Pure and Applied Functional Analysis
Volume 5, Number 4, 2020, 851–867

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

VECTOR-VALUED SOBOLEV MAPS AND TOPOLOGICAL SINGULARITIES

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Abstract. In this survey paper we review the analysis of topological singularities of Sobolev maps into manifolds and their applications to variational problems of Ginzburg-Landau type, focusing in particular to the sphere-valued case. We describe also recent developments in the vector-valued case and prospective applications to variational models of material science, such as the Landau-de Gennes one.

1. Topological singularities of sphere-valued Sobolev maps

1.1. Motivation: the Ginzburg-Landau functional. For the sake of motivation, consider the Ginzburg-Landau functional:

$$u \in W^{1,2}(\Omega, \mathbb{C}) \mapsto E_\varepsilon^{\mathrm{GL}}(u) := \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 \right\}, \qquad (\mathrm{GL}_\varepsilon)$$

where Ω is a smooth, bounded domain in $\mathbb{R}^d$, $d \geq 2$, and $\varepsilon > 0$ is a small parameter. Functionals of this form arise as variational models for the study of type-II superconductivity. In this context, $u(x)$ represents the magnetisation vector at a point $x \in \Omega$ and the energy favours configurations with $|u(x)| = 1$, which have a well-defined direction of magnetisation as opposed to the non-superconducting phase $u = 0$. Let $\mathbb{S}^1$ denote the unit circle in the complex plane $\mathbb{C}$.

As is well known since the fundamental monograph by Bethuel, Brezis and Hélein [8], minimisers u_ε subject to a (ε-independent) boundary condition $u_{\varepsilon|\partial\Omega} = u_{\mathrm{bd}} \in W^{1/2,2}(\partial\Omega, \mathbb{S}^1)$ satisfy the sharp energy bound $E_\varepsilon^{\mathrm{GL}}(u_\varepsilon) \leq C|\log\varepsilon|$ for some ε-independent constant C. In particular, u_ε takes values "close" to $\mathbb{S}^1$ when ε is small, in the sense that $\int_\Omega (1 - |u_\varepsilon|^2)^2 \leq C\varepsilon^2|\log\varepsilon|$. Despite the lack of uniform energy bounds, under suitable conditions on u_{bd}, minimisers u_ε converge to a limit map $u_0 \colon \Omega \to \mathbb{S}^1$, which is smooth except for a singular set of codimension two (see e.g. [8, 38, 43, 9, 36, 2, 11, 45]). Moreover, the singular set of u_0 is itself a minimiser — in a suitable sense — of some "weighted area" functional. The emergence of singularities in the limit map u_0 is related to topological obstructions, which may prevent the existence of a map in $W^{1,2}(\Omega, \mathbb{S}^1)$ that satisfies the boundary conditions.

It should be remarked that the logarithmic energy bound

$$E_\varepsilon^{\mathrm{GL}}(v_\varepsilon) \leq C\,|\log\varepsilon|$$

2010 *Mathematics Subject Classification.* 58C06, 49Q15, 49Q20.
Key words and phrases. Ginzburg-Landau type functionals, topological singularities, distributional Jacobian, manifold-valued maps, density of smooth maps, lifting.

does *not* guarantee compactness of the sequence $(v_\varepsilon)_{\varepsilon>0}$, in any Sobolev norm. Indeed, the maps $v_\varepsilon(x) := \exp(i\varphi(x)|\log\varepsilon|^{1/2})$, where $\varphi \in C^\infty(\Omega, \mathbb{R})$ is a fixed, non-constant function, satisfy $|v_\varepsilon| = 1$ and

$$E_\varepsilon^{\mathrm{GL}}(v_\varepsilon) = \frac{|\log\varepsilon|}{2}\int_\Omega |\nabla\varphi|^2 \leq C\,|\log\varepsilon|\,,$$

but $|\nabla v_\varepsilon| = \mathrm{O}(|\log\varepsilon|^{1/2})$ so the gradient diverges as $\varepsilon \to 0$. Actually, even for energy minimisers, no compactness can be expected even in L^1_{loc} (unless additional assumptions on the boundary datum are made). Indeed, Brezis and Mironescu [15] constructed a sequence of minimisers u_{ε_n}, on the unit ball $B^d \subseteq \mathbb{R}^d$ with $d \geq 2$, that satisfies $E_{\varepsilon_n}^{\mathrm{GL}}(u_{\varepsilon_n}) = \mathrm{o}(|\log\varepsilon_n|)$ as $\varepsilon_n \to 0$ and yet has no subsequence that converges a.e. on a set of positive measure, as there holds $\sup_{x\in B^d}|u_{\varepsilon_n}(x) - \exp(inx_1)| \to 0$.

In the previous examples, the lack of compactness is due to *oscillations of the phase* and not to topological obstructions. In fact, it is possible to isolate the topological contribution to the energy and prove compactness results on that part alone. This is usually achieved by the use of distributional Jacobians.

1.2. **The distributional Jacobian.** Let $d \geq k \geq 2$ be integers. Given a map $u = (u^1, \ldots, u^k)\colon \mathbb{R}^d \to \mathbb{R}^k$ of class C^2, we compute that

$$\mathrm{d}u^1 \wedge \ldots \wedge \mathrm{d}u^k = \frac{1}{k}\mathrm{d}\left(\sum_{i=1}^k (-1)^{i+1} u^i \widehat{\mathrm{d}u^i}\right), \tag{1.1}$$

where $\widehat{\mathrm{d}u^i} := \mathrm{d}u^1 \wedge \ldots \wedge \mathrm{d}u^{i-1} \wedge \mathrm{d}u^{i+1} \wedge \ldots \wedge \mathrm{d}u^k$. In case $d = k$, we can rewrite the identity (1.1) using vector calculus instead of differential forms. More precisely, when $d = k = 2$ we have

$$\det(\nabla u) = \frac{1}{2}\partial_1\left(u^1\partial_2 u^2 - u^2\partial_2 u^1\right) + \frac{1}{2}\partial_2\left(u^2\partial_1 u^1 - u^1\partial_1 u^2\right)$$

and if $d = k = 3$ then

$$\det(\nabla u) = \frac{1}{3}\operatorname{div}(u\cdot\partial_2 u \times \partial_3 u,\, u\cdot\partial_3 u\times\partial_1 u,\, u\cdot\partial_1 u\times\partial_2 u).$$

Similar — but more involved — reformulations are possible if $d = k > 3$.

The right-hand side of (1.1) is well-defined for any $u \in W^{1,k}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^k)$, while the left-hand side is well-defined (in the sense of distributions) even if $u \in (L^\infty \cap W^{1,k-1}_{\mathrm{loc}})(\mathbb{R}^d, \mathbb{R}^k)$. Therefore, we might use the left-hand side of (1.1) to define the *distributional Jacobian* of u:

$$\mathrm{J}u := \frac{1}{k}\mathrm{d}\left(\sum_{i=1}^k (-1)^{i+1} u^i \widehat{\mathrm{d}u^i}\right) \quad \text{for } u \in (L^\infty \cap W^{1,k-1})(\mathbb{R}^d, \mathbb{R}^k). \tag{1.2}$$

The rôle of the distributional Jacobian in connection with relaxation problems in the calculus of variations has been pointed out, for instance, by Ball [3] (distributional determinant in non-linear elasticity) and by Brezis, Coron and Lieb [14] (harmonic maps and minimal connections; see also Bethuel, Brezis and Coron [7]).

As a consequence of its definition (1.2), the Jacobian enjoys weak compactness properties. For instance, if $(u_j)_{j\in\mathbb{N}}$ is a sequence of maps such that

$$\begin{aligned} &\sup_{j\in\mathbb{N}} \|u_j\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^k)} < +\infty \quad \text{and} \\ &u_j \to u \text{ strongly in } W^{1,k-1}(\mathbb{R}^d,\mathbb{R}^k), \end{aligned} \tag{1.3}$$

then $\mathrm{J}u_j \rightharpoonup \mathrm{J}u$ in the distributional sense of $\mathscr{D}'(\mathbb{R}^d)$. The same conclusion holds if

$$\begin{aligned} &\sup_{j\in\mathbb{N}} \|u_j\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^k)} < +\infty \quad \text{and} \\ &u_j \to u \textit{ weakly} \text{ in } W^{1,p}(\mathbb{R}^d,\mathbb{R}^k) \text{ for } p > k-1. \end{aligned} \tag{1.4}$$

A quantitative continuity estimate for the Jacobian was proved by Brezis and Nguyen.

Theorem 1.1 ([16, Theorem 1]). *Let $d = k \geq 2$, $k-1 \leq p \leq +\infty$, and let $1 \leq q \leq +\infty$ be such that $(k-1)/p + 1/q = 1$. Then, for any u, $v \in (L^q \cap W^{1,p}_{\mathrm{loc}})(\mathbb{R}^k, \mathbb{R}^k)$ and any C^1 function $\varphi\colon \mathbb{R}^k \to \mathbb{R}$ supported in a ball $B \subseteq \mathbb{R}^k$, there holds*

$$|\langle \mathrm{J}u - \mathrm{J}v,\, \varphi\rangle| \leq C \|u - v\|_{L^q(B)} \left(\|\nabla u\|^{k-1}_{L^p(B)} + \|\nabla v\|^{k-1}_{L^p(B)}\right) \|\nabla\varphi\|_{L^\infty(B)}$$

where $C > 0$ is a constant that only depends on k.

Another important feature of the Jacobian is its ability to capture topological information. To understand why this is the case, we introduce the $(k-1)$-form

$$\omega_{\mathbb{S}^{k-1}}(y) := \frac{1}{k}\sum_{i=1}^{k} (-1)^{i+1} y^i \widehat{\mathrm{d}y^i} \qquad \text{for } y \in \mathbb{R}^k,$$

which is (the 1-homogeneous extension of) a volume form on $\mathbb{S}^{k-1}$. The cohomology class of $\omega_{\mathbb{S}^{k-1}}$, restricted to $\mathbb{S}^{k-1}$, generates the de Rham cohomology $H^{k-1}_{\mathrm{dR}}(\mathbb{S}^{k-1}) \simeq \mathbb{Z}$. Then, we may rewrite (1.2) as

$$\mathrm{J}u = \mathrm{d}u^*(\omega_{\mathbb{S}^{k-1}}). \tag{1.5}$$

Consider now a sphere-valued map $u\colon \mathbb{R}^k \to \mathbb{S}^{k-1}$, possibly with point singularities (e.g. $u(x) := x/|x|$), and let $B \subseteq \mathbb{R}^k$ be a ball whose boundary ∂B does not intersect any singularity of u. By formally integrating the identity (1.5) on B and applying Stokes' theorem, we obtain

$$\int_B \mathrm{J}u = \int_B \mathrm{d}u^*(\omega_{\mathbb{S}^{k-1}}) = \int_{\partial B} u^*(\omega_{\mathbb{S}^{k-1}}) = \alpha_k \deg(u,\, \partial B,\, \mathbb{S}^{k-1}),$$

where $\alpha_k = \mathrm{Vol}(\mathbb{S}^{k-1})/k$ is the volume of the unit ball of $\mathbb{R}^k$ and $\deg(u,\, \partial B,\, \mathbb{S}^{k-1})$ denotes the topological degree of $u_{|\partial B}\colon \partial B \to \mathbb{S}^{k-1}$. More precisely, we have the following property which was proven in [14]: suppose that $u \in W^{1,k-1}_{\mathrm{loc}}(\mathbb{R}^k,\, \mathbb{S}^{k-1})$ is smooth except for a finite number of points $x_1, \ldots, x_p$. Then, there holds

$$\mathrm{J}u = \alpha_k \sum_{i=1}^{p} d_i \delta_{x_i} \qquad \text{in } \mathscr{D}'(\mathbb{R}^k), \tag{1.6}$$

where $d_i \in \mathbb{Z}$ denotes the topological degree of u restricted to a small sphere around the point x_i. To prove this formula, one can approximate u with a sequence of smooth maps $u_\varepsilon\colon \mathbb{R}^k \to \mathbb{R}^k$ such that $u = u_\varepsilon$ out of small balls $B_\varepsilon(x_i)$ around the singularites. By constructing suitable approximations, one can compute $\mathrm{J}u_\varepsilon$ using Stokes' theorem as above, and make sure that $u_\varepsilon \to u$ strongly in $W^{1,k-1}$, so to pass to the limit using the continuity of J, (1.3). We refer the reader to [12] and references therein for a comprehensive treatment of the relation between the jacobian and the topological degree.

In a similar spirit, if $d \geq k \geq 2$ and $u \in W^{1,k-1}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{S}^{k-1})$ is smooth out of a smoothly embedded, closed, oriented $(d-k)$-manifold $M \subseteq \mathbb{R}^d$, then the distributional Jacobian $\mathrm{J}u$ may be identified with a vector-valued measure supported on M. Indeed, we have (see [37])

$$\star\mathrm{J}u = \alpha_k \Delta \boldsymbol{\tau}_M \mathscr{H}^{d-k} \llcorner M, \tag{1.7}$$

where Δ is an integer number and denotes the topological degree of u restricted to the boundary of a k-disk that intersects transversally M, while $\boldsymbol{\tau}_M$ is a unit, tangent $(d-k)$-vector field that orients M. Moreover,

$$\star\colon \Lambda^k\mathbb{R}^d \to \Lambda_{d-k}\mathbb{R}^d$$

is (a variant of) the Hodge star duality operator: for a k-covector ω, $\star\omega$ is defined as the unique $(d-k)$-vector such that

$$\langle \tau, \star\omega\rangle = \langle \omega \wedge \tau, \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_d\rangle \qquad \text{for any } (d-k)\text{-covector } \tau,$$

where $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ is a positively oriented, orthonormal basis for $\mathbb{R}^d$. If u is smooth, $\star\mathrm{J}u(x)$ is a simple $(d-k)$-vector that spans the kernel of the multilinear form $\mathrm{J}u(x) = \mathrm{d}u^1(x) \wedge \ldots \wedge \mathrm{d}u^k(x)$, i.e. the tangent space to the level surface of u at x.

We will come back to the link between Jacobian and level sets, which is made precise by the coarea formula, in Section 1.4 below. For the time being, we consider an example. Let $u\colon \mathbb{R}^k \to \mathbb{S}^{k-1}$ be defined by $u(x) := x/|x|$ for $x \in \mathbb{R}^k \setminus \{0\}$. This map has an isolated singularity at the origin, which coincides with the support of the distributional Jacobian by (1.6), but is also the boundary of any level set $u^{-1}(y)$, for $y \in \mathbb{S}^{k-1}$. This is no coincidence, and in fact the distributional Jacobian of $u \in W^{1,k-1}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{S}^{k-1})$, for $d \geq k$, may be characterised as the boundary of a generic level set $u^{-1}(y)$, for $y \in \mathbb{S}^{k-1}$ [1, Theorem 3.8]. This fact, combined with the boundary rectifiability theorem by Federer and Fleming [23], implies the following rectifiability result: if $d \geq k \geq 2$, $u \in W^{1,k-1}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{S}^{k-1})$, and *if* $\mathrm{J}u$ is a bounded measure, then $\mathrm{J}u$ may be written in the form (1.7), where M is a $(d-k)$-rectifiable set with orientation $\boldsymbol{\tau}_M$, and Δ is an integer-valued multiplicity function (see [37, Theorem 1.1] and [1, Theorem 5.6]).

1.3. **Jacobians and density of smooth maps in sphere-valued Sobolev spaces.** Let B^k denote the open unit ball in $\mathbb{R}^k$. The distributional Jacobian of $u \in W^{1,k-1}(B^k, \mathbb{S}^{k-1})$ is an obstruction to the strong approximability of u by smooth maps. Indeed, the Jacobian determinant of a smooth map $\varphi\colon B^k \to \mathbb{S}^{k-1}$ vanishes, because all the derivatives of φ are tangent to the sphere; therefore, by the continuity of the Jacobian (1.3), if u belongs to the strong $W^{1,k-1}$-closure of $(C^\infty \cap W^{1,k-1})(B^k, \mathbb{S}^{k-1})$ then $\mathrm{J}u = 0$.

The condition $\mathrm{J}u = 0$ turns out to be sufficient for strong approximability by smooth maps, and moreover, smooth maps are sequentially weakly dense in

$W^{1,k-1}(B^k, \mathbb{S}^{k-1})$: see [4] and, for the case $W^{1,p}(B^k, \mathbb{S}^{k-1})$ with $k-1 < p < k$, [10]. (These results do *not* generalise to higher-dimensional domains: for instance, smooth maps are not sequentially weakly dense in $W^{1,3}(B^4, \mathbb{S}^2)$ [6].) The proof of these facts is based on "removal of the singularities" via the so-called "dipole construction", as in [7]. We illustrate here the main ideas through an example.

Let us consider the map $u(x) := x/|x|$, and let $N := (1, 0, \ldots, 0)$. For $\varepsilon > 0$, we can construct a family of smooth maps $\varphi_\varepsilon \colon \partial B^k \to \mathbb{S}^{k-1}$ such that $\varphi_\varepsilon(x) = u(x)$ if $x^1 \leq 1 - \varepsilon$, $\deg(\varphi_\varepsilon, \partial B^k, \mathbb{S}^{k-1}) = 0$ and $|\nabla\varphi_\varepsilon| \leq C\varepsilon^{-1}$, where C is an ε-independent constant. These maps may be obtained by inserting a patch of degree -1 in a small geodesic disk, of radius proportional to ε, around the pole N of ∂B^k. The sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $W^{1,k-1}(\partial B^k)$, and because the topological degree of each φ_ε is zero, we can find a smooth extension $\psi_\varepsilon \colon \overline{B}^k \to \mathbb{S}^{k-1}$ of φ_ε such that

$$\|\nabla\psi_\varepsilon\|_{L^{k-1}(B^k)} \leq C.$$

Then, we define

$$u_\varepsilon(x) := \begin{cases} \varphi_\varepsilon\left(\dfrac{x}{|x|}\right) & \text{if } \varepsilon < |x| < 1 \\ \psi_\varepsilon\left(\dfrac{x}{\varepsilon}\right) & \text{if } |x| \leq \varepsilon. \end{cases}$$

The map u_ε is Lipschitz, and up to subsequences, we have the convergence

$$|\nabla u_\varepsilon|^{k-1}\,\mathrm{d}x \rightharpoonup^* |\nabla u|^{k-1}\,\mathrm{d}x + \alpha\mathscr{H}^1 \llcorner L \qquad \text{weakly}^* \text{ as measures as } \varepsilon \to 0,$$

where L is the line segment between the origin and N, and α is a positive number that depends on the sequence $(\varphi_\varepsilon)_{\varepsilon>0}$. In this example, we have "removed the singularity" at the origin by introducing an opposite singularity N at the boundary, then connecting the two along a straight line.

The defect measure (i.e. the weak limit of $|\nabla u_\varepsilon|^{k-1}\,\mathrm{d}x - |\nabla u|^{k-1}\,\mathrm{d}x$) can be characterised in terms of the Jacobian of u. We consider the case $k = 3$. By inspecting the definition (1.2), we see that $\langle \mathrm{J}u, \varphi\rangle$ is well-defined for any Lipschitz test function φ, so we can consider the dual norm:

$$\|\mathrm{J}u\|_{\mathrm{flat}} := \sup\left\{\langle \mathrm{J}u, \varphi\rangle \colon \varphi \in W^{1,\infty}_0(B^3, \mathbb{R}),\ \|\nabla\varphi\|_{L^\infty(B^3)} \leq 1\right\}.$$

If u is smooth outside a finite number of points $x_1, \ldots, x_p \in B^3$ and the topological degree of $u_{|\partial B^3}$ is zero then, using (1.6), we can write

$$\mathrm{J}u = \frac{4\pi}{3}\sum_{i=1}^{\ell}\delta_{p_i} - \frac{4\pi}{3}\sum_{i=1}^{\ell}\delta_{n_i}$$

where $(p_i, n_i)_{i=1}^{\ell}$ are points in B^3 (possibly with repetitions). Brezis, Coron and Lieb [14, Section IV] showed that

$$\begin{aligned} \|\mathrm{J}u\|_{\mathrm{flat}} &= \frac{4\pi}{3}\sup_{\varphi\colon \|\nabla\varphi\|_{L^\infty(B^3)}\leq 1}\left(\sum_{i=1}^{\ell}\varphi(p_i) - \sum_{i=1}^{\ell}\varphi(n_i)\right) \\ &= \frac{4\pi}{3}\inf_{\sigma}\sum_{i=1}^{\ell}\left|p_i - n_{\sigma(i)}\right|, \end{aligned}$$

where the infimum is taken over all permutations σ of $\{1, \ldots, \ell\}$. Because of this interpretation, $\|\mathrm{J}u\|_{\mathrm{flat}}$ was called the minimal connection for $\mathrm{J}u$ in [14].

Theorem 1.2 ([7]). *Let $u \in W^{1,2}(B^3, \mathbb{S}^2)$ be such that $u_{|\partial B^3} \in (C^0 \cap W^{1,2})(\partial B^3, \mathbb{S}^2)$ and $\deg(u, \partial B^3, \mathbb{S}^2) = 0$. Then, there holds*

$$\inf_{(u_j)_{j\in\mathbb{N}}} \liminf_{j\to+\infty} \frac{1}{2}\int_{B^3} |\nabla u_j|^2 = \frac{1}{2}\int_{B^3} |\nabla u|^2 + 3\,\|\mathrm{J}u\|_{\mathrm{flat}},$$

where the infimum is taken over all sequences of maps $u_j \in C^0(\overline{B}^3, \mathbb{S}^2)$ such that $u_j = u$ on ∂B^3 and $u_j \rightharpoonup u$ weakly in $W^{1,2}$.

More general relaxation results (higher dimensional domains, other target manifolds) have been obtained by Giaquinta, Modica and Souček [26] using the language and the framework of Cartesian currents.

1.4. **The oriented coarea formula.**

In Section 1.2, we have described some properties of the distributional Jacobian of a sphere-valued map $u \in W^{1,k-1}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{S}^{k-1})$. It turns out that the study of $\mathrm{J}u$ when $u \in (L^\infty \cap W^{1,k-1}_{\mathrm{loc}})(\mathbb{R}^d, \mathbb{R}^k)$ can be reduced to the previuous case. Indeed, for $y \in \mathbb{R}^k$, we define the map $u_y \colon \mathbb{R}^d \to \mathbb{S}^{k-1}$ by

$$u_y(x) := \frac{u(x) - y}{|u(x) - y|} \qquad \text{for } x \in \mathbb{R}^d \setminus u^{-1}(y).$$

If u is smooth and y is a regular value of u, then by the discussion of Section 1.2 we might expect $\mathrm{J}u_y$ to be a unit multiplicity rectifiable current supported on the smooth $(d-k)$-manifold $u^{-1}(y)$. The following property, sometimes referred to as the *oriented coarea formula*, relates $\mathrm{J}u_y$ and $\mathrm{J}u$.

Theorem 1.3 ([37, Theorem 1.2], [1]). *Let $d \geq k \geq 2$, and let $u \in (L^\infty \cap W^{1,k-1}_{\mathrm{loc}})(\mathbb{R}^d, \mathbb{R}^k)$. Then, for a.e. $y \in \mathbb{R}^k$ we have $u_y \in W^{1,k-1}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{S}^{k-1})$, $\mathrm{J}u_y$ is supported on a $(d-k)$-rectifiable set, and there holds*

$$\mathrm{J}u = \frac{1}{\alpha_k}\int_{\mathbb{R}^k} \mathrm{J}u_y \,\mathrm{d}y$$

in the sense of distributions. Here α_k denotes the volume of the unit ball in $\mathbb{R}^k$.

To pave the way for the discussion in Section 2, it will be useful to recall here why we have $u_y \in W^{1,k-1}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{S}^{k-1})$ for a.e. $y \in \mathbb{R}^k$. This proof is based on a trick that was used by Hardt, Kinderlehrer and Lin [30, Lemma 2.3]. The chain rule implies that $|\nabla u_y| \leq 2|u-y|^{-1}|\nabla u|$. W.l.o.g., we might restrict our attention to the case $|y| \leq M := \|u\|_{L^\infty(\mathbb{R}^d)} + 1$. By integrating over y in the ball $B^k_M \subseteq \mathbb{R}^k$ of radius M, and letting $B^d \subseteq \mathbb{R}^d$ be a ball, we obtain

$$\begin{aligned}\int_{B^k_M} \|\nabla u_y\|^{k-1}_{L^{k-1}(B^d)} \,\mathrm{d}y &\leq 2\int_{B^k_M}\left(\int_{B^d}\frac{|\nabla u(x)|^{k-1}}{|u(x)-y|^{k-1}}\,\mathrm{d}x\right)\mathrm{d}y \\ &= 2\int_{B^d}|\nabla u(x)|^{k-1}\left(\int_{B^k_M}\frac{\mathrm{d}y}{|u(x)-y|^{k-1}}\right)\mathrm{d}x \\ &\leq 2\int_{B^d}|\nabla u(x)|^{k-1}\left(\int_{B^k_{2M}}\frac{\mathrm{d}z}{|z|^{k-1}}\right)\mathrm{d}x \\ &=: C_{k,M}\,\|\nabla u\|^{k-1}_{L^{k-1}(B^d)}\end{aligned}$$

We have made the change of variable $z = u(x) - y$ in the inner integral, and used the fact that $z \mapsto |z|^{-p}$ is locally integrable on $\mathbb{R}^k$ for $p < k$. The constant $C_{k,M}$ depends also on M, hence on $\|u\|_{L^\infty(\mathbb{R}^d)}$.

1.5. **Applications to variational problems.** The theory of distributional Jacobians can be applied to the asymptotic analysis, as $\varepsilon \to 0$, of variational problems of the form (GL$_\varepsilon$). Let $\Omega \subseteq \mathbb{R}^d$ be a bounded, Lipschitz domain. For $1 \leq p < +\infty$, we define $W^{-1,p}(\Omega,\, \Lambda_{d-2}\mathbb{R}^d)$ as the dual of $W^{1,p'}_0(\Omega,\, \Lambda^{d-2}\mathbb{R}^d)$, where $p' := p/(p-1)$ is the Hölder conjugate of p. We have

Theorem 1.4 ([36, 2])**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded, Lipschitz domain with $d \geq 2$, and let $K > 0$ be a fixed constant. Then, the following properties hold.*

(i) Compactness and lower bound. *For any sequence $u_\varepsilon \in W^{1,2}(\Omega, \mathbb{C})$ such that $E^{\mathrm{GL}}_\varepsilon(u_\varepsilon) \leq K|\log\varepsilon|$, there exists a (non relabelled) subsequence and a $(d-2)$-current J such that $\star\mathrm{J}u_\varepsilon \to \pi J$ in $W^{-1,p}(\Omega,\, \Lambda_{d-2}\mathbb{R}^d)$ for every $p < d/(d-1)$. The current J has the structure of a $(d-2)$-rectifiable boundary in Ω with finite mass $|J|(\Omega) < +\infty$ and integer multiplicity. Moreover,*

$$\liminf_{\varepsilon\to 0}\frac{E^{\mathrm{GL}}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \geq \pi|J|(\Omega).$$

(ii) Upper bound. *For any $(d-2)$-rectifiable boundary J in Ω with finite mass and integer multiplicity, there exists a sequence $u_\varepsilon \in W^{1,2}(\Omega, \mathbb{C})$ such that $\star\mathrm{J}u_\varepsilon \to \pi J$ in $W^{-1,p}(\Omega,\, \Lambda_{d-2}\mathbb{R}^d)$ for every $p < d/(d-1)$ and*

$$\lim_{\varepsilon\to 0}\frac{E^{\mathrm{GL}}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} = \pi|J|(\Omega).$$

If the u_ε's are critical points of $E^{\mathrm{GL}}_\varepsilon$ with $E^{\mathrm{GL}}_\varepsilon(u_\varepsilon) \leq K|\log\varepsilon|$, and under suitable assumptions on the boundary data, the bounds on $\mathrm{J}u_\varepsilon$ make it possible to obtain compactness for the u_ε's themselves, by PDE arguments [9]. In this case, we have $u_\varepsilon \to u_0$ in $W^{1,p}$ for $p < d/(d-1)$, and $\pi J = \lim_\varepsilon \star\mathrm{J}u_\varepsilon = \star\mathrm{J}u_0$.

Ginzburg-Landau type functionals of k-growth in the gradient (i.e., the term $|\nabla u|^2$ in (GL$_\varepsilon$) is replaced by $|\nabla u|^k$, with $k \geq 2$ an integer) and Dirichlet boundary conditions have also been studied [2]. In this case, the $\mathrm{J}u_\varepsilon$'s concentrate on a rectifiable set of codimension k, whose cobordism class is determined by the domain and the boundary condition. Other energy regimes arise naturally for Ginzburg-Landau type functionals and are interesting for applications. In particular the energy regime $E_\varepsilon(u_\varepsilon) \approx |\log \varepsilon|^2$ corresponds to the onset of the mixed phase in type-II superconductors, and to the appearance of vortices in Bose-Einstein condensates. These situations have been extensively studied in the two-dimensional case, especially by Sandier and Serfaty in the case of superconductivity (see [45] and references therein).

2. Manifold-valued Sobolev maps and topological singularities

2.1. **Motivation: variational problems for material science.** There are other functionals, arising as variational models for material science, which share a common structure with the Ginzburg-Landau functional (GL_ε), i.e. they can be written in the form

$$u \in W^{1,k}(\Omega, \mathbb{R}^m) \mapsto E_\varepsilon(u) := \int_\Omega \left\{ \frac{1}{k} |\nabla u|^k + \frac{1}{\varepsilon^2} f(u) \right\}. \tag{2.1}$$

Here $f\colon \mathbb{R}^m \to \mathbb{R}$ is a non-negative, smooth potential that satisfies suitable coercivity and non-degeneracy conditions, and $\mathscr{N} := f^{-1}(0)$ is assumed to be a non-empty, smoothly embedded, compact, connected submanifold of $\mathbb{R}^m$ without boundary. The elements of $\mathscr{N}$ correspond to the ground states for the material, i.e. the local configurations that are most energetically convenient. An important example is the Landau-de Gennes model for nematic liquid crystals (in the so-called one-constant approximation of the uniaxial phase, see e.g. [22]). In this case, $k = 2$ and the distinguished manifold is a real projective plane $\mathscr{N} = \mathbb{R}\mathrm{P}^2$, whose elements describe the locally preferred direction of alignment of the constituent molecules (which might be schematically described as un-oriented rods).

As in the Ginzburg-Landau case, topological obstructions may imply the *lack* of an extension operator $W^{1-1/k,k}(\partial\Omega, \mathscr{N}) \to W^{1,k}(\Omega, \mathscr{N})$. As a consequence, minimisers u_ε subject to a Dirichlet boundary condition $u_\varepsilon = u_{\mathrm{bd}} \in W^{1-1/k,k}(\partial\Omega, \mathscr{N})$ may not satisfy uniform energy bounds with respect to ε. Compactness results in the spirit of the Ginzburg-Landau theory have been shown for minimisers of the Landau-de Gennes functional [39, 28, 17, 18]. However, some points that are understood in the Ginzburg-Landau theory — for instance, a variational characterisation of the singular set of the limit — are still missing, even for the Landau-de Gennes functional.

Unfortunately, the theory of Jacobians does not carry over directly to this setting. Consider the following simple example: let S be a $(d-k)$-plane that intersects Ω, and let $u\colon \Omega \setminus S \to \mathscr{N}$ be a map that is smooth everywhere, except at S. Then, each point of S can be encircled by a $(k-1)$-dimensional sphere $\Sigma \subseteq \Omega \setminus S$, and the (based) homotopy class of $u_{|\Sigma}\colon \Sigma \to \mathscr{N}$ defines an element of $\pi_{k-1}(\mathscr{N})$ which, roughly speaking, characterises the behaviour of the material configuration around the defect. (This is the basic idea of the topological classification of defects in ordered materials; see e.g. [40] for more details.) If $\pi_{k-1}(\mathscr{N})$ contains elements of finite order, these cannot be realised by integration of a differential form, so no notion of Jacobian that can be expressed as a differential form (such as (1.2)) is able to capture such homotopy classes of defects.

In the following sections, our aim is to construct an object that (i) brings topological information and (ii) enjoys compactness properties even when the distributional

Jacobian is not defined, in particular when $\pi_{k-1}(\mathscr{N})$ contains elements of finite order. A notion of "set of topological singularities" for a manifold-valued Sobolev map was already introduced by Pakzad and Rivière [42], using the language of flat chains. In [19], we carried out a different construction which we will sketch below; however, since the latter is also based on the formalism of flat chains, we first recall some basic definitions and facts about flat chains, following the approach in [48, 24, 47].

2.2. **Flat chains with coefficients in an abelian group.** Let $(\mathbf{G}, |\cdot|)$ be a normed abelian group, that is, an abelian group together with a non-negative function $|\cdot|\colon \mathbf{G} \to [0, +\infty)$ that satisfies

(i) $|g| = 0$ if and only if $g = 0$
(ii) $|-g| = |g|$ for any $g \in \mathbf{G}$
(iii) $|g + h| \leq |g| + |h|$ for any $g, h \in \mathbf{G}$.

In addition, we assume that

$$|g| \geq 1 \qquad \text{for any } \mathbf{G} \setminus \{0\}. \tag{2.2}$$

For $n \in \mathbb{Z}$, $1 \leq n \leq d$, a polyhedral n-chain with coefficients in $\mathbf{G}$ is a linear combination, with coefficients in $\mathbf{G}$, of compact, convex, oriented n-dimensional polyhedra in $\mathbb{R}^d$, modulo a suitable equivalence relation $\sim$. We define $\sim$ by requiring $-\sigma \sim \sigma'$ if the polyhedra σ' and σ only differ for the orientation, and $\sigma \sim \sigma_1 + \sigma_2$ if σ is obtained by gluing σ_1, σ_2 along a common face (with the correct orientation). The set of polyhedral n-chain with coefficients in $\mathbf{G}$ is a group, with a naturally defined addition operation, and is denoted $\mathbb{P}_n(\mathbb{R}^d; \mathbf{G})$. Every element $S \in \mathbb{P}_n(\mathbb{R}^d; \mathbf{G})$ can be represented as a finite sum

$$S = \sum_{i=1}^{p} g_i [\![\sigma_i]\!], \tag{2.3}$$

where $g_i \in \mathbf{G}$, the σ_i's are compact, convex, non-overlapping n-dimensional polyhedra, and $[\![\cdot]\!]$ denotes the equivalence class modulo the relation $\sim$ defined above. Thus, S may be identified with a finite collection of polyhedra as above, endowed with multiplicities in $\mathbf{G}$.

Polyhedral chains enjoy a notion of boundary: the boundary is a linear operator $\partial\colon \mathbb{P}_n(\mathbb{R}^d; \mathbf{G}) \to \mathbb{P}_{n-1}(\mathbb{R}^d; \mathbf{G})$, identified by its actions on single polyhedra, which satisfies $\partial(\partial S) = 0$ for any chain S. The mass of a polyhedral chain $S \in \mathbb{P}_n(\mathbb{R}^d; \mathbf{G})$, presented in the form (2.3), is defined by $\mathbb{M}(S) := \sum_i |g_i| \mathscr{H}^n(\sigma_i)$. The flat norm of a polyhedral n-dimensional chain S is defined by

$$\begin{aligned} \mathbb{F}(S) := \inf\Big\{ &\mathbb{M}(P) + \mathbb{M}(Q)\colon P \in \mathbb{P}_{n+1}(\mathbb{R}^d; \mathbf{G}), \\ &Q \in \mathbb{P}_n(\mathbb{R}^d; \mathbf{G}),\ S = \partial P + Q \Big\}. \end{aligned}$$

Thus, two chains S_1, S_2 are close with respect to the flat norm if $S_2 - S_1$ is, up to small errors, the boundary of a chain of small mass. It can be showed (see e.g. [24, Section 2]) that $\mathbb{F}$ indeed defines a norm on $\mathbb{P}_n(\mathbb{R}^d; \mathbf{G})$, in such a way that the group operation on $\mathbb{P}_n(\mathbb{R}^d; \mathbf{G})$ is $\mathbb{F}$-Lipschitz continuous. The completion of $(\mathbb{P}_n(\mathbb{R}^d; \mathbf{G}), \mathbb{F})$, as a metric space, will be denoted $\mathbb{F}_n(\mathbb{R}^d; \mathbf{G})$. It can be given the structure of a $\mathbf{G}$-module, and it is called the group of flat n-chain with coefficients in $\mathbf{G}$. Moreover, the mass $\mathbb{M}$ extends to a $\mathbb{F}$-lower semi-continuous functional $\mathbb{F}_n(\mathbb{R}^d;\, \mathbf{G}) \to [0,\, +\infty]$, still denoted $\mathbb{M}$, and it remains true that

$$\begin{aligned} \mathbb{F}(S) = \inf\Big\{ \mathbb{M}(P) + \mathbb{M}(Q) \colon P \in \mathbb{F}_{n+1}(\mathbb{R}^d;\, \mathbf{G}), \\ Q \in \mathbb{F}_n(\mathbb{R}^d;\, \mathbf{G}),\ S = \partial P + Q \Big\} \end{aligned} \tag{2.4}$$

for any $S \in \mathbb{F}_n(\mathbb{R}^d;\, \mathbf{G})$ [24, Theorem 3.1].

A flat chain S is said to be supported in a closed set $K \subseteq \mathbb{R}^d$ if, for any open set $U \supseteq K$, S is the $\mathbb{F}$-limit of a sequence of polyhedral chains supported in U. If M is a smooth n-dimensional manifold, respectively a n-rectifiable set, then we can define a chain $[\![M]\!]$ supported on M with constant multiplicity $1 \in \mathbf{G}$ by approximating M with polyhedral sets, considering the associated polyhedral chains (with unit multiplicity), and passing to the limit in the flat norm. The chain $[\![M]\!]$ is an example of a *smooth*, respectively, *rectifiable* chain. More generally, Equation (2.4) shows that the boundary of a n-rectifiable chain of finite mass is a $(n-1)$-flat chain; for instance, the "Koch's snowflake", which is a planar set of Hausdorff dimension greater than 1 that bounds a finite area, can be seen as the support of a 1-dimensional flat chain. In fact, under the assumption (2.2), *any* $(n-1)$-flat chain has the form (boundary of a rectifiable n-chain) + (rectifiable $(n-1)$-chain) [24, 47].

In case $\mathbf{G} = \mathbb{Z}$, rectifiable chains may be identified with rectifiable currents with integer multiplicity, by integration. The class of n-chains of finite mass with coefficients in $\mathbb{Z}$ may be interpreted as bounded measures with values in the space of n-vectors, and in general flat n-chains with coefficients in $\mathbb{Z}$ may be regarded as elements of $W^{1,\infty}_0(\mathbb{R}^d,\, \Lambda^n\mathbb{R}^d)'$.

Finally, we define the group of flat n-chains relative to an open set $\Omega \subseteq \mathbb{R}^d$ as the quotient group

$$\mathbb{F}_n(\Omega;\, \mathbf{G}) := \mathbb{F}_n(\mathbb{R}^d;\, \mathbf{G})/\{S \in \mathbb{F}_n(\mathbb{R}^d;\, \mathbf{G}) \colon S \text{ is supported in } \mathbb{R}^d \setminus \Omega\}.$$

The quotient norm may equivalently be rewritten as

$$\begin{aligned} \mathbb{F}_\Omega(S) := \inf \big\{ \mathbb{M}(P \llcorner \Omega) + \mathbb{M}(Q \llcorner \Omega) \colon P \in \mathbb{F}_{n+1}(\mathbb{R}^d;\, \mathbf{G}), \\ Q \in \mathbb{F}_n(\mathbb{R}^d;\, \mathbf{G}),\ S - \partial P - Q \text{ is supported in } \mathbb{R}^d \setminus \Omega \big\} \end{aligned} \tag{2.5}$$

where $P \llcorner \Omega$ denotes the restriction of P to Ω (see [19, Section 2] for more details).

2.3. **Sketch of the construction.** Let $\mathscr{N} \subseteq \mathbb{R}^m$ be a smoothly embedded manifold without boundary; let $k \geq 2$ be an integer. We make the following assumption on $\mathscr{N}$ and k:

(H) $\mathscr{N}$ is compact and $(k-2)$-connected, that is $\pi_0(\mathscr{N}) = \pi_1(\mathscr{N}) = \ldots = \pi_{k-2}(\mathscr{N}) = 0$. In case $k = 2$, we also assume that $\pi_1(\mathscr{N})$ is abelian.

The integer k is thus related to the topology of $\mathscr{N}$. The condition (H) guarantees that $k \leq \dim \mathscr{N} + 1$ and in case $\mathscr{N}$ is a sphere, we can indeed choose $k = \dim \mathscr{N} + 1$; however, the inequality may be strict in general. For instance, if $\mathscr{N}$ is a real projective plane, $\mathscr{N} \simeq \mathbb{R}\mathrm{P}^2$, then (H) is satisfied if and only if $k = 2$. Under the assumption (H), there is no topological obstruction associated with defects of codimension $< k$; $\mathscr{N}$-valued maps may have singularities of codimension $< k$, but these can be removed by local surgery. On the other hand, singularities of codimension k (or higher) may be associated with topological obstructions, and are classified by elements of $\pi_{k-1}(\mathscr{N})$. As a consequence of (H), the group $\pi_{k-1}(\mathscr{N})$ is abelian and may be endowed with a norm that satisfies (2.2) (see e.g. [19, Section 2.2]). It will be the coefficient group for our flat chains.

Let $d \geq k \geq 2$, and let $\Omega \subseteq \mathbb{R}^d$ be a bounded, smooth domain. The "set of topological singularities" of a map $u \in W^{1,k-1}(\Omega,\, \mathscr{N})$ has been constructed by Pakzad and Rivière [42], as a flat chain, by approximating u with maps $\tilde{u}\colon \mathbb{R}^d \to \mathscr{N}$ having "nice singularities", i.e. $\tilde{u}$ is smooth out of a polyhedral set. In [19], this construction was carried over using a different approach, namely, approximating u with smooth maps $v\colon \mathbb{R}^d \to \mathbb{R}^m$ and then reprojecting v onto $\mathscr{N}$. This approach is close in spirit to that presented in Section 1.4.

While it is impossible to construct a smooth projection of $\mathbb{R}^m$ onto a closed manifold $\mathscr{N}$, under the assumption (H) it is possible to construct a smooth projection $\varrho\colon \mathbb{R}^m \setminus \mathscr{X} \to \mathscr{N}$, where $\mathscr{X}$ is a finite union of manifolds of dimension $\leq m-k$. Moreover, we can make sure that

$$|\nabla\varrho(y)| \leq \frac{C}{\mathrm{dist}(y,\, \mathscr{X})} \qquad \text{for any } y \in \mathbb{R}^m \setminus \mathscr{X}. \tag{2.6}$$

(In Section 1.4, we used the radial projection $\varrho\colon \mathbb{R}^k \setminus \{0\} \to \mathbb{S}^{k-1}$, $\varrho(y) := y/|y|$.) The existence of such ϱ was obtained by Hardt and Lin as a consequence of more general results of topology [31, Lemma 6.1]; self-contained approaches are presented in [13, 32].

Take $u \in C^\infty(\mathbb{R}^d,\, \mathbb{R}^m)$. One could be tempted to identify the set of topological singularities of u with $u^{-1}(\mathscr{X})$, which is exactly the set where the reprojection $\varrho(u)$ fails to be well-defined, but $u^{-1}(\mathscr{X})$ may be very irregular even if u is smooth. However, Thom transversality theorem implies that, for a.e. $y \in \mathbb{R}^m$, the set $(u-y)^{-1}(\mathscr{X})$ is indeed a finite union of (possibly disconnected) manifolds of dimension $\leq d-k$. For each $(m-k)$-manifold $K \subseteq \mathscr{X}$, we consider the inverse image $(u-y)^{-1}(K)\cap\Omega$ and equip it with a multiplicity, i.e. the homotopy class of $\varrho(u-y)$ around $(u-y)^{-1}(K)$, which is an element of $\pi_{k-1}(\mathscr{N})$. By doing so, we define a smooth chain $\mathbf{S}_y(u) \in \mathbb{F}_{d-k}(\Omega;\, \pi_{k-1}(\mathscr{N}))$. We have disregarded the contributions coming from manifolds $K \subseteq \mathscr{X}$ of dimension $< m-k$: this is because no $S \in \mathbb{F}_{d-k}(\Omega;\, \pi_{k-1}(\mathscr{N}))$ can be supported on a set of dimension $< d-k$, unless $S = 0$ [46, Theorem 3.1].

The chain $\mathbf{S}_y(u)$ satisfies the following topological property. Take a smoothly embedded, oriented k-disk $D \subseteq \Omega$, such that ∂D does not intersect $(u-y)^{-1}(\mathscr{X})$ (hence, $\varrho(u-y)$ is well defined on ∂D). Generically, D intersects the support of $\mathbf{S}_y(u)$ at a finite number of points. By summing up the multiplicities of $\mathbf{S}_y(u)$ at the intersection points, with a sign accounting for the relative orientations of D and $\mathbf{S}_y(u)$, we define the so-called intersection index, denoted $\mathbb{I}(\mathbf{S}_y(u),\, [\![D]\!]) \in \pi_{k-1}(\mathscr{N})$ (see e.g. [19, Section 2.1] for more details). Then, a simple topological argument shows that

$$\mathbb{I}(\mathbf{S}_y(u),\, [\![D]\!]) = \text{homotopy class of } \varrho(u-y) \text{ on } \partial D. \tag{2.7}$$

In this sense, the chain $\mathbf{S}_y(u)$ carries topological information on u.

Thanks to the estimate (2.6) on $\nabla\varrho$, we can now integrate over $y \in \mathbb{R}^m$ and apply a strategy similar to that devised by Hardt, Kinderlehrer and Lin (sketched in Section 1.4). In particular, by applying the coarea formula, we obtain a continuity estimate on $\mathbf{S}_y(u)$ depending on the Sobolev norms of u. Then, by density, one can define $\mathbf{S}_y(u)$ in case u is a Sobolev map.

We let $X := (L^\infty \cap W^{1,k-1})(\Omega, \mathbb{R}^m)$ and endow this set with a topology, in such a way that a sequence $(u_j)_{j\in\mathbb{N}}$ converges to u in X if and only if $u_j \to u$ strongly in $W^{1,k-1}$ and $\sup_{j\in\mathbb{N}} \|u_j\|_{L^\infty} < +\infty$. We also consider the set $Y := L^1(\mathbb{R}^m, \mathbb{F}_{d-k}(\Omega; \pi_{k-1}(\mathscr{N})))$, whose elements are measurable maps $y \in \mathbb{R}^m \mapsto S_y \in \mathbb{F}_{d-k}(\Omega; \pi_{k-1}(\mathscr{N}))$ such that

$$\|S\|_Y := \int_{\mathbb{R}^m} \mathbb{F}_\Omega(S_y)\,\mathrm{d}y < +\infty.$$

The set Y is a complete normed $\pi_{k-1}(\mathscr{N})$-modulus, with respect to the norm $\|\cdot\|_Y$.

Theorem 2.1 ([19]). *Suppose that* (H) *is satisfied. Then, there exists a unique continuous map* $\mathbf{S}\colon X \to Y$ *such that, for any* $u \in X \cap C^\infty(\Omega, \mathbb{R}^m)$*, a.e.* $y \in \mathbb{R}^m$*, and any smoothly embedded, oriented k-disk* $D \subseteq \Omega$ *such that* $\partial D \cap (u-y)^{-1}(\mathscr{X}) = \emptyset$*, the property* (2.7) *holds. In addition, for any* u_0*,* $u_1 \in X$ *and a.e.* $y \in \mathbb{R}^m$*, we can write* $\mathbf{S}_y(u_1) - \mathbf{S}_y(u_0) = \partial R_y$ *in* Ω*, where* R_y *is a* $(d-k+1)$*-chain that satisfies*

$$\begin{aligned}\int_{\mathbb{R}^m} \mathbb{M}(R_y)\,\mathrm{d}y \le C(\max\{\|u_0\|_{L^\infty(\Omega)}, \|u_1\|_{L^\infty(\Omega)}\}) \\ \cdot \int_\Omega |u_1 - u_0| \left(|\nabla u_1|^{k-1} + |\nabla u_0|^{k-1}\right)\end{aligned} \tag{2.8}$$

and $C\colon \mathbb{R}^+ \to \mathbb{R}^+$ *is a locally bounded function that only depends on* $\mathscr{N}$*,* k*,* ϱ*,* $\mathscr{X}$*, and* Ω*. Finally, if* $u \in W^{1,k-1}(\Omega, \mathscr{N})$ *then for a.e.* y*,* $y' \in \mathbb{R}^m$ *such that* $\max\{|y|, |y'|\} < \mathrm{dist}(\mathscr{N}, \mathscr{X})$ *there holds*

$$\mathbf{S}_y(u) = \mathbf{S}_{y'}(u). \tag{2.9}$$

Actually, Property (2.7) holds for any $u \in X$, provided that both sides of the identity are suitably defined (we refer to [19, Section 2 and Theorem 3.1]). The inequality (2.8), together with (2.5), implies the continuity estimate

$$\begin{aligned}\|\mathbf{S}(u_1) - \mathbf{S}(u_0)\|_Y \le C(\max\{\|u_0\|_{L^\infty(\Omega)}, \|u_1\|_{L^\infty(\Omega)}\}) \\ \cdot \int_\Omega |u_1 - u_0| \left(|\nabla u_1|^{k-1} + |\nabla u_0|^{k-1}\right),\end{aligned}$$

which is analougous to Theorem 1.1. In particular, we have stability of $\mathbf{S}$ with respect to strong and weak convergence, as in (1.3)–(1.4). Therefore, some of the compensation compactness properties that are typical of the Jacobian are retained by $\mathbf{S}$. By choosing u_0 equal to a constant (so that $\mathbf{S}_y(u_0) = 0$ for a.e. y), we also see that $\mathbf{S}_y(u_1)$ may be written as a relative boundary: $\mathbf{S}_y(u_1) = \partial R_y$ inside Ω, where R_y is a $(d-k+1)$-flat chain that satisfies

$$\int_{\mathbb{R}^k} \mathbb{M}(R_y)\,\mathrm{d}y \le C(\|u_1\|_{L^\infty(\Omega)})\, \|\nabla u_1\|^{k-1}_{L^{k-1}(\Omega)}\,. \tag{2.10}$$

In case u is $\mathscr{N}$-valued, (2.9) states that the map $y \mapsto \mathbf{S}_y(u)$ is locally constant around the origin; we denote its constant value by $\mathbf{S}^{\mathrm{PR}}(u)$. The chain $\mathbf{S}^{\mathrm{PR}}(u)$ coincides with the topological singular set as introduced by Pakzad and Rivière in [42].

In the special case $\mathscr{N} = \mathbb{S}^{k-1} \subseteq \mathbb{R}^k$, we have $\pi_{k-1}(\mathbb{S}^{k-1}) \simeq \mathbb{Z}$ and so $\mathbf{S}_y(u)$ has an alternative description in terms of currents. If we make the choice $\mathscr{X} = \{0\} \subseteq \mathbb{R}^k$ and $\varrho(y) = y/|y|$, then Theorem 1.3 implies

$$\mathrm{J}u = \frac{1}{\alpha_k} \int_{\mathbb{R}^m} \mathbf{S}_y(u)\, \mathrm{d}y \qquad \text{for any } u \in (L^\infty \cap W^{1,k-1})(\Omega,\, \mathbb{R}^k),$$

where α_k is the volume of the unit k-disk and the integral in the right-hand side is intended in the sense of distributions. However, if $\pi_{k-1}(\mathscr{N})$ is a finite group (or, more generally, if it only contains elements of finite order), then there is no meaningful way to define the integral of $\mathbf{S}_y(u)$ with respect to the Lebesgue measure $\mathrm{d}y$, as $\pi_{k-1}(\mathscr{N}) \otimes \mathbb{R} = 0$.

It is worth noticing that the proof of our main result, Theorem 2.1, does not strictly rely upon the manifold structure of $\mathscr{N}$. What is needed, is the existence and regularity of the exceptional set $\mathscr{X}$ and the retraction ϱ, in order to be able to apply Thom transversality theorem. This suggests a possible extension to more general targets $\mathscr{N} \subseteq \mathbb{R}^m$ such as, for instance, finite simplicial complexes.

2.4. **Applications: density of smooth maps in manifold-valued function spaces.** We can apply the operator $\mathbf{S}$ to tackle some questions in the theory of manifold-valued function spaces. Here, we focus in particular on issues related to (strong or weak) density of smooth functions in manifold-valued Sobolev or BV spaces.

Just as the distributional Jacobian, the operator $\mathbf{S}$ is as an obstruction to strong approximability by smooth maps. Bethuel [5] showed that smooth maps are dense in $W^{1,p}(B^d,\, \mathscr{N})$, where B^d is the open unit ball in $\mathbb{R}^d$, if and only if $\pi_{\lfloor p \rfloor}(\mathscr{N}) = 0$ or $p \geq d$. In Section 1.3, we have seen that the strong $W^{1,p}$-closure of $C^\infty(\overline{B}^k,\, \mathbb{S}^{k-1})$ can be characterised using the distributional Jacobian. Pakzad and Rivière [42, Theorem II] generalised this result to other target manifolds. As a corollary of our construction, we recover Pakzad and Rivière's result.

Theorem 2.2. *Let $d \geq 2$ be an integer, let $1 \leq p < d$, and let $\mathscr{N}$ be a compact, smooth, $(\lfloor p \rfloor - 1)$-connected manifold without boundary. In case $1 \leq p < 2$, we also suppose that $\pi_1(\mathscr{N})$ is abelian. Then, there exists a continuous map*

$$\mathbf{S}^{\mathrm{PR}} \colon W^{1,p}(B^d,\, \mathscr{N}) \to \mathbb{F}_{d-\lfloor p \rfloor -1}(B^d;\, \pi_{\lfloor p \rfloor}(\mathscr{N}))$$

such that $\mathbf{S}^{\mathrm{PR}}(u) = 0$ if and only if u is a strong $W^{1,p}$-limit of smooth maps $\overline{B}^d \to \mathscr{N}$.

In contrast with the result by Pakzad and Rivière, we do not need to impose the technical restriction $\lfloor p \rfloor \in \{1,\, d-1\}$. The arguments in [42] rely on fine results in Geometric Measure Theory [25] (which require $\lfloor p \rfloor \in \{1,\, d-1\}$); instead, the proof of Theorem 2.2 follows directly from our main Theorem 2.1 and in particular on the integral estimate (2.10) for the mass of the connection. This control is then combined with the "removal of the singularities" results in [42]. It is worth mentioning that the theorem may fail if the domain is not a disk (see the counterexamples in [29] and the discussion in [42]).

We next drive our attention to manifold-valued BV-maps. Recall that the space $\mathrm{BV}(\Omega, \mathbb{R}^m)$, by definition, consists of those functions $u \in L^1(\Omega, \mathbb{R}^m)$ whose distributional derivative $\mathrm{D}u$ is a finite Radon measure. We say that a sequence u_j of BV-functions converges weakly to u if and only if $u_j \to u$ strongly in L^1 and $\mathrm{D}u_j \rightharpoonup^* \mathrm{D}u$ weakly* as elements of the dual $C_0(\Omega, \mathbb{R}^m)'$. We define $\mathrm{BV}(\Omega, \mathscr{N})$ as the set of maps $u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$ such that $u(x) \in \mathscr{N}$ for a.e. $x \in \Omega$. The space $\mathrm{BV}(\Omega, \mathbb{S}^1)$ has been extensively studied by Davila and Ignat [21, 33] (see also [34] for the case $\mathscr{N} = \mathbb{R}\mathrm{P}^n$).

Theorem 2.3. *Let $\mathscr{N}$ be a smooth, compact, connected manifold without boundary, with abelian $\pi_1(\mathscr{N})$. Then, $C^\infty(\overline{B}^d, \mathscr{N})$ is sequentially weakly dense in $\mathrm{BV}(B^d, \mathscr{N})$.*

A similar result has been obtained by Giaquinta and Mucci [27, Theorem 2.13], who worked in the framework of Cartesian currents. Giaquinta and Mucci need the additional assumption that $H_1(\mathscr{N})$ contains no element of finite order, in order to apply the formalism of currents. By working in the setting of flat chains, instead of currents, this assumption is not required any more, although we still need that $\pi_1(\mathscr{N})$ be abelian. In contrast with the scalar case, it may *not* be possible to construct approximating maps $u_j \in C^\infty(\overline{B}^d, \mathscr{N})$ in such a way that $|\mathrm{D}u_j|(B^d) \to |\mathrm{D}u|(B^d)$ (see [27]): this gap phenomenon is analougous to the one illustrated in Section 1.3 above.

The proof of Theorem 2.3, as that of Theorem 2.2, is based on the "removal of the singularity" by Bethuel, Brezis and Coron [7]: first we control the flat norm of the topological singular set by means of (2.10), then we remove the singularities using the results of [42]. The flat norm of the topological singular set coincides with the "minimal connection" of Bethuel, Brezis and Coron (see Section 1.3).

As remarked above, the techniques presented in this paper apply to quite general target manifolds, but not all. In particular, closed manifolds $\mathscr{N}$ with non-abelian $\pi_1(\mathscr{N})$ are excluded, because the theory of flat chains with coefficients in a group $\mathbf{G}$ requires $\mathbf{G}$ to be abelian. However, in the topological obstruction theory, this kind of restriction can be removed by using suitable technical tools (homology with local coefficients systems). This leaves a hope to extend, at least partially, some of the results to the case of non-abelian $\pi_1(\mathscr{N})$. Density (in the sense of biting convergence) of smooth maps in $W^{1,1}(\Omega, \mathscr{N})$ with non-abelian $\pi_1(\mathscr{N})$ has been proven by Pakzad [41].

We have not discussed applications of the operator $\mathbf{S}$ to variational problems, such as (2.1). We expect that the results presented in this section could be used as tools to obtain energy lower bounds for (2.1) in the spirit of [44, 35], or even Γ-convergence results along the lines of [2]. These questions will be addressed in a forthcoming work [20].

Acknowledgements

G. C.'s research was supported by the Basque Government through the BERC 2018-2021 program and by the Spanish Ministry of Economy and Competitiveness: MTM2017-82184-R. G. O. was partially supported by GNAMPA-INdAM.

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Manuscript received April 25 2019
revised June 14 2019

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