\mathbf{u}^R-\mathbf{u}^{R'})(t)\|_{\mathbb{V}_s}^{s'/s}\\ &\leq 2M^{s'/s}\sup_{t\in[0,\widetilde{T}]}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{\mathbb{H}}^{1-s'/s}\to 0,\end{aligned}\tag{3.21}$$

as $R,R'\to\infty$. Thus $\mathbf{u}^R\to\mathbf{u}$ strongly in $\mathrm{L}^\infty([0,\widetilde{T}];\mathbb{V}_{s'})$, for $0<s'<s$. □

By the divergence free condition and the algebra property of the $\mathbb{V}^{s'}$-norm, we also have

$$\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|_{\mathbb{V}_{s'-1}}=\|\nabla\cdot(\mathbf{u}\otimes\mathbf{v})\|_{\mathbb{V}_{s'-1}}\leq C\|\mathbf{u}\otimes\mathbf{v}\|_{\mathbb{V}^{s'}}\leq C\|\mathbf{u}\|_{\mathbb{V}_{s'}}\|\mathbf{v}\|_{\mathbb{V}_{s'}},\tag{3.22}$$

for $s'>5/2$.

Proposition 3.4. *For $0<s'<s$, we have*

$$\sup_{t\in[0,\widetilde{T}]}\|\mathcal{S}_R\mathrm{B}(\mathbf{u}^R(t))-\mathrm{B}(\mathbf{u}(t))\|_{\mathbb{V}_{s'-1}}\to 0,\ \textit{as } R\to\infty.\tag{3.23}$$

Proof. Using (2.7), (3.1), (3.22), Propositions 3.2 and 3.3, for $0<\varepsilon<1$, we have

$$\begin{aligned}
&\sup_{t\in[0,\widetilde{T}]}\|\mathcal{S}_R\mathrm{B}(\mathbf{u}^R(t))-\mathrm{B}(\mathbf{u}(t))\|_{\mathbb{V}_{s'-1}}\\
&\leq \sup_{t\in[0,\widetilde{T}]}\|\mathcal{S}_R\mathrm{B}(\mathbf{u}^R(t),(\mathbf{u}^R-\mathbf{u})(t))\|_{\mathbb{V}_{s'-1}}+\sup_{t\in[0,\widetilde{T}]}\|\mathcal{S}_R\mathrm{B}((\mathbf{u}^R-\mathbf{u})(t),\mathbf{u}(t))\|_{\mathbb{V}_{s'-1}}\\
&\quad+\sup_{t\in[0,\widetilde{T}]}\|\mathcal{S}_R\mathrm{B}(\mathbf{u}(t))-\mathrm{B}(\mathbf{u}(t))\|_{\mathbb{V}_{s'-1}}\\
&\leq 2C\sup_{t\in[0,\widetilde{T}]}\|\mathbf{u}^R(t)\|_{\mathbb{V}_{s'}}\sup_{t\in[0,\widetilde{T}]}\|(\mathbf{u}^R-\mathbf{u}^{R'})(t)\|_{\mathbb{V}_{s'}}+\frac{C}{R^\varepsilon}\sup_{t\in[0,\widetilde{T}]}\|\mathrm{B}(\mathbf{u}(t))\|_{\mathbb{V}_{s'-1+\varepsilon}}\\
&\leq 2CM\sup_{t\in[0,\widetilde{T}]}\|(\mathbf{u}^R-\mathbf{u}^{R'})(t)\|_{\mathbb{V}_{s'}}+\frac{CM^2}{R^\varepsilon}\to 0,
\end{aligned}\tag{3.24}$$

as $R\to\infty$. □

Proposition 3.5. *For $0<s'<s$, we have*

$$\sup_{t\in[0,\widetilde{T}]}\|(a*\mathrm{A}(\mathbf{u}^R-\mathbf{u}))(t)\|_{\mathbb{V}_{s'-2}}\to 0,\quad \textit{as}\quad R\to\infty.\tag{3.25}$$

Proof. Using Proposition 3.3, we have

$$\sup_{t\in[0,\widetilde{T}]}\|(a*\mathrm{A}(\mathbf{u}^R-\mathbf{u}))(t)\|_{\mathbb{V}_{s'-2}}\leq C(\widetilde{T})\sup_{t\in[0,\widetilde{T}]}\|(\mathbf{u}^R-\mathbf{u})(t)\|_{\mathbb{V}_{s'}}\to 0,\tag{3.26}$$

as $R\to\infty$. □

For the initial data convergence, by using (3.1), we obtain

$$\|\mathcal{S}_R\mathbf{u}_0-\mathbf{u}_0\|_{\mathbb{V}_{s'}}\leq\frac{C}{R^\varepsilon}\|\mathbf{u}_0\|_{\mathbb{V}_{s'+\varepsilon}}\leq\frac{C}{R^\varepsilon}\|\mathbf{u}_0\|_{\mathbb{V}_s}\to 0,\tag{3.27}$$

as $R\to\infty$. Now it remains to show the convergence of the time derivative. From (3.3), using (3.22), we have

$$\begin{aligned}
\sup_{t\in[0,\widetilde{T}]}\left\|\frac{\mathrm{d}\mathbf{u}^R(t)}{\mathrm{d}t}\right\|_{\mathbb{V}_{s-2}}&\leq\sup_{t\in[0,\widetilde{T}]}\|\mathcal{S}_R\mathrm{B}(\mathbf{u}^R(t))\|_{\mathbb{V}_{s-2}}+\sup_{t\in[0,\widetilde{T}]}\|(a*\mathrm{A}\mathbf{u}^R)(t)\|_{\mathbb{V}_{s-2}}\\
&\leq C\sup_{t\in[0,\widetilde{T}]}\|\mathbf{u}^R(t)\|^2_{\mathbb{V}_{s-1}}+C(\widetilde{T})\sup_{t\in[0,\widetilde{T}]}\|\mathbf{u}^R(t)\|_{\mathbb{V}_s}\\
&\leq C(M)<+\infty.
\end{aligned}\tag{3.28}$$

Thus, we can extract a subsequence $R^m\to+\infty$ such that

$$\frac{\mathrm{d}\mathbf{u}^{R^m}}{\mathrm{d}t}\xrightarrow{w^*}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\text{ in }\mathrm{L}^\infty([0,\widetilde{T}];\mathbb{V}_{s-2}).\tag{3.29}$$

Theorem 3.6 (Local Existence and Uniqueness)**.** *Suppose the initial velocity $\mathbf{u}_0\in\mathbb{V}_s$ for $s>5/2$ with $\|\mathbf{u}_0\|_{\mathbb{V}_s}\leq N_0$, for some $N_0>0$. Then there exists a time $\widetilde{T}$ depending only on N_0 such that the system (2.9) has a unique solution in the class*

$$\mathbf{u}\in\mathrm{C}([0,T];\mathbb{V}_s)\cap\mathrm{C}^1(0,T;\mathbb{V}_{s-2}),\tag{3.30}$$

at least for $T=\widetilde{T}(N_0)$.

Proof. Using the strong convergences discussed in Propositions 3.3, 3.4, and 3.5, we know that the time derivative converges strongly in $\mathrm{L}^{\infty}([0,\widetilde{T}];\mathbb{V}_{s'-2})$ and $\mathbf{u}(\cdot)$ solves (2.9) as an equality in $\mathrm{L}^{\infty}([0,\widetilde{T}];\mathbb{V}_{s'-2})$, for $0<s'<s$. We also know that $\mathrm{L}^{\infty}([0,\widetilde{T}];\mathbb{V}_s)\cong \mathrm{L}^1(0,\widetilde{T};\mathbb{V}_{-s})'$, and $\mathrm{L}^1(0,\widetilde{T};\mathbb{V}_{-s})$ is separable. Using the energy estimates in Proposition 3.2, and along with the Banach-Alaoglu theorem, we can extract a subsequence such that

$$\mathbf{u}^{R^m}\xrightarrow{w^*}\mathbf{u}\ \text{ in }\ \mathrm{L}^{\infty}([0,\widetilde{T}];\mathbb{V}_s), \tag{3.31}$$

which guarantees that the limit satisfies $\mathbf{u}\in \mathrm{L}^{\infty}([0,\widetilde{T}];\mathbb{V}_s)$. Also $\mathbf{u}\in \mathrm{C}_w([0,\widetilde{T}];\mathbb{V}_s)$, that is, $\mathbf{u}$ is continuous in the weak topology of $\mathbb{V}_s$. This can be proved in the following way. Let $\langle\phi,\mathbf{u}\rangle_{\mathbb{V}_{-s}\times\mathbb{V}_s}$ denote the duality pairing of $\mathbb{V}_{-s}$ and $\mathbb{V}_s$ through the $\mathbb{H}$-inner product. Since $\mathbf{u}^R\to\mathbf{u}$ in $\mathrm{L}^{\infty}([0,\widetilde{T}];\mathbb{V}_{s'})$, for any $0<s'<s$, it follows that $\left\langle\phi,\mathbf{u}^R\right\rangle_{\mathbb{V}_{-s'}\times\mathbb{V}_{s'}}\to\langle\phi,\mathbf{u}\rangle_{\mathbb{V}_{-s'}\times\mathbb{V}_{s'}}$ uniformly on $[0,\widetilde{T}]$, for any $\phi\in\mathbb{V}_{-s'}$. Using (3.19) and the fact that $\mathbb{V}_{-s'}$ is dense in $\mathbb{V}_{-s}$ for $s'<s$, by means of an $\varepsilon/3$ argument (see [10]), we have $\left\langle\phi,\mathbf{u}^R\right\rangle_{\mathbb{V}_{-s}\times\mathbb{V}_s}\to\langle\phi,\mathbf{u}\rangle_{\mathbb{V}_{-s}\times\mathbb{V}_s}$ uniformly on $[0,\widetilde{T}]$ for any $\phi\in\mathbb{V}_{-s}$. Thus, we have $\mathbf{u}\in\mathrm{C}_w([0,\widetilde{T}];\mathbb{V}_s)$. Uniqueness follows easily, since we have (2.13) and

$$|(\mathrm{B}(\mathbf{u}_1)-\mathrm{B}(\mathbf{u}_2),\mathbf{u}_1-\mathbf{u}_2)_{\mathbb{H}}|\leq C\|\nabla\mathbf{u}_1\|_{\mathbb{L}^{\infty}}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{H}}^2. \tag{3.32}$$

Let now prove $\mathbf{u}\in\mathrm{C}([0,T];\mathbb{V}_s)\cap\mathrm{C}^1(0,T;\mathbb{V}_{s-2})$. We first prove that $\mathbf{u}\in\mathrm{C}([0,T];\mathbb{V}_s)$. Since $\mathbf{u}\in\mathrm{C}_w([0,\widetilde{T}];\mathbb{V}_s)$, it suffices to show that the norm $\|\mathbf{u}(t)\|_{\mathbb{V}_s}$ is a continuous function of time. A similar calculation as in Proposition 3.1 yields

$$\sup_{t\in[0,\widetilde{T}]}\|\mathbf{u}(t)\|_{\mathbb{V}_s}-\|\mathbf{u}_0\|_{\mathbb{V}_s}\leq\frac{\|\mathbf{u}_0\|_{\mathbb{V}_s}^2C\widetilde{T}}{1-C\widetilde{T}\|\mathbf{u}_0\|_{\mathbb{V}_s}}. \tag{3.33}$$

From the fact that $\mathbf{u}\in\mathrm{C}_w([0,\widetilde{T}];\mathbb{V}_s)$, we have $\liminf\limits_{t\to0^+}\|\mathbf{u}(t)\|_{\mathbb{V}_s}\geq\|\mathbf{u}_0\|_{\mathbb{V}_s}$. Estimate (3.33) gives $\limsup\limits_{t\to0^+}\|\mathbf{u}(t)\|_{\mathbb{V}_s}\leq\|\mathbf{u}_0\|_{\mathbb{V}_s}$ also. In particular, $\lim\limits_{t\to0^+}\|\mathbf{u}(\cdot,s)\|_{\mathbb{V}_s}=\|\mathbf{u}_0\|_{\mathbb{V}_s}$ This gives us strong right continuity at $t=0$, and arguing similarly as in [11](see Theorem 3. 5, page 109-112), we finally obtain $\mathbf{u}\in\mathrm{C}([0,\widetilde{T}];\mathbb{V}_s)$. An estimate similar to (3.28) yields

$$\sup_{t\in[0,\widetilde{T}]}\left\|\frac{\mathrm{d}\mathbf{u}(t)}{\mathrm{d}t}\right\|_{\mathbb{V}_{s-2}}\leq C(M)<+\infty, \tag{3.34}$$

and this gives $\mathbf{u}\in\mathrm{C}^1(0,\widetilde{T};\mathbb{V}_{s-2})$. □

Remark 3.7. In order to obtain the pressure estimate, let us take divergence on the first equation in (2.2) and using the divergence free condition to find

$$\Delta p=-\sum_{i,j=1}^3\left(\frac{\partial\mathbf{u}_i}{\partial x_j}\right)\left(\frac{\partial\mathbf{u}_j}{\partial x_i}\right)=-(\nabla\mathbf{u})(\nabla\mathbf{u})^{\top}.$$

For $s > 5/2$, using the algebra property of $\mathbb{H}^s$-norm, we estimate $\|\Delta p\|_{\mathrm{L}^\infty([0,\widetilde{T}];\mathbb{H}^{s-1})}$ as

$$\begin{aligned}\sup_{t\in[0,\widetilde{T}]} \|\Delta p(t)\|_{\mathbb{H}^{s-1}} &= \sup_{t\in[0,\widetilde{T}]} \|(\nabla \mathbf{u}(t))(\nabla \mathbf{u}(t))^\top\|_{\mathbb{H}^{s-1}} \leq C \sup_{t\in[0,\widetilde{T}]} \|\nabla \mathbf{u}(t)\|^2_{\mathbb{H}^{s-1}} \\ &\leq C \sup_{t\in[0,\widetilde{T}]} \|\mathbf{u}(t)\|^2_{\mathbb{H}^s} \leq CM^2,\end{aligned}$$

and hence $p \in \mathrm{L}^\infty([0,\widetilde{T}];\mathbb{H}^{s+1})$.

4. Beale-Kato-Majda Blow-up criterion and its extension

The following theorem implies that if the solution fails to be regular past a certain time, then the vorticity must necessarily be unbounded.

Theorem 4.1 (Beale-Kato-Majda)**.** *Let $\mathbf{u}(\cdot)$ be the solution of Navier-Stokes equations with hereditary viscosity (2.9), and suppose that there exists a time T^* such that the solution cannot be continued in the class (3.30) to $T = T^*$. Assume also that T^* is the first such time. Then*

$$\int_0^{T^*} \|\omega(t)\|_{\mathbb{L}^\infty} \mathrm{d}t = \infty,$$

and in particular

$$\limsup_{t\uparrow T^*} \|\omega(t)\|_{\mathbb{L}^\infty} = \infty.$$

Corollary 4.2. *For the solution $\mathbf{u}(\cdot)$ of the Navier-Stokes equations with hereditary viscosity (2.9), suppose there are constants K_0 and T^* so that on any interval $[0,T]$ of existence of the solution in class (3.30), with $T < T^*$, the vorticity satisfies the a-priori estimate:*

$$\int_0^{T^*} \|\omega(t)\|_{\mathbb{L}^\infty} \mathrm{d}t \leq K_0.$$

Then the solution can be continued in the class (3.30) to the interval $[0,T^*]$.

Proof of Theorem 4.1. We first claim that

$$\limsup_{t\uparrow T^*} \|\mathbf{u}(t)\|_{\mathbb{V}_s} = \infty. \tag{4.1}$$

If (4.1) is not true, then we have $\|\mathbf{u}(t)\|_{\mathbb{V}_s} \leq C_0$ for some C_0 and all $0 < t < T^*$. Then by the local existence and uniqueness theorem (see Theorem 3.6), we can start a solution at any time t_1 with initial value $\mathbf{u}(t_1)$, and this solution will be regular for $t_1 \leq t \leq t_1 + T_0(C_0)$, with T_0 independent of t_1. If $t_1 > T^* - T_0$, then we have extended the original solution past time T^*, which is a contradiction to the choice of T^*.

In order to prove the theorem, we assume that

$$\int_0^{T^*} \|\omega(t)\|_{\mathbb{L}^\infty} \mathrm{d}t \leq K_0 < +\infty, \tag{4.2}$$

and show that

$$\|\mathbf{u}(t)\|_{\mathbb{V}_s} \leq C_0,\ 0 < t < T^*, \tag{4.3}$$

for some C_0 contradicting (4.1). Let us now estimate the $\mathbb{H}$-norm of $\omega(t)$ by multiplying (2.14) with $\omega(t)$. We have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_{\mathbb{H}}^2 + ((\mathbf{u}(t)\cdot\nabla)\omega(t)), \omega(t))_{\mathbb{H}} &= ((\omega(t)\cdot\nabla)\mathbf{u}(t), \omega(t))_{\mathbb{H}} \\ &\quad + ((a*\Delta\omega)(t), \omega(t))_{\mathbb{H}}\,.\end{aligned} \tag{4.4}$$

Since $\nabla\cdot\mathbf{u} = 0$, we have $((\mathbf{u}\cdot\nabla)\omega, \omega)_{\mathbb{H}} = 0$. Using the Cauchy-Schwarz inequality, Hölder's inequality and (2.16), we easily have

$$|((\omega\cdot\nabla)\mathbf{u}, \omega)_{\mathbb{H}}| \leq \|\,(\omega\cdot\nabla)\,\mathbf{u}\|_{\mathbb{H}}\|\omega\|_{\mathbb{H}} \leq \|\nabla\mathbf{u}\|_{\mathbb{H}}\|\omega\|_{\mathbb{L}^\infty}\|\omega\|_{\mathbb{H}} \leq C\|\omega\|_{\mathbb{L}^\infty}\|\omega\|_{\mathbb{H}}^2. \tag{4.5}$$

Let us integrate (4.4) from 0 to t and use (4.5) to find

$$\|\omega(t)\|_{\mathbb{H}}^2 \leq \|\omega_0\|_{\mathbb{H}}^2 + C\int_0^t \|\omega(r)\|_{\mathbb{L}^\infty}\|\omega(r)\|_{\mathbb{H}}^2\mathrm{d}r + \int_0^t ((a*\Delta\omega)(r), \omega(r))_{\mathbb{H}}\,\mathrm{d}r. \tag{4.6}$$

Using (2.13), we obtain

$$\|\omega(t)\|_{\mathbb{H}}^2 \leq \|\omega_0\|_{\mathbb{H}}^2 + C\int_0^t \|\omega(r)\|_{\mathbb{L}^\infty}\|\omega(r)\|_{\mathbb{H}}^2\mathrm{d}r. \tag{4.7}$$

An application of Gronwall's inequality in (4.7) yields

$$\|\omega(t)\|_{\mathbb{H}}^2 \leq \|\omega_0\|_{\mathbb{H}}^2 \exp\left(C\int_0^t \|\omega(r)\|_{\mathbb{L}^\infty}\mathrm{d}r\right), \tag{4.8}$$

and hence

$$\|\omega(t)\|_{\mathbb{H}} \leq K_1\|\omega_0\|_{\mathbb{H}}, \tag{4.9}$$

for all $0 < t < T^*$, where $K_1 = \exp(CK_0)$.

Now we operate with J^s in the system (2.9) and then taking inner product with $\mathrm{J}^s\mathbf{u}(\cdot)$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|_{\mathbb{V}_s}^2 = -\left(\mathrm{B}(\mathbf{u}(t)), \mathbf{u}(t)\right)_{\mathbb{V}_s} - ((a*\mathrm{A}\mathbf{u})(t), \mathbf{u}(t))_{\mathbb{V}_s}\,. \tag{4.10}$$

Cauchy-Schwarz inequality and (2.8) gives us

$$\left|(\mathrm{B}(\mathbf{u}), \mathbf{u})_{\mathbb{V}_s}\right| \leq C\|\nabla\mathbf{u}\|_{\mathbb{L}^\infty}\|\mathbf{u}\|_{\mathbb{V}_s}^2. \tag{4.11}$$

Integrating the equality (4.10) from 0 to t and using (4.11) to obtain

(4.12)

$$\|\mathbf{u}(t)\|_{\mathbb{V}_s}^2 \leq \|\mathbf{u}_0\|_{\mathbb{V}_s}^2 + C\int_0^t \|\nabla\mathbf{u}(r)\|_{\mathbb{L}^\infty}\|\mathbf{u}(r)\|_{\mathbb{V}_s}^2\mathrm{d}s - \int_0^t ((a*\mathrm{A}\mathbf{u})(r), \mathbf{u}(r))_{\mathbb{V}_s}\,\mathrm{d}r.$$

We use (2.13) in (4.12) to find

$$\|\mathbf{u}(t)\|_{\mathbb{V}_s}^2 \leq \|\mathbf{u}_0\|_{\mathbb{V}_s}^2 + C\int_0^t \|\nabla\mathbf{u}(r)\|_{\mathbb{L}^\infty}\|\mathbf{u}(r)\|_{\mathbb{V}_s}^2\mathrm{d}r,$$

and an application of Gronwall's inequality yields

$$\|\mathbf{u}(t)\|_{\mathbb{V}_s}^2 \leq \|\mathbf{u}_0\|_{\mathbb{V}_s}^2 \exp\left(C\int_0^t \|\nabla\mathbf{u}(r)\|_{\mathbb{L}^\infty}\mathrm{d}r\right). \tag{4.13}$$

From the estimate (1.1), we infer that

$$\|\nabla\mathbf{u}\|_{\mathbb{L}^\infty} \leq C\Big\{1+\|\omega\|_{\mathbb{L}^\infty}\log_e\left(\|\mathbf{u}\|_{\mathbb{V}_s}+e\right)\Big\}, \tag{4.14}$$

for $s>5/2$. Let us define $y(t)=\log_e\left(\|\mathbf{u}(t)\|_{\mathbb{V}_s}+e\right)$ and use (4.14) in (4.13) to find

$$y(t) \leq y(0)+C\int_0^t \left[1+\|\omega(r)\|_{\mathbb{L}^\infty}y(r)\right]\mathrm{d}r. \tag{4.15}$$

An application of Gronwall's inequality in (4.15) yields

$$\log_e\left(\|\mathbf{u}(t)\|_{\mathbb{V}_s}+e\right) \leq \left(\log_e\left(\|\mathbf{u}_0\|_{\mathbb{V}_s}+e\right)+Ct\right)\exp(CK_0), \tag{4.16}$$

for all $t\in[0,T^*)$ and (4.3) follows. $\square$

Theorem 4.3 (BMO space extension). *For the solution $\mathbf{u}(\cdot)$ of the system (2.9), suppose there are constants M_0, M_1 and T^* so that on any interval $[0,T]$ of existence of the solution in class (3.30), with $T<T^*$, the vorticity satisfies the a-priori estimate:*

$$\int_0^{T^*}\|\omega(t)\|_{\mathrm{BMO}}\mathrm{d}t \leq M_0,$$

or the deformation tensor $\mathrm{D}\mathbf{u}=\frac{1}{2}\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^\top\right)$ satisfies the a-priori estimate:

$$\int_0^{T^*}\|\mathrm{D}\mathbf{u}(t)\|_{\mathrm{BMO}}\mathrm{d}t \leq M_1.$$

Then the solution can be continued in the class (3.30) to the interval $[0,T^*]$.

Proof. The proof follows by combining (1.4) and (4.13), and arguing similarly in the previous theorem. By the boundedness of Riesz transforms in BMO (see [8]), there holds $\|\nabla\mathbf{u}\|_{\mathrm{BMO}}\leq C\|\mathrm{D}\mathbf{u}\|_{\mathrm{BMO}}$. Thus by using (1.4), we also have

$$\|\nabla\mathbf{u}\|_{\mathbb{L}^\infty} \leq C\left(1+\|\mathrm{D}\mathbf{u}\|_{\mathrm{BMO}}\log_e\left(e+\|\mathbf{u}\|_{\mathbb{V}_s}\right)\right). \tag{4.17}$$

Hence we obtain the required result by combining (4.17) and (4.13) as in the previous theorem. $\square$

Theorem 4.4 (Besov space extension). *For the solution $\mathbf{u}(\cdot)$ of the system (2.9), suppose there are constants L_0 and T^* so that on any interval $[0,T]$ of existence of the solution in class (3.30), with $T<T^*$, the vorticity satisfies the a-priori estimate:*

$$\int_0^{T^*}\|\omega(t)\|_{\dot{\mathrm{B}}^0_{\infty,\infty}}\mathrm{d}t \leq L_0.$$

Then the solution can be continued in the class (3.30) to the interval $[0,T^*]$.

Proof. The proof follows by combining (1.6) and (4.13), and arguing similarly in the Theorem 4.1. $\square$

Remark 4.5. 1. The constants M_0, M_1 and L_0 appearing in Theorems 4.3 and 4.4 depend on s also.

2. The authors in [1] remarked that "in fact the developments of this paper seem to indicate that the mathematical structure of the Navier-Stokes equation with hereditary viscosity is in some sense in between that of the Euler equation and the conventional Navier-Stokes equation." Thus one can expect that the blow-up criterion for the Navier-Stokes equation with hereditary viscosity will be in some sense "weaker" than that of the Euler equations. But the analysis in this paper (see Theorems 4.1, 4.3 and 4.4) shows that the blow-up criterion is the same as that of Euler equations ([3, 7–9]). This clearly shows that the Navier-Stokes equation with hereditary viscosity is more close to the Euler equations than Navier-Stokes equations. This is also due to the lack of some global estimates, compared to the case of Navier-Stokes equations, even though the hereditary term is positive.

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