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EXACT PENALIZATION AND OPTIMALITY CONDITIONS FOR CONSTRAINED DIRECTIONAL PARETO EFFICIENCY

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Abstract. We study a flexible approach to directional Pareto efficiency in vector optimization and we give a description of the possible concepts of minimality in the directional setting that can be obtained in this framework. Then, we select some of these concepts and we give exact penalization results and optimality conditions for them. Several comments concerning the possibilities to obtain corresponding results for other directional minimality concepts are presented as well.

1. Introduction and notation

In this paper we explore some new possibilities of analysis opened by the introduction of the concepts of directional Pareto efficiencies in [4] and the subsequent study concerning optimality conditions for these notions given in [7]. Directional Pareto efficiency is a natural and flexible concept that modifies the standard Pareto efficiency by considering in the input space only a set of directions starting from the point under consideration, instead of a whole ball around that point. Moreover, this is a concept that is best fitted for vectorial optimization, and in the scalar case where one has only two directions it corresponds, roughly speaking, to the minimality at the end-point of an interval. Several versions of the Fermat Theorem for this general concept were obtained in [4].

However, as already mentioned in the quoted paper, several threads of investigation for the directional efficiency can be followed, and the aim of this paper is to consider some of these threads. Therefore, this work is threefold. Firstly, we aim at introducing new concepts of directional efficiency that correspond to some of the Pareto efficiencies in the standard sense, but we will see that the directional approach offers the possibility to discuss concepts without counterpart in classical setting. Secondly, we devise a set of exact penalization results and we explore how the standard assumptions have to be modified in the case we consider here. Thirdly, we present necessary optimality conditions for directional efficiency based on an enhanced result obtained in [7] concerning the directional openness (see [6]) of an epigraphical set-valued map. We describe the relations of our results with the existing results in literature and we consider several possibilities to get similar assertions for other types of minima.

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We now present the way our paper is organized and, in this process, we present further, more precise, details of the results and the methods we propose. After the presentation of notation, we consider, in the second section, the main concepts under study. Besides the standard notions of Pareto minimality, we recall the directional Pareto efficiencies introduced in [4] where the minimality is understood along the directions of a set in input space. Then, the latter approach is slightly modified in order to consider in output space a set of directions that defines the ordering cone. The form of the part of the ball around the underlying point as it appears in these definitions gives us the impetus to present a brief study of the properties of such sets. On this basis, one can define an enlargement of the set of directions, a tool we subsequently use to define a concept of directional proper efficiency in input space. A situation when an approximate directional minimum can be seen as a proper approximate efficient point is given at the end of the section. In the next two sections, we choose the notion of directional Pareto efficiency and we present penalization results and necessary optimality conditions. Firstly, in Section 3, we consider Clarke type penalization and we discuss the nature of the Lipschitz condition that is necessary in our situation. Then, we devise a penalization result in the case of a problem where the constraint is given in a generalized functional form. In Section 4 we consider the latter problem and we use a result from [7] in order to formulate Fritz John optimality conditions in terms of Mordukhovich's differentiation objects on finite dimensional spaces. We remark that, in the particular case of the standard Pareto efficiency, the assumptions can be relaxed and we present a set of hypotheses that allows to obtain a Karush-Kuhn-Tucker type result. Moreover, we emphasize that similar results can be obtained for other types of minima described in Section 2. An exemplification of possible additional information concerning the multipliers that can be obtained for proper efficiency ends the section. The last section presents some concluding remarks of our study.

Let us now present the notation we use throughout this paper. We assume that X, Y, and Z are normed vector spaces over the real field $\mathbb{R}$ and on a product of normed vector spaces we consider the sum norm. By $B(x,\varepsilon)$ we denote the open ball with center x and radius $\varepsilon > 0$ and by B_X the open unit ball of X. In the same manner, $D(x,\varepsilon)$ and D_X denote the corresponding closed balls. The symbol S_X stands for the unit sphere of X. By the symbol X^* we denote the topological dual of X.

Let $F : X \rightrightarrows Y$ be a set-valued map. The graph of F is

$$\operatorname{Gr} F := \{(x,y) \in X \times Y \mid y \in F(x)\},$$

and the usual inverse of F is the set-valued map $F^{-1} : Y \rightrightarrows X$ given by $(y,x) \in \operatorname{Gr} F^{-1}$ iff $(x,y) \in \operatorname{Gr} F$. Consider a nonempty subset A of X. Then, the image of A through F is

$$F(A) := \{y \in Y \mid \exists x \in A,\ y \in F(x)\}.$$

Clearly, for $B \subset Y$,

$$F^{-1}(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

Another kind of inverse image of a set through F is defined as

$$F^{+1}(B) := \{x \in X \mid F(x) \subset B\}.$$

If F has nonempty values, one says that it is (globally) upper semicontinuous if and only if for any open set $B \subset Y$, $F^{+1}(B)$ is open. For more details concerning the continuity properties of set-valued maps, the reader is referred to monograph [12].

One says that F is Lipschitz-like around $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ if there are some neighborhoods U of $\overline{x}$ and V of $\overline{y}$, respectively, and $L > 0$ such that for all $x', x'' \in U$

$$F(x') \cap V \subset F(x'') + L \left\| x' - x'' \right\| D_Y.$$

A comprehensive study of this property in relation to other metric regularity properties of set-valued map is to be found in [14, Chapter 1].

The distance function associated to $A \subset X$ is $d_A : X \to \mathbb{R}$ given by

$$d_A(x) = d(x, A) := \inf_{a \in A} \|x - a\|,$$

while the topological interior, topological closure and conic hull of A are denoted, respectively, by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{cone} A$.

The dual cone associated to A is

$$A^+ := \{x^* \in X^* \mid x^*(a) \geq 0, \forall a \in A\}.$$

It is clear that $A^+ = (\operatorname{cl} A)^+ = (\operatorname{cone} A)^+$.

2. Concepts of directional efficiency

First of all, we recall the usual concepts related to Pareto efficiency for constrained vector optimization problems governed by set-valued maps. Let $K \subset Y$ be a proper (that is, $K \neq \{0\}$, $K \neq Y$) cone.

Take $F : X \rightrightarrows Y$ as a set-valued mapping, and let us consider the following geometrically constrained optimization problem with multifunctions:

$$(P) \quad \text{minimize } F(x), \text{ subject to } x \in A,$$

where $A \subset X$ is a nonempty set. If the cone K is convex and pointed (that is, $K \cap -K = \{0\}$), then it naturally induces a partial preorder relation (denoted $\leq_K$) on Y by $y_1 \leq_K y_2$ if and only if $y_2 - y_1 \in K$, and the minimality for the above problem is understood with respect to this relation.

The formal definition in the general setting we consider here (that is, K is not necessarily convex or pointed) reads as follows.

Definition 2.1. A point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local Pareto minimum point for F on A if there exists a neighborhood U of $\overline{x}$ such that

$$(F(U \cap A) - \overline{y}) \cap -K \subset K. \tag{2.1}$$

The vectorial notion described by (2.1) covers as well the situation where f is a function (in which case $\overline{y} = f(\overline{x})$ is not mentioned) and the situation of classical local minima in scalar case (in which case we drop the label "Pareto"). If K is pointed, then (2.1) reduces to

$$(F(U \cap A) - \overline{y}) \cap -K = \{0\}.$$

A weaker notion operates when the cone K is solid, that is when it has nonempty topological interior.

Definition 2.2. If $\operatorname{int} K \neq \emptyset$, the point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local weak Pareto minimum point for F on A if there exists a neighborhood U of $\overline{x}$ such that

$$(F(U \cap A) - \overline{y}) \cap -\operatorname{int} K = \emptyset.$$

These concepts and many others are intensively studied in literature: see [8], [10], [17], [9], [11], and the references therein.

In [4], a directional notion of efficiency was introduced and studied. We now briefly recall it. Let $L \subset S_X$ be a nonempty closed set.

Definition 2.3. One says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local directional Pareto minimum point for F on A with respect to (the set of directions) L if there exists a neighborhood U of $\overline{x}$ such that

$$(F(U \cap A \cap [\overline{x} + \operatorname{cone} L]) - \overline{y}) \cap -K \subset K. \tag{2.2}$$

Of course, this concept corresponds to the situation where the restriction has the special form, depending on $\overline{x}$, $A \cap (\overline{x} + \operatorname{cone} L)$. When $A = X$ in (2.2) then one says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a local directional Pareto minimum point for F with respect to L.

If $\operatorname{int} K \neq \emptyset$, one defines as well the weak counterpart of the above notion.

Definition 2.4. One says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a local weak directional Pareto minimum point for F on A with respect to (the set of directions) L if there exists a neighborhood U of $\overline{x}$ such that

$$(F(U \cap A \cap [\overline{x} + \operatorname{cone} L]) - \overline{y}) \cap -\operatorname{int} K = \emptyset.$$

In all these notions, if one takes $U = X$, then one gets the corresponding global concepts.

The main ideas we exploit in this work are coming from the fact that $K = \operatorname{cone} S_K$, where $S_K := S_Y \cap K$ and, in order to get necessary optimality conditions for the Pareto efficiency, it is sometimes enough to consider directions from S_K (see, for instance, [4, Proposition 3.16]). Another idea that leads us to consider the framework we are going to present next is to allow a greater flexibility to the order relation defining the efficiencies and it is inspired by the concept of directional regularity introduced and studied in [6].

Definition 2.5. Let $L \subset S_X$ and $M \subset S_Y$ be nonempty closed sets. One says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ is a $(L, M)-$local directional Pareto minimum point for F on A if there exists a neighborhood U of $\overline{x}$ such that

$$(F(U \cap A \cap [\overline{x} + \operatorname{cone} L]) - \overline{y}) \cap -\operatorname{cone} M \subset \operatorname{cone} M.$$

We denote the set of $(L, M)-$local directional Pareto minimum point for F on A by $\operatorname{Min}(F, A; L, M)$.

Remark 2.6. (i) Clearly, $\operatorname{cone} M$ is pointed if and only if $M \cap -M = \emptyset$ and in this case the above relation become

$$(F(U \cap A \cap [\overline{x} + \operatorname{cone} L]) - \overline{y}) \cap -\operatorname{cone} M = \{0\}.$$

(ii) For $U = X$ we speak about $(L, M)-$directional Pareto minimum point for F on A.

(iii) Basically, the notion of $(L, M)-$minimality replaces the focus on cone K by the possibility to work with the set of directions that generates the ordering cone.

Several examples of this type of minimality was given in [4]. Here is another example illustrating the set-valued case we consider in this work.

Example 2.7. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$,

$$F(x) := \begin{cases} \{x\} \times \left[-\sqrt{1-x^2}, \sqrt{1-x^2}\right], & \text{if } x \in [-1,1] \\ \mathbb{R}^2, \text{ otherwise.} \end{cases}$$

Take $A = [-1, 1]$ and $M = S_{\mathbb{R}^2} \cap \mathbb{R}^2_+$. Then it is easy to see that

$$\operatorname{Min}(F, A; \{-1, +1\}, M) = \left\{(x, y) \in \mathbb{R}^2 \mid x \in [-1, 0], y = -\sqrt{1-x^2}\right\},$$

$$\operatorname{Min}(F, A; \{+1\}, M) = \left\{(x, y) \in \mathbb{R}^2 \mid x \in [-1, 1], y = -\sqrt{1-x^2}\right\},$$

$$\operatorname{Min}(F, A; \{-1\}, M) = \left\{(x, y) \in \mathbb{R}^2 \mid x \in [-1, 0], y = -\sqrt{1-x^2}\right\}.$$

Observe as well that

$$\operatorname{Min}(F, A; \{-1, +1\}, M) = \operatorname{Min}(F, A; \{+1\}, M) \cap \operatorname{Min}(F, A; \{-1\}, M).$$

Actually, the following facts are easy to see.

Proposition 2.8. *Let $L_1, L_2 \subset S_X$, $M_1, M_2 \subset S_Y$ be closed sets.*
(i) If $L_1 \subset L_2$, then $\operatorname{Min}(F, A; L_2, M_1) \subset \operatorname{Min}(F, A; L_1, M_1)$.
(ii) If $M_1 \subset M_2$, then $\operatorname{Min}(F, A; L_1, M_2) \subset \operatorname{Min}(F, A; L_1, M_1)$.
(iii) One has

$$\begin{aligned} \operatorname{Min}(F, A; L_1, M_1) &= \bigcap_{l \in L_1, m \in M_1} \operatorname{Min}(F, A; \{l\}, \{m\}) \\ &= \bigcap_{l \in L_1} \operatorname{Min}(F, A; \{l\}, M_1) \\ &= \bigcap_{m \in M_1} \operatorname{Min}(F, A; L_1, \{m\}). \end{aligned}$$

The item *(iii)* suggests that one can consider single directions in S_X or S_Y for the study of $(L, M)-$minimality, a detail that will be exploited in Section 4.

Basically, in the above notation, a relation of the following form

$$(F(U \cap [\overline{x} + \operatorname{cone} L]) - \overline{y}) \cap - \operatorname{cone} M = \{0\},$$

can be equivalently written as

$$F(D(\overline{x}, \varepsilon) \cap [\overline{x} + \operatorname{cone} L]) \cap (\overline{y} - \operatorname{cone} M) = \{\overline{y}\},$$

for some $\varepsilon > 0$. Now, observe that

$$D(\overline{x}, \varepsilon) \cap [\overline{x} + \operatorname{cone} L] = \overline{x} + \varepsilon D_X \cap \operatorname{cone} L = \overline{x} + \varepsilon [0, 1] L$$

and this suggest a study of the set $D_L := [0, 1] L$. Observe that

$$L = S_X \cap \operatorname{cone} L \text{ and } \operatorname{cone} L = \operatorname{cone} D_L.$$

Further, for $\varepsilon > 0$ define the $\varepsilon-$enlargement of L as

$$L_\varepsilon = \{x \in S_X \mid d(x, L) \leq \varepsilon\}.$$

We record the following simple facts.

Proposition 2.9. *Let $\emptyset \neq L \subset S_X$ and $\varepsilon > 0$. Then:*

(i) *L is closed if and only if D_L is closed and if and only if* $\operatorname{cone} L$ *is closed;*
(ii) *L is compact if and only if D_L is compact;*
(iii) *if X is a Banach space and L is weakly compact, then D_L is weakly compact;*
(iv) *L_ε is closed;*
(v) $\operatorname{cone} L \backslash \{0\} \subset \operatorname{int} \operatorname{cone} L_\varepsilon$;
(vi) *if L is closed, then*

$$\bigcap_{\delta > 0} L_\delta = L, \quad \bigcap_{\delta > 0} D_{L_\delta} = D_L \quad \textit{and} \quad \bigcap_{\delta > 0} \operatorname{cone} L_\delta = \operatorname{cone} L;$$

(vii) *D_L is convex if and only if* $\operatorname{cone} L$ *is convex.*

Proof. (i), (ii) Standard arguments based on the characterization of closedness and compactness with sequences, as well as the continuity of the norm prove the theses.

(iii) We use the Eberlein–Šmulian Theorem that states that, on Banach spaces, the weak compactness can be characterized by sequences. Take $(x_n) \subset D_L$ a sequence which, by the definition of D_L, can be expressed as $(\alpha_n u_n)$ where $(\alpha_n) \subset [0, 1]$ and $(u_n) \subset L$. By the weak compactness of L and the Eberlein–Šmulian Theorem, (u_n) admits a weakly convergent subsequence to some $u \in L$. In turn, (α_n) is a bounded sequence of real numbers, so we can assume, without loss of generality, that it weakly converges, on the same subsequence, to an element $\alpha \in [0, 1]$. Consequently, (x_n) has a subsequence weakly converging towards $\alpha u \in D_L$. Therefore, D_L is weakly compact.

(iv) It is clear that L_ε is the intersection between S_X and a level set of a continuous function, whence it is closed.

(v) We show first that $L \subset \operatorname{int} \operatorname{cone} L_\varepsilon$. Take $u \in L$ and suppose, by way of contradiction, that for any natural number $n > 0$,

$$B(u, n^{-1}) \cap (X \backslash \operatorname{cone} L_\varepsilon) \neq \emptyset.$$

Then, there is a sequence $(u_n) \to u$ such that $u_n \notin \operatorname{cone} L_\varepsilon$ for all n. Since $\|u\| = 1$, we can suppose that $u_n \neq 0$. Therefore,

$$\|u_n\|^{-1} u_n \to u$$

and $\|u_n\|^{-1} u_n \in S_X$ for any n. Since $B(u, \varepsilon) \cap S_X \subset L_\varepsilon$, for n sufficiently large, $\|u_n\|^{-1} u_n \in L_\varepsilon$, whence $u_n \in \operatorname{cone} L_\varepsilon$, which is a contradiction. Now, for $v \in \operatorname{cone} L \setminus \{0\}$, there are $\alpha > 0$ and $u \in L$ with $v = \alpha u$. From the previous step, there is $\rho > 0$ such that $B(u, \rho) \subset \operatorname{cone} L_\varepsilon$. Then, we have $B(v, \alpha\rho) \subset \operatorname{cone} L_\varepsilon$, showing that $v \in \operatorname{int} \operatorname{cone} L_\varepsilon$.

(vi) The inclusion

$$L \subset \bigcap_{\delta > 0} L_\delta$$

is always true. Assume now that L is closed and suppose that there would be $u \in \bigcap_{\delta>0} L_\delta \setminus L$. Then, $d(u, L) := \mu > 0$. In particular, this means that $u \notin L_{2^{-1}\mu}$, which is a contradiction.

For the second equality, again, the inclusion

$$D_L \subset \bigcap_{\delta>0} D_{L_\delta}$$

is obvious. Since L is closed, then, according to (i), D_L is closed. If there exists $u \in \bigcap_{\delta>0} D_{L_\delta} \setminus D_L$, then $d(u, D_L) := \mu > 0$. We show that this implies than $u \notin D_{L_{3^{-1}\mu}}$, which is a contradiction. Indeed, otherwise, one can find $u' \in L_{3^{-1}\mu}$ and $t \in [0,1]$ such that $u = tu'$ and $v \in L$ such that $\|u' - v\| < 2 \cdot 3^{-1}\mu$. Clearly, $tv \in D_L$. This implies

$$\mu = d(u, D_L) < \|u - tv\| = \left\|tu' - tv\right\| = t\left\|u' - v\right\| \leq 2 \cdot 3^{-1}\mu < \mu.$$

The conclusion follows.

For the last equality, we proceed similarly. While

$$\operatorname{cone} L \subset \bigcap_{\delta>0} \operatorname{cone} L_\delta$$

is clear, take, by way of contradiction, $u \in \bigcap_{\delta>0} \operatorname{cone} L_\delta \setminus \operatorname{cone} L$. Then, $u \neq 0$ and $\|u\|^{-1} u \notin L$, whence $d(\|u\|^{-1} u, L) := \mu > 0$. If one has $u \in \operatorname{cone} L_{3^{-1}\mu}$, then $\|u\|^{-1} u \in L_{3^{-1}\mu}$, so one will have $v \in L$ with

$$\left\|\|u\|^{-1} u - v\right\| < 2 \cdot 3^{-1}\mu.$$

Therefore,

$$\mu = d(\|u\|^{-1} u, L) \leq \left\|\|u\|^{-1} u - v\right\| < 2 \cdot 3^{-1}\mu,$$

again a contradiction.

(vii) The obvious relations $\operatorname{cone} L = \operatorname{cone} D_L$ and $D_L = D_X \cap \operatorname{cone} L$ are enough in order to conclude. □

Remark 2.10. The converse of the item (iii) from Proposition 2.9 does not hold: for instance, in a reflexive Banach space, $L := S_X$ is not weakly compact, while $D_L = D_X$ has this property. The cones with weakly compact intersection with the unit ball are studied in [3] under the name of reflexive cones. Proposition 2.9 (iii) shows that if L is weakly compact, then $\operatorname{cone} L$ is a reflexive cone.

Remark 2.11. As seen in the items (v) and (vi), the cone generated by L_ε is a solid enlargement of $\operatorname{cone} L$. Another link with the existing literature concerning enlargements of cones is given by the following inclusions:

$$D_{L_\varepsilon} \subset \{x \in D_X \mid d(x, D_L) \leq \varepsilon \|x\|\}$$

and

$$\operatorname{cone} L_\varepsilon \subset \{x \in X \mid d(x, \operatorname{cone} L) \leq \varepsilon \|x\|\}.$$

Indeed, if $u \in D_{L_\varepsilon} \setminus \{0\}$ then clearly $u \in D_X$ and $\|u\|^{-1} u \in L_\varepsilon$. Hence, for $\delta > 0$ there is $v \in L$ such that $\left\| \|u\|^{-1} u - v \right\| < \varepsilon + \delta$, so $\|u\| v \in D_L$ and

$$d(u, D_L) \leq \left\| \|u\| \|u\|^{-1} u - \|u\| v \right\| = \|u\| \left\| \|u\|^{-1} u - v \right\| \leq (\varepsilon + \delta) \|u\| .$$

Letting $\delta \to 0$, one obtains the desired relation. For the second inclusion, take $u \in \operatorname{cone} L_\varepsilon$. Then, there are $\alpha \geq 0$ and $v \in L_\varepsilon$ such that $u = \alpha v$. In particular, $v \in D_{L_\varepsilon}$, whence, from the previous step, $d(v, D_L) \leq \varepsilon \|v\| = \varepsilon$ and

$$d(u, \operatorname{cone} L) = d(\alpha v, \operatorname{cone} L) \leq d(\alpha v, \alpha D_L) = \alpha d(v, D_L) \leq \alpha \varepsilon = \varepsilon \|u\| .$$

Therefore, the enlargement given by L_ε is smaller that the enlargement proposed in [13], which, for a given cone K, reads as follows:

$$K^\varepsilon := \{u \in X \mid d(u, K) \leq \varepsilon \|u\|\}.$$

Now, in view of the announced fact concerning the flexibility offered by our setting in the study of different degrees of Pareto efficiency, we discuss the possibility to devise some directional concepts for proper and approximate efficiency (see [17] for a discussion on the standard case). We mention that both proper and approximate minimality are very useful and studied in the framework of standard vector optimization (see [10] and [8], for details). In order to keep the presentation as concise as possible, we proceed with both these versions of minimality in the directional case at once and for the unconstrained problem

$$(P_u) \quad \text{minimize } F(x), \text{ subject to } x \in X.$$

A concept of properness in the sense of Henig (see [8, p. 110]) by an enlargement of cone M is easy to be written in our case. However, the directional character also in input space of the Pareto efficiency we consider here allows to define a concept of proper efficiency with respect to X.

Definition 2.12. Let $L \subset S_X$ and $M \subset S_Y$ be nonempty closed sets. One says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a $(L, M)-$local directional proper in X and approximate in Y Pareto minimum point for F (or for Problem (P_u)) if there exists a neighborhood U of $\overline{x}$, an element $v \in M$ and two constants $\varepsilon, \delta > 0$ such that

$$(F(U \cap (\overline{x} + \operatorname{cone} L_\varepsilon)) - \overline{y}) \cap (-\operatorname{cone} M - \delta v) = \emptyset.$$

The approximate character of the above concept is given by the presence of the term δv, while the properness is given by the enlargement $\operatorname{cone} L_\varepsilon$ (in accordance with Proposition 2.9). While the approximate character is similar with that in the case of standard vector and scalar optimization (see [1] and the references therein) the proper feature means that one can enlarge the set of direction in X with respect to which $(\overline{x}, \overline{y})$ is a minimum.

In this way, any interested reader can generate several concepts of efficiency and the study of such concepts is of potential interest. It is not our aim here to exhaustively describe such possibilities, but we illustrate the above concept by the next result where we give sufficient conditions to ensure that an approximate minimum can be seen as a proper approximate minimum.

Proposition 2.13. *Let $L \subset S_X$ and $M \subset S_Y$ be closed sets and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Suppose that there exist a neighborhood U of $\overline{x}$, an element $v \in M$ and constants $\delta > 0$ such that*

$$(F(U \cap (\overline{x} + \operatorname{cone} L)) - \overline{y}) \cap (-\operatorname{cone} M - \delta v) = \emptyset.$$

If F is upper semicontinuous and has nonempty values, and L is compact, then there exists $\varepsilon > 0$ such that

$$(F(U \cap (\overline{x} + \operatorname{cone} L_\varepsilon)) - \overline{y}) \cap (-\operatorname{cone} M - \delta v) = \emptyset.$$

Proof. The relation

$$(F(U \cap (\overline{x} + \operatorname{cone} L)) - \overline{y}) \cap (-\operatorname{cone} M - \delta v) = \emptyset,$$

can be written as: there exists $\rho > 0$ such that

$$F(\overline{x} + \rho[0,1]L) \subset \overline{y} + Y \setminus (-\operatorname{cone} M - \delta v),$$

that is

$$\overline{x} + \rho[0,1]L \subset F^{+1}\left(\overline{y} + Y \setminus (-\operatorname{cone} M - \delta v)\right).$$

Since the set $\overline{y} + Y \setminus (-\operatorname{cone} M - \delta v)$ is open, and F is upper semicontinuous with nonempty values the set in the right-hand side is open. According to Proposition 2.9 (ii), the compactness of L ensures the compactness of the left-hand set. Then there exists $\mu > 0$ such that

$$\bigcup_{\theta \in [0,1], u \in L} (\overline{x} + B(\rho\theta u, \mu)) \subset F^{+1}\left(\overline{y} + Y \setminus (-\operatorname{cone} M - \delta v)\right).$$

Now, in order to have the conclusion, it is enough to prove that there exists $\varepsilon > 0$ such that

$$\rho[0,1]L_\varepsilon \subset \bigcup_{\theta \in [0,1], u \in L} B(\rho\theta u, \mu)$$

which is implied by

$$\rho\theta L_\varepsilon \subset \bigcup_{u \in L} B(\rho\theta u, \mu),$$

for all $\theta \in (0,1]$. We show that the latter inclusion is true for $\varepsilon \in \left(0, \rho^{-1}\mu\right)$. Indeed, if it had not been the case, there would have existed $v \in \rho\theta L_\varepsilon \setminus \bigcup_{u \in L} B(\rho\theta u, \mu)$, so $(\rho\theta)^{-1}v \in L_\varepsilon$ and for all $u \in L$, $\|v - \rho\theta u\| \geq \mu$. Then, for $\delta \in (\varepsilon, \rho^{-1}\mu)$ there is $w \in L$ with $\left\|(\rho\theta)^{-1}v - w\right\| < \delta$, whence

$$\rho^{-1}\mu \leq (\rho\theta)^{-1}\mu \leq \left\|(\rho\theta)^{-1}v - w\right\| < \delta,$$

which is a contradiction. The proof is complete. □

3. Exact penalization results

In order to obtain necessary optimality conditions for problem (P), a fruitful technique is the exact penalization, that consists of adding a penalty term to the objective set-valued map F such that a given solution of the initial problem become solution for an unconstrained optimization problem for which we have more tools of investigation. Again, such results could be described as well for proper and/or approximate minimality. We restrict ourselves to the $(L,M)-$local directional Pareto minimality. Of course, an infinite penalization technique can be envisaged for the problems we study (see [5] and [14]), but in view of optimality conditions the Clarke penalization offers more precise conclusions. Therefore, the first result is a Clarke-type penalization (see [5] for the initial scalar version, and [15] for a vectorial form of this principle).

Theorem 3.1. *Let $L \subset S_X$ and $M \subset S_Y$ be nonempty closed sets such that $M \cap -M = \emptyset$ and $\operatorname{cone} M$ is convex. Let $(\overline{x},\overline{y}) \in \operatorname{Gr} F \cap (A \times Y)$ be a $(L,M)-$local directional Pareto minimum point for problem (P). Suppose that:*

(i) $A \cap [\overline{x} + \operatorname{cone} L]$ is locally closed at $\overline{x}$;

(ii) there exist $\ell > 0$, $e \in M$ and U a neighborhood of $\overline{x}$ such that, for every $(x', x'') \in (U \cap [\overline{x} + \operatorname{cone} L]) \times (U \cap A \cap [\overline{x} + \operatorname{cone} L])$,

$$F\left(x'\right) + \ell \left\|x' - x''\right\| e \subset F\left(x''\right) + \operatorname{cone} M.$$

Then, for every $\ell' > \ell$, $(\overline{x},\overline{y})$ is $(L,M)-$local directional Pareto minimum point for the unconstrained problem

$$\min \; F(x) + \ell' d\left(x, A \cap [\overline{x} + \operatorname{cone} L]\right) e.$$

Proof. As already mentioned, the condition $M \cap -M = \emptyset$ is equivalent to the fact that $\operatorname{cone} M$ is pointed. Since $(\overline{x},\overline{y})$ solves in the directional sense given above the problem (P), then there exists $\varepsilon > 0$ such that (see Remark 2.6)

$$(F\left(B\left(\overline{x},\varepsilon\right) \cap A \cap [\overline{x} + \operatorname{cone} L]\right) - \overline{y}) \cap -\operatorname{cone} M = \{0\}. \tag{3.1}$$

Without loss of generality, one can suppose that $B(\overline{x},\varepsilon) \subset U$ and $A \cap [\overline{x} + \operatorname{cone} L] \cap D(\overline{x},\varepsilon)$ is closed.

Fix $\ell' > \ell$ and let $\rho = \min\left(\dfrac{\varepsilon\ell}{\ell + \ell'}, \dfrac{\varepsilon}{3}\right)$. We proceed by contradiction, assuming that $(\overline{x},\overline{y})$ is not a $(L,M)-$local directional Pareto minimum point for the set-valued mapping

$$F(\cdot) + \ell' d\left(\cdot, A \cap [\overline{x} + \operatorname{cone} L]\right) e.$$

Then, one can find $x \in B\left(\overline{x},\rho\right) \cap [\overline{x} + \operatorname{cone} L]$ for which

$$\left(F\left(x\right) + \ell' d\left(x, A \cap [\overline{x} + \operatorname{cone} L]\right) e - \overline{y}\right) \cap -\operatorname{cone} M \neq \{0\}.$$

We infer that there exists $y \in F\left(x\right)$ satisfying the relation

$$\overline{y} - y - \ell' d\left(x, A \cap [\overline{x} + \operatorname{cone} L]\right) e \in \operatorname{cone} M \setminus \{0\}. \tag{3.2}$$

Now, we show there exists as well $u \in A \cap [\overline{x} + \operatorname{cone} L]$ such that

$$\|u - x\| \leq \frac{\ell'}{\ell} d\left(x, A \cap [\overline{x} + \operatorname{cone} L]\right).$$

Indeed, this assertion rests upon the following argument. If

$$d\left(x, A \cap [\overline{x} + \operatorname{cone} L]\right) > 0$$

the claim is obviously true since $d\left(\cdot, A \cap [\overline{x} + \operatorname{cone} L]\right)$ is an infimum. Otherwise, $d\left(x, A \cap [\overline{x} + \operatorname{cone} L]\right) = 0$, whence $x \in \operatorname{cl}\left(A \cap [\overline{x} + \operatorname{cone} L]\right)$, so there is a sequence $(x_n) \subset A \cap [\overline{x} + \operatorname{cone} L]$ such that $x_n \to x$. Since $x \in B(\overline{x}, \rho) \subset B(\overline{x}, \varepsilon)$, for n large enough, $x_n \in B(\overline{x}, \varepsilon) \subset D(\overline{x}, \varepsilon)$. Then $x \in \operatorname{cl}\left(A \cap [\overline{x} + \operatorname{cone} L] \cap D(\overline{x}, \varepsilon)\right) = A \cap [\overline{x} + \operatorname{cone} L] \cap D(\overline{x}, \varepsilon)$ and then we can take $u := x$. Consequently,

$$\|u - x\| \leq \frac{\ell'}{\ell} d\left(x, A \cap [\overline{x} + \operatorname{cone} L]\right) \leq \frac{\ell'}{\ell} \|x - \overline{x}\| \leq \frac{\ell'}{\ell}\rho. \tag{3.3}$$

Whence

$$\|u - \overline{x}\| \leq \|u - x\| + \|x - \overline{x}\| \leq \rho\left(\frac{\ell'}{\ell} + 1\right) < \varepsilon,$$

i.e.,

$$u \in B\left(\overline{x}, \varepsilon\right) \cap A \cap [\overline{x} + \operatorname{cone} L].$$

By hypothesis, we can guarantee the existence of $v \in F(u)$ such that $y - v + \ell \|x - u\| e \in \operatorname{cone} M$. Using (3.3) and the convexity of $\operatorname{cone} M$, we obtain that

$$y - v + \ell' d\left(x, A \cap [\overline{x} + \operatorname{cone} L]\right) e \in \operatorname{cone} M.$$

Finally, by adding the last relation and relation (3.2), we obtain

$$\overline{y} - v \in \operatorname{cone} M \setminus \{0\} + \operatorname{cone} M \subset \operatorname{cone} M \setminus \{0\}.$$

The relation above provides a contradiction with (3.1). Thus, the conclusion follows. □

Remark 3.2. In the framework of Clarke penalization, in general, the objective function is supposed to verify a vectorial Lipschitz property on an entire neighborhood of the minimum point. In this sequel, it is properly to use the following vectorial Lipschitz-like property: based on [15], if K is a cone, one says that F is $K-$Lipschitz around $\overline{x} \in X$ of rank $\ell > 0$ if there exist a neighborhood U of $\overline{x}$ and an element $e \in K \cap S_Y$ such that for every $x', x'' \in U$,

$$F\left(x'\right) + \ell_f \left\|x'' - x'\right\| e \in F\left(x''\right) + K.$$

For our purpose, it is not necessary to require the above relation on an entire neighborhood $\overline{x}$. Indeed, take $L = M := \{1\}, A := [0, +\infty), \overline{x} := 0$ and $\varepsilon > 0$. Let us define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x, & x \geq 0 \\ \varepsilon, & x < 0. \end{cases}$$

Then, for every $x' \in (-\varepsilon, \varepsilon)$ and $x'' \in [0, \varepsilon)$, we have that $f(x') + \|x' - x''\| - f(x'') \in [0, +\infty)$, so f satisfies the Lipschitz type condition needed in our result. However, the latter inclusion is no longer true if $x'' \in (-\varepsilon, \varepsilon)$.

The next penalization result we propose refers to the case of generalized functional constraints. More precisely, we consider the problem (P) where one replaces the set A with the set

$$\{x \in X \mid 0 \in G(x) + Q\} = G^{-1}(-Q),$$

where $G : X \rightrightarrows Z$ is a set-valued maps and $Q \subset Z$ is a proper convex closed cone. We denote the problem (P) with this type of constraint by (P_f) ("f" from functional). The epigraphical set-valued map, $\tilde{G} : X \rightrightarrows Z$ is given by $\tilde{G}(x) = G(x) + Q$. With this notation, the set of feasible points of (P_f) can be written as $\tilde{G}^{-1}(0)$.

In order to deal with the directional concepts of efficiency we consider the minimal time function used in [6] to introduce directional regularity notions for mappings.

Let $\emptyset \neq L \subset S_X$ and $\emptyset \neq \Omega \subset X$ be some sets (the set of directions and the target set). Then, the function

$$\begin{aligned} T_L(x, \Omega) &:= \inf \{t \geq 0 \mid \exists u \in L : x + tu \in \Omega\} \\ &= \inf \{t \geq 0 \mid (x + tL) \cap \Omega \neq \emptyset\} \end{aligned} \tag{3.4}$$

is called the directional minimal time function with respect to M.

Clearly, if $L = S_X$, then $T_M(\cdot, \Omega) = d(\cdot, \Omega)$. Moreover, we use the convention that $T_L(x, \emptyset) = \infty$ for every x and we denote in what follows $T_L(x, \{u\})$ by $T_L(x, u)$. Obviously,

$$[T_L(x, u) < +\infty] \iff [T_L(x, u) = \|u - x\| \text{ and } u - x \in \operatorname{cone} L] .$$

Let $F : X \rightrightarrows Y$ be a set-valued mapping and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, $\emptyset \neq L \subset S_X$, $\emptyset \neq M \subset S_Y$.

We recall the following concept of directional calmness introduced in [4]. One says that F is directionally calm at $(\overline{x}, \overline{y})$ with respect to L and M with constant $\alpha > 0$ if there exist some neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that, for every $x \in U$,

$$\sup_{y \in F(x) \cap V} T_M(y, F(\overline{x})) \leq \alpha T_L(\overline{x}, x). \tag{3.5}$$

We use as well the convention $\sup_{x \in \emptyset} := 0$.

In view of the properties of minimal time functions already mentioned, the directional calmness means that for every $\alpha' > \alpha$ and every $y \in F(x) \cap V$ with $x \in (\overline{x} + \operatorname{cone} L) \cap U$, there is a positive $t \leq \alpha' \|x - \overline{x}\|$ such that

$$(y + tM) \cap F(\overline{x}) \neq \emptyset.$$

This means that for every $\alpha' > \alpha$ and $x \in (\overline{x} + \operatorname{cone} L) \cap U$,

$$F(x) \cap V \subset F(\overline{x}) - \left[0, \alpha' \|x - \overline{x}\|\right] \cdot M.$$

We have the following result for the problem (P_f) which is inspired by a technique used in [16].

Theorem 3.3. *Let $L \subset S_X$ be a nonempty closed set such that $\operatorname{cone} L$ is convex, and $M \subset S_Y$ be a nonempty closed set such that $M \cap -M = \emptyset$ and $\operatorname{cone} M$ is convex. Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (G^{-1}(-Q) \times Y)$ be a $(L, M)-$local directional Pareto minimum point for problem (P_f). Suppose that:*

(i) *there exist $\ell > 0, e \in M$ and U a neighborhood of $\overline{x}$ such that, for every $(x', x'') \in (U \cap [\overline{x} + \operatorname{cone} L]) \times \left(U \cap G^{-1}(-Q) \cap [\overline{x} + \operatorname{cone} L]\right)$,*

$$F\left(x'\right) + \ell \left\|x' - x''\right\| e \subset F\left(x''\right) + \operatorname{cone} M.$$

(ii) $\tilde{G}^{-1}$ *is is directionally calm at* $(0, \overline{x})$ *with respect to* S_Q *and* L *with constant* $\alpha > 0$.

Then, for any $\alpha' > \alpha$ *the point* $((\overline{x}, 0), \overline{y})$ *is a* $((L, S_Q), M)-$*local directional Pareto minimum point on* $\operatorname{Gr} \tilde{G}$ *for the mapping*

$$(x, z) \rightrightarrows F(x) + \ell\alpha' \|z\| e.$$

Proof. Denote by ε a positive constant such that all the local properties from hypotheses hold on balls with radius ε around the underlying points. Take $\alpha' > \alpha$. We have to show that $((\overline{x}, 0), \overline{y})$ is a $((L, S_Q), M)-$local directional Pareto minimum point on $\operatorname{Gr} \tilde{G}$ for the multifunction $\Gamma : X \times Z \rightrightarrows Y$,

$$\Gamma(x, z) := F(x) + \ell\alpha' \|z\| e.$$

Let $(x, z) \in \operatorname{Gr} \tilde{G} \cap B((\overline{x}, 0), \min\{2^{-1}, (2\alpha')^{-1}\} \cdot \varepsilon)$ with $x \in \overline{x} + \operatorname{cone} L$ and $z \in 0 + \operatorname{cone} S_Q = Q$ and suppose, by way of contradiction, that there exists $y \in F(x)$ such that

$$\overline{y} - y - \ell\alpha' \|z\| e \in \operatorname{cone} M \setminus \{0\}. \tag{3.6}$$

By the calmness of $\tilde{G}^{-1}$ at $(0, \overline{x})$ with respect to S_Q and L with constant $\alpha > 0$ one has that

$$x \in \tilde{G}^{-1}(z) \cap (B(\overline{x}, 2^{-1}\varepsilon)) \subset \tilde{G}^{-1}(0) - \left[0, \alpha' \|z\|\right] \cdot L.$$

Consequently, there exists $u \in \tilde{G}^{-1}(0)$ such that

$$u - x \in \left[0, \alpha' \|z\|\right] \cdot L.$$

In particular, $u \in G^{-1}(-Q)$ so u is a feasible point for the problem (P_f), and

$$\begin{aligned} u &\in x + \left[0, \alpha' \|z\|\right] \cdot L \subset \overline{x} + \operatorname{cone} L + \left[0, \alpha' \|z\|\right] \cdot L \\ &\subset \overline{x} + \operatorname{cone} L. \end{aligned}$$

Moreover,

$$\|u - \overline{x}\| \leq \|u - x\| + \|x - \overline{x}\| \leq 2^{-1}\varepsilon + 2^{-1}\varepsilon < \varepsilon.$$

Therefore, from the Lipschitz property of F (assumption (i)) there exists $v \in F(u)$ such that

$$y + \ell \|x - u\| e - v \in \operatorname{cone} M,$$

whence, by the inequality $\|x - u\| \leq \alpha' \|z\|$, and since $e \in M$,

$$y + \ell\alpha' \|z\| e - v \in \operatorname{cone} M. \tag{3.7}$$

Now, we add relations (3.6) and (3.7) and we get that

$$\overline{y} - v \in \operatorname{cone} M \setminus \{0\},$$

which contradicts the minimality of $\overline{x}$. □

Combining Theorems 3.1 and 3.3, we get the following consequence that represents a genuine penalization result for the problem (P_f).

Corollary 3.4. *Let* $L \subset S_X$ *be a nonempty closed set such that* $\operatorname{cone} L$ *is convex, and* $M \subset S_Y$ *be a nonempty closed set such that* $M \cap -M = \emptyset$ *and* $\operatorname{cone} M$ *is convex. Let* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (G^{-1}(-Q) \times Y)$ *be a* $(L, M)-$*local directional Pareto minimum point for problem* (P_f). *Suppose that*

(i) *there exist* $\ell > 0, e \in M$ *and* U *a neighborhood of* $\overline{x}$ *such that, for every* $(x', x'') \in (U \cap [\overline{x} + \operatorname{cone} L]) \times \left(U \cap \tilde{G}^{-1}(-Q \cup Q) \cap [\overline{x} + \operatorname{cone} L]\right)$,

$$F(x') + \ell \left\|x' - x''\right\| e \subset F(x'') + \operatorname{cone} M.$$

(ii) $\tilde{G}^{-1}$ *is is directionally calm at* $(0, \overline{x})$ *with respect to* S_Q *and* L *with constant* $\alpha > 0$.

(iii) $\operatorname{Gr} \tilde{G} \cap ([\overline{x} + \operatorname{cone} L] \times Q)$ *is locally closed at* $(\overline{x}, 0)$.

Then, for any $\alpha' > \alpha$ *and* $\lambda > \max\{\ell, \ell\alpha'\}$, *the point* $((\overline{x}, 0), \overline{y})$ *is an unconstrained* $((L, S_Q), M)-$*local directional Pareto minimum point for the mapping*

$$X \times Z \ni (x, z) \rightrightarrows F(x) + \ell\alpha' \|z\| e + \lambda d((x, z), \operatorname{Gr} \tilde{G} \cap [(x + \operatorname{cone} L) \times Q])e.$$

Proof. Consider again $\Gamma : X \times Z \rightrightarrows Y$,

$$\Gamma(x, z) := F(x) + \ell\alpha' \|z\| e.$$

Since $G^{-1}(-Q) \subset \tilde{G}^{-1}(-Q \cup Q)$, by means of Theorem 3.3, $((\overline{x}, 0), \overline{y})$ is a $((L, S_Q), M)-$local directional Pareto minimum point for Γ on $\operatorname{Gr} \tilde{G}$. Take

$$\begin{aligned}\left((x', z'), (x'', z'')\right) \in\ & ((U \times Z) \cap ((\overline{x}, 0) + \operatorname{cone}(L \times S_Q))) \\ & \times \left((U \times Z) \cap \operatorname{Gr} \tilde{G} \cap ((\overline{x}, 0) + \operatorname{cone}(L \times S_Q))\right).\end{aligned}$$

Since $z'' \in \tilde{G}(x'') + \operatorname{cone} S_Q = \tilde{G}(x'')$ and $z'' \in Q$ (whence $x'' \in \tilde{G}^{-1}(Q) \subset \tilde{G}^{-1}(-Q \cup Q)$), it is not difficult to see that, under assumption (i),

$$\Gamma(x', z') + \max\{\ell, \ell\alpha'\} \left\|(x', z') - (x'', z'')\right\| \subset \Gamma(x'', z'') + \operatorname{cone} M.$$

This means that Γ satisfies the vectorial Lipschitz property needed for the objective mapping in Theorem 3.1, with the constant $\max\{\ell, \ell\alpha'\}$. Therefore, using also assumption (iii), by Theorem 3.1 we get the result. □

4. Optimality conditions for functional constrained problems

In this section we obtain necessary optimality conditions for problem (P_f). In the above notation, define $\tilde{F} : X \rightrightarrows Y$ by $\tilde{F}(x) := F(x) + \operatorname{cone} M$. Consider the mapping $\left(\tilde{F}, \tilde{G}\right) : X \rightrightarrows Y \times Z$,

$$\left(\tilde{F}, \tilde{G}\right)(x) := \tilde{F}(x) \times \tilde{G}(x).$$

We have the following result.

Proposition 4.1. *Let* $L \subset S_X$ *be a nonempty closed set, and* $M \subset S_Y$ *be a nonempty closed set such that* $M \cap -M = \emptyset$ *and* $\operatorname{cone} M$ *is convex. Let* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (G^{-1}(-Q) \times Y)$ *and take* $\bar{q} \in Q \cap -G(\overline{x})$. *If* $(\overline{x}, \overline{y})$ *is a* $(L, M)-$*local directional Pareto minimum point for problem* (P_f), *then there is a neighborhood* U *of* $\overline{x}$ *such that*

$$\left[\left(\tilde{F}, \tilde{G}\right)(U \cap [\overline{x} + \operatorname{cone} L]) - (\overline{y}, -\bar{q})\right] \cap (-\operatorname{cone} M \times -Q) \subset \{0\} \times -Q.$$

Consequently, for any neighborhoods V *of* $\overline{y}$ *and* W *of* $-\bar{q}$, *and any point* $(v,q) \in M \times (Q\backslash\{0\})$, *the inclusion*

$$(V \times W) \cap [(\overline{y}, -\bar{q}) - \operatorname{cone}\{v, q\}] \subset \left(\tilde{F}, \tilde{G}\right)(U \cap [\overline{x} + \operatorname{cone} L])$$

cannot hold.

Proof. In order to prove the first conclusion, let U be the neighborhood of $\overline{x}$ for which the local minimality condition holds. Take any $x \in U \cap [\overline{x} + \operatorname{cone} L]$, and $(y,z) \in \left(\tilde{F}, \tilde{G}\right)(x)$ such that

$$(y,z) - (\overline{y}, -\bar{q}) \in (-\operatorname{cone} M \times -Q).$$

Then, one can find $y' \in F(x)$ and $u \in \operatorname{cone} M$ such that

$$y' + u - \overline{y} \in -\operatorname{cone} M,$$

and there are $q' \in Q$, $z' \in G(x)$ such that

$$z' + q' + \bar{q} \in -Q.$$

The last relation leads to $z' \in -Q$, whence $x \in U \cap [\overline{x} + \operatorname{cone} L] \cap G^{-1}(-Q)$. Moreover,

$$y' - \overline{y} \in -\operatorname{cone} M - u \subset -\operatorname{cone} M.$$

Therefore, by assumption,

$$y' - \overline{y} = 0,$$

and this implies as well that

$$u \in -\operatorname{cone} M \cap \operatorname{cone} M,$$

so $u = 0$. Consequently, $y = \overline{y}$ and the thesis follows.

For the second conclusion, take some neighborhoods V of $\overline{y}$ and W of $-\bar{q}$, take $(v,q) \in M \times (Q\backslash\{0\})$ and suppose, by way of contradiction, that

$$(V \times W) \cap [(\overline{y}, -\bar{q}) - \operatorname{cone}\{v, q\}] \subset \left(\tilde{F}, \tilde{G}\right)(U \cap [\overline{x} + \operatorname{cone} L]).$$

Then,

$$(V \times W) \cap [(\overline{y}, -\bar{q}) - \operatorname{cone}\{v, q\}] - (\overline{y}, -\bar{q}) \subset \left(\tilde{F}, \tilde{G}\right)(U \cap [\overline{x} + \operatorname{cone} L]) - (\overline{y}, -\bar{q}),$$

whence, from the above step,

$$[(V \times W) - (\overline{y}, -\bar{q})] \cap -\operatorname{cone}\{v, q\} \cap -(\operatorname{cone} M \times Q) \subset \{0\} \times -Q.$$

This means

$$[(V \times W) - (\overline{y}, -\bar{q})] \cap -\operatorname{cone}\{v, q\} \subset \{0\} \times -Q,$$

and, in particular,

$$(V - \overline{y}) \cap -\operatorname{cone}\{v\} \subset \{0\},$$

which is not possible. The proof is complete. $\square$

In what follows, we use the above proposition together with a result which is an adapted version of [7, Theorem 3.7] where the authors obtain a directional openness result for an epigraphical set-valued map, by using some methods from [6]. Before presenting the mentioned result, we need some preparation concerning generalized differentiation constructions developed by Mordukhovich and his collaborators (see [14]).

Definition 4.2. Let X be a normed vector space, S be a non-empty subset of X and let $x \in S$, $\varepsilon \geq 0$. The set of $\varepsilon-$normals to S at x is

$$\widehat{N}_\varepsilon(S,x) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{S} x} \frac{x^*(u-x)}{\|u-x\|} \leq \varepsilon \right\}. \tag{4.1}$$

If $\varepsilon = 0$, the elements in the right-hand side of (4.1) are called Fréchet normals, and $\widehat{N}(S,x) := \widehat{N}_0(S,x)$ is the Fréchet normal cone to S at x.

Let $\overline{x} \in S$. The limiting (or Mordukhovich) normal cone to S at $\overline{x}$ is

$$N(S,\overline{x}) := \{x^* \in X^* \mid \exists \varepsilon_n \downarrow 0, x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}_{\varepsilon_n}(S,x_n), \forall n \in \mathbb{N}\}.$$

If X is an Asplund space, and S is locally closed at $\overline{x}$, the formula for the limiting normal cone takes a simpler form, namely:

$$N(S,\overline{x}) = \{x^* \in X^* \mid \exists x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}(S,x_n), \forall n \in \mathbb{N}\}. \tag{4.2}$$

Definition 4.3. Let $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x},\overline{y}) \in \operatorname{Gr} F$. Then, the Fréchet coderivative at $(\overline{x},\overline{y})$ is the set-valued map $\widehat{D}^*F(\overline{x},\overline{y}) : Y^* \rightrightarrows X^*$ given by

$$\widehat{D}^*F(\overline{x},\overline{y})(y^*) := \{x^* \in X^* \mid (x^*,-y^*) \in \widehat{N}(\operatorname{Gr} F,(\overline{x},\overline{y}))\}.$$

Similarly, the normal coderivative of F at $(\overline{x},\overline{y})$ is the set-valued map $D^*F(\overline{x},\overline{y}) : Y^* \rightrightarrows X^*$ given by

$$D^*F(\overline{x},\overline{y})(y^*) := \{x^* \in X^* \mid (x^*,-y^*) \in N(\operatorname{Gr} F,(\overline{x},\overline{y}))\}.$$

The version we use of Theorem 3.7 from [7] reads as follows.

Theorem 4.4. *Let X, Y be finite dimensional, $L \subset S_X$, $M \subset S_Y$ be nonempty closed sets such that* $\operatorname{cone} L, \operatorname{cone} M$ *are convex and $M \cap -M = \emptyset$. Take $v \in M$ and let $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x},\overline{y}) \in \operatorname{Gr} F$. Suppose that:*

(i) $\operatorname{Gr} F$ *is locally closed at* $(\overline{x},\overline{y})$;

(ii) *there exist $c > 0, r > 0$ such that for all $y^* \in (\operatorname{cone} M)^+$ such that $y^*(v) = 1$, and for every $z^* \in B(0,2c)$, $(x,y) \in \operatorname{Gr} F \cap [B(\overline{x},r) \times B(\overline{y},r)]$ and $x^* \in \hat{D}^*F(x,y)(y^*+z^*)$ there is $u \in L$ that satisfies*

$$-x^*(u) \geq c\,\|y^*+z^*\|.$$

Then there is $\varepsilon > 0$ such that for all $a \in (0,c)$ and $\rho \in (0,\varepsilon)$ one has

$$\begin{aligned} B(\overline{y},a\rho) \cap [\overline{y} - \operatorname{cone}\{v\}] &\subset F(B(\overline{x},\rho) \cap [\overline{x} + \operatorname{cone} L]) + \operatorname{cone} M \\ &= \tilde{F}(B(\overline{x},\rho) \cap [\overline{x} + \operatorname{cone} L]). \end{aligned}$$

With these tools at hand, we propose the following result concerning necessary optimality conditions of Fritz John type for $(L,M)-$local directional Pareto minimality of problem (P_f).

Theorem 4.5. *Let X, Y, Z be finite dimensional, $L \subset S_X$, $M \subset S_Y$ be nonempty closed sets such that* $\operatorname{cone} L, \operatorname{cone} M$ *are convex and $M \cap -M = \emptyset$. Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (G^{-1}(-Q) \times Y)$ and take $\bar{q} \in Q \cap -G(\overline{x})$. Assume that* $\operatorname{int} \operatorname{cone} M \neq \emptyset$ *and* $\operatorname{int} Q \neq \emptyset$ *and consider $(v, q) \in \operatorname{int} \operatorname{cone} M \times \operatorname{int} Q$. Suppose that*

(i) $\operatorname{Gr} F$ *and* $\operatorname{Gr} G$ *are locally closed at $(\overline{x}, \overline{y})$ and $(\overline{x}, -\bar{q})$, respectively;*

(ii) *F and G are Lipschitz-like around $(\overline{x}, \overline{y})$ and $(\overline{x}, -\bar{q})$, respectively.*

If $(\overline{x}, \overline{y})$ is a $(L, M)-$local directional Pareto minimum point for problem (P_f), then there exist $x^* \in L^+$, $y^* \in (\operatorname{cone} M)^+$, $z^* \in Q^+$ such that $(y^*, z^*)(v, q) = 1$ and

$$x^* \in D^*F(\overline{x}, \overline{y})(y^*) + D^*G(\overline{x}, -\bar{q})(z^*).$$

Proof. The $(L, M)-$local minimality of $(\overline{x}, \overline{y})$ and Proposition 4.1 allow us to state that the set-valued map (F, G) does not satisfy the openness conclusion from Theorem 4.4, where instead of $\overline{y}$ one has $(\overline{y}, -q)$, instead of v one has (v, q) and instead of $\operatorname{cone} M$ one has $\operatorname{cone} M \times Q$. Therefore, since the closedness of $\operatorname{Gr}(F, G)$ at $(\overline{x}, \overline{y}, -\bar{q})$ is assumed, the other assumption of Theorem 4.4 does not hold. Consequently, for every $n \in \mathbb{N} \setminus \{0\}$, there exist $(x_n, y_n, z_n) \in \operatorname{Gr}(F, G) \cap B\left((\overline{x}, \overline{y}, -\bar{q}), n^{-1}\right)$, $(y_n^*, z_n^*) \in (\operatorname{cone} M \times Q)^+$, with $(y_n^*, z_n^*)(v, q) = 1$, $(v_n^*, w_n^*) \in B(0, 2n^{-1}) \subset Y^* \times Z^*$ and

$$x_n^* \in \widehat{D}^*(F, G)(x_n, y_n, z_n)((y_n^*, z_n^*) + (v_n^*, w_n^*))$$

such that, for any $u \in L$,

$$-x_n^*(u) < n^{-1} \left\| (y_n^*, z_n^*) + (v_n^*, w_n^*) \right\|. \tag{4.3}$$

Clearly, $v_n^* \to 0$, $w_n^* \to 0$. Due to the fact that $(v, q) \in \operatorname{int} \operatorname{cone} M \times \operatorname{int} Q$, [8, Lemma 2.2.17] ensures that both sequences (y_n^*) and (z_n^*) are bounded. Since the spaces are finite dimensional, we do not restrict the generality if we assume that $y_n^* \to y^* \in (\operatorname{cone} M)^+$ and $z_n^* \to z^* \in Q^+$. According to the definition of the Fréchet coderivative,

$$(x_n^*, -(y_n^* + v_n^*), -(z_n^* + w_n^*)) \in \widehat{N}(\operatorname{Gr}(F, G), (x_n, y_n, z_n)).$$

Define

$$C_1 := \{(x, y, z) \in X \times Y \times Z \mid y \in F(x)\}$$
$$C_2 := \{(x, y, z) \in X \times Y \times Z \mid z \in G(x)\},$$

and observe that

$$\operatorname{Gr}(F, G) = C_1 \cap C_2.$$

Since X, Y, Z are finite dimensional, the approximate sum rule for the Fréchet normals holds (see, e.g., [2]) and, consequently, there are the points $(x_{in}, y_{in}, z_{in}) \in B((x_n, y_n, z_n), n^{-1}) \cap C_i$, $i = 1, 2$, such that

$$\begin{aligned} \widehat{N}(\operatorname{Gr}(F, G), (x_n, y_n, z_n)) \subset \widehat{N}(C_1, (x_{1n}, y_{1n}, z_{1n})) + \widehat{N}(C_2, (x_{2n}, y_{2n}, z_{2n})) \\ + n^{-1} D_{X^* \times Y^* \times Z^*}. \end{aligned}$$

Then, there exist $(x_{in}^*, -y_{in}^*, -z_{in}^*) \in \widehat{N}(C_i, (x_{in}, y_{in}, z_{in}))$, $i = 1, 2$, and $(u_n^*, p_n^*, q_n^*) \in n^{-1} D_{X^* \times Y^* \times Z^*}$ such that

$$(x_n^*, -y_n^* - v_n^*, -z_n^* - w_n^*) = (x_{1n}^* + x_{2n}^* + u_n^*, -y_{1n}^* - y_{2n}^* + p_n^*, -z_{1n}^* - z_{2n}^* + q_n^*).$$

Furthermore,

$$\widehat{N}(C_1,(x_{1n},y_{1n},z_{1n})) = \widehat{N}(\operatorname{Gr} F,(x_{1n},y_{1n}))\times\{0\},$$
$$\widehat{N}(C_2,(x_{2n},y_{2n},z_{2n})) = \left\{(x^*,y^*,z^*) \mid (x^*,z^*)\in \widehat{N}(\operatorname{Gr} G,(x_{2n},z_{2n})), y^*=0\right\},$$

so $y_{2n}^*=0$ and $z_{1n}^*=0$. Therefore,

$$y_{1n}^* = y_n^* + v_n^* + p_n^* \to y^*,$$

and

$$z_{2n}^* = z_n^* + w_n^* + q_n^* \to z^*.$$

Next, we observe that the sequences $(x_{1n}^*),(x_{2n}^*)$ are bounded. Indeed, since for all n,

$$x_{1n}^* \in \widehat{D}^*F(x_{1n},y_{1n})(y_{1n}^*)$$

and

$$x_{2n}^* \in \widehat{D}^*G(x_{2n},y_{2n})(z_{2n}^*),$$

due to the assumption (ii), and [14, Theorem 1.43 (i)] one gets the boundedness of these sequences.

Consequently, because X is finite dimensional, we can suppose again, without loss of generality, that $(x_{1n}^*),(x_{2n}^*)$ are convergent to some $x_1^*,x_2^*\in X^*$. Taking into account the convergence of the sequences (x_{in},y_{in},z_{in}) towards $(\overline{x},\overline{y},-\overline{q})$ for $i=1,2$, one gets

$$x_1^* \in D^*F\left(\overline{x},\overline{y}\right)(y^*),$$
$$x_2^* \in D^*G\left(\overline{x},-\overline{q}\right)(z^*).$$

Since $(x_{1n}^*+x_{2n}^*) = (x_n^*-u_n^*)$ from (4.3) one deduces, by passing to the limit, that for all $u\in L$,

$$-x_1^*(u)-x_2^*(u)\le 0,$$

whence $x_1^*+x_2^*\in L^+$. Obviously, $(y^*,z^*)(v,q)=1$ and the proof ends. □

Remark 4.6. In the particular case when $L=S_X$, the conclusion of the above theorem reduces to

$$0\in D^*F\left(\overline{x},\overline{y}\right)(y^*) + D^*G\left(\overline{x},-\overline{q}\right)(z^*),$$

for some $y^*\in(\operatorname{cone} M)^+$ and $z^*\in Q^+$ with $(y^*,z^*)(v,q)=1$. In this particular case, we obtain Fritz John multipliers without being necessary to consider that both F and G have the Lipschitz-like property.

Indeed, suppose, for instance, that F is Lipschitz-like around $(\overline{x},\overline{y})$. Then, for any $n\in\mathbb{N}\setminus\{0\}$ and $u\in L$, we have

$$-x_n^*(u) < n^{-1}\left\|(y_n^*,z_n^*)+(v_n^*,w_n^*)\right\|,$$

for some $(y_n^*,z_n^*)\in(\operatorname{cone} M\times Q)^+$, $(y_n^*,z_n^*)(v,q)=1$, $(v_n^*,w_n^*)\in B(0,2n^{-1})\subset Y^*\times Z^*$. Replacing $u\in L$ by $-u\in L$ in the last inequality and using the definition of operator norm, we get that, for any $u\in L$,

$$\left|x_n^*(u)\right|\le\left\|x_n^*\right\|\le n^{-1}\left\|(y_n^*,z_n^*)+(v_n^*,w_n^*)\right\|.$$

Letting $n\to+\infty$ and knowing as above that (y_n^*,z_n^*) is a bounded sequence, one gets that (x_n^*) has the limit equal to 0. In addition, using the same notations exposed in previous proof, there exist $(x_{in}^*, -y_{in}^*, -z_{in}^*) \in \widehat{N}(C_i, (x_{in}, y_{in}, z_{in}))$, $i = 1, 2$, and $(u_n^*, p_n^*, q_n^*) \in n^{-1} D_{X^* \times Y^* \times Z^*}$ such that

$$(x_n^*, -y_n^* - v_n^*, -z_n^* - w_n^*) = (x_{1n}^* + x_{2n}^* + u_n^*, -y_{1n}^* + p_n^*, -z_{2n}^* + q_n^*)$$

with $y_{1n}^* \to y^*$ and $z_{2n}^* \to z^*$. Now, with the help of [14, Theorem 1.43 (i)], one easily obtains that (x_{1n}^*) is bounded because F is Lipschitz-like and, for any natural number n,

$$x_{1n}^* \in \widehat{D}^* F(x_{1n}, y_{1n})(y_{1n}^*).$$

Hence, the sequence $(x_{2n}^*) = (x_n^* - x_{1n}^* - u_n^*)$ is bounded. Further, the argument follows the same path as in the above proof. So, we conclude that $x_1^* + x_2^* \in L^+ = S_X^+ = \{0\}$ and $(y^*, z^*)(v, q) = 1$.

Now, if we suppose that G is Lipschitz-like, then the only difference from the above argument is that we obtain the boundedness of (x_{2n}^*) instead of (x_{1n}^*). Then, the sequence $(x_{1n}^*) = (x_n^* - x_{2n}^* - u_n^*)$ is bounded and the conclusion follows as before.

Remark 4.7. The existence of $y^* \neq 0$ is guaranteed under the hypothesis that F is Lipschitz-like around $(\overline{x}, \overline{y})$ and G is metrically regular around $(\overline{x}, -\overline{q})$. Let us prove this statement arguing by contradiction. Assuming $y^* = 0$, based on [14, Theorem 1.44 (i)], then $D^* F\left(\overline{x}, \overline{y}\right)(y^*) = \{0\}$. Now, we always have, for every $y^* \in Y^*$,

$$y^* \in D^* G\left(\overline{x}, -\overline{q}\right)(z^*) \quad \Leftrightarrow \quad -z^* \in D^* G^{-1}\left(-\overline{q}, \overline{x}\right)(-y^*).$$

In particular, for $y^* = 0$, we get that $-z^* \in D^* G^{-1}\left(-\overline{q}, \overline{x}\right)(0)$. Since G is metrically regular around $(\overline{x}, -\overline{q})$, it follows that G^{-1} is Lipschitz-like around $(-\overline{q}, \overline{x})$, one can see [14, Theorem 1.49 (i)], and hence

$$-z^* \in D^* G^{-1}\left(-\overline{q}, \overline{x}\right)(0) = \{0\}.$$

This gives a contradiction, since $(y^*, z^*)(v, q) = 1$. Hence $y^* \neq 0$.

Similar results can be deduced by combining the facts described in the Sections 2, 3, and 4 for other types of minima such as approximate minima or proper minima. Naturally, some new technical aspects can arise when discussing such results. To illustrate this assertion, we present an example of such a situation. More precisely, when one studies proper efficiency in the sense of Definition 2.12, it is natural that the multiplier associated to the objective mapping lays in $L_\varepsilon^+ = (\operatorname{cone} L_\varepsilon)^+$, and extra information can be obtained from this. We have the following result.

Proposition 4.8. *If* $x^* \in (\operatorname{cone} L_\varepsilon)^+$, *then for every* $u \in L$,

$$x^*(u) \geq (\varepsilon + 2)^{-1} \varepsilon \left\| x^* \right\|.$$

Proof. We have seen in Proposition 2.9 (v) that $\operatorname{cone} L \setminus \{0\} \subset \operatorname{int} \operatorname{cone} L_\varepsilon$. Actually, we can show the more precise fact that for any $u \in L$, $D(u, \rho) \subset \operatorname{cone} L_\varepsilon$, where $\rho := (\varepsilon + 2)^{-1} \varepsilon$. Indeed, take $u \in L$ and $v \in D(u, \rho)$. Then, there is $z \in D(0, \rho)$ such that $v = u + z$ and

$$\begin{aligned}\left\| u - \|v\|^{-1} v \right\| &= \left\| u - \|u + z\|^{-1} (u + z) \right\| = \|u + z\|^{-1} \left\| \|u + z\| u - u - z \right\| \\ &\leq (1 - \rho)^{-1} \left\| \left(\|u + z\| - 1 \right) u - z \right\| \\ &\leq (1 - \rho)^{-1} \left(\left| \|u + z\| - 1 \right| + \|z\| \right) \leq 2(1 - \rho)^{-1} \|z\|\end{aligned}$$

$$\leq 2\rho(1-\rho)^{-1} = \varepsilon.$$

Therefore, $\|v\|^{-1} v \in L_\varepsilon$, so $v \in \operatorname{cone} L_\varepsilon$.

Now, taking $x^* \in (\operatorname{cone} L_\varepsilon)^+$, one has that for all $u \in L$, and $z \in D(0,\rho)$,

$$x^*(u+z) \geq 0,$$

that is

$$x^*(u) \geq -x^*(z).$$

The latter relation leads to

$$x^*(u) \geq \sup_{z\in D(0,\rho)} x^*(z) = \rho \|x^*\| ,$$

and this is the conclusion. $\square$

5. Concluding remarks

The variety of directional vector efficiency concepts that can be defined by the approach we propose in this paper covers many of the standard types of Pareto efficiency. Moreover, by considering in input space a set of directions that, under some circumstances, can be enlarged one can speak about degrees of directional Pareto efficiency. The exact penalization results (Section 3) and necessary optimality conditions (Section 4) are illustrations, for prototype of (L, M)-directional Pareto efficiency, of some principles and techniques that can be extended and specialized for other types of minimality as proper directional minimality and approximate directional minimality. More developments on these topis will be the subject of future research.

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