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SET OPTIMIZATION PROBLEMS ON ORDERED TOPOLOGICAL VECTOR SPACES

JINLU LI AND CHRISTIANE TAMMER

Abstract. In this paper, we study the properties of the preorders and partial orders equipped on topological vector spaces (t.v.s.), which are induced by proper closed and convex cones and pointed closed and convex cones, respectively. Then we consider three ordering relations on the power sets of ordered t.v.s.. They are called power preorder, upward power preorder and downward power preorder. We apply these preorders to prove some fixed point theorems on ordered t.v.s., which are used to prove the existence of extended Nash equilibrium for set valued mappings and the existence of vector Nash equilibrium for single valued mappings.

1. Introduction

Set optimization problems have been studied by many authors (see [2, 3, 9, 12, 17–22, 26]), which can be considered as the extensions of optimization problems and vector optimization problems from single valued mappings to set valued mappings (see [2–4, 7–11, 16]). In [15], the present authors studied the set optimization problems on preordered sets, in which neither algebraic structure nor topological structure is required. In the paper [15], three preorder relations are introduced in the power sets of preordered sets, which are used as the standards to define the orders of subsets of ordered sets. Based on these orders, one can study the set optimization problems for set-valued mappings. This paper is a continuation of [15] to study set optimization problems on ordered t.v.s.. More precisely, in this paper, the underlying spaces of considered set optimization problems are t.v.s. equipped with some order structures, such as, preorder, partial order and lattice ordering. We will show when we consider the set optimization problems on ordered t.v.s. as special cases of set optimization problems on preordered sets, some existence of equilibrium problems for set-valued mappings and single valued mappings can be immediately obtained from the results in [15]. On the other hand, by using the algebraic structures and the topological structures on the considered vector spaces, we will prove some new solvability theorem for Nash equilibrium problems of set-valued mappings on ordered t.v.s..

This paper is organized as follows: In Section 2, we recall some notations, concepts and fixed point theorems on preordered sets (see [1, 5, 13–15], for more details). In

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particular, we recall some fixed point theorems from [15], which will be used for the proof of the existence theorems in this paper.

In Sections 3 and 4, we study ordered topological spaces and ordered t.v.s.. We prove some chain-complete properties and universally inductive properties of compact subsets in ordered topological spaces.

In Sections 5 and 6, we introduce the concepts of order-semi continuities and order-convexities. By these concepts, we prove the existence of order-clustered extended Nash equilibrium and extended Nash equilibrium for set-valued mappings in ordered t.v.s..

In Section 7, we study the existence of vector Nash equilibrium for single valued mappings in ordered t.v.s., which are considered as applications of the results from Sections 5 and 6. In Section 8, we introduce the concept of order-cluster invariant mappings, which is a condition for mappings to have an extended Nash equilibrium reduced from order-clustered extended Nash equilibrium.

2. Preliminaries on preordered sets

2.1. **The order-clustered sets of preordered sets.** In this section, we recall some notations and concepts regarding to (general) preordered sets (see [1, 5, 13–15], for more details). These notations and concepts will be used for preordered topological spaces and preordered t.v.s. in the following sections.

In general, let $(Z, \succcurlyeq)$ be a preordered set, in which except the ordering relation $\succcurlyeq$ equipped on Z, neither algebraic structure nor topological structure is required. Consider $z_1, z_2 \in Z$. If $z_2 \succcurlyeq z_1$ and $z_1 \succcurlyeq z_2$ both hold, then it is denoted by $z_1 \backsim z_2$ and z_1 and z_2 are said to be $\succcurlyeq$-equivalent. For every $z \in Z$, we denote

$$[z]_{\succcurlyeq} := \{t \in Z : \; t \backsim z\}. \tag{2.1}$$

$[z]_{\succcurlyeq}$ is simply denoted by $[z]$ if there is no confusion caused. $[z]$ is called the $\succcurlyeq$-cluster in Z containing z (or simply called the cluster in Z containing z). The binary relation $\backsim$ induced by $\succcurlyeq$ is an equivalence relation on Z and the set of all clusters forms a partition of Z, which is written as

$$[Z]_{\succcurlyeq} = [Z] = \{[z] : \; z \in Z\}.$$

$[Z]$ is called the $\succcurlyeq$ cluster space (simply called the cluster space) of Z with respect to the preorder $\succcurlyeq$. For any nonempty subset $A \subseteq Z, (A, \succcurlyeq)$ is also a preordered set. We denote

$$[A]_{\succcurlyeq} = [A] = \{[z] : \; z \in A\}, \tag{2.2}$$

where $[z]$ is defined in (2.1), which is with respect to $\succcurlyeq$ on X (not with respect to the restriction of $\succcurlyeq$ to A).

For $z \in Z$, we identify z with $\{z\}$.

Furthermore, we have

$$z \in [z], \text{ for every } z \in Z, \text{ and } A \subseteq \cup[A] = \cup_{z \in A}[z], \text{ for any } A \subseteq Z.$$

The collection of preordering relations on a nonempty set Z includes partial orders as special cases. Notice that a preordered set $(Z, \succcurlyeq)$ is a partially ordered set if and only if

$$z = [z], \text{ for every } z \in Z, \quad \text{or} \quad A = [A], \text{ for every } A \subseteq Z.$$

In [25] and [15], the concept of fixed point for both of single valued mappings and set-valued mappings on partially ordered sets were generalized to order-clustered fixed point on preordered sets.

Let $(Z, \succcurlyeq)$ be a preordered set and A a nonempty subset of Z. Let $f : A \to A$ be a single valued mapping. A point $z \in A$ is called a $\succcurlyeq$-clustered fixed point (or simply, a clustered fixed point) of f if $z \in [f(z)]$.

Let $F : A \to 2^A \setminus \{\emptyset\}$ be a set-valued mapping. A point $z \in A$ is called a $\succcurlyeq$-clustered fixed point (or simply, a clustered fixed point) of F if $z \in [v]$, for some $v \in F(z)$. By the notation (2.1), it is $z \in F(z)$.

We use the notation $\mathfrak{F}(F)$ for the set of $\succcurlyeq$-clustered fixed points of F and $\mathcal{F}(F)$ for the set of fixed points of F. In general, we have

$$\mathcal{F}(F) \subseteq \mathfrak{F}(F).$$

If $(Z, \succcurlyeq)$ is a partially ordered set, then $\mathcal{F}(F) = \mathfrak{F}(F)$.

2.2. **Ordering relations on power sets.** Let $(Z, \succcurlyeq)$ be a preordered set. In [15], based on $\succcurlyeq$, three preordering relations are defined on the power set 2^Z, which are called the power preorder, upward power preorder and the downward power preorder. These set relations are well known in set optimization, see Jahn and Ha [9], Kuroiwa [17], [18], [21]. For earlier references, see Nishnianidze [23], Young [27].

We recall these definitions below for easy reference.

(a) The power preorder, denoted by $\succcurlyeq^P$. For any $A, B \in 2^Z$, we say that $A \preccurlyeq^P B$ if the following two conditions are satisfied:
   (U) upward condition: for any $a \in A$, there is $b \in B$ such that $b \succcurlyeq a$;
   (D) downward condition: for any $b \in B$, there is $a \in A$ such that $a \preccurlyeq b$.
(b) The upward power preorder, denoted by $\succcurlyeq^U$. For any $A, B \in 2^Z$, we say that $A \preccurlyeq^U B$ if the upward condition (U) is satisfied.
(c) The downward power preorder, denoted by $\succcurlyeq^D$. For any $A, B \in 2^Z$, we say that $A \preccurlyeq^D B$ if the downward condition (D) is satisfied.

Remark 2.1. Observe that

$$A \preceq^D B \quad \Longleftrightarrow \quad B \preceq'^U A,$$

where $a \preceq' b$ if $b \preceq a$. So, the results on fixed points for $\preceq^D$ can be deduced from those related to $\preceq^U$ while those for $\succcurlyeq^P$ putting together those for $\succcurlyeq^U$ and $\succcurlyeq^D$.

2.3. **Order-clustered fixed point theorems.** We recall some order-clustered fixed point theorems proved in [15], where we are using the following chain-completeness concept. Let $(Z, \succcurlyeq)$ be a (general) preordered set and let A be a nonempty subset of Z. A is said to be $\succcurlyeq$-chain-complete (or chain-complete) whenever every $\succcurlyeq$-chain $\{z_\alpha\}$ in A has at least one least $\succcurlyeq$-upper bound (it may not be unique). The set of the least $\succcurlyeq$-upper bounds of this given chain is a $\succcurlyeq$-cluster in Z and it is denoted by $\vee\{z_\alpha\}$. A is said to be $\succcurlyeq$-re-chain-complete (simply said to be re-chain-complete), if every $\succcurlyeq$-chain $\{z_\alpha\}$ in A has at least one greatest $\succcurlyeq$-lower bound (it may not be unique). The set of the greatest $\succcurlyeq$-lower bounds of this given chain is a $\succcurlyeq$-cluster in Z and it is denoted by $\wedge\{z_\alpha\}$. If A is both chain-complete and re-chain-complete, then it is said to be bi-chain-complete.

Hence, a nonempty subset A in a preordered set $(Z, \succcurlyeq)$ is chain-complete (re-chain-complete), whenever, for any chain $\{z_\alpha\}$ in A, one has $\vee\{z_\alpha\} \neq \emptyset$ ($\wedge\{z_\alpha\} \neq \emptyset$).

For notions of inductivities, see Section 3.2.

Theorem 2.2 ([15, Theorem 2.1])**.** *Let $(Z, \succcurlyeq)$ be a $\succcurlyeq$-chain-complete preordered set with the $\succcurlyeq$-cluster space $[Z]$. Let $F : Z \to 2^Z \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that F satisfies the following conditions:*

(i) *F is order increasing with respect to $\succcurlyeq$ and $\succcurlyeq^U$;*
(ii) *for every fixed $z \in Z$, $[F(z)]$ is universally inductive in $([Z], \succcurlyeq^U)$;*
(iii) *there is $z_1 \in Z$ such that $\{z_1\} \preccurlyeq^U F(z_1)$.*

Then, F has a $\succcurlyeq$-clustered fixed point. Moreover, we have

(a) *$\mathfrak{F}(F)$ is a nonempty inductive preordered subset of Z;*
(b) *F has a $\succcurlyeq$-maximal clustered fixed point u_1 with $u_1 \succcurlyeq z_1$.*

Theorem 2.3 ([15, Theorem 2.3])**.** *Let $(Z, \succcurlyeq)$ be a bi-chain-complete preordered set with cluster space $[Z]$. Let $F : Z \to 2^Z \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that F satisfies the following conditions:*

(i) *F is order increasing with respect to $\succcurlyeq$ and $\succcurlyeq^P$;*
(ii) *for every fixed $z \in Z$, $[F(z)]$ is bi-universally inductive in $\big([Z], \succcurlyeq^P\big)$;*
(iii) *there are $z_1, z_2 \in Z$ such that $\{z_1\} \preccurlyeq^U F(z_1)$ and $\{z_2\} \succcurlyeq^D F(z_2)$.*

Then, F has a $\succcurlyeq$-clustered fixed point. Moreover, we have

(a) *$\mathfrak{F}(F)$ is a nonempty bi-inductive preordered subset of Z;*
(b) *F has a $\succcurlyeq$-maximal clustered fixed point u_1 with $u_1 \succcurlyeq z_1$;*
(c) *F has a $\succcurlyeq$-minimal clustered fixed point u_2 with $u_2 \preccurlyeq z_2$.*

In particular, if $(Z, \succcurlyeq)$ is a partially ordered set in [15, Theorem 2.3], which is considered as a special case of preordered sets, then we obtain some fixed point theorems on partially ordered sets proved by Fujimoto [6], Li [13, 14] and others.

Corollary 2.4. *Let $(Z, \succcurlyeq)$ be a bi-chain-complete partially ordered set. Let $F : Z \to 2^Z$ be a set- valued mapping. Suppose that F satisfies the following conditions:*

(i) *F is order increasing with respect to $\succcurlyeq$ and $\succcurlyeq^P$;*
(ii) *for every fixed $z \in Z, F(z)$ is bi-universally inductive in $(Z, \succcurlyeq)$;*
(iii) *there are $z_1, z_2 \in Z$ such that $\{z_1\} \preccurlyeq^U F(z_1)$ and $\{z_2\} \succcurlyeq^D F(z_2)$.*

Then, F has a fixed point. Moreover, we have

(a) *$\mathcal{F}(F)$ is a nonempty bi-inductive subset of $(Z, \succcurlyeq)$;*
(b) *F has a $\succcurlyeq$-maximal fixed point u_1 with $u_1 \succcurlyeq z_1$;*
(c) *F has a $\succcurlyeq$-minimal fixed point u_2 with $u_2 \preccurlyeq z_2$.*

2.4. **The induced partial order on the $\succcurlyeq$-cluster space.** On the $\succcurlyeq$-cluster space $[Z]_{\succcurlyeq} = [Z] = \{[z] : z \in Z\}$, we defined an order relation $\hat{\succcurlyeq}$ as follows: for any $[z_1]$, $[z_2] \in [Z]$,

$$[z_1] \mathrel{\hat{\succcurlyeq}} [z_2] \quad \text{if and only if} \quad z_1 \succcurlyeq z_2. \tag{2.3}$$

One can check that the relation $\hat{\succcurlyeq}$ on the $\succcurlyeq$-cluster space $[Z]$ is well defined and it is said to be induced by $\succcurlyeq$ on Z. It has some useful properties, which are listed as a proposition below.

Proposition 2.5. *Let $(Z, \succcurlyeq)$ be a preordered set with the $\succcurlyeq$-cluster space $[Z]$. Let $\hat{\succcurlyeq}$ be the order relation on $[Z]$ induced by $\succcurlyeq$ on Z. One has*

(a) *$\hat{\succcurlyeq}$ is a partial order on $[Z]$;*

(b) *As restricted to $[Z]$, the preorders $\succcurlyeq^P$, $\succcurlyeq^U$, $\succcurlyeq^D$ all are partial orders and coincide with the partial order $\hat{\succcurlyeq}$ on $[Z]$. That is,*

$$\succcurlyeq^P |_{[Z]} = \succcurlyeq^U |_{[Z]} = \succcurlyeq^D |_{[Z]} = \hat{\succcurlyeq}.$$

Proof. The proof of the assertion is straightforward and it is omitted here. $\square$

3. Ordered topological spaces

3.1. **Chain-completeness on preordered topological spaces.** In this section, we discuss the chain-completeness and universally inductivities of ordered topological spaces, in which a topological structure and an ordering relation both are equipped. It prepares for us to study ordered t.v.s., in which, in addition to the topological structure and the algebraic structure, an ordering relation is equipped in the following sections.

Let (Z, τ) be a topological space equipped with an ordering relation $\succcurlyeq$ on Z. In this paper, $\succcurlyeq$ is said to be a preorder (partial order) on Z if it is a preordering relation (partial ordering relation) on Z and, in addition, it satisfies the following condition:

(O_4) For every $u \in Z$, the $\succcurlyeq$-intervals $(u] := \{z \in Z : z \preccurlyeq u\}$ and $[u) := \{z \in Z : z \succcurlyeq u\}$ all are τ-closed.

Then, $(Z, \tau, \succcurlyeq)$ is called a preordered (partially ordered) topological space. It is simply written as $(Z, \succcurlyeq)$, in where the topology τ on Z is hidden and its existence is understood. It follows immediately that, in a preordered (partially ordered) topological space $(Z, \tau, \succcurlyeq)$, for any $z \in Z$, the cluster $[z]$ is always τ-closed.

In [14], Li proved that every nonempty compact subset in a partially ordered topological space is both chain-complete and universally inductive. In this subsection, we extend these properties to preordered topological space. The proof is similar to that in [14].

Proposition 3.1. *Every nonempty compact subset in a preordered Hausdorff topological space is bi-chain-complete.*

Proof. Let $(Z, \succcurlyeq)$ be a preordered Hausdorff topological space and let A be a nonempty compact subset of Z. We first prove that A is $\succcurlyeq$-chain-complete. Take an arbitrary $\succcurlyeq$-chain $\{z_\alpha\} \subseteq A$. We show that $\vee\{z_\alpha\} \neq \emptyset$. At first, we prove that

$$(\cap_\alpha [z_\alpha)) \cap A = (\cap_\alpha \{z \in Z : z \succcurlyeq z_\alpha\}) \cap A \neq \emptyset. \tag{3.1}$$

Assume controversially that $\cap_\alpha \{z \in Z : z \succcurlyeq z_\alpha\} \cap A = \emptyset$. For every index α, we write

$$[z_\alpha)^C := \{z \in Z : z \not\succcurlyeq z_\alpha\} = \{z \in Z : z \prec z_\alpha \text{ or } z \not\sim z_\alpha\}.$$

Then, $[z_\alpha)^C$ is open, for every index α. Since $\cup_\alpha([z_\alpha)^C) = (\cap_\alpha [z_\alpha))^C$, from the controversial assumption of (3.1), it yields that $\{[z_\alpha)^C\}$ is an open cover of A. Since A is compact, then A has a finite cover contained in $\{[z_\alpha)^C\}$. There is a positive integer n such that

$$\{[z_i)^C : i = 1, 2, \ldots, n\} \subseteq \{[z_\alpha)^C\}$$

is an open cover of A. Without loss of the generality, we suppose that $z_1 \preccurlyeq z_2 \preccurlyeq \ldots \preccurlyeq z_n$. It implies that $[z_1)^C \subseteq [z_2)^C \subseteq \ldots \subseteq [z_n)^C$. We obtain that $A \subseteq [z_n)^C$. Since $z_n \in A$, it implies $z_n \not\succcurlyeq z_n$. It is a contradiction. That proves (3.1).

Let $B = \overline{\{z_\alpha\}}$ be the topological closure of the chain $\{z_\alpha\} \subseteq A$. Then, B is a compact subset contained in A. It is clear that $\{z_\alpha\}$ is also a chain in B. Similarly, to the proof of (3.1), we can prove that

$$(\cap_\alpha[z_\alpha)) \cap B = (\cap_\alpha\{z \in Z : \ z \succcurlyeq z_\alpha\}) \cap B \neq \emptyset.$$

It is clear that

$$(\cap_\alpha[z_\alpha)) \cap B \subseteq (\cap_\alpha[z_\alpha)) \cap A.$$

Take an arbitrary point $z \in (\cap_\alpha[z_\alpha)) \cap B$. In the case that $z \in \{z_\alpha\}$, then $z \in \vee\{z_\alpha\}$. Otherwise, we have that $z \in \overline{\{z_\alpha\}}$ and z is a $\succcurlyeq$-upper bound of the chain $\{z_\alpha\}$. Let y be an arbitrary given $\succcurlyeq$-upper bound of the chain $\{z_\alpha\}$. Then, $\{z_\alpha\} \subseteq (y]$. Since $z \in \overline{\{z_\alpha\}}$ and $(y]$ is closed, it implies $z \in (y]$. That is, $z \preccurlyeq y$. From the fact that z is a $\succcurlyeq$-upper bound of the chain $\{z_\alpha\}$, it implies $z \in \vee\{z_\alpha\}$. Hence, A is $\succcurlyeq$-chain-complete.

Similarly, to the above proof, we can show that $\wedge\{z_\alpha\} \neq \emptyset$. So, A is also re-chain-complete. □

3.2. **Universally inductivities on preordered topological spaces.** A nonempty subset A of a preordered set $(Z, \succcurlyeq)$ is said to be universally inductive in $(Z, \succcurlyeq)$ if, for any given $\succcurlyeq$-chain $\{x_\alpha\} \subseteq Z$ satisfying that every element $x_\beta \in \{x_\alpha\}$ has a $\succcurlyeq$-upper bound in A, then this $\succcurlyeq$-chain $\{x_\alpha\}$ has a $\succcurlyeq$-upper bound in A.

A is said to be re-universally inductive in $(Z, \succcurlyeq)$ if, for any given $\succcurlyeq$-chain $\{x_\alpha\} \subseteq Z$ satisfying that every element $x_\beta \in \{x_\alpha\}$ has a $\succcurlyeq$-lower bound in A, then this $\succcurlyeq$-chain $\{x_\alpha\}$ has a $\succcurlyeq$-lower bound in A. A is said to be bi-universally inductive in $(Z, \succcurlyeq)$ if, A is both universally inductive and re-universally inductive in $(Z, \succcurlyeq)$.

The empty subset $\emptyset$ of a preordered set $(Z, \succcurlyeq)$ is automatically bi-universally inductive in $(Z, \succcurlyeq)$.

Proposition 3.2. *Every compact subset in a preordered Hausdorff topological space is bi-universally inductive.*

Proof. Let $(Z, \succcurlyeq)$ be a preordered Hausdorff topological space and let A be a nonempty compact subset of Z. We first show that A is universally inductive in $(Z, \succcurlyeq)$. Take an arbitrary chain $\{z_\alpha\} \subseteq Z$ satisfying that, for every element $z_\beta \in \{z_\alpha\}$, there is a point $u_\beta \in A$ such that $z_\beta \preccurlyeq u_\beta$. It implies that

$$[z_\alpha) \cap A \neq \emptyset, \text{ for every index } \alpha.$$

Then, this proposition will be proved if the following is true,

$$(\cap_\alpha[z_\alpha)) \cap A = (\cap_\alpha\{z \in Z : \ z \succcurlyeq z_\alpha\}) \cap A \neq \emptyset. \tag{3.2}$$

Assume controversially that $\cap_\alpha\{z \in Z : \ z \succcurlyeq z_\alpha\} \cap A = \emptyset$. For every index α, we define

$$[z_\alpha)^C := \{z \in Z : \ z \not\succcurlyeq z_\alpha\} = \{z \in Z : \ z \prec z_\alpha \text{ or } z \not\sim z_\alpha\}.$$

Then, $[z_\alpha)^C$ is open, for every index α. Since $\cup_\alpha([z_\alpha)^C) = (\cap_\alpha[z_\alpha))^C$, from the controversial assumption to (3.2), it yields that $\{[z_\alpha)^C\}$ is an open cover of A. Since A is compact, then A has a finite cover contained in $\{[z_\alpha)^C\}$. There is a positive integer n such that

$$\{[z_i)^C : \; i = 1, 2, \ldots, n\} \subseteq \{[z_\alpha)^C\}$$

and $\{[z_i)^C : \; i = 1, 2, \ldots, n\}$ is an open cover of A. Without loss of the generality, we suppose that $z_1 \preccurlyeq z_2 \preccurlyeq \ldots z_n$. It implies that $[z_i)^C \subseteq [z_2)^C \subseteq \ldots \subseteq [z_n)^C$. We obtain that $A \subseteq [z_n)^C$. Since $u_n \in A$, it implies $u_n \in [z_n)^C$. It contradicts to the selection of u_n that satisfies $z_n \preccurlyeq u_n$. So, (3.2) is proved. It implies that the whole chain $\{z_\alpha\}$ has a $\succcurlyeq$-upper bound in $(A, \succcurlyeq)$. Hence, A is universally inductive in $(Z, \succcurlyeq)$. Similarly, to the above proof, we can show that, for an arbitrary $\succcurlyeq$-chain $\{z_\alpha\} \subseteq Z$ satisfying that, for every element $z_\beta \in \{z_\alpha\}$, there is a point $v_\beta \in A$ such that $v_\beta \preccurlyeq z_\beta$, then the whole chain $\{z_\alpha\}$ has a $\succcurlyeq$-lower bound in $(A, \succcurlyeq)$. Hence A is re-universally inductive in $(Z, \succcurlyeq)$. It implies that A is bi-universally inductive in $(Z, \succcurlyeq)$. □

By [15, Propositions 2.1 and 2.2], as consequences of Propositions 3.1 and 3.2, and using the partial order $\hat{\succcurlyeq}$ on the $\succcurlyeq$-cluster space $[Z]$ in Proposition 2.5, we obtain the following corollaries. They will be used in the proofs of the existence theorems.

Corollary 3.3. *Let $(Z, \succcurlyeq)$ be a preordered Hausdorff topological space with the cluster space $[Z]$ equipped with the partial order $\hat{\succcurlyeq}$. Let A be a nonempty compact subset of Z. Then,*

$$([A], \hat{\succcurlyeq}) \;\; \textit{is} \;\; \hat{\succcurlyeq}\textit{-bi-chain-complete.}$$

Consequently, from Proposition 2.5, one has

$$([A], \succcurlyeq^N) \;\; \textit{is} \;\; \succcurlyeq^N\textit{-bi-chain-complete for} \;\; N = P, U, D.$$

Corollary 3.4. *Let $(Z, \succcurlyeq)$ be a preordered Hausdorff topological space with the cluster space $[Z]$ equipped with the partial order $\hat{\succcurlyeq}$. Let A be a nonempty compact subset of Z. Then,*

$$([A], \hat{\succcurlyeq}) \;\; \textit{is -bi-universally inductive in} \;\; ([Z], \hat{\succcurlyeq}).$$

Consequently, from Proposition 2.5, one has

$$([A], \succcurlyeq^N) \;\; \textit{is -bi-universally inductive in} \;\; ([Z], \succcurlyeq^N) \; \textit{for} \; N = P, U, D.$$

3.3. **Order-clustered fixed point theorems in preordered Hausdorff topological spaces.** By [15, Theorem 2.1 and Corollary 2.2] and from Corollaries 3.3 and 3.4, we obtain the following order-clustered fixed point theorems in preordered Hausdorff topological spaces.

The order monotonicity of mappings play important role for the existence of fixed point of the considered mapping. We recall the related definitions below.

In general, let $(S_1, \succcurlyeq_1)$ and $(S_2, \succcurlyeq_2)$ be preordered sets. A mapping $g : S_1 \to S_2$ is said to be $\succcurlyeq_1$-$\succcurlyeq_2$ increasing (decreasing), (or order increasing (decreasing) with respect to $\succcurlyeq_1$ and $\succcurlyeq_2$), whenever, for any $s, t \in S_1$,

$$s \preccurlyeq_1 t \text{ implies } g(s) \preccurlyeq_2 g(t) \quad (g(s) \succcurlyeq_2 g(t)). \tag{3.3}$$

Theorem 3.5. *Let $(Z, \succcurlyeq)$ be a preordered Hausdorff topological space with the cluster space $[Z]$. Let A be a nonempty compact subset of Z. Let $F : A \to 2^A \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that F satisfies the following conditions:*

(i) *F is $\succcurlyeq$-$\succcurlyeq^U$-increasing;*
(ii) *for every fixed $z \in A$, $F(z)$ is closed in A;*
(iii) *there is $z_1 \in A$ such that $\{z_1\} \succcurlyeq^U F(z_1)$.*

Then, F has a $\succcurlyeq$-clustered fixed point. That is, there are $z^*, u \in A$ with $u \backsim z^*$ such that

$$u \in F(z^*).$$

Moreover, we have

(a) *$\mathfrak{F}(F)$ is a nonempty inductive preordered subset of $(A, \succcurlyeq)$;*
(b) *F has a $\succcurlyeq$-maximal clustered fixed point u_1 with $u_1 \succcurlyeq z_1$.*

Theorem 3.6. *Let $(Z, \succcurlyeq)$ be a preordered Hausdorff topological space with the cluster space $[Z]$. Let A be a nonempty compact subset of Z. Let $F : A \to 2^A \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that F satisfies the following conditions:*

(i) *F is $\succcurlyeq$-$\succcurlyeq^P$-increasing;*
(ii) *for every fixed $z \in A$, $F(z)$ is closed in A;*
(iii) *there are $z_1, z_2 \in A$ such that $\{z_1\} \preccurlyeq^U F(z_1)$ and $\{z_2\} \succcurlyeq^D F(z_2)$.*

Then, F has a $\succcurlyeq$-clustered fixed point. Moreover, we have

(a) *$\mathfrak{F}(F)$ is a nonempty bi-inductive preordered subset of $(A, \succcurlyeq)$;*
(b) *F has a $\succcurlyeq$-maximal clustered fixed point u_1 with $u_1 \succcurlyeq z_1$;*
(c) *F has a $\succcurlyeq$-minimal clustered fixed point u_2 with $u_2 \preccurlyeq z_2$.*

4. Ordered topological vector spaces

4.1. **Proper cones and pointed cones in t.v.s.** Let (Z, τ) be a t.v.s. equipped with an ordering relation $\succcurlyeq$ on Z. We are using the notion $u \succ v$ if it implies $u \neq v$. In this paper, $\succcurlyeq$ is said to be a (nontrivial) preorder (partial order) on Z if it is a preordering relation (partial ordering relation) on Z and, in addition, it satisfies the following conditions:

(O_1) $u \succcurlyeq v$ implies $u + w \succcurlyeq v + w$, for $u, v, w \in Z$;
(O_2) $u \succcurlyeq v$ implies $\alpha u \succcurlyeq \alpha v$, for $u, v \in Z$ and $\alpha \geq 0$;
(O_3) there are distinct points $u, v \in Z$ satisfying $u \succ v$;
(O_4) for every $u \in Z$, the $\succcurlyeq$-intervals $(u] = \{z \in Z : \ z \preccurlyeq u\}$ and $[u) = \{z \in Z : z \succcurlyeq u\}$ all are τ-closed.

Then, $(Z, \succcurlyeq)$ is called a preordered (partially ordered) topological vector space.

Remark 4.1. Let Z be a vector space equipped with a preorder (partial order) relation $\succcurlyeq$ on Z. If $\succcurlyeq$ satisfies conditions (O_1–O_3), then $(Z, \succcurlyeq)$ is called a preordered (partially ordered) vector space. In this paper, we concentrate to ordered t.v.s..

Since ordering relations on t.v.s. satisfying conditions ($O_1 - O_4$) are totally determined by the corresponding cones in the given spaces, we recall some concepts regarding to cones which are used in this paper. Let K be a nonempty subset of a t.v.s. Z. K is called a cone in Z if K satisfies the following condition:

For any $z \in K$ and for any real number $\lambda \geq 0$, $\lambda z \in K$.

A cone K in Z is called a proper cone, if K also satisfies the following condition:

$$K \neq \{0\} \text{ and } K \neq Z \text{ (it means that } K \text{ is nontrivial).}$$

In addition to the above two conditions, if a proper cone K satisfies $K \cap (-K) = \{0\}$, then K is called a pointed cone in Z. Every pointed cone is a proper cone.

Let K be a proper closed and convex cone in Z. As usual, an ordering relation $\succcurlyeq_K$ on Z is defined by

$$z_2 \succcurlyeq_K z_1 \text{ if and only if } z_2 - z_1 \in K, \text{ for } z_1, z_2 \in Z. \tag{4.1}$$

We consider the preordered topological vector spaces as special cases of preordered sets. If there is no confusion caused, we simply write $\succcurlyeq_K$ as $\succcurlyeq$ in the following sequel. By (4.1), this proper closed and convex cone K defines a preorder on Z satisfying conditions $(O_1 - O_4)$, that is said to be induced by K (sometimes, we write this ordering notation by $\succcurlyeq_K$ to indicate that this ordering relation is induced by K. If there is no confusion caused, it is simply written as $\succcurlyeq$). So, $(Z, \succcurlyeq)$ is a preordered t.v.s. induced by the proper closed and convex cone K. Moreover, if K is a pointed closed and convex cone in Z, then the ordering relation $\succcurlyeq$ becomes a partial order on Z and $(Z, \succcurlyeq)$ is a partially ordered t.v.s. induced by the pointed closed and convex cone K.

Observations 4.2. For any given t.v.s. Z, through the definition (4.1),

(i) there is a one to one correspondence between the set of proper closed and convex cones in Z and the set of preordering relations on Z satisfying conditions $(O_1 - O_4)$;

(ii) there is a one to one correspondence between the set of pointed closed and convex cones in Z and the set of partial ordering relations on Z satisfying conditions $(O_1 - O_4)$.

Let $(Z, \succcurlyeq)$ be a preordered topological vector space, in which the preorder $\succcurlyeq$ on Z is induced by a proper closed and convex cone K in Z. Let

$$E_K := K \cap (-K) = \{z \in K : \ -z \in K\}.$$

One can easily check that E_K is a closed subspace of Z. E_K is called the $\succcurlyeq$-cluster kernel or the K-cluster kernel (simply called the cluster kernel), or it is called the $\backsim$-equivalence space (simply called the equivalence space) of the preorder $\succcurlyeq$ (or the cone K) on Z.

The preorder $\succcurlyeq$ on Z induced by K is a partial order if and only if the $\succcurlyeq$-cluster kernel E_K is the singleton $\{0\}$ (E_K is trivial if $E_K \in \{\{0\}, Z\}$).

One can see that, for any $z_1, z_2 \in Z$, $z_1 \backsim z_2$ if and only if $z_2 - z_1 \in E_K$. It implies that in the preordered t.v.s. $(Z, \succcurlyeq)$, the $\succcurlyeq$-clusters are represented by the $\succcurlyeq$-cluster kernel as following

$$[z]_{\succcurlyeq} = [z] = z + E_K, \text{ for every } z \in Z. \tag{4.2}$$

It follows that every $\succcurlyeq$-cluster $[z]$ in $(Z, \succcurlyeq)$ is an "affine space" in Z passing through the point z and "parallel" to the closed subspace E_K in Z. As the set of $\succcurlyeq$-clusters in $(Z, \succcurlyeq)$, from (2.2), the cluster space of $(Z, \succcurlyeq)$ induced by K has the following presentation:

$$[Z]_{\succcurlyeq} = [Z] = \{z + E_K : \ z \in Z\}. \tag{4.3}$$

Let A be a nonempty subset of a preordered vector space $(Z, \succcurlyeq)$, in which the preorder $\succcurlyeq$ on Z is induced by a proper convex cone K. Let $F : A \to 2^A \setminus \{\emptyset\}$ be a set-valued mapping. Then, every $\succcurlyeq$-clustered fixed point of F (recalled in Section 2) is also called a K-clustered fixed point of F.

4.2. **Ordering relations on power sets of ordered t.v.s.** Let $(Z, \succcurlyeq)$ be a preordered t.v.s. induced by a proper closed and convex cone K in Z. In the literature of set optimization theory on ordered t.v.s., some authors define an ordering relation $\succsim$ on the power set 2^Z as follows: for any $A, B \in 2^Z$,

$$B \succsim A \text{ if and only if } B - A \subseteq K. \tag{4.4}$$

As we discussed about this ordering relation on the power set of Z in [15], we think that this definition is "too strong" which may not be applicable in the real word. It is because that, following (4.4), $B_2 \succsim B_1$ if and only if

$$x_2 - x_1 \in K, \text{ for every } x_2 \in B_2, \text{ and } x_1 \in B_1.$$

It is a very strong condition. (As a general example, which may not be in preordered topological vector spaces, suppose that during a game competition, let A and B be two distinct teams. Then that team B is "better" than team A in the sense of definition (4.4) means that every given player in team B is "better" than every other player in team A. One can see that it is "too strong"). So, in [15] (recalled in Section 2 in this paper), some more applicable definitions of ordering relations among the subsets of ordered sets (see Jahn and Ha [9], Kuroiwa [17–19, 21]) were used, which are called the power preorder, upward power preorder and downward power preorder. They are denoted by $\succcurlyeq^P$, $\succcurlyeq^U$ and $\succcurlyeq^D$, respectively. These ordering relations on power sets should be more useful than the ordering $\succsim$ defined in (4.4) in the set optimization theory and vector optimization theory on ordered t.v.s..

Let $(Z, \succcurlyeq)$ be a preordered t.v.s. induced by a proper closed and convex cone K in Z. Then, the power preorder, upward power preorder and downward power preorder on 2^Z denoted by $\succcurlyeq^P$, $\succcurlyeq^U$ and $\succcurlyeq^D$ (see Jahn and Ha [9], Kuroiwa [17–19, 21], [15], recalled in Section 2 in this paper) can be redefined in terms of the proper closed and convex cone K in Z as follows: For any $A, B \in 2^Z$,

(a) (power preorder $\succcurlyeq^P$) $A \preccurlyeq^P B$ if and only if
 - (U) (upward condition) for any $a \in A$, there is $b \in B$ such that $b \succcurlyeq a$, that is, $b - a \in K$;
 - (D) (downward condition) for any $b \in B$, there is $a \in A$ such that $a \preccurlyeq b$, that is, $b - a \in K$.

 In terms of the cone K, it is rewritten as: $A \preccurlyeq^P B$ if and only if
 - (U)' (upward condition) $(B - a) \cap K \neq \emptyset$, for every $a \in A$,
 - (D)' (downward condition) $(b - A) \cap K \neq \emptyset$, for every $b \in B$,

(b) (upward preorder $\succcurlyeq^U$) $A \preccurlyeq^U B$ if and only if the upward condition (U) or (U)' holds,

(c) (downward preorder $\succcurlyeq^D$) $A \preccurlyeq^D B$ if and only if the downward condition (D) or (D)' holds.

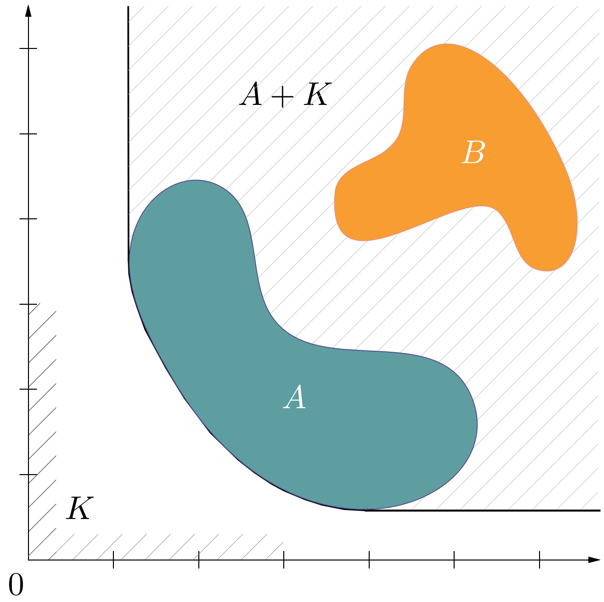

FIGURE 1. $A \preccurlyeq^D B$ $(B \subseteq A + K)$

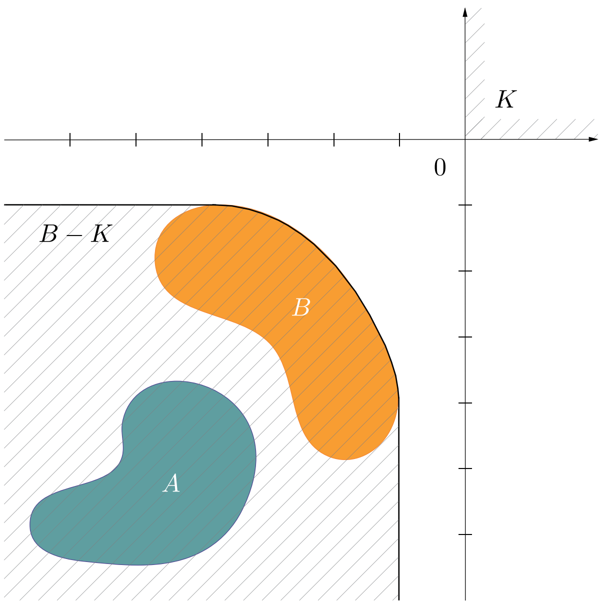

FIGURE 2. $A \preccurlyeq^U B$ $(A \subseteq B - K)$

From Subsection 2.2 in [15], one gets the following results immediately

Lemma 4.2. *Let $(Z, \succcurlyeq)$ be a preordered t.v.s. induced by a proper closed and convex cone K in Z. Then, $(2^Z, \succcurlyeq^P)$, $(2^Z, \succcurlyeq^U)$ and $(2^Z, \succcurlyeq^D)$ all are preordered sets; and $([Z], \succcurlyeq^P)$, $([Z], \succcurlyeq^U)$ and $([Z], \succcurlyeq^D)$ all are equivalent partially ordered sets.*

4.3. **The cluster space of a preordered t.v.s. is a partially ordered vector space.** Let $(Z, \succcurlyeq)$ be a preordered t.v.s. induced by a proper closed and convex cone K in Z. In this subsection, we define addition and scalar multiplication on the $\succcurlyeq$-cluster space $[Z]$, so that it becomes a vector space. For any $[z_1], [z_2] \in [Z]$ and any real number λ, we define

$$[z_1] + [z_2] = [z_1 + z_2] \text{ and } \lambda[z_1] = [\lambda z_1]. \tag{4.5}$$

Lemma 4.3. *Let $(Z, \succcurlyeq)$ be a preordered t.v.s. induced by a proper closed and convex cone K in Z. With the two addition and scalar multiplication operators on $[Z]$ defined in (4.5), one has*

(a) *$[Z]$ is a vector space;*
(b) *$([Z], \succcurlyeq^K), ([Z], \succcurlyeq^U)$ and $([Z], \succcurlyeq^D)$ all are equivalent partially ordered vector spaces. $[0] = E_K$ is the origin of $[Z]$.*

Proof. The preorder $\succcurlyeq$ on Z satisfies conditions $(O_1 - O_2)$. It implies that the equivalence relation $\backsim$ on Z corresponding to $\succcurlyeq$ has the following linearity properties.

(O_1)' $u_1 \backsim v_1$ and $u_2 \backsim v_2$ imply $u_1 + u_2 \backsim v_1 + v_2$, for u_1, u_2, v_1 and $v_2 \in Z$;
(O_2)' $u \backsim v$ implies $\alpha u \backsim \alpha v$, for $u, v \in Z$ and $\alpha \geq 0$.

Since the K-cluster kernel E_K is a subspace of Z, in terms of the proper closed and convex cone K in Z, (O_1)' and (O_2)' can be rewritten as:

(O_1)" $u_1 - v_1 \in E_K$ and $u_1 - v_1 \in E_K$ imply $(u_1 + u_2) - (v_1 + v_2) \in E_K$, for u_1, u_2, v_1 and $v_2 \in Z$;
(O_2)" $u - v \in E_K$ implies $\alpha u - \alpha v \in E_K$, for $u, v \in Z$ and $\alpha \geq 0$.

By these properties, one can check that the addition operator and scalar multiplication operator defined in (4.5) are well defined on $[Z]$; and $[Z]$ is a vector space. From Lemma 4.2, $\succcurlyeq^P, \succcurlyeq^U$ and $\succcurlyeq^D$ all are equivalent partial orders on $[Z]$. To ensure that $([Z], \succcurlyeq^K), ([Z], \succcurlyeq^U)$ and $([Z], \succcurlyeq^D)$ all are partially ordered vector space (see Remark 4.1), we need to check that the partial ordering relations $\succcurlyeq^P, \succcurlyeq^U$ and $\succcurlyeq^D$ on $[Z]$ all satisfy conditions (O_1), (O_2) and (O_3). It is straightforward to check, and it is omitted here. □

4.4. **Order-clustered fixed point theorems on ordered t.v.s.** By Theorems 3.5 and 3.6 in the previous section, we immediately obtain some order-clustered fixed point theorems on preordered (partially ordered) t.v.s., in which the conditions are directly presented by the cones. So that these theorems can be easily used in vector optimization theory and vector variational inequalities. For this purpose, we explain the order-clustered fixed point in t.v.s. with respect to the proper closed and convex cones.

Let Z be a t.v.s. and K a proper closed and convex cone in Z with the K-cluster kernel E_K, which induces a preorder $\succcurlyeq$ on Z. Let A be a nonempty subset of Z and let $F : A \to 2^A \setminus \{\emptyset\}$ be a set-valued mapping. A point $z \in A$ is a K-clustered fixed point of F if there is $u \in F(z)$ such that $z^* - u \in E_K$.

Theorem 4.4. *Let $(Z, \succcurlyeq)$ be a preordered Hausdorff t.v.s. induced by a proper closed and convex cone K in Z with the K-cluster kernel E_K. Let A be a nonempty compact subset of Z. Let $F : A \to 2^A \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that F satisfies the following conditions:*

(i) *for any $t_1, t_2 \in A$, $t_2 - t_1 \in K$ implies that*

$$(F(t_2) - z_1) \cap K \neq \emptyset, \text{ for every } z_1 \in F(t_1), \tag{$(U)'$}$$

and

$$(z_2 - F(t_1)) \cap K \neq \emptyset, \text{ for every } z_2 \in F(t_2). \tag{$(D)'$}$$

(ii) *for every $z \in A$, $F(z)$ is closed;*

(iii) *there are $s_1, s_2 \in A$ such that*

$$(F(s_1) - s_1) \cap K \neq \emptyset \text{ and } (s_2 - F(s_2)) \cap K \neq \emptyset. \tag{4.6}$$

Then, F has a K-clustered fixed point. That is, there are $z^* \in A$ and $u \in F(z^*)$ such that

$$z^* - u \in E_K. \tag{4.7}$$

Moreover, we have

(a) *$\mathfrak{F}(F)$ is a nonempty bi-inductive preordered subset of $(A, \succcurlyeq)$;*

(b) *F has a $\succcurlyeq$-maximal K-clustered fixed point u_1 with $u_1 - s_1 \in K$;*

(c) *F has a $\succcurlyeq$-minimal K-clustered fixed point u_2 with $s_2 - u_2 \in K$.*

Proof. For the preordered Hausdorff t.v.s. $(Z, \succcurlyeq)$ induced by the proper closed and convex cone K, from Proposition 3.1, $(A, \succcurlyeq)$ is $\succcurlyeq$-chain complete.

Let $\succcurlyeq^P$, $\succcurlyeq^U$ and $\succcurlyeq^D$ be the power, upward power and downward power preorders, respectively, on the power set 2^Z. The conditions (U)' and (D)' in condition (i) in this theorem imply that F is order-increasing with respect to $\succcurlyeq$ and $\succcurlyeq^P$. So, F satisfies condition (i) in Theorem 2.3 in [15] recalled in in Section 2. From condition (ii) in this theorem, for every $z \in A$, $F(z)$ is a closed subset of a compact set A. So, it is compact. From Proposition 3.2, $F(z)$ is bi-universally inductive in $(A, \succcurlyeq)$. By Proposition 2.2 in [15], $[F(z)]$ is bi-universally inductive in $([A], \succcurlyeq^P)$. Then, F satisfies condition (ii) in Theorem 2.3 in [15]. Conditions (4.6) in condition (iii) in this theorem imply that $\{s_1\} \preccurlyeq^U F(s_1)$ and $\{s_2\} \succcurlyeq^D F(s_2)$. Hence F satisfies all conditions in Theorem 2.3 in [15]. Then, F has a $\succcurlyeq$-clustered fixed point. The conclusions (a), (b) and (c) immediately follow from (a), (b) and (c) of Theorem 2.3 in [15], respectively. $\square$

Next, we consider Banach spaces which are considered as special cases of Hausdorff t.v.s.. Let $(Z, \|\cdot\|, \tau)$ be a Banach space with norm $\|\cdot\|$ and weak topology τ. Let K be a proper convex cone in Z. Then, K is closed with respect to the τ-topology if and only if K is weakly closed with respect to the τ-topology. Let $\succcurlyeq$ be the ordering relation on Z induced by K, then $(Z, \|\cdot\|, \succcurlyeq)$ is a preordered t.v.s. if and only if $(Z, \tau, \succcurlyeq)$ is a preordered topological vector space. Applying Theorem 4.4 to preordered reflexive Banach spaces, we have the following result.

Corollary 4.5. *Let $(Z, \|\cdot\|)$ be a reflexive Banach space and K a proper closed and convex cone in Z. Let A be a nonempty norm-bounded closed convex subset of*

Z. Let $F : A \to 2^A \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that F satisfies the conditions (i) and (iii) in Theorem 4.4, and

(ii)' *$F(z)$ is weakly closed, for every $z \in A$.*

Then, the conclusions of Theorem 4.4 remain true.

Proof. Let τ be the weak topology on Z. Then, K is weakly closed with respect to the τ-topology. So, $(Z, \tau, \succcurlyeq)$ is a preordered τ-topological vector space. Since A is a nonempty norm-bounded closed convex subset in this reflexive Banach space Z, then A is weakly compact (τ- compact). By applying the weak topology τ on Z, this corollary follows from Theorem 4.4 immediately. □

5. The solvability of extended equilibrium problems in ordered t.v.s.

5.1. **The semicontinuity and convexity with respect to ordered power sets.** In this subsection, we introduce the concepts of order-semicontinuity and order-convexity, which will be used in the proof of the existence of extended equilibrium for set-valued mappings in ordered t.v.s. by Fan-KKM theorem.

Definition 5.1. Let X and Y be t.v.s.. Let $(Z, \succcurlyeq)$ be a preordered t.v.s. induced by a proper closed and convex cone K in Z. Let $\succcurlyeq^P, \succcurlyeq^U$ and $\succcurlyeq^D$ be the power preorder, upward power preorder and downward power preorder on 2^Z, respectively. Let C, D be nonempty closed and convex subsets of X and Y, respectively. Let $f : C \to 2^Z \setminus \{\emptyset\}$, $g : C \to 2^Z \setminus \{\emptyset\}$ and $T : C \times D \to 2^Z \setminus \{\emptyset\}$ be set-valued mappings.

(i) g is said to be $\succcurlyeq^U$-lower semicontinuous on D with respect to T, whenever the set
$$\{(x, y) \in C \times D : \; g(y) \preccurlyeq^U T(x, y)\}$$
is a closed subset in $C \times D$ with respect to the product topology of the topologies of X and Y. Corresponding to the cone K, it is equivalently restated as:
$$\{(x, y) \in C \times D : \; (T(x, y) - z) \cap K \neq \emptyset, \text{ for every } z \in g(y)\}$$
is a closed subset in $C \times D$ with respect to the product topology of the topologies of X and Y.

(ii) f is said to be $\succcurlyeq^D$-upper semicontinuous on C with respect to T , whenever the set
$$\{(x, y) \in C \times D : \; f(x) \succcurlyeq^D T(x, y)\}$$
is a closed subset in $C \times D$ with respect to the product topology of the topologies of X and Y. Corresponding to the cone K, it is equivalently restated as:
$$\{(x, y) \in C \times D : \; (z - T(x, y)) \cap K \neq \emptyset, \text{ for every } z \in f(x)\}$$
is a closed subset in $C \times D$ with respect to the product topology of the topologies of X and Y.

(iii) g is said to be $\succcurlyeq^U$ -convex on D , whenever, for given arbitrary positive integer k, for any $y_1, y_2, \ldots, y_k \in D$, and for any $0 \le \alpha_i \le 1$, $i = 1, 2, \ldots, k$, with $\sum_{i=1}^{k} \alpha_i = 1$, there is at least one $j = 1, 2, \ldots, k$, such that

$$g(\sum_{i=1}^{k} \alpha_i y_i) \preccurlyeq^U g(y_j).$$

Corresponding to the cone K, it is equivalently restated as:

$$(g(y_j) - z) \cap K \neq \emptyset, \text{ for every } z \in g(\sum_{i=1}^{k} \alpha_i y_i), \text{ for some } j = 1, 2, \ldots, k.$$

(iv) f is said to be $\succcurlyeq^D$-concave on C, whenever, for given arbitrary positive integer k, for any $x_1, x_2, \ldots, x_k \in C$, and for any $0 \leq \alpha_i \leq 1$, $i = 1, 2, \ldots, k$, with $\sum_{i=1}^{k} \alpha_i = 1$, there is at least one $j = 1, 2, \ldots, k$, such that

$$f(\sum_{i=1}^{k} \alpha_i x_i) \succcurlyeq^D f(x_j).$$

Corresponding to the cone K, it is equivalently restated as:

$$(z - f(x_j)) \cap K \neq \emptyset, \text{ for every } z \in f(\sum_{i=1}^{k} \alpha_i x_i), \text{ for some } j = 1, 2, \ldots, k.$$

(v) $T : C \times D \to 2^Z \setminus \{\emptyset\}$ is said to be convex saddle, whenever, for any given positive integer k and k points $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k) \in C \times D$, and for any $0 \leq \alpha_i \leq 1$, $i = 1, 2, \ldots, k$, with $\sum_{i=1}^{k} \alpha_i = 1$, there is at least one $1 \leq j \leq k$, such that

$$T(x_j, \sum_{i=1}^{k} \alpha_i y_i) \preccurlyeq^U T(\sum_{i=1}^{k} \alpha_i x_i, \sum_{i=1}^{k} \alpha_i y_i) \preccurlyeq^D T(\sum_{i=1}^{k} \alpha_i x_i, y_j).$$

Corresponding to the cone K, it is equivalently restated as:

$$(T(\sum_{i=1}^{k} \alpha_i x_i, \sum_{i=1}^{k} \alpha_i y_i) - z_1) \cap K \neq \emptyset, \text{ for every } z_1 \in T(x_j, \sum_{i=1}^{k} \alpha_i y_i),$$

and

$$(z_2 - T(\sum_{i=1}^{k} \alpha_i x_i, \sum_{i=1}^{k} \alpha_i y_i)) \cap K \neq \emptyset, \text{ for every } z_2 \in T(\sum_{i=1}^{k} \alpha_i x_i, y_j).$$

(vi) $T : C \times D \to 2^Z \setminus \{\emptyset\}$ is said to be quasi-convex saddle, whenever, for any given positive integer k and k points $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k) \in C \times D$, and for any $0 \leq \alpha_i \leq 1$, $i = 1, 2, \ldots, k$, with $\sum_{i=1}^{k} \alpha_i = 1$, there is at least a pair $1 \leq m, n \leq k$, such that

$$T(x_m, \sum_{i=1}^{k} \alpha_i y_i) \preccurlyeq^U T(\sum_{i=1}^{k} \alpha_i x_i, \sum_{i=1}^{k} \alpha_i y_i) \preccurlyeq^D T(\sum_{i=1}^{k} \alpha_i x_i, y_n).$$

Corresponding to the cone K, it is equivalently restated as:

$$(T(\sum_{i=1}^{k} \alpha_i x_i, \sum_{i=1}^{k} \alpha_i y_i) - z_1) \cap K \neq \emptyset, \text{ for every } z_1 \in T(x_m, \sum_{i=1}^{k} \alpha_i y_i),$$

and

$$(z_2 - T(\sum_{i=1}^{k} \alpha_i x_i, \sum_{i=1}^{k} \alpha_i y_i)) \cap K \neq \emptyset, \text{ for every } z_2 \in T(\sum_{i=1}^{k} \alpha_i x_i, y_n).$$

Remark 5.2. (a) On page 199 in [7], the order-lower semicontinuous and order-upper semicontinuous are defined for order monotone Cauchy sequences. Since the order monotone sequences are not sufficient for the proof of the main theorem in this paper, so we define the order- lower semicontinuities for arbitrary Cauchy sequences.

(b) On page 319 in [12], saddle points are defined in ordered spaces. That is a directly extension of the concept of saddle points from the ordinary order in real analysis to nonlinear order in order theory. The concept of convex saddle in part (v) of Definition 5.1 is a global concept, which deals any combination of a finite number (at least two) of points in the domain of the considered mappings.

It is clear that a mapping $T : C \times D \to 2^Z \setminus \{\emptyset\}$ is convex saddle implies that T is quasi-convex saddle. We provide an example below to demonstrate the concept of "quasi-convex saddle property" is indeed more general than the concept of convex saddle property.

Example 5.3. In part (v) of Definition 5.1, take X, Y and Z to be the real number set $\mathbb{R}$ with ordinary real order $\geq$. Let $T : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a real valued function defined by

$$T(x, y) := -x^2 + y^2, \text{ for every } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

The point $(0, 0)$ is the unique saddle point of T. Take an arbitrary given positive integer k, arbitrary distinct k points $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k) \in \mathbb{R} \times \mathbb{R}$ and any given $0 \leq \alpha_i \leq 1$, $i = 1, 2, \ldots, k$, with $\sum_{i=1}^{k} \alpha_i = 1$. For any $1 \leq j \leq k$, we have

$$T(\sum_{i=1}^{k} \alpha_i x_i, \sum_{i=1}^{k} \alpha_i y_i) = -(\sum_{i=1}^{k} \alpha_i x_i)^2 + (\sum_{i=1}^{k} \alpha_i y_i)^2;$$

$$T(x_j, \sum_{i=1}^{k} \alpha_i y_i) = -x_j^2 + (\sum_{i=1}^{k} \alpha_i y_i)^2;$$

$$T(\sum_{i=1}^{k} \alpha_i x_i, y_j) = -(\sum_{i=1}^{k} \alpha_i x_i)^2 + y_j^2.$$

In case if $(\sum_{i=1}^{k} \alpha_i x_i, \sum_{i=1}^{k} \alpha_i y_i) = (0, 0)$, which is the saddle point of F, then for any $1 \leq j \leq k$, the following inequalities hold:

$$T(x_j, \sum_{i=1}^{k} \alpha_i y_i) \leq T(\sum_{i=1}^{k} \alpha_i x_i, \sum_{i=1}^{k} \alpha_i y_i) \leq T(\sum_{i=1}^{k} \alpha_i x_i, y_j). \tag{5.1}$$

Take $k = 2$ and take two points $(3, 0)$ and $(2, 1)$. Let $0 < \alpha_1,\ \alpha_2 < 1$. One can check that there does not exist $j = 1, 2$ such that (5.1) holds. It implies that T does not have the convex saddle property on $\mathbb{R} \times \mathbb{R}$. Furthermore, we can check that, for any closed intervals C and D in $\mathbb{R}$ both containing 0 as an interior point, T does not have the convex saddle property on $C \times D$. In Example 5.6 below, we will show that T has the quasi-convex saddle property on $\mathbb{R} \times \mathbb{R}$.

5.2. **Existence of extended equilibrium with convex saddle property.** From Example 5.3, we see that the convex saddle property is much stronger than the quasi-convex saddle property; and there are some mappings which do not have convex saddle property and have equilibrium. In this subsection, we prove an existence theorem of extended equilibrium for some set-valued mappings with the convex saddle property.

In this subsection and throughout the following subsections in Section 5, $(Z, \succcurlyeq)$ denotes a preordered t.v.s. induced by a proper closed and convex cone K in Z. Based on the preorder $\succcurlyeq$ on Z, $\succcurlyeq^P, \succcurlyeq^U$ and $\succcurlyeq^D$ denote the power, upward power and downward power preorders on 2^Z, respectively.

Theorem 5.4. *Let X and Y be Hausdorff topological vector spaces (H.t.v.s.). Let $(Z, \succcurlyeq)$ be a preordered topological vector space. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty closed and convex subsets and let $T : C \times D \to 2^Z \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that T satisfies the following conditions:*

(i) *for every fixed $(x, y) \in C \times D$, $T(x, \cdot)$ is $\succcurlyeq^U$-lower semicontinuous on D and $T(\cdot, y)$ is $\succcurlyeq^D$-upper semicontinuous on C both with respect to $T(\cdot, \cdot)$;*
(ii) *T is convex saddle on $C \times D$;*
(iii) *there is $(x_0, y_0) \in C \times D$ such that $\{(s, t) \in C \times D : \ T(x_0, t) \preccurlyeq^U T(s, t) \preccurlyeq^D T(s, y_0)\}$ is compact.*

Then, there is $(x^*, y^*) \in C \times D$ such that

$$T(x, y^*) \preccurlyeq^U T(x^*, y^*) \preccurlyeq^D T(x^*, y), \ \text{for every } (x, y) \in C \times D. \tag{5.2}$$

Proof. Define a mapping $F : C \times D \to 2^{C \times D} \setminus \{\emptyset\}$ by

$$F(x, y) = \{(s, t) \in C \times D : T(x, t) \preccurlyeq^U T(s, t) \preccurlyeq^D T(s, y)\}, \ \text{for } (x, y) \in C \times D.$$

Since for $(x, y) \in C \times D$, $(x, y) \in F(x, y)$, F is well defined from $C \times D$ to $2^{C \times D} \setminus \{\emptyset\}$. Since C and D are nonempty closed and convex subsets of X and Y, respectively, then $C \times D$ is a closed and convex subset of $X \times Y$ with respect to the product topology. We first prove that, for any $(x, y) \in C \times D$, $F(x, y)$ is a closed subset in $C \times D$.

From Definition 5.1 and condition (i) in this theorem, for any $(x, y) \in C \times D$, we have that

$$\{(s, t) \in C \times D : T(x, t) \preccurlyeq^U T(s, t)\} \quad \text{and} \quad \{(s, t) \in C \times D : T(s, t) \preccurlyeq^D T(s, y)\}$$

both are closed subsets in $C \times D$ with respect to the product topology of the topologies of X and Y. As intersection of closed subsets, it implies that $F(x, y)$ is closed in $C \times D$.

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ be k points in $C \times D$, for some positive integer $k > 1$. Take an arbitrary linear combination of these k points:

$$(s, t) = \alpha_1(x_1, y_1) + \alpha_2(x_2, y_2) + \cdots + \alpha_k(x_k, y_k),$$

where $\alpha_i \geq 0$, for $i = 1, 2, \dots, k$ satisfying $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$. From condition (ii) in this theorem, $T : C \times D \to 2^Z \setminus \{\emptyset\}$ is convex saddle. Then, there is $1 \leq j \leq k$ such that

$$T(x_j, \sum_{i=1}^{k} \alpha_i y_i) \preccurlyeq^U T(\sum_{i=1}^{k} \alpha_i x_i, \sum_{i=1}^{k} \alpha_i y_i) \preccurlyeq^D T(\sum_{i=1}^{k} \alpha_i x_i, y_j).$$

That is,

$$T(x_j, t) \preccurlyeq^U T(s, t) \preccurlyeq^D T(s, y_j).$$

It implies that $(s, t) \in F(x_j, y_j)$. Hence, F is a KKM mapping. From condition (iii) of this theorem, $F(x_0, y_0)$ is compact. So, applying the Fan-KKM theorem, one has

$$\cap\{F(x, y) : \ (x, y) \in C \times D\} \neq \emptyset.$$

Take an arbitrary $(x^*, y^*) \in \cap\{F(x, y) : \ (x, y) \in C \times D\}$. (x^*, y^*) satisfies (5.2). $\square$

Observations. In game theory, the point $(x^*, y^*) \in C \times D$ in (5.2) is called an extended (or generalized) Nash equilibrium of some two-person strategic games with the utility function T. We examine the conclusion of Theorem 5.4 as a result in game theory.

If the utility function T satisfies the conditions in Theorem 5.4, then there is a strategy profile (x^*, y^*) satisfying (5.2). The first order inequality in (5.2) means that as the second player selects his strategy $y^* \in D$ to play, then for any strategy $x \in C$ for the first player to select, for any given element u in the utility set $T(x, y^*)$, there is an utility $w \in T(x^*, y^*)$ such that $u \preccurlyeq w$. The second inequality in (5.2) means that, as the first player selects his strategy $x^* \in C$ to play, then for any strategy $y \in D$ for the second player to select, for any given element v in the utility set $T(x^*, y)$, there is an utility $w \in T(x^*, y^*)$ such that $w \preccurlyeq v$.

5.3. **On the quasi-convex saddle property.** In Example 5.3, the real valued function

$$T(x, y) = -x^2 + y^2, \text{ for every } (x, y) \in \mathbb{R} \times \mathbb{R}$$

does not have the convex saddle property and it has an equilibrium (a saddle point) $(0, 0)$. So, in Theorem 5.4, the convex saddle property is a sufficient condition for the existence of an equilibrium (a saddle point) of a set-valued mapping, which is not a necessary condition.

On the other hand, the function T in Example 5.3 has the quasi-convex saddle property and it has a saddle point. It raises the question: in addition to the convex saddle property, what is the condition to assure the existence of an equilibrium (a saddle point) of a mapping?

Like the proof of Theorem 5.4, we can prove the following results regarding to the quasi-convex saddle property. It will be used for the proof of the existence theorem.

Lemma 5.5. *Let X and Y be H.t.v.s.. Let $(Z, \succcurlyeq)$ be a preordered topological vector space. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty closed and convex subsets and let $T : C \times D \to 2^Z \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that F satisfies the following conditions:*

(i) *for every fixed* $(x,y) \in C \times D, T(x,\cdot)$ *is* $\succcurlyeq^U$*-lower semicontinuous on* D *and* $T(\cdot,y)$ *is* $\succcurlyeq^D$*-upper semicontinuous on* C *both with respect to* $T(\cdot,\cdot)$*;*

(ii) F *is quasi-convex saddle on* $C \times D$ *;*

(iii) *there is* $(x_0,y_0) \in C \times D$ *such that the following two subsets are compact*

$$\{(s,t) \in C \times D : T(x_0,t) \preccurlyeq^U T(s,t)\} \text{ and } \{(s,t) \in C \times D : T(s,t) \preccurlyeq^D T(s,y_0)\}.$$

Then, there are $(x',y'),(x'',y'') \in C \times D$ *such that*

$$T(x,y') \preccurlyeq^U T(x',y') \text{ and } T(x'',y'') \preccurlyeq^D T(x'',y), \text{ for every } (x,y) \in C \times D. \tag{5.3}$$

Proof. Define mappings $F_1 : C \times D \to 2^{C\times D}$ and $F_2 : C \times D \to 2^{C\times D}$ by, for $(x,y) \in C \times D$,

$$F_1(x,y) = \{(s,t) \in C \times D : T(x,t) \preccurlyeq^U T(s,t)\},$$

and

$$F_2(x,y) = \{(s,t) \in C \times D : T(s,t) \preccurlyeq^D T(s,y)\}.$$

Since for every $(x,y) \in C \times D, (x,y) \in F_1(x,y)$ and $(x,y) \in F_2(x,y)$, F_1 and F_2 both are well defined from $C \times D$ to $2^{C\times D} \setminus \{\emptyset\}$. Rest of the proof is similar to the proof of Theorem 5.4. □

Example 5.6. Let T be defined in Example 5.3. We can check that, for any closed intervals C and D in $\mathbb{R}$ both containing 0 as an interior point, T does not have the convex saddle property on $C \times D$. On the other hand, one can check that T has the quasi-convex saddle property on $\mathbb{R} \times \mathbb{R}$ and F satisfies all conditions in Lemma 5.6. Moreover, we have that, for every $y' \in \mathbb{R}$, the point $(0,y')$ satisfies

$$T(x,y') \preccurlyeq^U T(0,y'), \text{ for every } (x,y) \in \mathbb{R} \times \mathbb{R}.$$

And, for every $x'' \in \mathbb{R}$, the point $(x'',0)$ satisfies

$$T(x'',0) \preccurlyeq^D T(x'',y), \text{ for every } (x,y) \in \mathbb{R} \times \mathbb{R}.$$

Lemma 5.5 does not guarantee the existence of a saddle point of T (even through T has one saddle point $(0,0)$).

6. Vector extended equilibrium problems with set-valued mappings

6.1. **Definitions of vector extended Nash equilibrium problems with set-valued mappings.** Let $(X,\succcurlyeq_X)$ and $(Y,\succcurlyeq_Y)$ be preordered t.v.s. induced by proper closed and convex cones $I \subseteq X$ and $J \subseteq Y$. Let E_I , E_J be the I-cluster kernel and the J-cluster kernel of $(X,\succcurlyeq_X)$ and $(Y,\succcurlyeq_Y)$, respectively.

Let $\succcurlyeq_{XY}$ be the component-wise preorder of $\succcurlyeq_X$ and $\succcurlyeq_Y$ on $X \times Y$ satisfying that, for any $(x_1,y_1),(x_2,y_2) \in X \times Y$,

$$(x_2,y_2) \succcurlyeq_{XY} (x_1,y_1) \text{ if and only if } x_2 \succcurlyeq_X x_1 \text{ and } y_2 \succcurlyeq_Y y_1.$$

Then, $(X \times Y,\succcurlyeq_{XY})$ is a preordered t.v.s. with the product topology and the preorder $\succcurlyeq_{XY}$ is induced by the proper closed and convex cone $I \times J$ in $X \times Y$. The $\succcurlyeq_{XY}$-cluster kernel of $(X \times Y,\succcurlyeq_{XY})$ is $E_I \times J$ satisfying

$$E_{I\times J} = E_I \times E_J.$$

In particular, the preordered t.v.s. $(X,\succcurlyeq_X)$ and $(Y,\succcurlyeq_Y)$ are partially ordered t.v.s., then I and J both are pointed closed and convex cones in X and Y, respectively, and the $\succcurlyeq$-cluster kernels E_I and E_J are singletons. In this case, $I \times J$ is a pointed closed and convex cone in $X \times Y$ and $(X \times Y, \succcurlyeq_{XY})$ is a partially ordered t.v.s. as well.

If $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ both are preordered H.t.v.s., then $(X \times Y, \succcurlyeq_{XY})$ is also a preordered Hausdorff topological vector space.

Let $(Z, \succcurlyeq)$ be a preordered topological vector space, in which the preorder $\succcurlyeq$ on Z is induced by a proper closed and convex cone K in Z. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty subsets and let $T : C \times D \to 2^Z$ be a set-valued mapping.

Definition 6.1. A point $(x_0, y_0) \in C \times D$ is called an $E_{I\times J}$-clustered extended Nash equilibrium for a set-valued mapping $T : C \times D \to 2^Z$, if there is $(x_1, y_1) \in C \times D$ with $(x_0, y_0) \backsim_{XY} (x_1, y_1)$ such that

$$T(x, y_0) \preccurlyeq^U T(x_1, y_0) \text{ and } T(x_0, y_1) \preccurlyeq^D T(x_0, y), \text{ for all } (x, y) \in C \times D.$$

Corresponding to the cone K, it is equivalently restated as: there is $(x_1, y_1) \in C \times D$ with

$$(x_0, y_0) - (x_1, y_1) \in E_{I\times J}$$

such that, for all $(x, y) \in C \times D$,

$$(T(x_1, y_0) - z) \cap K \neq \emptyset, \text{ for every } z \in T(x, y_0)$$

and

$$(z - T(x_0, y_1)) \cap K \neq \emptyset, \text{ for every } z \in T(x_0, y).$$

The set of all order-clustered extended Nash equilibriums of T is denoted by $\mathcal{E}(T)$.

For the preordered t.v.s. $(Z, \succcurlyeq)$, as usual, we let $\succcurlyeq^P$, $\succcurlyeq^U$ and $\succcurlyeq^D$ be the power preorder, upward power preorder and downward power preorder on 2^Z corresponding to $\succcurlyeq$, respectively.

With respect to the set-valued mapping $T : C \times D \to 2^Z$, we define two set-valued mappings $\Phi : C \to 2^D$ and $\Psi : D \to 2^C$ as follows:

$$\begin{aligned}\Phi(s) &= \{t \in D : \ T(s, t) \text{ is a } \succcurlyeq^D \text{-lower bound of } \{T(s, y) : y \in D\}\} \\ &= \{t \in D : (z - T(s, t)) \cap K \neq \emptyset, \text{ for every } z \in T(s, y), \text{ for every } y \in D\}, \\ &\quad \text{for any } s \in C.\end{aligned}$$

$$\begin{aligned}\Psi(t) &= \{s \in C : T(s, t) \text{ is a } \succcurlyeq^U \text{-upper bound of } \{T(x, t) : x \in C\}\} \\ &= \{s \in C : (T(s, t) - z) \cap K \neq \emptyset, \text{ for every } z \in T(x, t), \text{ for every } x \in C\}, \\ &\quad \text{for any } t \in D.\end{aligned}$$

Regarding to the order monotonicity which is defined in (3.3), it becomes the following form with respect to the cones that induce the orders.

Remark 6.2. Let $T : C \times D \to 2^Z \setminus \{\emptyset\}$ be a set-valued mapping. Corresponding to the cones, by the definition of order monotonicity in (3.3), we have

(a) For every $x \in C$, $T(x, \cdot) : D \to 2^Z \setminus \{\emptyset\}$ is $\succcurlyeq_Y$-$\succcurlyeq^U$ increasing if and only if, for any t_1, $t_2 \in D$, $t_2 - t_1 \in J$ implies

$$(T(x, t_2) - z) \cap K \neq \emptyset, \quad \text{for every} \quad z \in T(x, t_1).$$

(b) For every $y \in D$, $T(\cdot, y) : C \to 2^Z \setminus \{\emptyset\}$ is $\succcurlyeq_X$-$\succcurlyeq^D$ decreasing if and only if, for any s_1, $s_2 \in C$, $s_2 - s_1 \in I$ implies

$$(z - T(s_2, y)) \cap K \neq \emptyset, \quad \text{for every} \quad z \in T(s_1, y).$$

Corresponding to [15, Definition 3.5], we interpreter the concepts of order-upper and order-lower consistency in terms of the closed and convex cones. These concepts are related to the concept of closed graph of mappings with vector set-values.

Definition 6.3. Let $T : C \times D \to 2^Z$ be a set-valued mapping.

(a) T is said to be $\succcurlyeq^U$ -upper consistent on C, whenever, for any $t_1, t_2 \in D$, if,

$$T(x, t_1) \preccurlyeq^U T(x, t_2), \text{ for every } x \in C, \text{ implies } \Psi(t_1) \subseteq \Psi(t_2).$$

More precisely, for some $s \in C$, that $T(s, t_1)$ is a $\succcurlyeq^U$ -upper bound of $\{T(x, t_1) : x \in C\}$ implies that $T(s, t_2)$ is a $\succcurlyeq^U$ -upper bound of $\{T(x, t_2) : x \in C\}$.
It means that, if $T(x, t_1) \preccurlyeq^U T(x, t_2)$, for every $x \in C$, then, for some $s \in C$,

$$s \in \Psi(t_1) \quad \Longrightarrow \quad s \in \Psi(t_2).$$

Corresponding to the cone K, it is equivalently restated as: if for every $x \in C$, the following inclusion holds

$$(T(x, t_2) - z) \cap K \neq \emptyset, \text{ for every } z \in T(x, t_1),$$

then, for some $s \in C$,

$$(T(s, t_1) - z) \cap K \neq \emptyset, \text{ for every } z \in T(u, t_1), \text{ for every } u \in C,$$

implies

$$(T(s, t_2) - z) \cap K \neq \emptyset, \text{ for every } z \in T(u, t_2), \text{ for every } u \in C.$$

(b) T is said to be $\succcurlyeq^D$-lower consistent on D, whenever, for any $s_1, s_2 \in C$, if

$$T(s_1, y) \succcurlyeq^D T(s_2, y), \text{ for every } y \in D, \text{ implies } \Phi(s_1) \subseteq \Phi(s_2).$$

More precisely, for some $t \in D$, that $T(s_1, t)$ is a $\succcurlyeq^D$-lower bound of $\{T(s_1, y) : y \in D\}$ implies that $T(s_2, t)$ is a $\succcurlyeq^D$-lower bound of $\{T(s_2, y) : y \in D\}$.
It means that, if $T(s_1, y) \succcurlyeq^D T(s_2, y)$, for every $y \in D$, then, for some $t \in D$,

$$t \in \Phi(s_1)) \quad \Longrightarrow \quad t \in \Phi(s_2).$$

Corresponding to the cone K, it is equivalently restated as: if, for every $y \in D$, the following inclusion holds

$$(z - T(s_2, y)) \cap K \neq \emptyset, \text{ for every } z \in T(s_1, y),$$

then, for some $t \in D$,

$$(z - T(s_1, t)) \cap K \neq \emptyset, \text{ for every } z \in T(s_1, v), \text{ for every } v \in D,$$

implies

$$(z - T(s_2, t)) \cap K \neq \emptyset, \text{ for every } z \in T(s_2, v), \text{ for every } v \in D.$$

6.2. **Existence of order-clustered extended Nash equilibrium.** Now, we prove an existence theorem of order-clustered Nash equilibrium for some set-valued mappings on preordered t.v.s..

Theorem 6.4. *Let $(X, \succcurlyeq_I)$ and $(Y, \succcurlyeq_J)$ be preordered H.t.v.s. induced by proper closed and convex cones $I \subseteq X$ and $J \subseteq Y$. Let $(Z, \succcurlyeq)$ be a preordered vector space induced by a proper closed and convex cone K in Z. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty compact subsets and let $T : C \times D \to 2^Z \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that T satisfies the following conditions:*

(i) *for every $x \in C$, $T(x, \cdot) : D \to 2^Z \setminus \{\emptyset\}$ satisfies that, for any $t_1, t_2 \in D$, $t_2 - t_1 \in J$ implies*

$$(T(x, t_2) - z) \cap K \neq \emptyset, \text{ for every } z \in T(x, t_1);$$

(ii) *for every $y \in D$, $T(\cdot, y) : C \to 2^Z \setminus \{\emptyset\}$ satisfies that, for any $s_1, s_2 \in D$, $s_2 - s_1 \in I$ implies*

$$(z - T(s_2, y)) \cap K \neq \emptyset, \text{ for every } z \in T(s_1, y);$$

(iii) *T is $\succcurlyeq^U$-upper consistent on C and $\succcurlyeq^D$-lower consistent on D;*

(iv) *for every $(s, t) \in C \times D$, $\Phi(s)$ is closed in D and $\Psi(t)$ is closed in C;*

(v) *there are $(s_0, t_0), (u_0, v_0) \in C \times D$ with $v_0 \in \Phi(s_0)$ and $u_0 \in \Psi(t_0)$ such that*

$$u_0 - s_0 \in I \text{ and } v_0 - t_0 \in J.$$

Then, T has an $E_{I\times J}$-clustered extended Nash equilibrium. That is, there are $(x_0, y_0), (x_1, y_1) \in C \times D$ with

$$x_0 - x_1 \in E_I \text{ and } y_0 - y_1 \in E_J.$$

such that, for all$(x, y) \in C \times D$, the following inclusions hold

$$(T(x_1, y_0) - z) \cap K \neq \emptyset, \text{ for every } z \in T(x, y_0) \tag{6.1}$$

and

$$(z - T(x_0, y_1)) \cap K \neq \emptyset, \text{ for every } z \in T(x_0, y). \tag{6.2}$$

Moreover, one has

(a) *$\mathcal{E}(T)$ is a nonempty inductive subset in $(C \times D, \succcurlyeq_{XY})$;*

(b) *T has a $\succcurlyeq_{XY}$-maximal $E_{I\times J}$-clustered extended Nash equilibrium $(x_0, y_0) \succcurlyeq_{XY} (s_0, t_0)$.*

Proof. One can prove this theorem by using Theorems 3.1 and 4.4 in [18]. In here, we give a direct proof. Notice that $\succcurlyeq_I$, $\succcurlyeq_J$ and $\succcurlyeq$ are preorders on X, Y and Z, which are equivalently represented by the proper closed and convex cones $I \subseteq X$, $J \subseteq Y$ and $K \subseteq Z$, respectively. Since C and D are compact, so $C \times D$ is compact in the Hausdorff t.v.s. $X \times Y$ with respect to the product topology. From Proposition 3.1, $(C \times D, \succcurlyeq_{XY})$ is $\succcurlyeq_{XY}$-chain complete.

By using the preordering relations $\succcurlyeq_I$, $\succcurlyeq_J$ and $\succcurlyeq$, condition (i) in this theorem means that, for every $x \in C$, $T(x, \cdot) : D \to 2^Z \setminus \{\emptyset\}$ is order-increasing on D with respect to $\succcurlyeq_Y$ and $\succcurlyeq^U$, which is condition (i) in [15, Theorem 3.1]. Similarly, condition (ii) in this theorem implies condition (ii) in [15, Theorem 3.1]. From condition (iv) in this theorem, for every $(s,t) \in C \times D$, $\Phi(s)$ and $\Psi(t)$ are closed in the compact sets D and C, respectively. So, they are compact. From Proposition 3.2, $\Phi(s)$ is universally inductive in $(D, \succcurlyeq_J)$ and $\Psi(t)$ is universally inductive in $(C, \succcurlyeq_I)$. So, T satisfies conditions (iv) and (v) in [15, Theorem 3.1]. Condition (v) in this theorem implies that T satisfies condition (vi) in [15, Theorem 3.1]. Hence, T satisfies all conditions in [15, Theorem 3.1].

Then, by using the preordering relations $\succcurlyeq_I$, $\succcurlyeq_J$ and $\succcurlyeq$, T has a $\succcurlyeq_{XY}$-clustered extended Nash equilibrium. That is, there is $(x_0, y_0) \in C \times D$ satisfying that there exists $(x_1, y_1) \in C \times D$ with $(x_0, y_0) \backsim_{XY} (x_1, y_1)$ such that

$$T(x, y_0) \preccurlyeq^U T(x_1, y_0) \text{ and } T(x_0, y_1) \preccurlyeq^D T(x_0, y), \text{ for all } (x, y) \in C \times D. \tag{6.3}$$

We see that $(x_0, y_0) \backsim_{XY} (x_1, y_1)$ is equivalent to $x_0 - x_1 \in E_I$ and $y_0 - y_1 \in E_J$, and (6.3) is equivalent to (6.1), (6.2). When the preordering relations $\succcurlyeq_I$, $\succcurlyeq_J$ and $\succcurlyeq$ are equivalently represented by their corresponding cones I, J and K, respectively, rest of the conclusions of this theorem follow from the conclusions of [15, Theorem 3.1] immediately. □

6.3. **Existence of extended Nash equilibrium.** When we consider a special case in Theorem 6.4 proved in the previous subsection that $(X, \succcurlyeq_I)$ and $(Y, \succcurlyeq_J)$ are partially ordered H.t.v.s., we obtain an existence theorem for extended Nash equilibrium, which is a different version of Theorem 3.2 in [18].

Theorem 6.5. *Let $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ be partially ordered H.t.v.s. induced by pointed closed and convex cones $I \subseteq X$ and $J \subseteq Y$. Let $(Z, \succcurlyeq)$ be a preordered vector space induced by a proper closed and convex cone K in Z. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty compact subsets and let $T : C \times D \to 2^Z \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that T satisfies the conditions (i-v) in Theorem 6.4 Then, T has an extended Nash equilibrium. That is, there is $(x_0, y_0) \in C \times D$ such that, for every $(x, y) \in C \times D$, the following inclusions hold*

$$(T(x_0, y_0) - z) \cap K \neq \emptyset, \text{ for every } z \in T(x, y_0)$$

and

$$(z - T(x_0, y_0)) \cap K \neq \emptyset, \text{ for every } z \in T(x_0, y).$$

Moreover, one has

(a) *$\mathcal{E}(T)$ is a nonempty inductive subset in $(C \times D, \succcurlyeq_{XY})$;*
(b) *T has a $\succcurlyeq_{XY}$-maximal extended Nash equilibrium $(x_0, y_0) \succcurlyeq_{XY} (s_0, t_0)$.*

6.4. **Existence of extended Nash equilibrium in ordered Banach spaces.** Notice that in a Banach space, a nonempty convex cone is closed with respect to the norm topology if and only if it is closed with the weak topology. Then, we have the following consequences of Theorems 6.4 and 6.5.

Corollary 6.6. *Let $(X, \succcurlyeq_I)$ and $(Y, \succcurlyeq_J)$ be preordered Banach spaces induced by proper closed and convex cones $I \subseteq X$ and $J \subseteq Y$. Let $(Z, \succcurlyeq)$ be a preordered vector space induced by a proper closed and convex cone K in Z. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty weakly compact subsets and let $T : C \times D \to 2^Z \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that T satisfies conditions (i-iii) and (v) in Theorem 6.4 and* (iv)' *for every* $(s,t) \in C \times D$, $\Phi(s)$ *is weakly closed in* D *and* $\Psi(t)$ *is weakly closed in* C.

Then, T *has an* $E_{I\times J}$*-clustered extended Nash equilibrium; and all conclusions of Theorem 6.4 hold.*

In particular, if X and Y are reflexive Banach spaces, we have

Corollary 6.7. *Let* $(X, \succcurlyeq_I)$ *and* $(Y, \succcurlyeq_J)$ *be preordered reflexive Banach spaces induced by proper closed and convex cones* $I \subseteq X$ *and* $J \subseteq Y$. *Let* $(Z, \succcurlyeq)$ *be a preordered vector space induced by a proper closed and convex cone* K *in* Z. *Let* $C \subseteq X$ *and* $D \subseteq Y$ *be nonempty norm-bounded closed and convex subsets and let* $T : C \times D \to 2^Z \setminus \{\emptyset\}$ *be a set-valued mapping. Suppose that* T *satisfies conditions (i-iii) and (v) in Theorem 6.4 and condition (iv)' in Corollary 6.6. Then,* T *has an* $E_{I\times J}$*-clustered extended Nash equilibrium; and all conclusions of Theorem 6.4 hold.*

If $(X, \succcurlyeq_I)$ and $(Y, \succcurlyeq_J)$ are partially ordered Banach spaces, then we have the following existence results for extended Nash equilibrium.

Corollary 6.8. *Let* $(X, \succcurlyeq_I)$ *and* $(Y, \succcurlyeq_J)$ *be partially ordered Banach spaces induced by pointed closed and convex cones* $I \subseteq X$ *and* $J \subseteq Y$. *Let* $(Z, \succcurlyeq)$ *be a preordered vector space induced by a proper closed and convex cone* K *in* Z. *Let* $C \subseteq X$ *and* $D \subseteq Y$ *be nonempty weakly compact subsets and let* $T : C \times D \to 2^Z \setminus \{\emptyset\}$ *be a set-valued mapping. Suppose that* T *satisfies conditions (i-iii) and (v) in Theorem 6.4 and condition (iv)' in Corollary 6.6. Then,* T *has an extended Nash equilibrium; and all conclusions of Theorem 6.5 hold.*

Corollary 6.9. *Let* $(X, \succcurlyeq_I)$ *and* $(Y, \succcurlyeq_J)$ *be partially ordered reflexive Banach spaces induced by pointed closed and convex cones* $I \subseteq X$ *and* $J \subseteq Y$. *Let* $(Z, \succcurlyeq)$ *be a preordered vector space induced by a proper closed and convex cone* K *in* Z. *Let* $C \subseteq X$ *and* $D \subseteq Y$ *be nonempty bounded closed and convex subsets and let* $T : C \times D \to 2^Z \setminus \{\emptyset\}$ *be a set-valued mapping. Suppose that* T *satisfies conditions (i-iii) and (v) in Theorem 6.4 and condition (iv)' in Corollary 6.6. Then,* T *has an extended Nash equilibrium; and all conclusions of Theorem 6.5 hold.*

7. Applications to vector equilibrium problems with single valued mappings

7.1. **Vector Nash equilibrium problems for single valued mappings on ordered topological vector spaces.** In this section, like [15, Section 4], we discuss the existence of vector Nash equilibrium for single valued mappings as applications of the existence theorem proved in the previous section.

Let $(Z, \succcurlyeq)$ be a preordered t.v.s. induced by a proper closed and convex cone K in Z with power, upward power and downward power preorders $\succcurlyeq^P$, $\succcurlyeq^U$ and $\succcurlyeq^D$ on 2^Z, respectively. In general, $(2^Z, \succcurlyeq^P)$, $(2^Z, \succcurlyeq^U)$ and $(2^Z, \succcurlyeq^D)$ are different preordered sets with the same underlying set 2^Z.

Write $\mathcal{Z} = \{\{z\} \in 2^Z : z \in Z\}$. Then, $\mathcal{Z} \subseteq 2^Z$. When the preorders $\succcurlyeq^P$, $\succcurlyeq^U$ and $\succcurlyeq^D$ are restricted on $\mathcal{Z}$, they are equivalent to $\succcurlyeq$. We write

$$(\mathcal{Z}, \succcurlyeq^P) = (\mathcal{Z}, \succcurlyeq^U) = (\mathcal{Z}, \succcurlyeq^D) = (\mathcal{Z}, \succcurlyeq). \tag{7.1}$$

By the connections (7.1), we discuss the solvability of vector Nash equilibrium problems for single valued mappings as applications of the existence theorems of order extended Nash equilibrium for set-valued mappings proved in the previous section.

Let $(X, \succcurlyeq_I)$, $(Y, \succcurlyeq_J)$ and $(Z, \succcurlyeq_K)$ be preordered t.v.s. induced by proper closed and convex cones $I \subseteq X$, $J \subseteq Y$ and $K \subseteq Z$. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty subsets and let $T : C \times D \to Z$ be a single-valued mapping. With respect to the mapping T, the optimization set-valued mappings $\Phi : C \to 2^D$ and $\Psi : D \to 2^C$, for single valued mappings become the following forms:

$$\begin{aligned}\Phi(s) &= \{t \in D :\ T(s,t) \text{ is a } \succcurlyeq_K \text{-lower bound of } \{T(s,y) : y \in D\}\}, \text{ for any } s \in C\\ &= \{t \in D :\ T(s,y) - T(s,t) \in K, \text{ for every } y \in D\}, \text{ for any } s \in C.\end{aligned}$$

$$\begin{aligned}\Psi(t) &= \{s \in C :\ T(s,t) \text{ is a } \succcurlyeq_K \text{-upper bound of } \{T(x,t) :\ x \in C\}\}, \text{ for any } t \in D.\\ &= \{s \in C :\ T(s,t) - T(x,t) \in K, \text{ for every } x \in C\}, \text{ for any } t \in D.\end{aligned}$$

Considering single-valued mappings as special cases of set-valued mappings in Definition 6.3, the vector upper consistent and vector lower consistent properties become the following forms.

(a) T is said to be $\succcurlyeq$-upper consistent on C, whenever, for any $t_1, t_2 \in D$, if

$$T(x, t_1) \preccurlyeq_K T(x, t_2), \text{ for every } x \in C,$$

then, $\Psi(t_1) \subseteq \Psi(t_2)$. That is, for $s \in C$, $T(s, t_1)$ is a $\succcurlyeq_K$-upper bound of $\{T(x, t_1) :\ x \in C\}$ implies that $T(s, t_2)$ is a $\succcurlyeq$-upper bound of $\{T(x, t_2) : x \in C\}$.

Corresponding to the cone K, it is restated as: if

$$T(x, t_2) - T(x, t_1) \in K, \text{ for every } x \in C,$$

then, for some $s \in C$,

$$T(s,t_1) - T(x,t_1) \in K, \text{ for all } x \in C, \text{ implies } T(s,t_2) - T(x,t_2) \in K, \text{ for all } x \in C.$$

(b) T is said to be $\succcurlyeq$-lower consistent on D, whenever, for any $s_1, s_2 \in C$, if

$$T(s_1, y) \succcurlyeq_K T(s_2, y), \text{ for every } y \in D,$$

then, $\Phi(s_1) \subseteq \Phi(s_2)$. That is, for $t \in D$, $T(s_1, t)$ is a $\succcurlyeq_K$-lower bound of $\{T(s_1, y) :\ y \in D\}$ implies that $T(s_2, t)$ is a $\succcurlyeq_K$-lower bound of $\{T(s_2, y) : y \in D\}$.

Corresponding to the cone K, it is restated as: if

$$T(s_1, y) - T(s_2, y) \in K, \text{ for every } y \in D,$$

then, for some $t \in D$,

$$T(s_1,y) - T(s_1,t) \in K, \text{ for all } y \in D, \text{ implies } T(s_2,y) - T(s_2,t) \in K, \text{ for all } y \in D.$$

Recall: $E_{I\times J}$-clustered vector Nash equilibrium for single valued mappings:

A point $(x_0, y_0) \in C \times D$ is called an $E_{I\times J}$-clustered vector Nash equilibrium for a set-valued mapping $T : C \times D \to Z$, if there is $(x_1, y_1) \in C \times D$ with

$$(x_0, y_0) \backsim_{XY} (x_1, y_1)$$

such that $T(x, y_0) \preccurlyeq T(x_1, y_0)$ and $T(x_0, y_1) \preccurlyeq T(x_0, y)$, for all $(x, y) \in C \times D$.

Corresponding to the cone K, it is restated as:

$$(x_0, y_0) - (x_1, y_1) \in E_{I\times J}$$

such that

$$T(x_1, y_0) - T(x, y_0) \in K \text{ and } T(x_0, y) - T(x_0, y_1) \in K, \text{ for all } (x, y) \in C \times D,$$

The set of all order-clustered vector Nash equilibriums of this single valued mapping T is denoted by $\mathcal{E}(T)$ as well.

As a consequence of Theorem 6.4, we have the following result for order-clustered Nash equilibrium for single valued mappings. Since Theorem 6.4 is a different version of Theorem 3.1 in [18], one may consider the following two corollaries as different versions of Corollaries 4.1 and 4.2 in [18] in the present context.

Corollary 7.1. *Let $(X, \succcurlyeq_I)$ and $(Y, \succcurlyeq_J)$ be preordered H.t.v.s. induced by proper closed and convex cones $I \subseteq X$ and $J \subseteq Y$. Let $(Z, \succcurlyeq_K)$ be a preordered vector space induced by a proper closed and convex cone K in Z. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty compact subsets and let $T : C \times D \to Z \setminus \{\emptyset\}$ be a single valued mapping. Suppose that T satisfies the following conditions:*

(i) *for every $x \in C$, $T(x, \cdot) : D \to Z$ satisfies that, for any $t_1, t_2 \in D$,*

$$t_2 - t_1 \in J \text{ implies } T(x, t_2) - T(x, t_1) \in K;$$

(ii) *for every $y \in D$, $T(\cdot, y) : C \to Z$ satisfies that, for any $s_1, s_2 \in D$,*

$$s_2 - s_1 \in I \text{ implies } T(s_1, y) - T(s_2, y) \in K;$$

(iii) *T is $\succcurlyeq$-upper consistent on C and $\succcurlyeq_K$-lower consistent on D;*

(iv) *for every $(s, t) \in C \times D$, $\Phi(s)$ is closed in D and $\Psi(t)$ is closed in C;*

(v) *there are $(s_0, t_0), (u_0, v_0) \in C \times D$ with $v_0 \in \Phi(s_0)$ and $u_0 \in \Psi(t_0)$ such that*

$$u_0 - s_0 \in I \text{ and } v_0 - t_0 \in J.$$

Then, T has an $E_{I\times J}$-clustered vector Nash equilibrium. That is, there are (x_0, y_0), $(x_1, y_1) \in C \times D$ with

$$x_0 - x_1 \in E_I \text{ and } y_0 - y_1 \in E_J.$$

such that

$$T(x_1, y_0) - T(x, y_0) \in K \text{ and } T(x_0, y) - T(x_0, y_1) \in K, \text{ for all } (x, y) \in C \times D.$$

Moreover, one has

(a) *$\mathcal{E}(T)$ is a nonempty inductive subset in $(C \times D, \succcurlyeq_{XY})$;*

(b) *T has a $\succcurlyeq_{XY}$-maximal clustered vector Nash equilibrium $(x_0, y_0) \succcurlyeq_{XY} (s_0, t_0)$.*

If the considered ordered spaces in Corollary 7.1 are partially ordered spaces, by Theorem 6.5, we have the following result for the existence of vector Nash equilibrium for single valued mappings.

Corollary 7.2. *Let $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ be partially ordered H.t.v.s. induced by pointed closed and convex cones $I \subseteq X$ and $J \subseteq Y$. Let $(Z, \succcurlyeq)$ be a preordered vector space induced by a proper closed and convex cone K in Z. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty compact subsets and let $T : C \times D \to Z \setminus \{\emptyset\}$ be a single valued mapping. Suppose that T satisfies the conditions (i-v) in Corollary 7.2. Then, T has a vector Nash equilibrium. That is, there is $(x_0, y_0) \in C \times D$ such that*

$$T(x_0, y_0) - T(x, y_0) \in K \text{ and } T(x_0, y) - T(x_0, y_0) \in K, \text{ for all } (x, y) \in C \times D.$$

Moreover, one has

(a) *$\mathcal{E}(T)$ is a nonempty inductive subset in $(C \times D, \succcurlyeq_{XY})$;*
(b) *T has a $\succcurlyeq_{XY}$-maximal vector Nash equilibrium $(x_0, y_0) \succcurlyeq_{XY} (s_0, t_0)$.*

7.2. **Vector equilibrium problems for single valued mappings on ordered Banach spaces.** From Corollaries 7.1 and 7.2, (like Corollaries 6.6 to 6.9) we obtain results of existence of vector Nash equilibrium for single valued mappings on ordered Banach spaces.

Corollary 7.3. *Let $(X, \succcurlyeq_I)$ and $(Y, \succcurlyeq_J)$ be preordered Banach spaces induced by proper closed and convex cones $I \subseteq X$ and $J \subseteq Y$. Let $(Z, \succcurlyeq_K)$ be a preordered vector space induced by a proper closed and convex cone K in Z. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty weakly compact subsets and let $T : C \times D \to Z \setminus \{\emptyset\}$ be a single valued mapping. Suppose that T satisfies conditions (i-iii) and (v) in Corollary 7.1 and*

(iv)" *for every $(s, t) \in C \times D$, $\Psi(s)$ is weakly closed in D and $\Psi(t)$ is weakly closed in C.*

Then, T has an $E_{I \times J}$-clustered vector Nash equilibrium; and all conclusions of Corollary 7.1 hold.

If X and Y are reflexive Banach spaces, we have

Corollary 7.4. *Let $(X, \succcurlyeq_I)$ and $(Y, \succcurlyeq_J)$ be preordered reflexive Banach spaces induced by proper closed and convex cones $I \subseteq X$ and $J \subseteq Y$. Let $(Z, \succcurlyeq_K)$ be a preordered vector space induced by a proper closed and convex cone K in Z. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty norm bounded closed and convex subsets and let $T : C \times D \to Z \setminus \{\emptyset\}$ be a single valued mapping. Suppose that T satisfies conditions (i-iii) and (v) in Theorem 6.4 and condition (iv)" in Corollary 7.3. Then, T has an $E_{I \times J}$-clustered vector Nash equilibrium; and all conclusions of Corollary 7.1 hold.*

Regarding to the applications of Corollary 7.2 to ordered Banach spaces, if $(X, \succcurlyeq_I)$ and $(Y, \succcurlyeq_J)$ are partially ordered Banach spaces, then we have the following existence results for vector Nash equilibrium.

Remark 7.5. In Corollaries 7.3 and 7.4, if the ordering relations $\succcurlyeq_I$ and $\succcurlyeq_J$ are partial orders; that is, $I \subseteq X$ and $J \subseteq Y$ both are pointed closed and convex cones, then T has a vector Nash equilibrium; and all conclusions of Corollary 7.1 hold, where $\mathcal{E}(T)$ is replaced by $E(T)$.

8. Order-cluster invariant mappings

From the contents of Sections 6 and 7, one sees that order-clustered extended Nash equilibrium for set-valued mappings and order-clustered vector Nash equilibrium for single valued mappings will become extended Nash equilibrium and vector Nash equilibrium if the considered ordered t.v.s. are partially ordered. To obtain extended Nash equilibrium and vector Nash equilibrium, excepting the restriction of the underlying spaces, one can change the conditions for the considered mappings.

In economic theory and game theory, sometimes, the utilities will remain the same on different but order equivalent possible outcomes. It leads us to introduce the concepts of order-cluster invariant mappings on ordered sets (see [15, Definition 3.4]). We recall this definition on ordered t.v.s..

Definition 8.1. Let $(X, \succcurlyeq_I)$, $(Y, \succcurlyeq_J)$ and $(Z, \succcurlyeq_K)$ be a preordered t.v.s. induced by closed and convex cones $I \subseteq X$, $J \subseteq Y$ and $K \subseteq Z$, respectively. $E_{I\times J}$ is the $\succcurlyeq_{I\times J}$-cluster kernel or the $I \times J$-cluster kernel of the product ordered t.v.s. $(X \times Y, \succcurlyeq_{I\times J})$. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty subsets. A set-valued mapping $T : C \times D \to 2^Z \setminus \{\emptyset\}$ is said to be $\succcurlyeq_{XY}$-cluster invariant or $E_{I\times J}$-cluster invariant if, for any points $(x_1, y_1), (x_2, y_2) \in C \times D$,

$$(x_2, y_2) - (x_1, y_1) \in E_{I\times J}$$

implies that, for every $m, n = 1, 2$, one has

(U)” $(T(x_m, y_m) - z_n) \cap K \neq \emptyset$, for every $z_n \in T(x_n, y_n)$,

and

(D)” $(z_m - T(x_n, y_n)) \cap K \neq \emptyset$, for every $z_m \in T(x_m, y_m)$.

This mapping $T : C \times D \to 2^Z$ is said to be strongly $\succcurlyeq_{XY}$-cluster invariant if, for any points $(x_1, y_1), (x_2, y_2) \in C \times D$,

$$(x_2, y_2) \backsim_{XY} (x_1, y_1) \text{ implies } T(x_2, y_2) = T(x_1, y_1).$$

$$(x_2, y_2) - (x_1, y_1) \in E_{I\times J} \text{ implies that } T(x_2, y_2) = T(x_1, y_1).$$

A single-valued mapping $T : C \times D \to Z$ is said to be $\succcurlyeq_{XY}$-cluster invariant if, for any points $(x_1, y_1), (x_2, y_2) \in C \times D$,

$$(x_2, y_2) - (x_1, y_1) \in E_{I\times J} \text{ implies } T(x_2, y_2) - T(x_1, y_1) \in E_K.$$

That is equivalent to, for every $m, n = 1, 2$, one has

$$T(x_m, y_m) - T(x_n, y_n) \in K.$$

Notice that if $I \subseteq X$, $J \subseteq Y$ and $K \subseteq Z$, are pointed closed and convex cones, then the induced ordered spaces $(X, \succcurlyeq_I)$, $(Y, \succcurlyeq_J)$ and $(Z, \succcurlyeq)$ become partially ordered topological vector spaces. If follows that, for any nonempty subsets $C \subseteq X$ and $D \subseteq Y$, any single-valued mapping $T : C\times D \to Z$ is automatically $\succcurlyeq_{XY}$-cluster invariant.

Proposition 8.2. *Let the preordered H.t.v.s. $(X, \succcurlyeq_I)$ and $(Y, \succcurlyeq_J)$, the preordered t.v.s. $(Z, \succcurlyeq)$ and the set-valued mapping $T : C \times D \to 2^Z \setminus \{\emptyset\}$ be given in Theorem 6.4. Suppose that, in addition to condition (i-v) in Theorem 6.4, T satisfies the following condition:*

(vi) *T is $\succcurlyeq_{XY}$-cluster invariant.*

Then, T has a vector Nash equilibrium. That is, there is $(x_0, y_0) \in C \times D$ such that, for all $(x, y) \in C \times D$, the following inclusions hold

$$(T(x_0, y_0) - z) \cap K \neq \emptyset, \text{ for every } z \in T(x, y_0) \tag{8.1}$$

and

$$(z - T(x_0, y_0)) \cap K \neq \emptyset, \text{ for every } z \in T(x_0, y). \tag{8.2}$$

Moreover, one has

(a) *$E(T)$ is a nonempty inductive subset in $(C \times D, \succcurlyeq_{IJ})$;*
(b) *T has a $\succcurlyeq_{XY}$-maximal vector Nash equilibrium $(x_0, y_0) \succcurlyeq_{IJ} (s_0, t_0)$.*

Proof. From Theorem 6.4, there is $(x_0, y_0) \in C \times D$ satisfying that there exists $(x_1, y_1) \in C \times D$ with

$$(x_0, y_0) \backsim_{XY} (x_1, y_1) \text{ i.e., } x_0 - x_1 \in E_I \text{ and } y_0 - y_1 \in E_J.$$

such that, for all $(x, y) \in C \times D$, the following inclusions hold

$$(T(x_1, y_0) - z) \cap K \neq \emptyset, \text{ for every } z \in T(x, y_0), \tag{8.3}$$

and

$$(z - T(x_0, y_1)) \cap K \neq \emptyset, \text{ for every } z \in T(x_0, y). \tag{8.4}$$

From $(x_0, y_0) \backsim_{XY} (x_1, y_1)$, we get $(x_1, y_0) \backsim_{XY} (x_0, y_0) \backsim_{XY} (x_0, y_1)$. Since T is $\succcurlyeq_{XY}$-cluster invariant, it follows that

$$T(x_1, y_0) \backsim^P T(x_0, y_0) \backsim^P T(x_0, y_1). \tag{8.5}$$

Substituting (8.5) into (8.3) and (8.4), we obtain (8.1), (8.2). That is

$$T(x, y_0) \preccurlyeq^U T(x_0, y_0) \preccurlyeq^D T(x_0, y), \text{ for all } (x, y) \in C \times D.$$

□

For single valued mappings, we have

Proposition 8.3. *Let the preordered H.t.v.s. $(X, \succcurlyeq_I)$ and $(Y, \succcurlyeq_J)$, the preordered t.v.s. $(Z, \succcurlyeq)$ and the single mapping $T : C \times D \to Z \setminus \{\emptyset\}$ be given in Corollary 7.1. Suppose that, in addition to condition (i-v) in Corollary 7.1, T satisfies the following condition:*

(vi) *T is $\succcurlyeq_{XY}$-cluster invariant.*

Then, T has a vector Nash equilibrium. That is, there is $(x_0, y_0) \in C \times D$ such that

$$T(x_0, y_0) - T(x, y_0) \in K \text{ and } T(x_0, y) - T(x_0, y_0) \in K, \text{ for all } (x, y) \in C \times D,$$

Moreover, one has

(a) *$E(T)$ is a nonempty inductive subset in $(C \times D, \succcurlyeq_{XY})$;*
(b) *T has a $\succcurlyeq_{XY}$-maximal vector Nash equilibrium $(x_0, y_0) \succcurlyeq_{XY} (s_0, t_0)$.*

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J. Li
Department of Mathematics Shawnee State University Portsmouth, Ohio 45662 USA
E-mail address: jli@shawnee.edu

C. Tammer
Martin-Luther-University Halle-Wittenberg, Germany
E-mail address: christiane.tammer@mathematik.uni-halle.de