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A STRONG CONVERGENCE THEOREM FOR QUASI-CONTRACTIVE MAPPINGS AND INVERSE STRONGLY MONOTONE MAPPINGS

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Abstract. We prove a strong convergence theorem for finding a common element of the set of fixed points of a quasi-contractive mapping and the set of solutions of a variational inequality problem for an α-inverse strongly monotone mapping. We construct an iterative scheme for finding a common fixed point of a strictly pseudo-contractive mapping and a quasi-contractive mapping in a Hilbert space. We also consider the problem of finding a common element of the set of fixed points of a quasi-contractive mapping and the set of zeros of an inverse strongly monotone mapping.

1. Introduction

Let D be a nonempty, closed and convex subset of a real Hilbert space H and let P_D be the metric projection of H onto D. A mapping $A : D \to H$ is called

- monotone if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in D$.
- α-strongly monotone, if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2 \ \textit{for all}\ x, y \in D.$$

- α-inverse strongly monotone, if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2 \ \textit{for all}\ x, y \in D$$

- L-Lipschitz if there exits $L \geq 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in D$.

If $L < 1$ then A is called a contraction, if $L = 1$ then A is called nonexpansive. We denote by $F(A)$ the set of fixed points of A. Note that, any α-inverse strongly monotone mapping A is Lipschitz that is $\|Ax - Ay\| \leq \frac{1}{\alpha}\|x - y\|$ for all $x, y \in D$. It is also known that, if $\delta \leq 2\alpha$, then $I - \delta A$ is a nonexpansive mapping of D into H [18]. Let D be a nonempty closed convex subset of H and let $A : D \to H$ be a nonlinear mapping, the variational inequality problem (*in short*, VIP) is to find a point $x^\star \in K$ such that $\langle x - x^\star, Ax^\star\rangle \geq 0$ for all $x \in D$ [7, 9]. The set of solutions of the variational inequality problem is denoted by $VI(D, A)$. If $D = H$, then $VI(H, A) = A^{-1}(0)$, where $A^{-1}(0) = \{x \in H : Ax = 0\}$. The ideas of variational inequalities are used as tools to solve new problems in various fields of applied mathematics, engineering, contact problems in elasticity, nonlinear optimization and economics equilibrium, etc. For example, see [3, 4, 5, 6, 10, 12, 13, 15, 19, 22, 29, 31].

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The theory of variational inequalities was introduced by Stampacchia [26] and since then, has been extensively studied. For example, see [1, 2, 11, 30]. In 2005, Iiduka and Takahashi [18] suggested an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for an α-inverse strongly monotone mapping and got a strongly convergence theorem.

$$x_{n+1} = \lambda_n x + (1-\lambda_n) T P_D(x_n - \delta_n A x_n) \tag{1.1}$$

where $x_1 = x \in D$, $\lambda_n \in [0,1)$ with $\lim_{n\to\infty} \lambda_n = 0, \sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ and $\delta_n \in [c,d]$ for some $c,d \in (0,2\alpha)$ with $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$. They proved that the sequence $\{x_n\}$ generated by (1.1) converges strongly to $P_{F(T)\cap VI(K,A)}x$.

A mapping $T : D \to D$ is called quasi-contractive whenever there exists $\xi \in (0,1)$ such that

$$\|Tx - Ty\| \leq \xi(max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|Tx-y\|\})$$

for all $x, y \in D$ ([20, 21, 23, 28]).

In this paper, we introduce a new implicit iterative process for finding a common element of the set of fixed points of a quasi-contractive mapping and the set of solutions of a variational inequality problem for an α-inverse strongly monotone mapping. We obtain a strong convergence theorems for the following iterative method : suppose $x_1 = x \in D$ and

$$x_{n+1} = \lambda_n x + (1-\lambda_n)[\beta_n T P_D(x_n - \delta_n A x_n) + (1-\beta_n) T x_n],$$

where $\lambda_n \in [0,1)$ with $\lim_{n\to\infty} \lambda_n = 0, \sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ and $\delta_n \in [c,d]$ for some $c,d \in (0,2\alpha)$ with $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$ and $\beta_n \in [0,1]$ with $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

We construct an iterative scheme for finding a common fixed point of a strictly pseudo-contractive mapping and a quasi-contractive mapping in a Hilbert space. We also consider the problem of finding a common element of the set of fixed points of a quasi-contractive mapping and the set of zeros of an inverse strongly monotone mapping.

2. Preliminaries

Let D be a nonempty closed convex subset of Hilbert space H, with inner product $\langle .,. \rangle$ and nom $\|.\|$ respectively. For each element $x \in H$, there exists a unique nearest point in D, denoted by $P_D x$, such that $\|x - P_D x\| \leq \|x - y\|$ for all $y \in D$. $P_D : H \to D$ is called the metric projection and satisfies the following properties [14, 16, 17]

(a) $\|P_D x - P_D y\| \leq \|x - y\|$ for all $x, y \in H$.

(b) $\langle P_D - P_D y, x - y \rangle \geq 0$ for all $x, y \in H$.

(c) $\langle P_D - P_D y, x - y \rangle \geq \|P_D x - P_D y\|^2$ for all $x, y \in H$.

(d) $\langle x - P_D x, y - P_D x\rangle \leq 0$ for all $x \in H, y \in D$.

(e) $\|x - y\|^2 \geq \|x - P_D x\|^2 + \|y - P_D x\|^2$ for all $x \in H, y \in D$.

Let A be a monotone mapping of D into H. In the context of the variational inequality problem, this implies from the property (e) that

$$q \in VI(D, A) \Leftrightarrow q = P_D(q - \delta Aq) \ for\ some\ \delta > 0.$$

Let $x \in X$ and $\{x_n\}$ be a sequence in X. We use the symbols $x_n \to x$ and $x_n \rightharpoonup x$ for the strong and weakly convergences, respectively. It is known that H satisfies Opiĺs condition [16], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Let $T : D \to D$ be a nonexpansive mapping, I be the identity mapping. If $A = I - T$, then A is $\frac{1}{2}-$inverse strongly monotone and $F(T) = VI(D, A)$ [19]. A set-valued mapping $T' : H \to 2^H$ is called monotone if for all $x, y \in H, f \in T'x$ and $h \in T'y$ imply $\langle x - y, f - h\rangle \geq 0$. A monotone mapping $T' : H \to 2^H$ is maximal if the graph $G(T')$ of T' is not properly contained in the graph of any other monotone mapping. Also, a monotone mapping $T' : H \to 2^H$ is maximal if and only if, for $(x, f) \in H \times H, \langle x - y, f - h\rangle \geq 0$ for every $(y, h) \in G(T')$ implies $f \in T'x$. Let $A : D \to H$ be an inverse strongly monotone mapping and let $N_D u$ be the normal cone to D at $u \in D$, i.e., $N_D u = \{v \in H : \langle u - w, v\rangle \geq 0, \forall w \in D\}$. Define

$$T'u := \begin{cases} Au + N_D u & u \in D \\ \emptyset & u \notin D \end{cases}$$

It is known that T' is maximal monotone and $0 \in T'u$ if and only if $u \in VI(D, A)$; [24, 25].

Lemma 2.1. *Let X be a reflexive Banach space that satisfies the Opial's condition, D a nonempty, closed and convex subset of X, $\xi \in (0, \frac{1}{2})$ and $T : D \to D$ a mapping, satisfying $\|Tx - Ty\| \leq \xi(max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|Tx - y\|\}$ for all $x, y \in D$. Then T has a fixed point.*

Proof. Fix $x_0 \in X$ and let $x_n = T^n x_0, n \in \mathbb{N}$. Observe that $\|x_n - x_{n+1}\| = \|Tx_{n-1} - Tx_n\|$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $q = x_n$ is a fixed point of T. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned} &\|x_n - x_{n+1}\| \\ &\leq \xi(max\{\|x_{n-1} - x_n\|, \|x_{n-1} - Tx_{n-1}\|, \|x_n - Tx_n\|, \|x_{n-1} - Tx_n\|, \|Tx_{n-1} - x_n\|\}) \end{aligned}$$

hence

$$\|x_n - x_{n+1}\| \leq \xi(max\{\|x_{n-1} - x_n\|, \|x_n - x_{n+1}\|, \|x_{n-1} - x_{n+1}\|, 0\}).$$

Set

$$L(x_{n-1}, x_n) = max\{\|x_{n-1} - x_n\|, \|x_n - x_{n+1}\|, \|x_{n-1} - x_{n+1}\|, 0\}.$$

Clearly, we have at least one of the following four cases
(1) If $L(x_{n-1}, x_n) = \|x_n - x_{n+1}\|$, then $\|x_n - x_{n+1}\| \leq \xi \|x_n - x_{n+1}\| \Rightarrow \|x_n - x_{n+1}\| = 0$. Hence

$$\|x_n - x_{n+1}\| \leq \xi \|x_n - x_{n-1}\|.$$

(2) If $L(x_{n-1}, x_n) = \|x_{n-1} - x_{n+1}\|$, then $\|x_n - x_{n+1}\| \leq \xi \|x_{n-1} - x_{n+1}\|$. It followes that

$$\|x_n - x_{n+1}\| \leq \xi(\|x_{n-1} - x_n\| + \|x_n - x_{n+1}\|)$$

actually

$$\|x_n - x_{n+1}\| \leq \frac{\xi}{1-\xi} \|x_{n-1} - x_n\|.$$

(3) If $L(x_{n-1}, x_n) = \|x_{n-1} - x_n\|$, then $\|x_n - x_{n+1}\| \leq \xi \|x_n - x_{n-1}\|$.
(4) If $L(x_{n-1}, x_n) = 0$, then $\|x_n - x_{n+1}\| \leq 0$ and hence $\|x_n - x_{n+1}\| \leq \xi \|x_{n-1} - x_n\|$.
Therefore, we have

$$\|x_n - x_{n+1}\| \leq max\{\xi, \frac{\xi}{1-\xi}\} \|x_n - x_{n-1}\| = \frac{\xi}{1-\xi} \|x_n - x_{n-1}\|.$$

Since $\frac{\xi}{1-\xi} \in (0, 1)$, we have

$$\|x_n - x_{n+1}\| \leq \frac{\xi}{1-\xi} \|x_n - x_{n-1}\| \leq (\frac{\xi}{1-\xi})^n \|x_1 - x_0\|$$

$$0 \leq \lim_{n\to\infty} \sup \|x_n - x_{n+1}\| \leq \lim_{n\to\infty} \sup (\frac{\xi}{1-\xi})^n \|x_0 - x_1\| = 0.$$

Thus $\lim_{n\to\infty} \sup \|x_n - Tx_n\| = 0$. By the reflexivity of X, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p \in D$. By the Opial's condition, we have

$$\liminf_{k\to\infty} \|x_{n_k} - p\| < \liminf_{k\to\infty} \|x_{n_k} - Tp\|. \tag{2.1}$$

Observe that

$$\|x_{n_k} - Tp\| \leq \|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tp\|$$

which implies that

$$\|x_{n_k} - Tp\| \leq \|x_{n_k} - Tx_{n_k}\| + \xi L(x_{n_k}, p)$$

where

$$L(x_{n_k}, p) = max\{\|x_{n_k} - p\|, \|x_{n_k} - Tx_{n_k}\|, \|p - Tp\|, \|x_{n_k} - Tp\|, \|p - Tx_{n_k}\|\}.$$

If $L(x_{n_k}, p) = \|x_{n_k} - p\|$, then $\|Tx_{n_k} - Tp\| \leq \xi \|x_{n_k} - p\|$. Therefore

$$\liminf_{k\to\infty} \|x_{n_k} - Tp\| \leq \liminf_{k\to\infty} \xi \|x_{n_k} - p\| \leq \liminf_{k\to\infty} \|x_{n_k} - p\|. \tag{2.2}$$

If $L(x_{n_k}, p) = \|p - Tp\|$, then $\|Tx_{n_k} - Tp\| \leq \xi \|p - Tp\| \leq \xi(\|p - x_{n_k}\| + \|x_{n_k} - Tp\|)$ which implies that

$$\liminf_{k\to\infty} \|x_{n_k} - Tp\| \leq \liminf_{k\to\infty} \frac{\xi}{1-\xi} \|x_{n_k} - p\| \leq \liminf_{k\to\infty} \|x_{n_k} - p\|. \tag{2.3}$$

If $L(x_{n_k}, p) = \|p - x_{n_k}\|$, then

$$\|Tx_{n_k} - Tp\| \leq \xi \|p - Tx_{n_k}\| \leq \xi(\|p - x_{n_k}\| + \|x_{n_k} - Tx_{n_k}\|.$$

Hence

$$\liminf_{k\to\infty}\|x_{n_k}-Tp\| \le \liminf_{k\to\infty}\xi\|x_{n_k}-p\| \le \liminf_{k\to\infty}\|x_{n_k}-p\|. \tag{2.4}$$

From (2.1), (2.2), (2.3) and (2.4) we conclude that $(I-T)p=0$. If $L(x_{n_k},p)=\|x_{n_k}-Tx_{n_k}\|$, then $\|Tx_{n_k}-Tp\|\le\xi\|x_{n_k}-Tx_{n_k}\|$. This implies that

$$\liminf_{k\to\infty}\|x_{n_k}-Tp\| \le \liminf_{k\to\infty}(1+\xi)\|x_{n_k}-Tx_{n_k}\|=0.$$

If $L(x_{n_k},p)=\|x_{n_k}-Tp\|$, then $\|Tx_{n_k}-Tp\|\le\xi\|x_{n_k}-Tp\|$ so that

$$\liminf_{k\to\infty}\|x_{n_k}-Tp\| \le \liminf_{k\to\infty}\xi\|x_{n_k}-Tp\| \Rightarrow \liminf_{k\to\infty}\|x_{n_k}-Tp\|=0.$$

From (3.4) we obtain that $\liminf_{k\to\infty}\|x_{n_k}-p\| = \liminf_{k\to\infty}\|x_{n_k}-Tp\| = 0$. Therefore, $p-Tp=0$. □

3. Main results

In this section we shall prove a strong convergence theorem for quasi-contractive mappings and α-inverse strongly monoton mapping.

Theorem 3.1. *Let D be a nonempty closed convex subset of a real Hilbert space H, $A: D\to H$ an α-inverse strongly monotone mapping and $T: D\to D$ a quasi-contractive mapping with $F(T)\cap VI(D,A)\neq\emptyset, \xi\in(0,\frac{1}{2})$. Suppose $x_1=x\in D$ and $\{x_n\}$ is given by*

$$x_{n+1}=\lambda_n x+(1-\lambda_n)[\beta_n TP_D(x_n-\delta_n Ax_n)+(1-\beta_n)Tx_n], \tag{3.1}$$

for every $n=1,2,3,...$, where $\lambda_n\in[0,1)$ with $\lim_{n\to\infty}\lambda_n=0, \sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$, $\sum_{n=1}^{\infty}\lambda_n=\infty$ and $\delta_n\in[a,b]$ for some $a,b\in(0,2\alpha)$ with $\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n|<\infty$ and $\beta_n\in(0,1]$ with $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty, \lim_{n\to\infty}\beta_n=1$. Then $\{x_n\}$ converges strongly to $P_{F(T)\cap VI(D,A)}x$.

Proof. Let $q\in F(T)\cap VI(D,A)$. we show that $\|Tx-q\|\le\|x-q\|$ for all $x\in D, \xi\in(0,\frac{1}{2})$. Note that

$$\begin{aligned}\|Tx-q\|=\|Tx-Tq\| &\le \xi max\{\|x-q\|,\|x-Tx\|,\|q-Tq\|,\|x-Tq\|,\|q-Tx\|\}\\ &\le \xi max\{\|x-q\|,\|x-Tx\|,0,\|q-Tx\|\}.\end{aligned}$$

We consider the following four cases:
(1) $\|Tx-Tq\|\le\xi\|x-q\|\le\|x-q\|$.
(2) $\|Tx-Tq\|\le\xi\|x-Tx\|\le\xi(\|x-q\|+\|q-Tx\|)\Rightarrow\|Tx-Tq\|\le\frac{\xi}{1-\xi}\|x-q\|\le\|x-q\|$.
(3) $\|Tx-Tq\|\le\xi\|q-Tq\|=0\le\|x-q\|$.
(4) $\|Tx-Tq\|\le\xi\|q-Tx\|\le\xi(\|q-Tq\|+\|Tq-Tx\|)=0\le\|x-q\|$.
Put $z_n=P_D(x_n-\delta_n Ax_n)$, for $n\ge1$. Since $q=P_D(q-\delta_n Aq)$ and $\delta_n\in[0,2\alpha]$,

$$\begin{aligned}\|z_n-q\|^2 &= \|P_D(x_n-\delta_n Ax_n)-P_D(q-\delta_n Aq)\|^2\\ &\le \|(x_n-q)-\delta_n(Ax_n-Aq)\|^2\\ &\le \|x_n-q\|^2-2\delta_n\langle Ax_n-Aq,x_n-q\rangle+\delta_n^2\|Ax_n-Aq\|^2\\ &\le \|x_n-q\|^2-2\delta_n\alpha\|Ax_n-Aq\|^2+\delta_n^2\|Ax_n-Aq\|^2\end{aligned}$$

$$= \|x_n - q\|^2 + \delta_n(\delta_n - 2\alpha)\|Ax_n - Aq\|^2 \le \|x_n - q\|^2, \tag{3.2}$$

which gives that

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\lambda_n(x - q) + (1 - \lambda_n)[\beta_n(Tz_n - q) + (1 - \beta_n)(Tx_n - q)]\| \\
&\le \lambda_n\|x - q\| + (1 - \lambda_n)[\beta_n\|Tz_n - q\| + (1 - \beta_n)\|Tx_n - q\|] \\
&\le \lambda_n\|x - q\| + (1 - \lambda_n)[\beta_n\|z_n - q\| + (1 - \beta_n)\|x_n - q\|] \\
&= \lambda_n\|x - q\| + (1 - \lambda_n)\|x_n - q\| \\
&\le max\{\|x - q\|, \|x_1 - q\|\} \\
&= \|x - q\|.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded, and so are $\{z_n\}, \{Tz_n\}, \{Ax_n\}, \{Tx_n\}$. Since $I - \delta_n A$ is nonexpansive, we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|P_D(x_{n+1} - \delta_{n+1}Ax_{n+1}) - P_D(x_n - \delta_n Ax_n)\| \\
&\le \|(x_{n+1} - \delta_{n+1}Ax_{n+1}) - (x_n - \delta_n Ax_n)\| \\
&\le \|(x_{n+1} - \delta_{n+1}Ax_{n+1}) - (x_n - \delta_{n+1}Ax_n) + (\delta_n - \delta_{n+1})Ax_n\| \\
&\le \|(x_{n+1} - \delta_{n+1}Ax_{n+1}) - (x_n - \delta_{n+1}Ax_n)\| + |\delta_n - \delta_{n+1}|\|Ax_n\| \\
&\le \|x_{n+1} - x_n\| + |\delta_n - \delta_{n+1}|\|Ax_n\|. \qquad (3.3)
\end{aligned}$$

Set $L(x_n, x_{n-1}) = max\{\|x_n - x_{n-1}\|, \|x_n - Tx_n\|, \|x_{n-1} - Tx_{n-1}\|, \|x_n - Tx_{n-1}\|, \|x_{n-1} - Tx_n\|\}$. From (3.1) we obtain

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \|\lambda_n x + (1 - \lambda_n)[\beta_n TP_D(x_n - \delta_n Ax_n) + (1 - \beta_n)Tx_n] - \lambda_{n-1}x \\
&\quad - (1 - \lambda_{n-1})[\beta_{n-1}TP_D(x_{n-1} - \delta_{n-1}Ax_{n-1}) + (1 - \beta_{n-1})Tx_{n-1}]\| \\
&\le \|(\lambda_n - \lambda_{n-1})[x - (\beta_{n-1}Tz_{n-1} + (1 - \beta_{n-1})Tx_{n-1})] \\
&\quad + (1 - \lambda_n)[(\beta_n Tz_n + (1 - \beta_n)Tx_n) - (\beta_{n-1}Tz_n + (1 - \beta_{n-1})Tx_{n-1})] \\
&\le |\lambda_n - \lambda_{n-1}|[\beta_{n-1}\|x - Tz_{n-1}\| + (1 - \beta_{n-1})\|x - Tx_{n-1}\|] \\
&\quad + (1 - \lambda_n)\|(\beta_n Tz_n + (1 - \beta_n)Tx_n) - (\beta_{n-1}Tz_{n-1} + (1 - \beta_{n-1})Tx_{n-1})\| \\
&\le |\lambda_n - \lambda_{n-1}|[\beta_{n-1}\|x - Tz_{n-1}\| + (1 - \beta_{n-1})\|x - Tx_{n-1}\|] \\
&\quad + (1 - \lambda_n)[\beta_n\|Tz_n - Tz_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tz_{n-1}\| \\
&\quad + (1 - \beta_n)\|Tx_n - Tx_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\|] \\
&\le |\lambda_n - \lambda_{n-1}|[\beta_{n-1}\|x - Tz_{n-1}\| + (1 - \beta_{n-1})\|x - Tx_{n-1}\|] \\
&\quad + (1 - \lambda_n)[\beta_n L(z_n, z_{n-1}) + |\beta_n - \beta_{n-1}|\|Tz_{n-1}\| \\
&\quad + (1 - \beta_n)L(x_n, x_{n-1}) + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\|]
\end{aligned}$$

which gives that

$$\begin{aligned}
\|x_{n+1} - x_n\| \le M|\lambda_n - \lambda_{n-1}| &+ (1 - \lambda_n)(max\{L(z_n, z_{n-1}), L(x_n, x_{n-1})\}) \\
&+ 2M|\beta_n - \beta_{n-1}|
\end{aligned}$$

where M is constant such that

$$M \ge max\Big\{\sup_{n\ge1}\{\|Tx_{n-1}\|\}, \sup_{n\ge1}\{\|Tz_{n-1}\|\}, \sup_{n\ge1}\{\|x - Tz_{n-1}\|\}, \sup_{n\ge1}\{\|x - Tx_{n-1}\|\}\Big\}.$$

Inductively, we have

$$\|x_{n+m+1}-x_{n+m}\| \leq M\sum_{i=n}^{n+m}|\lambda_i-\lambda_{i-1}|+\prod_{i=n}^{n+m}(1-\lambda_i)max\{L(z_n,z_{n-1}),L(x_n,x_{n-1})\}$$
$$+2M\sum_{i=n}^{n+m}|\beta_i-\beta_{i-1}|.$$

Hence, for $n,m\in\mathbb{N}$

$$\limsup_{m\to\infty}\|x_{m+1}-x_m\| \leq M\sum_{i=n}^{\infty}|\lambda_i-\lambda_{i-1}|+\prod_{i=n}^{\infty}(1-\lambda_i)max\{L(z_n,z_{n-1}),L(x_n,x_{n-1})\}$$
$$+2M\sum_{i=n}^{\infty}|\beta_i-\beta_{i-1}|.$$

Since

$$\sum_{n=1}^{\infty}|\lambda_n-\lambda_{n-1}|<\infty,\sum_{n=1}^{\infty}|\beta_n-\beta_{n-1}|<\infty \; and \; \prod_{i=n}^{n+m}(1-\lambda_i)\leq \exp(-\sum_{i=n}^{n+m}(1-\lambda_i))\to 0,$$

it followe that $\limsup_{n\to\infty}\|x_{n+1}-x_n\|\leq 0$ and hence $\limsup_{n\to\infty}\|x_{n+1}-x_n\|=0$. From (3.3) and $\sum_{n=1}^{\infty}|\delta_n-\delta_{n-1}|<\infty$, we get $\lim_{n\to\infty}\|z_{n+1}-z_n\|=0$.
For $q\in F(T)\cap VI(D,A)$, from (3.2) we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\lambda_n(x-q)+(1-\lambda_n)[\beta_n(Tz_n-q)+(1-\beta_n)(Tx_n-q)]\|^2\\
&\leq \lambda_n\|x-q\|^2+(1-\lambda_n)[\beta_n\|Tz_n-q\|^2+(1-\beta_n)\|Tx_n-q\|^2]\\
&\leq \lambda_n\|x-q\|^2+(1-\lambda_n)[\beta_n\|z_n-q\|^2+(1-\beta_n)\|x_n-q\|^2]\\
&\leq \lambda_n\|x-q\|^2+(1-\lambda_n)[\beta_n(\|x_n-q\|^2+\delta_n(\delta_n-2\alpha)\|Ax_n-Aq\|^2)\\
&\qquad +(1-\beta_n)\|x_n-q\|^2]\\
&\leq \lambda_n\|x-q\|^2+\|x_n-q\|^2+(1-\lambda_n)\beta_n a(b-2\alpha)\|Ax_n-Aq\|^2.
\end{aligned}$$

And hence

$$\begin{aligned}
-(1-\lambda_n)\beta_n a(b-2\alpha)\|Ax_n-Aq\|^2 &\leq \lambda_n\|x-q\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2\\
&= \lambda_n\|x-q\|^2+(\|x_n-q\|-\|x_{n+1}-q\|)\\
&\qquad \times(\|x_n-q\|+\|x_{n+1}-q\|)\\
&\leq \lambda_n\|x-q\|^2+(\|x_n-q\|+\|x_{n+1}-q\|)\\
&\qquad \times\|x_n-x_{n+1}\|
\end{aligned}$$

Since $\lambda_n\to 0$ and $\|x_n-x_{n+1}\|\to 0$, we obtain $\|Ax_n-Aq\|\to 0$. By properties of metric projection, we have

$$\begin{aligned}
\|z_n-q\|^2 &= \|P_D(x_n-\delta_nAx_n)-P_D(q-\delta_nAq)\|^2\\
&\leq \langle (x_n-\delta_nAx_n)-(q-\delta_nAq), z_n-q\rangle\\
&= \frac{1}{2}\{\|z_n-q\|^2+\|(x_n-\delta_nAx_n)-(q-\delta_nAq)\|^2\\
&\quad -\|(x_n-\delta_nAx_n)-(q-\delta_nAq)-(z_n-q)\|^2\}\\
&\leq \frac{1}{2}\{\|z_n-q\|^2+\|x_n-q\|^2-\|(x_n-z_n)-\delta_n(Ax_n-Aq)\|^2\}
\end{aligned}$$

$$= \frac{1}{2}\{\|z_n - q\|^2 + \|x_n - q\|^2 - \|x_n - z_n\|^2 - \delta_n\|Ax_n - Aq\|^2 + 2\delta_n\langle x_n - z_n, Ax_n - Aq\rangle\}.$$

Therefore,we obtain that

$$\|z_n - q\|^2 \leq \|x_n - q\|^2 - \|x_n - z_n\|^2 + 2\delta_n\langle x_n - z_n, Ax_n - Aq\rangle - \delta_n\|Ax_n - Aq\|^2$$

thus

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|\lambda_n(x - q) + (1 - \lambda_n)[\beta_n(Tz_n - q) + (1 - \beta_n)(Tx_n - q)]\|^2 \\ &\leq \lambda_n\|x - q\|^2 + (1 - \lambda_n)[\beta_n\|Tz_n - q\|^2 + (1 - \beta_n)\|Tx_n - q\|^2] \\ &\leq \lambda_n\|x - q\|^2 + (1 - \lambda_n)[\beta_n\|z_n - q\|^2 + (1 - \beta_n)\|x_n - q\|^2] \\ &\leq \lambda_n\|x - q\|^2 + (1 - \lambda_n)[\beta_n(\|x_n - q\|^2 - \|x_n - z_n\|^2 - \delta_n\|Ax_n - Aq\|^2 \\ &\quad + 2\delta_n\langle x_n - z_n, Ax_n - Aq\rangle) + (1 - \beta_n)\|x_n - q\|^2].\end{aligned}$$

It follows that

$$\begin{aligned}(1 - \lambda_n)\|x_n - z_n\|^2 &\leq \lambda_n\|x - q\|^2 + (\|x_n - q\| + \|x_{n+1} - q\|)\|x_n - x_{n+1}\| \\ &\quad + (1 - \lambda_n)(2\delta_n\langle x_n - z_n, Ax_n - Aq\rangle - \delta_n\|Ax_n - Aq\|^2).\end{aligned}$$

Since $\lambda_n \to 0, \|x_n - x_{n+1}\| \to 0$ and $\|Ax_n - Aq\| \to 0$, we obtain $\|x_n - z_n\| \to 0$. Since $\lambda_n \to 0, \beta_n \to 1$ and

$$\|x_{n+1} - Tz_n\| \leq \lambda_n\|x - Tz_n\| + (1 - \lambda_n)(1 - \beta_n)\|Tx_n - Tz_n\|,$$

we obtain $\|x_{n+1} - Tz_n\| \to 0$. From $\|x_n - Tz_n\| \leq \|x_{n+1} - Tz_n\| + \|x_n - x_{n+1}\|$, we have $\|x_n - Tz_n\| \to 0$. Observe also that $\|z_n - Tz_n\| \leq \|x_n - Tz_n\| + \|x_n - z_n\|$, so we have $\|z_n - Tz_n\| \to 0$. We show that

$$\limsup_{n\to\infty}\langle x - y_0, Tz_n - y_0\rangle \leq 0,$$

where $y_0 = P_{F(T)\cap VI(D,A)}x$. To prove it, choose a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle x - y_0, Tz_n - y_0\rangle = \lim_{k\to\infty}\langle x - y_0, Tz_{n_k} - y_0\rangle.$$

Since $\{z_{n_k}\}$ is bounded, there exists a subsequence $\{z_{n_{k_j}}\}$ of $\{z_{n_k}\}$ which converges weakly to p. Whithout loss of generality, we assume that $z_{n_k} \rightharpoonup p$. Since $\|Tz_n - z_n\| \to 0$, we get $Tz_{n_k} \rightharpoonup p$. We show that $p \in F(T) \cap VI(D,A)$. First, we prove $p \in VI(D,A)$. Let

$$T'u := \begin{cases} Au + N_D u & u \in D \\ \emptyset & u \notin D \end{cases}$$

where $N_D u$ is the normal cone to D at $u \in D$, i.e., $N_D u = \{v \in H : \langle u - w, v\rangle \geq 0, \forall w \in D\}$. Then T' is maximal monotone and $0 \in T'u$ if and only if $u \in VI(D,A)$; [24]. Let $G(T')$ be the graph of T, let $(u,v) \in G(T')$. Since $v - Au \in N_D u$ and $z_n \in D$, we have $\langle u - z_n, v - Au\rangle \geq 0$. By property of metric projection, from $z_n = P_D(x_n - \delta_n Ax_n)$, we have $\langle u - z_n, z_n - (x_n - \delta_n Ax_n)\rangle \geq 0$ and hence

$$\langle u - z_n, \frac{z_n - x_n}{\delta_n} + Ax_n\rangle \geq 0.$$

From $\langle u - z, v - Au\rangle \geq 0$ for all $z \in D$ and $z_{n_k} \in D$, we have

$$\begin{aligned}\langle u - z_{n_k}, v\rangle &\geq \langle u - z_{n_k}, Au\rangle \\ &\geq \langle u - z_{n_k}, Au\rangle - \langle u - z_{n_k}, \frac{z_{n_k} - x_{n_k}}{\delta_{n_k}} + Ax_{n_k}\rangle \\ &= \langle u - z_{n_k}, Au - Az_{n_k}\rangle - \langle u - z_{n_k}, \frac{z_{n_k} - x_{n_k}}{\delta_{n_k}}\rangle + \langle u - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle \\ &\geq \langle u - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle - \langle u - z_{n_k}, \frac{z_{n_k} - x_{n_k}}{\delta_{n_k}}\rangle.\end{aligned}$$

Thus, we obtain $\langle u - p, v\rangle \geq 0$ as $k \to \infty$. Since T' is maximal monotone, we have $p \in T'^{-1}0$ and hence $p \in VI(D, A)$. Further, we prove $p \in F(T)$. By the Opial's condition, we have

$$\liminf_{k\to\infty} \|z_{n_k} - p\| < \liminf_{k\to\infty} \|z_{n_k} - Tp\|. \tag{3.4}$$

Observe that

$$\|z_{n_k} - Tp\| \leq \|z_{n_k} - Tz_{n_k}\| + \|Tz_{n_k} - Tp\|$$

which implies that

$$\|z_{n_k} - Tp\| \leq \|z_{n_k} - Tz_{n_k}\| + \xi L(z_{n_k}, p)$$

where

$$L(z_{n_k}, p) = max\{\|z_{n_k} - p\|, \|z_{n_k} - Tz_{n_k}\|, \|p - Tp\|, \|z_{n_k} - Tp\|, \|p - Tz_{n_k}\|\}.$$

If $L(z_{n_k}, p) = \|z_{n_k} - p\|$, then $\|Tz_{n_k} - Tp\| \leq \xi\|z_{n_k} - p\|$. Therefore

$$\liminf_{k\to\infty} \|z_{n_k} - Tp\| \leq \liminf_{k\to\infty} \xi\|z_{n_k} - p\| \leq \liminf_{k\to\infty} \|z_{n_k} - p\| \tag{3.5}$$

If $L(z_{n_k}, p) = \|p - Tp\|$, then $\|Tz_{n_k} - Tp\| \leq \xi\|p - Tp\| \leq \xi(\|p - z_{n_k}\| + \|z_{n_k} - Tp\|)$. This implies that

$$\liminf_{k\to\infty} \|z_{n_k} - Tp\| \leq \liminf_{k\to\infty} \frac{\xi}{1-\xi}\|z_{n_k} - p\| \leq \liminf_{k\to\infty} \|z_{n_k} - p\|. \tag{3.6}$$

If $L(z_{n_k}, p) = \|p - z_{n_k}\|$, then $\|Tz_{n_k} - Tp\| \leq \xi\|p - Tz_{n_k}\| \leq \xi(\|p - z_{n_k}\| + \|z_{n_k} - Tz_{n_k}\|$. Hence

$$\liminf_{k\to\infty} \|z_{n_k} - Tp\| \leq \liminf_{k\to\infty} \xi\|z_{n_k} - p\| \leq \liminf_{k\to\infty} \|z_{n_k} - p\|. \tag{3.7}$$

From (3.4), (3.5), (3.6) and (3.7), we conclude that $(I - T)p = 0$.
If $L(z_{n_k}, p) = \|z_{n_k} - Tz_{n_k}\|$, then $\|Tz_{n_k} - Tp\| \leq \xi\|z_{n_k} - Tz_{n_k}\|$. This implies that

$$\liminf_{k\to\infty} \|z_{n_k} - Tp\| \leq \liminf_{k\to\infty} (1 + \xi)\|z_{n_k} - Tz_{n_k}\| = 0$$

If $L(z_{n_k}, p) = \|z_{n_k} - Tp\|$, then $\|Tz_{n_k} - Tp\| \leq \xi\|z_{n_k} - Tp\|$. Hence

$$\liminf_{k\to\infty} \|z_{n_k} - Tp\| \leq \liminf_{k\to\infty} \xi\|z_{n_k} - Tp\| \Rightarrow \liminf_{k\to\infty} \|z_{n_k} - Tp\| = 0.$$

From (3.4) we obtain that $\liminf_{k\to\infty} \|z_{n_k} - p\| = \liminf_{k\to\infty} \|z_{n_k} - Tp\| = 0$. Therefore, $p - Tp = 0$. Thus, we obtain $p \in F(T)$. Therefore, $p \in F(T) \cap VI(D, A)$. Hence, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle x - y_0, Tz_n - y_0\rangle &= \lim_{k\to\infty} \langle x - y_0, Tz_{n_k} - y_0\rangle \\ &= \langle x - y_0, p - y_0\rangle \leq 0.\end{aligned}$$

Thus, for $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\langle x - y_0, Tz_n - y_0\rangle \le \varepsilon,\ \lambda_n\|x - y_0\|^2 \le \varepsilon\ \forall n \ge k.$$

Also, for $n \ge k$

$$(1-\beta_n)\langle x - y_0, Tx_n - y_0\rangle \le (1-\beta_n)\|x-y_0\|\|Tx_n - y_0\| < \varepsilon.$$

For $n \ge k$, we have

$$\begin{aligned}
\|x_{n+1} - y_0\|^2 &= \|\lambda_n x + (1-\lambda_n)(\beta_n Tz_n + (1-\beta_n)Tx_n) - y_0\|^2 \\
&= \lambda_n^2\|x - y_0\|^2 + 2\lambda_n(1-\lambda_n)\langle x - y_0, (\beta_n Tz_n + (1-\beta_n)Tx_n) - y_0\rangle \\
&\quad +(1-\lambda_n)^2\|(\beta_n Tz_n + (1-\beta_n)Tx_n) - y_0\|^2 \\
&\le \lambda_n\varepsilon + 2\lambda_n(1-\lambda_n)(\beta_n\varepsilon + (1-\beta_n)\langle x - y_0, Tx_n - y_0\rangle \\
&\quad +(1-\lambda_n)[\beta_n\|Tz_n - y_0\|^2 + (1-\beta_n)\|Tx_n - y_0\|^2] \\
&\le 3\lambda_n\varepsilon + (1-\lambda_n)\|x_n - y_0\|^2 \\
&= 3\varepsilon(1-(1-\lambda_n)) + (1-\lambda_n)\|x_n - y_0\|^2.
\end{aligned}$$

Inductively, we have

$$\|x_{n+1} - y_0\|^2 \le 3\varepsilon(1 - \prod_{i=k}^{n}(1-\lambda_i)) + \prod_{i=k}^{n}(1-\lambda_i)\|x_k - y_0\|^2.$$

Hence, we obtain

$$\limsup_{n\to\infty}\|x_{n+1} - y_0\|^2 \le 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we gives that $\limsup_{n\to\infty}\|x_{n+1} - y_0\|^2 \le 0$ and hence $x_n \to y_0$. □

Utilizing Theorems 3.1, we conclude the following corollary.

Corollary 3.2. *Let D be a nonempty closed convex subset of a real Hilbert space H, $A : D \to H$ an α-inverse strongly monotone mapping with $VI(D, A) \neq \emptyset$. Suppose $x_1 = x \in D$ and $\{x_n\}$ is given by*

$$x_{n+1} = \lambda_n x + (1-\lambda_n)[\beta_n P_D(x_n - \delta_n Ax_n) + (1-\beta_n)x_n],$$

for every $n = 1, 2, 3, ...,$ where $\lambda_n \in [0, 1)$ with $\lim_{n\to\infty}\lambda_n = 0, \sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty}\lambda_n = \infty$ and $\delta_n \in [a, b]$ for some $a, b \in (0, 2\alpha)$ with $\sum_{n=1}^{\infty}|\delta_{n+1} - \delta_n| < \infty$ and $\beta_n \in (0, 1]$ with $\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty, \lim_{n\to\infty}\beta_n = 1$. Then $\{x_n\}$ converges strongly to $P_{VI(D,A)}x$.

4. Applications

Utilizing Theorem 3.1, we prove strong convergence in a real Hilbert space H. A mapping $T' : D \to D$ is called strictly pseudo-contractive if there exists $0 \leq k < 1$ such that

$$\|T'x - T'y\|^2 \leq \|x - y\|^2 + k\|(I - T')x - (I - T')y\|^2$$

for all $x, y \in D$. For such a case, T' is called k-strictly pseudo-contractive. If $k = 0$, then T' is nonexpansive. Let $T' : D \to D$ be strictly pseudo-contractive. If $A = I - T'$, then A is $\frac{1-k}{2}$−inverse strongly monotone and $F(T') = VI(D, A)$ [8, 18, 27].

Theorem 4.1. *Let D be a nonempty closed convex subset of a real Hilbert space H, $\xi \in (0, \frac{1}{2})$, $T' : D \to D$ a k-strictly pseudo-contractive mapping and $T : D \to D$ a quasi-contractive mapping with $F(T) \cap F(T') \neq \emptyset$. Suppose $x_1 = x \in D$ and $\{x_n\}$ is given by*

$$x_{n+1} = \lambda_n x + (1 - \lambda_n)[\beta_n T((1 - \delta_n)x_n + \delta_n T'x_n) + (1 - \beta_n)Tx_n],$$

for every $n = 1, 2, 3, ...$, where $\lambda_n \in [0, 1)$ with $\lim_{n\to\infty} \lambda_n = 0, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\delta_n \in [a, b]$ for some $a, b \in (0, 1 - k)$ with $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$ and $\beta_n \in (0, 1]$ with $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \lim_{n\to\infty} \beta_n = 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)\cap F(T')}x$.

Proof. Let $A = I - T$. Then, A is $\frac{1-k}{2}$-inverse strongly monotone. We have $F(T') = VI(D, A)$ and $P_D(x_n - \delta_n Ax_n) = (1 - \delta_n)x_n + \delta_n T'x_n$; ([11]). Thus, by Theorem (3.1), we obtain the desired result. □

Theorem 4.2. *Let D be a nonempty closed convex subset of a real Hilbert space H, $A : H \to H$ an α-inverse strongly monotone mapping and $T : D \to D$ a quasi-contractive mapping with $F(T) \cap A^{-1}0 \neq \emptyset, \xi \in (0, \frac{1}{2})$. Suppose $x_1 = x \in D$ and $\{x_n\}$ is given by*

$$x_{n+1} = \lambda_n x + (1 - \lambda_n)[\beta_n TP_D(x_n - \delta_n Ax_n) + (1 - \beta_n)Tx_n], \tag{4.1}$$

for every $n = 1, 2, 3, ...$, where $\lambda_n \in [0, 1)$ with $\lim_{n\to\infty} \lambda_n = 0, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\delta_n \in [a, b]$ for some $a, b \in (0, 2\alpha)$ with $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$ and $\beta_n \in (0, 1]$ with $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \lim_{n\to\infty} \beta_n = 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)\cap A^{-1}0}x$.

Proof. We have $A^{-1}0 = VI(H, A)$. Putting $P_H = I$, by Theorem 3.1, we obtain the desired result. □

Example 4.3. Let $H = \mathbb{R}$ with Euclidean norm and usual Euclidean inner product. Let $D := [-1, 2]$ and $T : D \to D$ be defined by

$$Tx := \begin{cases} \frac{1}{3} & -1 \leq x < 0 \\ 0 & 0 \leq x < 1 \\ \frac{x}{5} & 1 \leq x \leq 2 \end{cases}$$

for all $x \in D$. Then, T is quasi-contractive, with $\xi = \frac{10}{21}$, $F(T) = \{0\}$, but T is not nonexpansive and continuous. Since for $x = \frac{5}{6}$, $y = 1$, $\frac{1}{6} = \|x-y\| < \|Tx-Ty\| = \frac{1}{5}$. If $x \in [-1, 0)$ and $y \in [0, 1)$, then

$$\frac{1}{3} = \|Tx - Ty\| < \frac{10}{21}\|x - y\|.$$

If $x \in [-1, 0)$ and $y \in [1, 2]$, then

$$max\|Tx - Ty\| = \frac{1}{15} < \frac{10}{21}\|x - y\|.$$

If $x \in [0, 1)$ and $y \in [1, 2]$, then

$$max\|Tx - Ty\| = \frac{2}{5} < \frac{10}{21}\|Tx - y\|.$$

Define $A : D \to \mathbb{R}$ by

$$Ax := \begin{cases} 0 & x \in [-1, 0] \\ \frac{x}{2} & x \in [0, 1) \\ \frac{\sqrt{x}}{2} & x \in [1, 2]. \end{cases}$$

Clearly, $VI(D, A) = [-1, 0]$, A is a monotone. Now, we prove that A is a α-inverse strongly monotone for $\alpha \le 2$. If $x, y \in [-1, 0)$, then

$$\langle Ax - Ay, x - y\rangle = 0 \ge 2\|Ax - Ay\|^2 = 0.$$

If $x, y \in [0, 1)$, then

$$\langle\frac{x}{2} - \frac{y}{2}, x - y\rangle \ge 2\|\frac{x}{2} - \frac{y}{2}\|^2.$$

If $x \in [1, 2]$, then

$$\langle\frac{\sqrt{x}}{2} - \frac{\sqrt{y}}{2}, x - y\rangle = 2\|\frac{\sqrt{x}}{2} - \frac{\sqrt{y}}{2}\|^2(\sqrt{x} + \sqrt{y}) \ge 2\|\frac{\sqrt{x}}{2} - \frac{\sqrt{y}}{2}\|^2.$$

If $x \in [-1, 0)$ and $y \in [0, 1)$, then

$$\langle 0 - \frac{y}{2}, x - y\rangle = 2\|\frac{y}{2}\|^2 + \langle\frac{y}{2}, -x\rangle > 2\|\frac{y}{2}\|^2.$$

If $x \in [-1, 0)$ and $y \in [1, 2]$, then

$$\langle 0 - \frac{\sqrt{y}}{2}, x - y\rangle = 2\|\frac{\sqrt{y}}{2}\|^2 + \langle\frac{\sqrt{y}}{2}, -x\rangle > 2\|\frac{\sqrt{y}}{2}\|^2.$$

If $x \in [0, 1)$ and $y \in [1, 2]$, then

$$\langle\frac{x}{2} - \frac{\sqrt{y}}{2}, x - y\rangle = 2\langle\frac{\sqrt{y}}{2} - \frac{x}{2}, \frac{y}{2} - \frac{x}{2}\rangle \ge 2\langle\frac{\sqrt{y}}{2} - \frac{x}{2}, \frac{\sqrt{y}}{2} - \frac{x}{2}\rangle = 2\|\frac{\sqrt{y}}{2} - \frac{x}{2}\|^2.$$

Hence, $\langle Ax - Ay, x - y\rangle \ge \|Ax - Ay\|^2$ for all $x \in [-1, 2]$ and thus A is α-inverse strongly monotone for $\alpha \le 2$, $0 \in F(T) \cap VI(D, A)$.

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