Pure and Applied Functional Analysis
Volume 5, Number 3, 2020, 807–816

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

A NONINTERSECTION PROPERTY FOR SOLUTIONS OF DISCRETE TIME OPTIMAL CONTROL PROBLEMS

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Abstract. In this work we study the structure of overtaking optimal solutions of discrete time optimal control problems arising in models of economic dynamics and show that they have a nonintersection property.

1. Introduction and the main result

The study of the existence and the structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [2, 4–14, 18, 19, 25, 28, 30, 33–35, 38, 46, 48] and the references mentioned therein. These problems arise in engineering [1, 26, 44], in models of economic growth [10, 15, 20–24, 29, 32, 39, 41–45, 50], in the game theory [16, 37, 47], in optimal control with PDE [17, 36, 40, 49] in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [3] and in the theory of thermodynamical equilibrium for materials [27, 31]. In this paper we study the infinite horizon problem related to a discrete-time optimal control system describing the Robinson-Solow-Srinivasan model and establish a nonintersection property for their optimal solutions.

It should be mentioned that discrete-time optimal control problems arising in economic dynamics usually are studied under assumptions that all their good programs converge to a turnpike which is an interior point of the set of admissible pairs [45, 47]. In this paper we study a large class of control systems for which the turnpike is not necessarily an interior point of the set of admissible pairs. This makes the situation more difficult and less understood.

One of the main topics in the infinite horizon optimal control theory is to study the existence and properties of solutions of problems over an infinite horizon using different optimality criteria. In the present paper, studying infinite horizon problems, we deal with the notion of good programs introduced by D. Gale in [15] which is of great usage in optimal control and economic dynamics (see, for example, [10, 44, 45] and the references mentioned therein) and with the notion of overtaking optimal program [10, 15, 41, 44, 45, 49, 50].

2010 *Mathematics Subject Classification.* 49J99.

Key words and phrases. Good program, infinite horizon problem, overtaking optimal program, turnpike.

2. Overtaking optimal programs

Let R^1 (R^1_+) be the set of real (non-negative) numbers and let R^n be the n-dimensional Euclidean space with non-negative orthant

$$R^n_+ = \{x = (x_1, \dots, x_n) \in R^n : \ x_i \geq 0, \ i = 1, \dots, n\}.$$

For every pair of vectors $x = (x_1, \dots, x_n), \ y = (y_1, \dots, y_n) \in R^n$, define their inner product by

$$xy = \sum_{i=1}^{n} x_i y_i$$

and let $x >> y$, $x > y$, $x \geq y$ have their usual meaning. Namely, for a given pair of vectors $x = (x_1, \dots, x_n), \ y = (y_1, \dots, y_n) \in R^n$, we say that $x \geq y$, if $x_i \geq y_i$ for all $i = 1, \dots, n$, $x > y$ if $x \geq y$ and $x \neq y$, and $x >> y$ if $x_i > y_i$ for all $i = 1, \dots, n$.

Let $e(i)$, $i = 1, \dots, n$, be the ith unit vector in R^n, and e be an element of R^n_+ all of whose coordinates are unity. For every $x \in R^n$, denote by $\|x\|$ its Euclidean norm in R^n.

Let $a = (a_1, \dots, a_n) >> 0$, $b = (b_1, \dots, b_n) >> 0$, $d \in (0, 1)$,

$$c_i = b_i/(1 + da_i), \ i = 1, \dots, n. \tag{2.1}$$

We assume the following:

$$\text{There exists } \sigma \in \{1, \dots, n\} \text{ such that for all}$$
$$i \in \{1, \dots, n\} \setminus \{\sigma\}, \ c_\sigma > c_i. \tag{2.2}$$

A sequence $\{x(t), y(t)\}_{t=0}^\infty$ is called a program if for each integer $t \geq 0$

$$(x(t), y(t)) \in R^n_+ \times R^n_+, \ x(t+1) \geq (1-d)x(t),$$
$$0 \leq y(t) \leq x(t), \ a(x(t+1) - (1-d)x(t)) + ey(t) \leq 1. \tag{2.3}$$

Let T_1, T_2 be integers such that $0 \leq T_1 < T_2$. A pair of sequences

$$(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1})$$

is called a program if $x(T_2) \in R^n_+$ and for each integer t satisfying $T_1 \leq t < T_2$ relations (2.3) is valid.

Assume that $w : [0, \infty) \to R^1$ is a continuous strictly increasing concave and differentiable function which represents the preferences of the planner.

Define

$$\Omega = \{(x, x') \in R^n_+ \times R^n_+ : \ x' - (1-d)x \geq 0$$
$$\text{and } a(x' - (1-d)x) \leq 1\} \tag{2.4}$$

and a correspondence $\Lambda : \Omega \to R^n_+$ given by

$$\Lambda(x, x') = \{y \in R^n_+ : \ 0 \leq y \leq x \text{ and } ey \leq 1 - a(x' - (1-d)x)\}. \tag{2.5}$$

For every $(x, x') \in \Omega$ set

$$u(x, x') = \max\{w(by) : \ y \in \Lambda(x, x')\}. \tag{2.6}$$

A golden-rule stock is $\widehat{x} \in R^n_+$ such that $(\widehat{x}, \widehat{x})$ is a solution to the problem: maximize $u(x, x')$ subject to

(i) $x' \geq x$; (ii) $(x, x') \in \Omega$.

It was shown in [20] that there exists a unique golden-rule stock

$$\widehat{x} = (1/(1 + da_\sigma))e(\sigma). \tag{2.7}$$

It is not difficult to see that $\widehat{x}$ is a solution to the problem

$$w(by) \to \max,\ y \in \Lambda(\widehat{x}, \widehat{x}).$$

Set

$$\widehat{y} = \widehat{x}. \tag{2.8}$$

For $i = 1, \dots, n$ set

$$\widehat{q}_i = a_i b_i/(1 + da_i),\ \widehat{p}_i = w'(b\widehat{x})\widehat{q}_i. \tag{2.9}$$

It was shown in [20] that

$$w(b\widehat{x}) \geq w(by) + \widehat{p}x' - \widehat{p}x \tag{2.10}$$

for every $(x, x') \in \Omega$ and for every $y \in \Lambda(x, x')$.

A program $\{x(t), y(t)\}_{t=0}^{\infty}$ is good if there is a real number M such that

$$\sum_{t=0}^{T}(w(by(t)) - w(b\widehat{y})) \geq M \text{ for every nonnegative integer } T.$$

A program $\{x(t), y(t)\}_{t=0}^{\infty}$ bad if

$$\lim_{T\to\infty} \sum_{t=0}^{T}(w(by(t)) - w(b\widehat{y})) = -\infty.$$

The following result was proved in [20].

Proposition 2.1. *Every program which is not good is bad.*

The following two results were obtained in [42]. They show that an asymptotic turnpike property holds for our infinite horizon problem.

Theorem 2.2. *Assume that the function w is strictly concave. Then for every good program $\{x(t), y(t)\}_{t=0}^{\infty}$,*

$$\lim_{t\to\infty} (x(t), y(t)) = (\widehat{x}, \widehat{x}).$$

Set

$$\xi_\sigma = 1 - d - (1/a_\sigma). \tag{2.11}$$

Theorem 2.3. *Assume that $\xi_\sigma \neq -1$. Then*

$$\lim_{t\to\infty} (x(t), y(t)) = (\widehat{x}, \widehat{x})$$

for every good program $\{x(t), y(t)\}_{t=0}^{\infty}$.

We use a notion of an overtaking optimal program introduced by Gale [15] and von Weizsacker [41]. This optimality criterion is used in optimal control [10, 44, 45].

A program $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ is overtaking optimal if

$$\limsup_{T\to\infty} \sum_{t=0}^{T}[w(by(t)) - w(by^*(t))] \leq 0$$

for every program $\{x(t), y(t)\}_{t=0}^{\infty}$ which satisfies $x(0) = x^*(0)$.

The following existence result was also obtained in [42].

Theorem 2.4. *Assume that for every good program* $\{x(t), y(t)\}_{t=0}^{\infty}$,

$$\lim_{t\to\infty} (x(t), y(t)) = (\widehat{x}, \widehat{x}).$$

Then for every point $x_0 \in R^n_+$ *there is an overtaking optimal program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *such that* $x(0) = x_0$.

The final result of this section was obtained in [21].

Theorem 2.5. *Assume that at least one of the following conditions holds:*

(a) *w is strictly concave.*

(b) $\xi_\sigma \neq -1$.

Let $\epsilon > 0$. *Then there exists* $\delta > 0$ *such that for each overtaking optimal program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *satisfying* $\|x(0) - \widehat{x}\| \leq \delta$ *the following inequality holds:*

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \leq \epsilon$$

for all integers $t \geq 0$.

3. A turnpike result

In this section we present two auxiliary results which play an important role in our study. The first of them was obtained in [21].

Proposition 3.1. *Let* $\epsilon > 0$. *Then there exists* $\delta > 0$ *such that for each* $x, x' \in R^n_+$ *satisfying*

$$\|x - \widehat{x}\|, \|x' - \widehat{x}\| \leq \delta$$

there exist $\bar{x} \geq x'$, $y \in R^n_+$ *such that*

$$(x, \bar{x}) \in \Omega,\ y \in \Lambda(x, \bar{x}),$$

$$\|y - \widehat{x}\| \leq \epsilon,\ \|\bar{x} - \widehat{x}\| \leq \epsilon.$$

It is easy to see that the following auxiliary result holds.

Proposition 3.2. *Assume that* T_1, T_2 *are nonnegative integers,* $T_1 < T_2$,

$$(\{x(t)\}_{t=T_1}^{T_2},\ \{y(t)\}_{t=T_1}^{T_2-1})$$

is a program and $u \in R^n_+$. *Then* $(\{x(t) + (1-d)^{t-T_1}u\}_{t=T_1}^{T_2},\ \{y(t)\}_{t=T_1}^{T_2-1})$ *is also a program.*

Let $z \in R^n_+$ and $T \geq 1$ be a natural number. Set

$$U(z, T) = \sup\{\sum_{t=0}^{T-1} w(by(t)) :\ (\{x(t)\}_{t=0}^{T},\ \{y(t)\}_{t=0}^{T-1}) \text{ is a program such that } x(0) = z\}.$$

Note that $U(z, T)$ is a finite number [23].

Let $x_0, x_1 \in R^n_+$, T_1, T_2 be integers, $0 \le T_1 < T_2$. Define

$$U(x_0, x_1, T_1, T_2) = \sup\{\sum_{t=T_1}^{T_2-1} w(by(t)) : (\{x(t)\}_{t=T_1}^{T_2}, \ \{y(t)\}_{t=T_1}^{T_2-1})$$
$$\text{is a program such that } x(T_1) = x_0, \ x(T_2) \ge x_1\}.$$

(Here we suppose that a supremum over empty set is $-\infty$.) Note that

$$U(x_0, x_1, T_1, T_2) < \infty$$

[23].

The next turnpike result was obtained in [23].

Theorem 3.3. *Assume that each good program $\{u(t), v(t)\}_{t=0}^{\infty}$ converges to the golden-rule stock $(\widehat{x}, \widehat{x})$:*

$$\lim_{t\to\infty} (u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let M, ϵ be positive numbers and $\Gamma \in (0, 1)$. Then there exist a natural number L and a positive number γ such that for each integer $T > 2L$, each $z_0, z_1 \in R^n_+$ satisfying $z_0 \le Me$ and $az_1 \le \Gamma d^{-1}$ and each program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ which satisfies

$$x(0) = z_0, \ x(T) \ge z_1, \quad \sum_{t=0}^{T-1} w(by(t)) \ge U(z_0, z_1, 0, T) - \gamma$$

there are integers τ_1, τ_2 such that

$$\tau_1 \in [0, L], \ \tau_2 \in [T - L, T],$$
$$\|x(t) - \widehat{x}\|, \ \|y(t) - \widehat{x}\| \le \epsilon \text{ for all } t = \tau_1, \dots, \tau_2 - 1.$$

Moreover if $\|x(0) - \widehat{x}\| \le \gamma$ then $\tau_1 = 0$ and if $\|x(T) - \widehat{x}\| \le \gamma$ then $\tau_2 = T$.

4. The main result

Assume that each good program $\{u(t), v(t)\}_{t=0}^{\infty}$ converges to the golden-rule stock $(\widehat{x}, \widehat{x})$:

$$\lim_{t\to\infty} (u(t), v(t)) = (\widehat{x}, \widehat{x}). \tag{4.1}$$

Theorem 4.1. *Assume that $\{x(t), y(t)\}_{t=0}^{\infty}$ is an overtaking optimal program, T_0 is a natural number and that*

$$x(0) = x(T_0). \tag{4.2}$$

Then for all integers $t \ge 0$,

$$x(t) = y(t) = \widehat{x}.$$

Proof. In view of (4.2), there exists a program $\{\tilde{x}(t), \tilde{y}(t)\}_{t=0}^{\infty}$ such that

$$\tilde{x}(t) = x(t), \ \tilde{y}(t) = y(t), \ t = 0, \dots, T_0 - 1 \tag{4.3}$$

and that for all integers $t \ge 0$,

$$\tilde{x}(t + T_0) = \tilde{x}(t), \ \tilde{y}(t + T_0) = \tilde{y}(t). \tag{4.4}$$

Proposition 2.1 implies that there are two cases:

(1) the program $\{\tilde{x}(t), \tilde{y}(t)\}_{t=0}^{\infty}$ is good;

(2) the program $\{\tilde{x}(t), \tilde{y}(t)\}_{t=0}^{\infty}$ is bad.

Assume that case (2) holds. By (4.3) and (4.4),

$$\begin{aligned} -\infty &= \lim_{T\to\infty} \sum_{t=0}^{T} (w(b\tilde{y}(t)) - w(b\widehat{x})) \\ &= \lim_{k\to\infty} \sum_{t=0}^{T_0 k-1} (w(b\tilde{y}(t)) - w(b\widehat{x})) \\ &= \lim_{k\to\infty} k \sum_{t=0}^{T_0-1} (w(b\tilde{y}(t)) - w(b\widehat{x})) \\ &= \lim_{k\to\infty} k \sum_{t=0}^{T_0-1} (w(by(t)) - w(b\widehat{x})), \end{aligned}$$

where k is a natural number. Therefore

$$\sum_{t=0}^{T_0-1} w(by(t)) < T_0 w(b\widehat{x}). \tag{4.5}$$

In view of (4.5), there exists a positive number γ such that

$$\gamma < T_0 w(b\widehat{x}) - \sum_{t=0}^{T_0-1} w(by(t)). \tag{4.6}$$

There exists $\gamma_0 \in (0, \gamma)$ such that

$$2\gamma_0 e(\sigma) \le \widehat{x} \tag{4.7}$$

and

$$w(b(\widehat{x} - \gamma_0 e(\sigma))) \ge w(\widehat{x}) - \gamma/8. \tag{4.8}$$

Proposition 3.1 implies that there exists $\delta \in (0, \gamma_0)$ such that the following property holds:

(P) for each $x, x' \in R^n_+$ satisfying

$$\|x - \widehat{x}\|, \|x' - \widehat{x}\| \le \delta$$

there exist $\bar{x} \ge x'$, $y \in R^n_+$ such that

$$(x, \bar{x}) \in \Omega,\ y \in \Lambda(x, \bar{x}),$$

$$\|y - \widehat{x}\| \le \gamma_0,\ \|\bar{x} - \widehat{x}\| \le \gamma_0$$

and

$$w(b(\widehat{x} + \delta e)) \le w(\widehat{x}) + \gamma/8.$$

Note [42] that every overtaking optimal program is good. Therefore in view of (4.1),

$$\lim_{t\to\infty} x(t) = \widehat{x},\ \lim_{t\to\infty} y(t) = \widehat{x}$$

and there exists a natural number T_1 such that for all integers $t \ge T_1$,

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \le \delta. \tag{4.9}$$

Set

$$x^{(1)}(t) = x(t+T_0),\ t = 0, \dots, T_0 + T_1, \tag{4.10}$$

$$y^{(1)}(t) = y(t+T_0),\ t = 0, \dots, T_0 + T_1 - 1. \tag{4.11}$$

Property (P), (4.9) and (4.10) imply that there exist

$$y^{(1)}(T_0 + T_1),\ x^{(1)}(T_0 + T_1 + 1) \in R^n_+$$

such that

$$(x^{(1)}(T_0 + T_1), x^{(1)}(T_0 + T_1 + 1)) \in \Omega, \tag{4.12}$$

$$y^{(1)}(T_0 + T_1) \in \Lambda(x^{(1)}(T_0 + T_1), x^{(1)}(T_0 + T_1 + 1)), \tag{4.13}$$

$$\|y^{(1)}(T_0 + T_1) - \widehat{x}\| \le \gamma_0,\ \|x^{(1)}(T_0 + T_1 + 1) - \widehat{x}\| \le \gamma_0, \tag{4.14}$$

$$x^{(1)}(T_0 + T_1 + 1) \ge \widehat{x}. \tag{4.15}$$

In view of (4.12) and (4.13), $(\{x^{(1)}(t)\}_{t=0}^{T_0+T_1+1}, \{y^{(1)}(t)\}_{t=0}^{T_0+T_1})$ is a program. For all integers $t \in \{T_0 + T_1 + 1, \dots, 2T_0 + T_1\}$, set

$$y^{(1)}(t) = \widehat{y}, \tag{4.16}$$

$$x^{(1)}(t+1) = \widehat{x} + (1-d)^{t-T_0-T_1}(x^{(1)}(T_0 + T_1 + 1) - \widehat{x}). \tag{4.17}$$

By Proposition 3.2 and (4.15)-(4.17), $(\{x^{(1)}(t)\}_{t=0}^{2T_0+T_1+1}, \{y^{(1)}(t)\}_{t=0}^{2T_0+T_1})$ is a program. It follows from (4.15) and (4.17) that

$$x^{(1)}(2T_0 + T_1 + 1) \ge \widehat{x}. \tag{4.18}$$

Property (P) and (4.9) imply that there exist

$$\bar{x} \ge x(2T_0 + T_1 + 2),\ y \in R^n_+ \tag{4.19}$$

such that

$$(\widehat{x}, \bar{x}) \in \Omega,\ y \in \Lambda(\widehat{x}, \bar{x}), \tag{4.20}$$

$$\|\bar{x} - \widehat{x}\| \le \gamma_0,\ \|y - \widehat{x}\| \le \gamma_0. \tag{4.21}$$

Set

$$y^{(1)}(2T_0 + T_1 + 1) = y, \tag{4.22}$$

$$x^{(1)}(2T_0 + T_1 + 2) = \bar{x} + (1-d)(x^{(1)}(2T_0 + T_1 + 1) - \widehat{x}). \tag{4.23}$$

It follows from (4.18), (4.20), (4.22) and (4.23) that

$$(x^{(1)}(2T_0 + T_1 + 1), x^{(1)}(2T_0 + T_1 + 2)) \in \Omega,$$

$$y^{(1)}(2T_0 + T_1 + 1) \in \Lambda(x^{(1)}(2T_0 + T_1 + 1), x^{(1)}(2T_0 + T_1 + 2)).$$

Thus $(\{x^{(1)}(t)\}_{t=0}^{2T_0+T_1+2}, \{y^{(1)}(t)\}_{t=0}^{2T_0+T_1+1})$ is a program. In view of (4.2), (4.10), (4.18), (4.19) and (4.23),

$$x^{(1)}(0) = x(T_0) = x(0), \tag{4.24}$$

$$x^{(1)}(2T_0 + T_1 + 2) \ge \bar{x} \ge x(2T_0 + T_1 + 2). \tag{4.25}$$

Since the program $\{x(t), y(t)\}_{t=0}^{\infty}$ is overtaking optimal Proposition 3.2, (4.6), (4.11), (4.24) and (4.25) imply that

$$\begin{aligned}
0 &\le \sum_{t=0}^{2T_0+T_1+1} w(by(t)) - \sum_{t=0}^{2T_0+T_1+1} w(by^{(1)}(t)) \\
&= \sum_{t=0}^{T_0-1} w(by(t)) + \sum_{t=T_0}^{2T_0+T_1+1} w(by(t)) \\
&\quad - \sum_{t=0}^{T_0+T_1-1} w(by^{(1)}(t)) - w(by^{(1)}(T_0+T_1)) \\
&\quad - T_0 w(b\widehat{y}) - w(by^{(1)}(2T_0+T_1+1)) \\
&\le -\gamma + w(by(2T_0+T_1)) + w(by(2T_0+T_1+1)) \\
&\quad - w(by^{(1)}(T_0+T_1)) - w(by^{(1)}(2T_0+T_1+1)). \qquad (4.26)
\end{aligned}$$

Property (P), (4.8), (4.9) and (4.26) imply that

$$\begin{aligned}
0 &\le -\gamma + 2w(b(\widehat{x}+\delta e)) - 2w(b(\widehat{x}-\gamma_0 e(\sigma))) \\
&\le -\gamma + 2w(b(\widehat{x})) + \gamma/4 - (2w(b(\widehat{x})) - \gamma/4) \le -\gamma/2,
\end{aligned}$$

a contradiction. The contradiction we have reached proves that case (2) does not hold. Thus the program $\{\tilde{x}(t), \tilde{y}(t)\}_{t=0}^{\infty}$ is good. In view of (4.1),

$$x(t) = \widehat{x},\ t = 0, \dots, T_0.$$

Together with Theorem 3.3, (4.1) and the inequality $a\widehat{x} < d^{-1}$ this implies that

$$x(t) = y(t) = \widehat{x},\ t = 0, 1, \dots.$$

Theorem 4.1 is proved.

$\square$

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Manuscript received October 16 2019
revised December 2 2019

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