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ON THE TRANSFINITE MEAN VALUE INTERPOLATION OF DYKEN AND FLOATER

MICHEL C. DELFOUR AND ANDRÉ GARON

ABSTRACT. The object of this paper is the *Transfinite Mean Value Interpolation* (TMI) introduced by Dyken and Floater [3] and its generalization (k-TMI) by Delfour and Garon [2] for a special family of vector weight functions. Mathematically, it amounts to construct a continuous extension of a continuous function f defined on the compact locally Lipschitzian boundary Γ of an open subset Ω of $\mathbb{R}^n$ to Ω, to its complement Ω^c, or to $\mathbb{R}^n$. The extension property has been proved for Ω convex and for n-polytopes which are not necessarily convex. In general, it requires an additional local boundedness condition on Γ, but an explicit characterization of such Γ's is not yet available. In this paper we first prove that, if f is Lipschitz continuous on Γ, no additional condition on Γ is required for the continuous interpolation from Γ to all $\mathbb{R}^n$. In a second part, we prove that for $m \geq 1$ and $k > n+m$ the partial derivatives of the *enhanced* (m, k)-TMI introduced by Delfour and Garon [2] continuously interpolate the corresponding partial derivatives of f up to order m, when f and its partial derivatives up to order m are Lipschitz continuous in a tubular neighbourhood of Γ. Our construction completely solves the problem raised by Floater and Schulz [5] in 2008.

1. Introduction

The *Transfinite Mean Value Interpolation* (TMI) was introduced by Dyken and Floater [3] in 2009 and Bruvoll and Floater [1] in 2010 in the context of Imaging and finite elements mesh adaptation. Given an open subset Ω of $\mathbb{R}^n$ with compact locally Lipschitzian boundary Γ and a continuous function f on Γ, they introduced the infinitely continuously differentiable function

$$\hat{F}(y) \overset{\text{def}}{=} \frac{\int_\Gamma f(\xi) \frac{(\xi-y)\cdot n_\Omega(\xi)}{\|\xi-y\|^{n+1}}\, d\Gamma}{\int_\Gamma \frac{(\xi-y)\cdot n_\Omega(\xi)}{\|\xi-y\|^{n+1}}\, d\Gamma}, \quad y \in \Omega\backslash\Gamma, \tag{1.1}$$

where $n_\Omega(\xi)$ is the unit exterior normal to Ω, and some conditions on Γ to make it a continuous extension of f from Γ to Ω. This problem was generalized by Delfour and Garon [2] from the exponent $(n+1)$ to a real exponent $k > n$ (k-TMI) for the special family of vector weight functions $x/\|x\|^k$ with a relaxation of some of the conditions in [3]. Conditions were also given for the continuous interpolation to the complement $\Omega^c = \mathbb{R}^n\backslash\overline{\Omega}$ of Ω and to the whole space $\mathbb{R}^n$.

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In order to establish that $\hat{F}$ continuously interpolates f, the limit

$$\lim_{\substack{y\to x\\ y\in\Omega\setminus\Gamma}} \frac{\int_\Gamma f(\xi)\frac{(\xi-y)\cdot n_\Omega(\xi)}{\|\xi-y\|^k}\,d\Gamma}{\int_\Gamma \frac{(\xi-y)\cdot n_\Omega(\xi)}{\|\xi-y\|^k}\,d\Gamma} = f(x)$$

must exists for all $x \in \Gamma$ and all continuous functions f. This is true for Ω convex ([3]) and for n-polytopes ([7]) that are not necessarily convex, but, in general, some non-trivial additional conditions on Γ seem to be required and a complete explicit characterization of such Γ's is still not available.

In this paper we first show that the continuous interpolation from Γ to $\mathbb{R}^n$ is obtained for Lipschitz continuous functions $f : \Gamma \to \mathbb{R}^p$, $p \geq 1$, without additional assumptions on Γ. This indicates that there is some trade-off between the properties of f and Γ. The proof is not trivial and requires a theorem of independent interest for the mean value interpolation of a bounded continuous function $f : \overline{O} \to \mathbb{R}^p$, where O is an open subset of $\mathbb{R}^n$ with compact locally Lipschitzian boundary Γ.

In a second part, we consider the *enhanced* (m, k)-TMI

$$\hat{F}(y) \stackrel{\text{def}}{=} \int_\Gamma \left[f(\xi) + \sum_{\ell=1}^{m} \sum_{\alpha\in\mathbb{N}^n,\, |\alpha|=\ell} \frac{1}{\alpha!}\partial^\alpha f(\xi)(y-\xi)^\alpha \right] \frac{\frac{y-\xi}{\|y-\xi\|^k}\cdot n_\Omega(\xi)}{\phi(y)}\,d\Gamma,$$

$$\phi(y) \stackrel{\text{def}}{=} \int_\Gamma \varphi(y-\xi)\cdot n_\Omega(\xi)\,d\Gamma, \quad y \in \mathbb{R}^n\setminus\Gamma,$$

introduced in [2] for an integer $m \geq 1$, a real number $k > n + m$, and a function f such that f and its partial derivatives up to order m are Lipschitz continuous in a tubular neighbourhood of Γ, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a *multi-index*, $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$ for $x \in \mathbb{R}^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$, and $\alpha! = \alpha_1!\,\alpha_2!\ldots\alpha_n!$. We prove that the partial derivatives of $\hat{F}$ up to order m continuously interpolate the corresponding partial derivatives of f up to order m. Our construction solves the problem raised by Floater and Schulz [5] in 2008.

2. Interpolation from an open subset O of $\mathbb{R}^n$

Theorem 2.1. *Let O be an open subset of $\mathbb{R}^n$ with compact locally Lipschitzian boundary ∂O, $k > n$ a real number, and $g : \overline{O} \to \mathbb{R}^p$, $p \geq 1$, a bounded continuous function. Then the function*

$$\widetilde{G}(y) \stackrel{\text{def}}{=} \begin{cases} g(y), & y \in \overline{O}, \\ \dfrac{\int_O g(\xi)\,\frac{1}{\|\xi-y\|^k}\,d\xi}{\int_O \frac{1}{\|\xi-y\|^k}\,d\xi}, & y \in \mathbb{R}^n\setminus\overline{O}, \end{cases} \tag{2.1}$$

is bounded and continuously interpolates g from $\overline{O}$ to $\mathbb{R}^n$.

Proof. For $k > n$, the following integral is finite since for $y \in \mathbb{R}^n\setminus\overline{O}$

$$\begin{aligned} 0 < \int_O \frac{1}{\|\xi-y\|^k}\,d\xi &\leq \int_{\mathbb{R}^n\setminus B_{d_{\partial O}(y)}(y)} \frac{1}{\|y-\xi\|^k}\,d\xi \\ &= \int_{d_{\partial O}(y)}^\infty \frac{1}{\rho^k}\,\beta_n\,\rho^{n-1}\,d\rho = \frac{\beta_n}{k-n}\frac{1}{d_{\partial O}(y)^{k-n}}, \end{aligned} \tag{2.2}$$

where β_n is the surface area of the unit sphere in $\mathbb{R}^n$. Hence, the integral

$$G(y) \overset{\text{def}}{=} \frac{\int_O g(\xi)\, \frac{1}{\|\xi-y\|^k}\, d\xi}{\int_O \frac{1}{\|\xi-y\|^k}\, d\xi}, \quad y \in \mathbb{R}^n \backslash \overline{O},$$

is well-defined and finite for g bounded and continuous. For $x \in \partial O$,

$$G(y) - g(x) = \frac{\int_O [g(\xi)-g(x)]\, \frac{1}{\|\xi-y\|^k}\, d\xi}{\int_O \frac{1}{\|\xi-y\|^k}\, d\xi}, \quad y \in \mathbb{R}^n \backslash \overline{O},$$

and for $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $\xi \in \overline{O}$ such that $\|\xi - x\| < \delta$, $\|g(\xi) - g(x)\| < \varepsilon$. So, we have the following estimate

$$\begin{aligned} \|G(y) - g(x)\| \leq & \frac{\int_{O\cap B_\delta(x)} \|g(\xi)-g(x)\|\, \frac{1}{\|\xi-y\|^k}\, d\xi}{\int_O \frac{1}{\|\xi-y\|^k}\, d\xi} \\ & + \frac{\int_{O\backslash B_\delta(x)} \|g(\xi)-g(x)\|\, \frac{1}{\|\xi-y\|^k}\, d\xi}{\int_O \frac{1}{\|\xi-y\|^k}\, d\xi} \\ \leq & \, \varepsilon + 2 \sup_{\zeta\in\overline{O}} \|g(\zeta)\| \frac{\int_{O\backslash B_\delta(x)} \frac{1}{\|\xi-y\|^k}\, d\xi}{\int_O \frac{1}{\|\xi-y\|^k}\, d\xi}. \end{aligned}$$

By assumption, g is bounded in $\overline{O}$. For $k > n$ and ∂O compact and locally Lipschitzian, the denominator of the second term goes to infinity as $y \to x$ (cf. [2, Thm. 2.7]).[1] Its numerator is bounded for $\|\xi - x\| \geq \delta$ and $\|y - x\| < \delta/2$. Indeed, for $\|y - x\| < \delta/2$,

$$\begin{aligned} \int_{O\backslash B_\delta(x)} \frac{1}{\|\xi-y\|^k}\, d\xi \leq & \int_{\mathbb{R}^n\backslash B_\delta(x)} \frac{1}{\|\xi-y\|^k}\, d\xi \\ \leq & \int_{\mathbb{R}^n\backslash B_\delta(x)} \frac{1}{(\|\xi-x\|-\delta/2)^k}\, d\xi. \\ = & \int_\delta^{+\infty} \frac{1}{(\rho-\delta/2)^k}\, \beta_n\, \rho^{n-1}\, d\rho \\ = & \frac{\delta^n}{\delta^k}\beta_n \int_1^{+\infty} \left(\frac{\rho}{\rho-1/2}\right)^k \rho^{n-1-k}\, d\rho \\ \leq & \frac{2^k}{\delta^{k-n}}\, \beta_n \int_1^{+\infty} \rho^{n-1-k}\, d\rho = \frac{2^k}{\delta^{k-n}}\, \frac{\beta_n}{k-n} < \infty. \end{aligned}$$

Therefore, $G(y) \to g(x)$ as $y \to x$ and the continuous bounded function $\widetilde{G}$ defined in (2.1) continuously interpolates g from $\overline{O}$ to $\mathbb{R}^n$. □

[1]From the proof of [2, Thm. 2.7] for an open subset Ω of $\mathbb{R}^n$ with compact locally Lipschitzian boundary

$$\int_O \frac{1}{\|\xi-y\|^k}\, d\xi \geq \frac{c(\theta)}{d_{\partial O}(y)^{k-n}}\, \frac{1}{n\, 2^k}.$$

where $c(\theta)$) is the n-volume of the conical sector of angle θ and radius 1.

3. The k-TMI for a Lipschitz function $f : \Gamma \to \mathbb{R}^p$

In view of the computations of Dyken and Floater [3] to establish that the $(n+1)$-TMI is an interpolation, some assumptions on the fluctuations of the boundary Γ and/or of the function f are needed. The limit

$$\lim_{y\to x,\, y\in\Omega} \int_\Gamma f(\xi)\, \frac{\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)}{\int_\Gamma \frac{\zeta-y}{\|\zeta-y\|^k}\cdot n_\Omega(\zeta)\, d\Gamma}\, d\Gamma = f(x)$$

must exist for all continuous functions $f : \Gamma \to \mathbb{R}^p$ and all $x \in \Gamma$.

In [2] the following local boundedness condition was used for H^{n-1}-almost all[2] $x \in \Gamma$: there exists $\delta = \delta(x) > 0$ and $c(x) > 0$ such that

$$\forall y \in B_\delta(x) \cap \Omega, \quad \frac{\int_\Gamma \left| \frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\right| d\Gamma}{\int_\Gamma \frac{\zeta-y}{\|\zeta-y\|^k}\cdot n_\Omega(\zeta)\, d\Gamma} \le c(x). \tag{3.1}$$

This is true for Ω convex ([3]) and for n-polytopes ([7] for all $y \in B_\delta(x)\backslash\Gamma$) that are not necessarily convex, but, in general, some additional conditions on Γ seem to be required. Condition (3.1) forces the linear functionals

$$f \mapsto \int_\Gamma f(\xi)\, \frac{\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)}{\int_\Gamma \frac{\zeta-y}{\|\zeta-y\|^k}\cdot n_\Omega(\zeta)\, d\Gamma}\, d\Gamma : C^0(\Gamma) \to \mathbb{R} \tag{3.2}$$

to be continuous and uniformly bounded for all $y \in B_\delta(x) \cap \Omega$.

It turns out that, under an additional condition on f, the continuous interpolation occurs in $\mathbb{R}^n$ when Γ is only compact and locally Lipschitzian. Let $C^{0,1}(\Gamma;\mathbb{R}^p)$, $p \ge 1$ an integer, denote the vector space of Lipschitz continuous functions from Γ to $\mathbb{R}^p$.

Theorem 3.1. *Let Ω be an open subset of $\mathbb{R}^n$ with compact locally Lipschitzian boundary Γ, $k > n$ a real number, and $f \in C^{0,1}(\Gamma;\mathbb{R}^p)$, $p \ge 1$, with Lipschitz constant $c(f;\Gamma)$. Then*

$$\hat{F}(y) \overset{\text{def}}{=} \int_\Gamma f(\xi)\, \frac{\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)}{\int_\Gamma \frac{\zeta-y}{\|\zeta-y\|^k}\cdot n_\Omega(\zeta)\, d\Gamma}\, d\Gamma, \quad y \in \mathbb{R}^n\backslash\Gamma, \tag{3.3}$$

continuously interpolates f from Γ to $\mathbb{R}^n$.

Proof. We want to use the divergence theorem to change the integrals over Γ into integrals over $\Omega^c = \mathbb{R}^n\backslash\overline{\Omega}$ and apply Theorem 2.1. To do that we need a bounded continuous extension of f to $\overline{\Omega^c} = \mathbb{R}^n\backslash\Omega$. Given a Lipschitz continuous function $f : \Gamma \to \mathbb{R}^p$, there exists a Lipschitzian extension to $\mathbb{R}^n$ with Lipschitz constant $c(\bar{f}) \le \sqrt{m}\, c(f;\Gamma)$ ([4, Thm. 1, p. 80]) but their extension is not bounded in $\overline{\Omega^c}$ when Ω^c is not bounded. In order to apply Theorem 2.1, we need to modify $\bar{f}$ away from Γ to obtain a function bounded in $\mathbb{R}^n$. We use the following *cut-off function* for some fixed $h > 0$

$$r \mapsto s(r) \overset{\text{def}}{=} \left\{\begin{matrix} 1 - r/h, & 0 \le r < h \\ 0, & r \ge h \end{matrix}\right\} : [0,\infty) \to \mathbb{R} \tag{3.4}$$

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[2] H^{n-1} denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

with Lipschitz constant $1/h$ and the Lipschitz function

$$y \mapsto s(d_\Gamma(y)) : \mathbb{R}^n \to \mathbb{R}, \quad d_\Gamma(x) \overset{\text{def}}{=} \inf_{y\in\Gamma} \|y-x\|. \tag{3.5}$$

Since d_Γ is also Lipschitzian of constant one, we have

$$|s(d_\Gamma(y)) - s(d_\Gamma(y'))| \le \|y-y'\|/h, \quad \nabla_y(s(d_\Gamma(y)) = s'(d_\Gamma(y))\,\nabla_y d_\Gamma(y).$$

Finally, define the extension

$$x \mapsto \tilde f(x) \overset{\text{def}}{=} \bar f(x)\, s(d_\Gamma(x)) : \mathbb{R}^n \to \mathbb{R}^p. \tag{3.6}$$

Since Γ is compact, the tubular neighbourhood $\{x \in \mathbb{R}^n : d_\Gamma(x) \le h\}$ is compact, the support $\operatorname{supp} \tilde f \subset \{x \in \mathbb{R}^n : d_\Gamma(x) \le h\}$ is compact, $\tilde f$ is continuous and bounded in $\mathbb{R}^n$, and $\bar f$ is bounded in $\{x \in \mathbb{R}^n : d_\Gamma(x) \le h\}$. It remains to show that $\tilde f$ is Lipschitz in $\mathbb{R}^n$. For x such that $d_\Gamma(x) \le h$

$$\begin{aligned}
\tilde f(y) - \tilde f(x) &= [\bar f(y) - \bar f(x)]\, s(d_\Gamma(y)) + \bar f(x)\,[s(d_\Gamma(y)) - s(d_\Gamma(x))] \\
\|\tilde f(y) - \tilde f(x)\| &\le \|\bar f(y) - \bar f(x)\| \sup_{y\in\mathbb{R}^n} |s(d_\Gamma(y))| \\
&\quad + \left(\sup_{d_\Gamma(x)\le h} \|\bar f(x\|\right) |s(d_\Gamma(y)) - s(d_\Gamma(x))| \\
&\le c(\bar f)\,\|y-x\| \underbrace{\sup_{y\in\mathbb{R}^n} |s(d_\Gamma(y))|}_{\le 1} + \sup_{d_\Gamma(z)\le h} \|\bar f(z)\|\,\frac{1}{h}\,\|y-x\|.
\end{aligned}$$

By interchanging the role of x and y, for y such that $d_\Gamma(y) \le h$

$$\|\tilde f(y) - \tilde f(x)\| \le \left[c(\bar f) + \sup_{d_\Gamma(z)\le h} \|\bar f(z)\|\,\frac{1}{h}\right] \|y-x\|.$$

For $d_\Gamma(x) > h$ and $d_\Gamma(y) > h$, $\tilde f(y) - \tilde f(x) = 0$. Finally,

$$\|\tilde f(y) - \tilde f(x)\| \le c(\tilde f)\,\|y-x\|, \quad c(\tilde f) \overset{\text{def}}{=} c(\bar f) + \sup_{d_\Gamma(z)\le h} \|\bar f(z)\|\,\frac{1}{h}.$$

So $\tilde f$ is Lipschitzian and bounded in $\mathbb{R}^n$. Since $\tilde f = \bar f = f$ on Γ, we can replace $f : \Gamma \to \mathbb{R}^p$ by $\tilde f : \mathbb{R}^n \to \mathbb{R}^p$ in the definition (3.3) of $\hat F$.

It is now sufficient to prove the theorem for $p = 1$. Since for $k > n$ and $y \in \Omega$, the function $\xi \mapsto 1/\|\xi - y\|^k$ is integrable in the complement $\Omega^c = \mathbb{R}^n \backslash \overline{\Omega}$, use the divergence theorem for the numerator and the denominator

$$\begin{aligned}
\hat F(y) &= \frac{-\int_{\Omega^c} \operatorname{div}_\xi \left[\tilde f(\xi)\, \frac{\xi-y}{\|\xi-y\|^k}\right] d\xi}{-\int_{\Omega^c} \operatorname{div}_\xi \frac{\xi-y}{\|\xi-y\|^k}\, d\xi} \\
&= \frac{\int_{\Omega^c} \nabla \tilde f(\xi) \cdot \frac{\xi-y}{\|\xi-y\|^k}\, d\xi}{\int_{\Omega^c} \operatorname{div}_\xi \frac{\xi-y}{\|\xi-y\|^k}\, d\xi} + \frac{\int_{\Omega^c} \tilde f(\xi) \operatorname{div}_\xi \frac{\xi-y}{\|\xi-y\|^k}\, d\xi}{\int_{\Omega^c} \operatorname{div}_\xi \frac{\xi-y}{\|\xi-y\|^k}\, d\xi}.
\end{aligned}$$

But $\operatorname{div}_\xi \frac{\xi-y}{\|\xi-y\|^k} = (n-k)/\|x-\xi\|^k$ and

$$\hat{F}(y) = \frac{\int_{\Omega^c} \nabla\tilde{f}(\xi)\cdot(\xi-y)\,\frac{1}{\|\xi-y\|^k}\,d\xi}{(n-k)\int_{\Omega^c}\frac{1}{\|\xi-y\|^k}\,d\xi} + \frac{(n-k)\int_{\Omega^c}\tilde{f}(\xi)\,\frac{1}{\|\xi-y\|^k}\,d\xi}{(n-k)\int_{\Omega^c}\frac{1}{\|\xi-y\|^k}\,d\xi}.$$

As $\tilde{f}$ is continuous and bounded in $\mathbb{R}^n\backslash\Omega$, the second integral converges to $\tilde{f}(x) = f(x)$ as $y \to x \in \Gamma$ from Theorem 2.1. As for the first integral, it goes to zero as $y \to x \in \Gamma$. The gradient $\nabla\tilde{f}$ is bounded almost everywhere by the constant $c(\tilde{f})$ and is zero on the set $\{\xi \in \mathbb{R}^n : d_\Gamma(\xi) \geq h\}$. Therefore,

$$\left|\frac{\int_{\Omega^c}\nabla\tilde{f}(\xi)\cdot(\xi-y)\cdot\frac{1}{\|\xi-y\|^k}\,d\xi}{(n-k)\int_{\Omega^c}\frac{1}{\|\xi-y\|^k}\,d\xi}\right| \leq \frac{c(\tilde{f})}{k-n}\,\frac{\int_{\{\xi\in\Omega^c:d_\Gamma(\xi)<h\}}\frac{1}{\|\xi-y\|^{k-1}}\,d\xi}{\int_{\Omega^c}\frac{1}{\|\xi-y\|^k}\,d\xi}.$$

Since Γ is compact, $\{\xi \in \mathbb{R}^n\backslash\Omega : d_\Gamma(\xi) \leq h\}$ is compact and there exists a sufficiently small $\delta > 0$ and a sufficiently large $R > 0$ such that, for all $y \in B_\delta(x)$, $\{\xi \in \mathbb{R}^n\backslash\Omega : d_\Gamma(\xi) \leq h\} \subset B_R(y)$. So

$$\begin{aligned}\int_{\{\xi\in\Omega^c:d_\Gamma(\xi)<h\}}\frac{1}{\|\xi-y\|^{k-1}}\,d\xi &\leq \int_{B_R(y)\backslash B_{d_\Gamma(y)}(y)}\frac{1}{\|\xi-y\|^{k-1}}\,d\xi\\ &= \int_{d_\Gamma(y)}^R \frac{1}{\rho^{k-1}}\,\beta_n\,\rho^{n-1}\,d\rho\\ &= \frac{\beta_n}{k-1-n}\left[\frac{1}{d_\Gamma(y)^{k-1-n}} - \frac{1}{R^{k-1-n}}\right].\end{aligned}$$

Finally, from [2] the denominator is greater or equal to $1/(c\,d_\Gamma(y)^{k-n})$ and

$$\begin{aligned}\frac{\int_{\{\xi\in\Omega^c:d_\Gamma(\xi)<h\}}\frac{1}{\|\xi-y\|^{k-1}}\,d\xi}{\int_{\Omega^c}\frac{1}{\|\xi-y\|^k}\,d\xi} &\leq \frac{\beta_n}{k-1-n}\left[\frac{1}{d_\Gamma(y)^{k-1-n}} - \frac{1}{R^{k-1-n}}\right]c\,d_\Gamma(y)^{k-n}\\ &= \frac{\beta_n\,c}{k-1-n}\left[d_\Gamma(y) - \frac{d_\Gamma(y)^{k-n}}{R^{k-1-n}}\right] \to 0\end{aligned}$$

as $y \to x \in \Gamma$ for $k > n$. So, $\hat{F}$ continuously interpolates f from Γ to Ω. The proof that $\hat{F}$ continuously interpolates f from Γ to Ω^c is similar. $\square$

An interesting consequence of the construction in the proof of the last theorem is that it provides a new way to compute the k-TMI interpolant. We need the function $\tilde{g}(x) = s(d_\Gamma(x))$ which is equal to 1 on Γ for the denominator. Now

$$\begin{aligned}\frac{\int_\Gamma f(\xi)\,\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\,d\Gamma}{\int_\Gamma\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\,d\Gamma} &= \frac{\int_\Gamma \tilde{f}(\xi)\,\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\,d\Gamma}{\int_\Gamma\tilde{g}(\xi)\frac{\zeta-y}{\|\zeta-y\|^k}\cdot n_\Omega(\zeta)\,d\Gamma}\\ &= \frac{\int_{\{\xi\in\Omega^c:d_\Gamma(\xi)<h\}}\left[(n-k)\,\tilde{f}(\xi)+\nabla\tilde{f}(\xi)\cdot(\xi-x)\right]\frac{1}{\|\xi-y\|^k}\,d\xi}{\int_{\{\xi\in\Omega^c:d_\Gamma(\xi)<h\}}\left[(n-k)\,\tilde{g}(\xi)+\nabla\tilde{g}(\xi)\cdot(\xi-x)\right]\frac{1}{\|\xi-y\|^k}\,d\xi}\\ &= \frac{\int_{\{\xi\in\Omega^c:d_\Gamma(\xi)<h\}}\left[\tilde{f}(\xi)+\frac{1}{k-n}\,\nabla\tilde{f}(\xi)\cdot(x-\xi)\right]\frac{1}{\|\xi-y\|^k}\,d\xi}{\int_{\{\xi\in\Omega^c:d_\Gamma(\xi)<h\}}\left[\tilde{g}(\xi)+\frac{1}{k-n}\,\nabla\tilde{g}(\xi)\cdot(x-\xi)\right]\frac{1}{\|\xi-y\|^k}\,d\xi}.\end{aligned}$$

For $k = n + 1$ the formula is similar to the enhanced (m, k)-TBI with $E = \{\xi \in \mathbb{R}^n \backslash \Omega : d_\Gamma(\xi) \leq h\}$ and $m = 1$ ([2, sec. 4.1]) but E has dimension n

$$\frac{\int_\Gamma f(\xi)\, \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, d\Gamma}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, d\Gamma} = \frac{\int_{\{\xi\in\Omega^c : d_\Gamma(\xi)<h\}} \left[\tilde{f}(\xi) + \nabla\tilde{f}(\xi)\cdot(x-\xi)\right] \frac{1}{\|\xi-y\|^k}\, d\xi}{\int_{\{\xi\in\Omega^c : d_\Gamma(\xi)<h\}} \left[\tilde{g}(\xi) + \nabla\tilde{g}(\xi)\cdot(x-\xi)\right] \frac{1}{\|\xi-y\|^k}\, d\xi}.$$

Both volume formulae do not require a knowledge of the normal.

In a finite element set up in dimension $n = 2$ with triangular elements, f and 1 can be approximated by piecewise linear functions through each boundary node. Construct a layer of triangles next to Γ of thickness roughly h in Ω^c. Construct on each triangle linear functions $\tilde{f}$ and $\tilde{g}$ with value 0 at the nodes in Ω^c and matching f and 1 at the boundary nodes. So, the gradient is constant in each triangle and the above formula is easy to implement. The parameter h is arbitrary but is bounded above by some constant $\bar{h}$ that depends on the locally Lipschizian compact boundary Γ.

4. The Enhanced (m, k)-TMI

The Enhanced (m, k)-TMI introduced in [6] is solving a problem raised by M. S. Floater and C. Schulz [5] in 2008. We have already proved that it preserves $P^{m+1}(\mathbb{R}^n)$, the space of polynomials on $\mathbb{R}^n$ of degree less than or equal to $m+1$ and $P^{m+1}(\mathbb{R}^n : \mathbb{R}^p)$, the space of polynomials on $\mathbb{R}^n$ into $\mathbb{R}^p$ of degree less than or equal to $m+1$ for $p > 1$. We now complete the picture with the following theorem for the interpolation of the partial derivatives of f.

Theorem 4.1. *Let Ω be an open subset of $\mathbb{R}^n$ with compact locally Lipschitzian boundary Γ, $m \geq 1$ an integer, $k > n+m$ a real number, and a function f such that f and its partial derivatives up to order m are Lipschitz continuous in a tubular neighbourhood $\{y \in \mathbb{R}^n : d_\Gamma(y) \leq h\}$ of Γ of thickness $h > 0$. Then the partial derivatives of the function*

$$\hat{F}(y) \stackrel{\text{def}}{=} \int_\Gamma \left[f(\xi) + \sum_{\ell=1}^{m} \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=\ell}} \frac{1}{\alpha!} \partial^\alpha f(\xi)(y-\xi)^\alpha \right] \frac{\frac{y-\xi}{\|y-\xi\|^k} \cdot n_\Omega(\xi)}{\phi(y)}\, d\Gamma, \tag{4.1}$$

$$\phi(y) \stackrel{\text{def}}{=} \int_\Gamma \frac{y-\xi}{\|y-\xi\|^k} \cdot n_\Omega(\xi)\, d\Gamma, \quad y \in \mathbb{R}^n \backslash \Gamma, \tag{4.2}$$

continuously interpolate the corresponding partial derivatives of f up to order m. By applying the theorem component by component, the results also hold for a vector function $f : \{y \in \mathbb{R}^n : d_\Gamma(y) \leq h\} \to \mathbb{R}^p$, $p > 1$.

Remark 4.2. Note that the assumption on f is verified for any polynomial vector function $f : \mathbb{R}^n \to \mathbb{R}^p$. □

Proof. We give the proof for $p = 1$ and $m = 1$. The cases $m > 1$ use the same technique but the number of terms increases beyond what is reasonable. By definition

$$\begin{aligned}
&\hat{F}(y) - f(y)\\
=&\frac{1}{\phi(y)} \int_\Gamma [f(\xi) + \nabla f(\xi)\cdot(y-\xi) - f(y)] \, \frac{y-\xi}{\|y-\xi\|^k} \cdot n_\Omega(\xi)\, d\Gamma\\
=&-\frac{1}{\phi(y)} \int_{\Omega^c} \operatorname{div}_\xi \left([f(\xi) + \nabla f(\xi)\cdot(y-\xi) - f(y)] \, \frac{y-\xi}{\|y-\xi\|^k} \right) d\xi\\
=&-\frac{1}{\phi(y)} \int_{\Omega^c} [f(\xi) + \nabla f(\xi)\cdot(y-\xi) - f(y)] \operatorname{div}_\xi \left(\frac{y-\xi}{\|y-\xi\|^k} \right) d\xi\\
&-\frac{1}{\phi(y)} \int_{\Omega^c} D^2 f(\xi)\,(y-\xi)\cdot \frac{y-\xi}{\|y-\xi\|^k}\, d\xi\\
=&-\frac{1}{\phi(y)} \int_{\Omega^c} [f(\xi) + \nabla f(\xi)\cdot(y-\xi) - f(y)] \, (n-k)\, \frac{1}{\|y-\xi\|^k}\, d\xi\\
&-\frac{1}{\phi(y)} \int_{\Omega^c} D^2 f(\xi)\,(y-\xi)\cdot(y-\xi)\, \frac{1}{\|y-\xi\|^k}\, d\xi.
\end{aligned}$$

For the gradients

$$\begin{aligned}
&\nabla\hat{F}(y) - \nabla f(y)\\
=&-\frac{1}{\phi(y)} \int_{\Omega^c} [\nabla f(\xi) - \nabla f(y)] \, (n-k)\, \frac{1}{\|y-\xi\|^k}\, d\xi\\
&-\frac{1}{\phi(y)} \int_{\Omega^c} [f(\xi) + \nabla f(\xi)\cdot(y-\xi) - f(y)] \, (n-k)\,(-k) \frac{y-\xi}{\|y-\xi\|^{k+2}}\, d\xi\\
&-\frac{1}{\phi(y)} \int_{\Omega^c} 2\, D^2 f(\xi)(y-\xi)\, \frac{1}{\|y-\xi\|^k}\, d\xi\\
&-\frac{1}{\phi(y)} \int_{\Omega^c} D^2 f(\xi)\,(y-\xi)\cdot(y-\xi)\,(-k) \frac{y-\xi}{\|y-\xi\|^{k+2}}\, d\xi\\
&-\frac{\nabla\phi(y)}{\phi(y)^2} \int_\Gamma [f(\xi) + \nabla f(\xi)\cdot(y-\xi) - f(y)] \, \frac{y-\xi}{\|y-\xi\|^k} \cdot n_\Omega(\xi)\, d\Gamma.
\end{aligned}$$

Since ∇f is Lipschitzian there exists a constant c such that

$$\|\nabla f(\xi) - \nabla f(y)\| \le c\,\|\xi - y\|, \quad \|D^2 f(\xi)\| \le c \text{ a.e.,} \tag{4.3}$$

and there exists $\theta \in (0,1)$ such that

$$f(y) = f(\xi) + \nabla f(\xi)\cdot(y-\xi) + \frac{1}{2}\int_0^1 D^2 f(\xi + \theta(y-\xi))(y-\xi)\cdot(y-\xi)\, d\theta$$

Since $D^2 f$ is bounded a.e.

$$f(\xi) + \nabla f(\xi)\cdot(y-\xi) - f(y) = -\frac{1}{2}\int_0^1 D^2 f(\xi + \theta(y-\xi))(y-\xi)\cdot(y-\xi)\, d\theta$$

$$|f(\xi) + \nabla f(\xi)\cdot(y-\xi) - f(y)| \le c\,\|y-\xi\|^2. \tag{4.4}$$

The first four terms on the right-hand side of $\nabla \hat{F}(y) - \nabla f(y)$ are bounded by

$$c' \frac{1}{|\phi(y)|} \int_{\Omega^c} \frac{1}{\|y-\xi\|^{k-1}}\, d\xi = c' \frac{1}{|\phi(y)|} \frac{1}{k-1-n} \int_{\Omega^c} \operatorname{div}_\xi \frac{y-\xi}{\|y-\xi\|^{k-1}}\, d\xi$$
$$= \frac{c'}{k-1-n} \frac{\int_\Gamma \frac{y-\xi}{\|y-\xi\|^{k-1}} \cdot n_\Omega(\xi)\, d\Gamma}{\int_\Gamma \frac{y-\xi}{\|y-\xi\|^{k}} \cdot n_\Omega(\xi)\, d\Gamma} \le c'' d_\Gamma(y)$$

for some generic constants since

$$0 < \int_\Gamma \frac{\xi - y}{\|y-\xi\|^{k-1}} \cdot n_\Omega(\xi)\, d\Gamma \le c'\, d_\Gamma(y)^{n-(k-1)} \text{ if } k-1 > n$$
$$\int_\Gamma \frac{\xi - y}{\|y-\xi\|^{k}} \cdot n_\Omega(\xi)\, d\Gamma \ge c'\, d_\Gamma(y)^{n-k} \text{ if } k > n$$

from [2, Thms. 2.1 and 2.7]. Hence, this requires $k > n+1$ to make the first four terms go to zero as $y \to x \in \Gamma$.

We now look at the fifth term

$$-\frac{\nabla \phi(y)}{\phi(y)^2} \int_\Gamma [f(\xi) + \nabla f(\xi)\cdot(y-\xi) - f(y)]\, \frac{y-\xi}{\|y-\xi\|^k} \cdot n_\Omega(\xi)\, d\Gamma$$

For the first factor involving $\phi(y)$

$$\phi(y) = \int_\Gamma \frac{y-\xi}{\|y-\xi\|^k} \cdot n_\Omega(\xi)\, d\Gamma$$
$$= \int_{\Omega^c} \operatorname{div}_\xi \frac{y-\xi}{\|y-\xi\|^k}\, d\xi = -(k-n) \int_{\Omega^c} \frac{1}{\|y-\xi\|^k}\, d\xi$$
$$\nabla_y \phi(y) = k\,(k-n) \int_{\Omega^c} \frac{y-k}{\|y-\xi\|^{k+2}}\, d\xi$$

and from Delfour-Garon [2, Thms. 2.1 and 2.7]

$$|\phi(y)| \ge c\, d_\Gamma(y)^{n-k}$$
$$\|\nabla_y \phi(y)\| \le k\,(k-n) \int_{\Omega^c} \frac{1}{\|y-\xi\|^{k+1}}\, d\xi \le c\, d_\Gamma(y)^{n-(k+1)}$$

for $k > n$ and $k > n-1$, respectively, Finally, for $k > n$

$$\left\| \frac{\nabla \phi(y)}{\phi(y)^2} \right\| \le c\, \frac{d_\Gamma(y)^{n-(k+1)}}{d_\Gamma(y)^{2(n-k)}} = c\, d_\Gamma(y)^{k-n-1}. \tag{4.5}$$

For the second factor of the fifth term

$$\int_\Gamma [f(\xi) + \nabla f(\xi)\cdot(y-\xi) - f(y)]\, \frac{y-\xi}{\|y-\xi\|^k} \cdot n_\Omega(\xi)\, d\Gamma,$$

the direct use of the divergence theorem necessitates the stronger condition $k > n+2$ instead of the condition $k > n+1$ for the first four terms. To get around this, we use the tubular neighbourhood $\{y \in \mathbb{R}^n : d_\Gamma(y) \le h\}$ of Γ and the truncation $s(d_\Gamma(y))$

introduced in (3.4)-(3.5) in the proof of Theorem 3.1:

$$\begin{aligned}
&\int_\Gamma [f(\xi)+\nabla f(\xi)\cdot(y-\xi)-f(y)]\,\frac{y-\xi}{\|y-\xi\|^k}\cdot n_\Omega(\xi)\,s(d_\Gamma(y))\,d\Gamma\\
&= -\int_{\Omega^c}\left([f(\xi)+\nabla f(\xi)\cdot(y-\xi)-f(y)]\,(k-n)\,\frac{1}{\|y-\xi\|^k}\right)\,s(d_\Gamma(y))\,d\xi\\
&\quad -\int_{\Omega^c}\left(D^2f(\xi)\,(y-\xi)\cdot(y-\xi)\,\frac{1}{\|y-\xi\|^k}\right)\,s(d_\Gamma(y))\,d\xi\\
&\quad -\int_{\Omega^c}[f(\xi)+\nabla f(\xi)\cdot(y-\xi)-f(y)]\,\frac{y-\xi}{\|y-\xi\|^k}\cdot\nabla_y(s(d_\Gamma(y)))\,d\Gamma
\end{aligned}$$

and, using the inequalities (4.3) and (4.4) and the fact that $s(d_\Gamma(y)) = 0$ and $\nabla_y(s(d_\Gamma(y))) = 0$ for $d_\Gamma(y) \geq h$,

$$\begin{aligned}
&\left|\int_\Gamma [f(\xi)+\nabla f(\xi)\cdot(y-\xi)-f(y)]\,\frac{y-\xi}{\|y-\xi\|^k}\cdot n_\Omega(\xi)\,d\Gamma\right|\\
&\leq c\int_{\Omega^c\cap\{y\in\mathbb{R}^n:d_\Gamma(y)<h\}}\frac{1}{\|y-\xi\|^{k-2}}\,d\xi.
\end{aligned}$$

Since Γ is compact, $\{\xi\in\mathbb{R}^n\backslash\Omega : d_\Gamma(\xi)\leq h\}$ is compact and there exists a sufficiently small $\delta>0$ and a sufficiently large $R>0$ such that, for all $y\in B_\delta(x)$, $\{\xi\in\mathbb{R}^n\backslash\Omega : d_\Gamma(\xi)\leq h\}\subset B_R(y)$. So

$$\begin{aligned}
\int_{\{\xi\in\Omega^c:d_\Gamma(\xi)<h\}}\frac{1}{\|\xi-y\|^{k-2}}\,d\xi &\leq \int_{B_R(y)\backslash B_{d_\Gamma(y)}(y)}\frac{1}{\|\xi-y\|^{k-2}}\,d\xi\\
&=\int_{d_\Gamma(y)}^R\frac{1}{\rho^{k-2}}\,\beta_n\,\rho^{n-1}\,d\rho\\
&=\frac{1}{k-n-2}\left[\frac{1}{d_\Gamma(y)^{k-2-n}}-\frac{1}{R^{k-2-n}}\right].
\end{aligned}$$

Finally, using inequality (4.5),

$$\begin{aligned}
&\left|\frac{\nabla\phi(y)}{\phi(y)^2}\int_\Gamma [f(\xi)+\nabla f(\xi)\cdot(y-\xi)-f(y)]\,\frac{y-\xi}{\|y-\xi\|^k}\cdot n_\Omega(\xi)\,d\Gamma\right|\\
&\leq c\,d_\Gamma(y)^{k-n-1}\left[\frac{1}{d_\Gamma(y)^{k-2-n}}-\frac{1}{R^{k-2-n}}\right]=c\left[d_\Gamma(y)+\frac{d_\Gamma(y)^{k-n-1}}{R^{k-2-n}}\right]
\end{aligned}$$

that goes to zero as $y\to x\in\Gamma$ if $k>n+1$. □

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M. C. Delfour
Département de mathématiques et de statistique, Université de Montréal, CP 6128, succ. Centre-ville, Montréal (Qc), Canada H3C 3J7
E-mail address: `delfour@crm.umontreal.ca`

A. Garon
Département de Génie mécanique, École Polytechnique de Montréal, C.P. 6079, succ. Centre-ville, Montréal (Qc), Canada H3C 3A7
E-mail address: `andre.garon@polymtl.ca`