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# $N$-LAPLACIAN ELLIPTIC EQUATIONS IN EXTERIOR DOMAINS VIA KELVIN TRANSFORM

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Abstract. Let $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$ be the exterior of the closed unit ball in $\mathbb{R}^N$. We prove existence and enclosure results for the following boundary value problem

$$-\Delta_N u = a(x)g(u) \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega = \partial B(0,1),$$

where the positive coefficient $a$ satisfies a certain integrability condition. We are looking for solutions in the Beppo-Levi space $D_0^{1,N}(\Omega)$ which is the completion of $C_c^\infty(\Omega)$ with respect to the $\|\nabla\cdot\|_{N,\Omega}$-norm. Unlike in the situation of the corresponding $p$-Laplacian equation in $\mathbb{R}^N$ with $N > p > 1$, the treatment of the borderline case $N = p$ considered here is more subtle, and the behavior of solutions is qualitatively significantly different. Our main tool in studying the above problem will be the Kelvin transform, which allows us to establish a one-to-one mapping between solutions of the problem above and solutions of an associated $N$-Laplacian problem in the ball $B(0,1)$. Moreover, a Brezis-Nirenberg type result concerning $D_0^{1,N}(\Omega)$ versus $C^1_{\mathrm{loc}}(\overline{\Omega})$ minimizers of the energy functional related to the problem above is proved.

## 1. Introduction

Let $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$ be the exterior of the closed unit ball $B = B(0,1)$ in $\mathbb{R}^N$ with $N \geq 2$, and let $X = D_0^{1,N}(\Omega)$ be the Beppo-Levi space, which is the completion of $\mathcal{D} = C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_X^N = \int_\Omega |\nabla u|^N \, dx.$$

Our main aim is to prove multiplicity, regularity, and enclosure results for quasilinear elliptic problems in the exterior domain $\Omega$ of the form

$$-\Delta_N u = a(x)g(u) \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega = \partial B, \tag{1.1}$$

where $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the $N$-Laplacian, $a : \Omega \to \mathbb{R}$ is a positive function satisfying some integrability condition, and $g : \mathbb{R} \to \mathbb{R}$ is some continuous function to be specified later.

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**Definition 1.1.** The function $u \in X$ is called a (weak) solution of (1.1) if the following is satisfied:

$$\int_\Omega |\nabla u|^{N-2} \nabla u \, \nabla \varphi \, dx = \int_\Omega a(x) g(u) \, \varphi \, dx, \quad \forall \, \varphi \in \mathcal{D} \text{ (resp. } X). \tag{1.2}$$

In [1] it is shown that the Beppo-Levi space $D_0^{1,p}(\Omega)$ on the exterior domain $\Omega$ is a well defined reflexive Banach space for any $1 < p \leq N$. However, unlike in the situation of the corresponding $p$-Laplacian equation in $\mathbb{R}^N$ with $1 < p < N$, the treatment of the borderline case $N = p$ considered here is more subtle, and the behavior of solutions is qualitatively significantly different. The main reason for this is that the underlying solution space $X = D_0^{1,N}(\Omega)$ for $N = p$ is qualitatively different from the space $X = D_0^{1,p}(\Omega)$ for $1 < p < N$, which is readily seen by the following characterization. As for the borderline case $N = p$ in [14, Theorems I.2.7, I.2.16] it is shown that $X$ coincides with $Y := \hat{D}_0^{1,N}(\Omega)$ which is given by

$$\hat{D}_0^{1,N}(\Omega) = \left\{ \begin{array}{c} u \in L^{1,N}(\Omega) : u \in L^N(\Omega \cap B_R), \ \forall \, R > 1, \\ \text{and } \eta \, u \in W_0^{1,N}(\Omega) \ \text{ for any } \eta \in C_c^\infty(\mathbb{R}^N) \end{array} \right\}, \tag{1.3}$$

where $B_R = B(0, R)$ is the open ball of radius $R$ centered at the origin, and

$$L^{1,N}(\Omega) = \{u \in L^1_{\text{loc}}(\Omega) : \nabla u \in [L^N(\Omega)]^N\},$$

and $W_0^{1,N}(\Omega)$ denotes the usual Sobolev space of $N$-integrable functions on $\Omega$ with zero traces on $\partial\Omega$. Note that $\eta \, u \in W_0^{1,N}(\Omega)$ for any $\eta \in C_c^\infty(\mathbb{R}^N)$ implies $u|_{|x|=1} = 0$ in the sense of traces (here $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^N$). In case $1 < p < N$, due to the Sobolev embedding, $D_0^{1,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$, where $p^* = \frac{Np}{N-p}$ denotes the critical Sobolev exponent, which yields the following characterization of $D_0^{1,p}(\Omega)$

$$D_0^{1,p}(\Omega) = \{u \in L^{1,p}(\Omega) : u \in L^{p^*}(\Omega)\}. \tag{1.4}$$

In view of (1.4), $u \in D_0^{1,p}(\Omega)$ is $p^*$-integrable for $1 < p < N$, while due to (1.3), $u$ need not be $q$-integrable for any $1 \leq q < \infty$ in case $N = p$. For example, in case $N = p = 2$, the function

$$u(x) = 1 - \frac{1}{|x|^2}, \quad x \in \Omega = \mathbb{R}^2 \setminus \overline{B(0,1)},$$

is easily seen to belong to $Y = \hat{D}_0^{1,2}(\Omega)$, but $u \notin L^q(\Omega)$ for any $q$ with $1 \leq q < \infty$.

To underline the borderline situation considered here we remark that for any unbounded domain $\hat{\Omega}$ in $\mathbb{R}^N$ different from $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$, the space $D_0^{1,N}(\hat{\Omega})$ may not even be a function space. For example, in case $N = p = 2$ and $\hat{\Omega} = \mathbb{R}^2$, $D_0^{1,2}(\mathbb{R}^2)$ cannot be realized as a space of functions (as for counterexamples see [15, Section 2.7] or [9]), which is why we consider the exterior domain $\Omega$.

Our present work is motivated by the recent paper [3], where (1.1) is considered for the semilinear case $N = p = 2$, and by [11] where the author shows that $p$-harmonicity in a domain $G \subset \mathbb{R}^N$ is preserved under Kelvin transform if $N = p$. We extend the results obtained in [3] for the semilinear case $N = p = 2$ to the quasilinear case $N = p \geq 3$, which is by no means a straightforward generalization, since for $N \geq 3$ the $N$-Laplacian equation (1.1) requires a number of novel arguments and considerations compared with the treatment of the corresponding Laplacian equation in case $N = 2$ which, in addition, enjoys a Hilbert space setting.

As in [3] our approach in studying problem (1.1) is based on Kelvin transform. The strategy in treating problem (1.1) is to show first that the Kelvin transform provides an isometric, order-preserving isomorphism between the space $X = D_0^{1,N}(\Omega)$ on the exterior domain $\Omega$ and the Sobolev space $W := W_0^{1,N}(B)$ on the ball $B$. Then we establish a one-to-one correspondence between solutions $u$ of problem (1.1) and solutions $\hat{u}$ of an associated $N$-Laplacian problem in the ball $B(0,1)$ of the form

$$-\Delta_N \hat{u} = b(x)g(\hat{u}) \text{ in } B, \quad \hat{u} = 0 \text{ on } \partial B, \tag{1.5}$$

which is related to (1.1) via Kelvin transform, where the coefficient $b$ is given in terms of the Kelvin transform of the coefficient $a$. Moreover, a Brezis-Nirenberg type result concerning $D_0^{1,N}(\Omega)$ versus $C^1_{\mathrm{loc}}(\overline{\Omega})$ local minimizers of the energy functional related to problem (1.1) is proved.

The paper is organized as follows: In Section 2, we prove existence and regularity results for some eigenvalue problems in balls, and formulate our main result. In Section 3, the Kelvin transform is introduced, and a number of calculus rules as well as important mapping properties of the Kelvin transform are proved. In particular, the one-to-one correspondence between solutions of (1.1) and (1.5) is shown. In Section 4, a novel Brezis-Nirenberg type result is proved, which is a $W_0^{1,N}(B)$ versus $C^1_{\mathrm{loc}}(\overline{B}\setminus\{0\})$ local minimizer result of the energy functional related to (1.5). Finally, our main result will be proved in Section 5.

## 2. Hypotheses, preliminaries and the main result

As introduced in Section 1, throughout the paper we use the notation $X = D_0^{1,N}(\Omega) = Y$ with $N \geq 2$ and $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$, where $Y$ is characterized by (1.3). The coefficient $a$ and the nonlinearity $g$ in (1.1) are supposed to satisfy the following hypotheses:

**(Ha)** $a : \Omega \to \mathbb{R}_+$ is measurable, $a(x) > 0$ a.e. in $\Omega$ and $a \in L^\infty_{\mathrm{loc}}(\overline{\Omega})$ satisfying for some $r > 1$

$$\int_\Omega (a(y))^r |y|^{2N(r-1)}\, dy < \infty. \tag{2.1}$$

**(Hg)** $g : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies
- (i) $|g(s)| \leq c(1 + |s|^{\tilde{q}})$, for some $\tilde{q} > 1$,
- (ii) $\lim_{s\to 0} \frac{g(s)}{|s|^{N-2}s} = \mu > 0$,
- (iii) $-\infty \leq \lim_{|s|\to\infty} \frac{g(s)}{|s|^{N-2}s} = \nu$.

A typical example of a coefficient satisfying hypothesis (Ha) is as follows.

**Example 2.1.** Let $a : \Omega \to \mathbb{R}_+$ be any measurable function satisfying $0 < a(x) \leq c\frac{1}{|x|^{N+\alpha}}$ with $\alpha > 0$ and some positive constant $c$. To verify (Ha) we only need to show that (2.1) holds. Let $c > 0$ be a generic constant whose value may change from line to line, and using spherical coordinates we get

$$\begin{aligned}\int_\Omega (a(y))^r |y|^{2N(r-1)}\,dy &\le c\int_\Omega \Big(\frac{1}{|y|^{N+\alpha}}\Big)^r |y|^{2N(r-1)}\,dy\\ &\le c\int_1^\infty \varrho^{-(N+\alpha)r}\varrho^{2N(r-1)}\varrho^{N-1}\,d\varrho\\ &\le c\int_1^\infty \varrho^{Nr-\alpha r-N-1}\,d\varrho<\infty,\end{aligned}$$

provided $Nr-\alpha r-N<0$ which is true if $r>1$ and $\alpha>0$ are related to each other by $\alpha>N\frac{r-1}{r}$.

With $B=B(0,1)$ we define $b: B\to\mathbb{R}$ by

$$b(x)=\frac{1}{|x|^{2N}}a\big(\frac{x}{|x|^2}\big), \tag{2.2}$$

and for $R>1$ with $B_R=B(0,R)$ we introduce the function $b_R: B_R\to\mathbb{R}$ defined by

$$b_R(x)=\begin{cases} b(x) & \text{if } \ x\in B,\\ 0 & \text{if } \ x\in B_R\setminus B.\end{cases} \tag{2.3}$$

The functions $b$ and $b_R$ have the following properties.

**Corollary 2.2.** *Assume hypothesis (Ha). Then $b$ and $b_R$ enjoy the following properties (Hb) and (Hb$_R$), respectively:*

- **(Hb)** *$b: B\to\mathbb{R}_+$ is measurable, $b(x)>0$ a.e. in $B$, $b\in L^\infty_{\mathrm{loc}}(\overline{B}\setminus\{0\})$ and $b\in L^r(B)$ for some $r>1$ with $r$ as in (Ha).*
- **(Hb$_R$)** *$b_R: B_R\to\mathbb{R}_+$ is measurable, $b(x)\ge 0$ a.e. in $B_R$, $b_R\in L^\infty_{\mathrm{loc}}(\overline{B}_R\setminus\{0\})$ and $b_R\in L^r(B_R)$ for some $r>1$ with $r$ as in (Ha).*

*Proof.* We only need to check that $b\in L^r(B)$ for some $r>1$, as the rest is obvious. By straightforward calculation and applying the change of variable given through the inversion mapping $y=\frac{x}{|x|^2}$ we get

$$\int_B b(x)^r\,dx=\int_B\frac{1}{|x|^{2Nr}}a\Big(\frac{x}{|x|^2}\Big)^r dx=\int_\Omega |y|^{2Nr}a(y)^r\frac{1}{|y|^{2N}}\,dy<\infty.$$

A detailed discussion on the inversion mapping and its derivative used in the calculation before will be given in Section 3, see Lemma 3.2, formula (3.2). □

Let us consider next the following eigenvalue problems:

$$-\Delta_N u=\lambda\, b(x)|u|^{N-2}u \quad \text{in } B, \quad u=0 \quad \text{on } \partial B, \tag{2.4}$$

and

$$-\Delta_N u=\lambda\, b_R(x)|u|^{N-2}u \quad \text{in } B_R, \quad u=0 \quad \text{on } \partial B_R, \tag{2.5}$$

Let $W:=W_0^{1,N}(B)$ and $W_R:=W_0^{1,N}(B_R)$ denote the usual Sobolev spaces on $B$ and $B_R$ with homogeneous boundary values on $\partial B$ and $\partial B_R$, equipped with norms $\|u\|_W=\|\nabla u\|_{N,B}$ and $\|u\|_{W_R}=\|\nabla u\|_{N,B_R}$, respectively. Here and in what follows we denote the $L^q(D)$-norm over $D\subset\mathbb{R}^N$ by $\|\cdot\|_{q,D}$ or simply $\|\cdot\|_q$ if there is no ambiguity regarding $D$. By $q'$ we denote the Hölder conjugate of $q$, i.e., $1/q + 1/q' = 1$.

The following results concerning (2.4) and (2.5) can readily be deduced from [8, Lemma 3.1, Lemma 3.2, Theorem 3.1].

**Corollary 2.3.** (i) *Under (Hb) there exists the first eigenvalue $\lambda_1 > 0$ of the eigenvalue problems (2.4) that can be characterized by*

$$\lambda_1 = \inf\Big\{\|\nabla u\|_{N,B}^N = \|u\|_W^N : \int_B b(x)|u|^N\,dx = 1\Big\}$$

*The first eigenvalue $\lambda_1$ is simple, and the corresponding eigenfunctions do not change sign. Let $\varphi_1 \in W$ be the nonnegative eigenfunction corresponding to $\lambda_1$. Then $\varphi_1 \in L^\infty(B)$.*

(ii) *Analogously, under (Hb$_R$) there exists the first eigenvalue $\lambda_{1,R} > 0$ of the eigenvalue problems (2.5) that can be characterized by*

$$\lambda_{1,R} = \inf\Big\{\|\nabla u\|_{N,B_R}^N = \|u\|_{W_R}^N : \int_{B_R} b_R(x)|u|^N\,dx = 1\Big\}$$

*The first eigenvalue $\lambda_{1,R}$ is simple, and the corresponding eigenfunctions do not change sign. Let $\varphi_{1,R} \in W_R$ be the nonnegative eigenfunction corresponding to $\lambda_{1,R}$. Then $\varphi_{1,R} \in L^\infty(B_R)$.*

(iii) $\lambda_{1,R} \le \lambda_1$.

*Proof.* As $b \in L^r(B)$ with $r > 1$ and $b(x) > 0$ in $B$ we may apply the theory developed in [8, Section 3.2], and thus from [8, Lemma 3.1, Theorem 3.1] it follows the existence of the first eigenvalue $\lambda_1$ and the corresponding nonnegative eigenfunction $\varphi_1$, and by [8, Lemma 3.2] we get $\varphi_1 \in L^\infty(B)$.

By definition of the weight function $b_R$ of the eigenvalue problems (2.5) we see that $b_R \in L^r(B_R)$ with $r > 1$, and $\operatorname{meas}\{x \in B_R : b_R(x) > 0\} = \operatorname{meas} B > 0$, which allows us to apply similar arguments as before which shows (ii). Finally, from the definition of $b_R$ we get

$$\int_{B_R} b_R(x)|u|^N\,dx = \int_B b(x)|u|^N\,dx,$$

which in view of the variational characterization of $\lambda_1$ and $\lambda_{1,R}$ implies $\lambda_{1,R} \le \lambda_1$. □

Next we are going to study further regularity properties of the nonnegative eigenfunction $\varphi_1 : B \to \mathbb{R}$. To this end let us introduce the following subspace $V$ of $W$: For $R_0 > 1$ fixed, we define

$$V = \{v \in W : v \in C(\overline{B}) \cap C^1(\overline{A_{R_0}})\}, \text{ with } A_{R_0} = B \cap \Big\{x \in \mathbb{R}^N : |x| > \frac{1}{R_0}\Big\}, \tag{2.6}$$

which is a Banach space under the norm $\|\cdot\|_V$ given by

$$\|v\|_V = \|v\|_{C(\overline{B})} + \|v\|_{C^1(\overline{A_{R_0}})} + \|v\|_W.$$

The positive cone $V_+$ of the space $V$ is given by

$$V_+ = \{v \in V : v \ge 0 \text{ in } \overline{B}\}.$$

One readily verifies that the interior of $V_+$ is nonempty and is characterized as follows:

$$\operatorname{int}(V_+) = \{v \in V_+ : v(x) > 0 \text{ for } x \in B,\ \frac{\partial v(x)}{\partial n} < 0 \text{ for } x \in \partial B\},$$

where $\frac{\partial v(x)}{\partial n} = \frac{\partial v(x)}{\partial x}$ is the outward normal derivative at $x \in \partial B$.

**Remark 2.4.** We remark that the interior of the positive cone $W_+$ of $W$ given by $W_+ = \{u \in W : u \geq 0\}$ is empty while the interior of $V_+$ is nonempty. This fact along with the Brezis-Nirenberg type result, which is proved in Section 4, will play a crucial role in proving the existence of local minimizers of the energy functional associated to problem (1.1).

**Lemma 2.5.** *Assume hypothesis (Hb) and let $\varphi_1$ be the nonnegative eigenfunction corresponding to the first eigenvalue $\lambda_1$ of (2.4). Then $\varphi_1 \in \operatorname{int}(V_+)$.*

*Proof.* By Corollary 2.3 (i), we have $\varphi_1 \in W \cap L^\infty(B)$ and $\varphi_1(x) \geq 0$. Thus in view of (Hb) the right-hand side function of (2.4), which is $x \mapsto \lambda_1\, b(x)\varphi_1(x)^{N-1} =: \tilde{b}(x)$, satisfies $\tilde{b}(x) \geq 0$ and $\tilde{b} \in L^\infty_{\text{loc}}(\overline{B} \setminus \{0\})$ and $\tilde{b} \in L^r(B)$ for some $r > 1$. Since $\varphi_1$ is a solution of

$$-\Delta_N \varphi_1 = \tilde{b}(x) \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B,$$

we may apply an elliptic regularity result given by [7, Corollary 7.1]), which implies that $\varphi_1$ is Hölder continuous in $\overline{B}$, that is $\varphi_1 \in C^\alpha(\overline{B})$, $0 < \alpha < 1$. From Harnack inequality (e.g. [7, Theorem 9.1]) we infer that $\varphi_1(x) > 0$ for $x \in B$. As $\tilde{b}$ is, in particular, bounded in the annulus $A_{2R_0}$, that is $\tilde{b} \in L^\infty(A_{2R_0})$, by regularity results (see e.g. [6, 10]) we get $\varphi_1 \in C^{1,\alpha}(\overline{A_{R_0}})$, which shows, in particular, that $\varphi_1 \in V_+$. Since $\varphi_1(x) > 0$ for $x \in B$, the proof is complete provided $\varphi_1$ satisfies $\frac{\partial \varphi_1(x)}{\partial n} < 0$ for $x \in \partial B$. The latter, however, is a consequence of a boundary point lemma proved in [13, Theorem 5.5.1], because $\varphi_1$ is, in particular, a $C^1$ solution in $\overline{A_{R_0}}$ with $\varphi_1(x) > 0$ in $A_{R_0}$ and $\varphi_1(x) = 0$ for $x \in \partial B \subset \partial A_{R_0}$. □

**Remark 2.6.** Lemma 2.5 applies accordingly to the nonnegative eigenfunction $\varphi_{1,R}$ corresponding to the first eigenvalue $\lambda_{1,R}$ of the eigenvalue problem (2.5) with $B$ replaced by $B_R$. We remark that the restriction $\varphi_{1,R}|_{\overline{B}}$ is bounded away from zero by a positive constant $c_m$, that is,

$$\min_{x \in \overline{B}} \varphi_{1,R}(x) = c_m > 0.$$

The main result of this paper reads as follows.

**Theorem 2.7.** *Assume (Ha) and (Hg)with $\lambda_1 < \mu < \infty$ and $-\infty \leq \nu < \lambda_{1,R}$ where $\lambda_1$ and $\lambda_{1,R}$ are the first eigenvalues of (2.4) and (2.5), respectively. Then the following holds true.*

(i) *The exterior problem (1.1) has a positive solution $u_+ \in X$ and a negative solution $u_- \in X$.*

(ii) *If, in addition, $s \mapsto g(s)$ is nondecreasing, then there is a positive solution $\tilde{u}_+ \in X$ and a negative solution $\tilde{u}_- \in X$ that can be characterized as local minima of the associated energy functional $J : X \to \mathbb{R}$ given by*

$$J(u) = \frac{1}{N}\int_\Omega |\nabla u|^N \, dx - \int_\Omega a(x)G(u)\, dx, \tag{2.7}$$

*where $G(s) = \int_0^s g(t)\, dt$ is the primitive of $g$.*

**Remark 2.8.** A few comments on Theorem 2.7, whose proof will be given in Section 5, are in order. We note that the conditions for existence of local minimizers in $X$ are given in terms of parameters $\mu$ and $\nu$ that are related to eigenvalue problems on bounded domains. As said in the introduction, the link to bounded domain problems is the one-to-one correspondence between solutions u of the exterior problem (1.1) and the bounded domain problem (1.5) via Kelvin transform, which will be seen in Section 3. Also the fact that the energy functional $J$ is well defined will be proved in the next section.

**Remark 2.9.** Without the condition of $s \mapsto g(s)$ being nondecreasing, the statement of Theorem 2.7 can even be made more precise in that one can show the existence of a smallest positive and a greatest negative solution of (1.1), which, however, are not necessarily local minimizer of $J$, but of some related truncated functional.

## 3. Kelvin transform and equivalence results

Let $B \subset \mathbb{R}^N$ be the unit ball. The mapping $x \mapsto \frac{x}{|x|^2} =: \hat{x}$ is the inversion through the sphere $\partial B$, which provides a bijection from $\mathbb{R}^N \setminus \overline{B}$ onto $B \setminus \{0\}$, and vice versa, since $\hat{\hat{x}} = x$. If $u$ is $N$-harmonic in $B$, that is, $\Delta_N u = 0$ in $B$, so is $\hat{u}(x) = u\Big(\frac{x}{|x|^2}\Big)$ in the reflected domain $\mathbb{R}^N \setminus \overline{B}$, cf. [11]. This gives rise to the following definition of the Kelvin transform.

**Definition 3.1.** Let $u : B \to \mathbb{R}$. The Kelvin transform of $u$ denoted by $(Ku)(x) = \hat{u}(x)$ is defined by

$$(Ku)(x) = u\Big(\frac{x}{|x|^2}\Big), \quad x \in \Omega = \mathbb{R}^N \setminus \overline{B}.$$

Let us next prove some calculus rules related to the inversion mapping.

**Lemma 3.2.** *Let $\hat{x}(x) = \frac{x}{|x|^2}$ be the inversion mapping. Then the Frechet derivative $x \mapsto D\hat{x}(x)$ is given by*

$$D\hat{x}(x) = \frac{1}{|x|^2} I - \frac{2}{|x|^4} T, \tag{3.1}$$

*where $I$ is the unit $N$-matrix and $T$ is the following $N \times N$ matrix*

$$T = x\, x^t = \begin{pmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_N \\ x_2x_1 & x_2^2 & \cdots & x_2x_N \\ \vdots & \vdots & & \vdots \\ x_Nx_1 & x_Nx_2 & \cdots & x_N^2 \end{pmatrix}$$

*and the absolute value of the determinant of $D\hat{x}$, i.e. $|\det(D\hat{x}(x))|$, is equal to*

$$|\det(D\hat{x}(x))| = \frac{1}{|x|^{2N}}. \tag{3.2}$$

*Moreover, for any $\xi, \eta \in \mathbb{R}^N$ we get*

$$\langle D\hat{x}(x)\xi, D\hat{x}(x)\eta\rangle = \frac{1}{|x|^4}\langle \xi, \eta\rangle, \tag{3.3}$$

*where $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{R}^N$.*

*Proof.* Formula (3.1) follows by straightforward calculation. As for (3.2) we note that $D\hat{x}$ is a symmetric matrix, which yields

$$D\hat{x}(x)D\hat{x}(x) = \frac{1}{|x|^4}I - \frac{4}{|x|^6}T + \frac{4}{|x|^8}T^2 = \frac{1}{|x|^4}I,$$

where we have used $T^2 = |x|^2 T$ in the last equation. This readily results in (3.2). Now (3.3) follows directly as

$$\langle D\hat{x}(x)\xi, D\hat{x}(x)\eta\rangle = \langle D\hat{x}(x)D\hat{x}(x)\xi, \eta\rangle = \frac{1}{|x|^4}\langle \xi, \eta\rangle.$$

$\square$

**Lemma 3.3.** *Let $\varphi \in C_c^\infty(B)$ (with $B$ the unit ball in $\mathbb{R}^N$) and let $\hat{\varphi}(x) = \varphi\Big(\frac{x}{|x|^2}\Big)$ its Kelvin transform. Then the gradient of $\hat{\varphi}(x)$ can be calculated by*

$$\nabla\hat{\varphi}(x) = \frac{1}{|x|^2}\nabla\varphi\Big(\frac{x}{|x|^2}\Big) - \frac{2}{|x|^4}\Big\langle \nabla\varphi\Big(\frac{x}{|x|^2}\Big), x\Big\rangle x, \quad \forall\ x \in \Omega. \tag{3.4}$$

*In particular,*

$$|\nabla\hat{\varphi}(x)| = \frac{1}{|x|^2}\Big|\nabla\varphi\Big(\frac{x}{|x|^2}\Big)\Big|. \tag{3.5}$$

*Proof.* Clearly $\hat{\varphi} \in C^\infty(\Omega)$ (note: $\hat{\varphi}$ does not necessarily have compact support in $\Omega$) with $\hat{\varphi} = 0$ in a neighborhood of $\partial B$. Applying the chain rule we get by using the inversion mapping $\hat{x}(x) = \frac{x}{|x|^2}$ and (3.1)

$$\begin{aligned} \nabla\hat{\varphi}(x) &= \nabla\varphi(\hat{x}(x))D\hat{x}(x) = \nabla\varphi(\hat{x}(x))\Big(\frac{1}{|x|^2}I - \frac{2}{|x|^4}T\Big) \\ &= \frac{1}{|x|^2}\nabla\varphi(\hat{x}(x)) - \frac{2}{|x|^4}\langle\nabla\varphi(\hat{x}(x)), x\rangle x, \end{aligned}$$

and

$$\langle\nabla\hat{\varphi}(x), \nabla\hat{\varphi}(x)\rangle = \frac{1}{|x|^4}|\nabla\varphi(\hat{x}(x))|^2,$$

which proves (3.4) and (3.5). $\square$

Before proving one of our main tools used in treating the exterior problem (1.1), which is given by Theorem 3.5 below, let us first prove some technical lemma.

**Lemma 3.4.** *For any $u \in X$ and $R > 1$ there exists some positive constant $c(N, R)$ such that*

$$\int_{\Omega\cap B_R} |u|^N \, dx \le c(N,R) \int_\Omega |\nabla u|^N \, dx, \tag{3.6}$$

*where $B_R = B(0, R)$.*

*Proof.* From [1, Lemma 2.2], for any $u \in X$ with $w \in L^1\big((1,\infty);[s\log s]^{N-1}\big)$ we get the inequality

$$\int_\Omega |u|^N \, w(|x|)\, dx \le \|w\|_{L^1\left((1,\infty);[s\log s]^{N-1}\right)} \int_\Omega |\nabla u|^N \, dx, \tag{3.7}$$

where $w \in L^1\big((1,\infty);[s\log s]^{N-1}\big)$ means

$$\int_1^\infty w(s)[s\log s]^{N-1}\, ds < \infty.$$

In particular, $w(s) = \frac{1}{s^2[s\log s]^{N-1}}$ is a possible choice, which yields

$$\|w\|_{L^1\left((1,\infty);[s\log s]^{N-1}\right)} = 1,$$

and thus from (3.7) we get

$$\int_\Omega |u|^N \frac{1}{|x|^2[|x|\log|x|]^{N-1}}\, dx \le \int_\Omega |\nabla u|^N \, dx. \tag{3.8}$$

The left-hand side of (3.8) can be estimated below as

$$\begin{aligned}
\int_\Omega |u|^N \frac{1}{|x|^2[|x|\log|x|]^{N-1}}\, dx &\ge \int_{\Omega\cap B_R} |u|^N \frac{1}{|x|^2[|x|\log|x|]^{N-1}}\, dx \\
&\ge \int_{\Omega\cap B_R} |u|^N \frac{1}{R^2[R\log R]^{N-1}}\, dx \\
&\ge \frac{1}{R^{2N}} \int_{\Omega\cap B_R} |u|^N \, dx,
\end{aligned}$$

which proves the lemma with $c(R,N) = R^{2N}$. □

**Theorem 3.5.** *The Kelvin transform $K : W = W_0^{1,N}(B) \to X$ defined by $\hat{u}(x) = (Ku)(x) = u\Big(\frac{x}{|x|^2}\Big)$ provides an order-preserving, isometric isomorphism from $W$ to $X$, and $K^{-1} = K$.*

*Proof.* Let $\varphi \in C_c^\infty(B)$, then its Kelvin transform $\hat{\varphi}(x) = \varphi\Big(\frac{x}{|x|^2}\Big)$ belongs to $C^\infty(\Omega)$ with $\hat{\varphi} = 0$ in a neighborhood of $\partial B$, and respectively, if $\varphi \in C_c^\infty(\Omega)$ then its Kelvin transform $\hat{\varphi} \in C_c^\infty(B)$, and in either case by applying Lemma 3.3 we can make use of formula (3.4) and (3.5), where $\hat{x}(x) = \frac{x}{|x|^2}$. Let $B_R = B(0,R)$, $R > 1$, and let $\varphi \in C_c^\infty(B)$, then by using (3.2) we can estimate as follows

$$\begin{aligned}
\int_{\Omega\cap B_R} |\hat{\varphi}(y)|^N \, dy &= \int_{\Omega\cap B_R} \Big|\varphi\Big(\frac{y}{|y|^2}\Big)\Big|^N dy = \int_{B\cap\{|x|>\frac{1}{R}\}} |\varphi(x)|^N \frac{1}{|x|^{2N}}\, dx \\
&\le R^{2N} \int_B |\varphi(x)|^N \, dx, \qquad (3.9)
\end{aligned}$$

and from (3.5) we obtain by using (3.2) and applying the change of variable

$$\int_\Omega |\nabla\hat\varphi(y)|^N\,dy = \int_\Omega \frac{1}{|y|^{2N}}\Big|\nabla\varphi\Big(\frac{y}{|y|^2}\Big)\Big|^N\,dy = \int_B |\nabla\varphi(x)|^N\,dx. \tag{3.10}$$

From (3.9) and (3.10) we see that $\hat\varphi \in Y = X$, where $Y$ is characterized through (1.3). The latter shows that $K : C_c^\infty(B) \to Y$ is a linear and bounded operator, and thus $K$ has a unique extension to $W$ due to the density of $C_c^\infty(B)$ in $W$, which is denoted by $\tilde K$. From (3.10) it follows $\|K\varphi\|_X = \|\hat\varphi\|_X = \|\varphi\|_W$, which shows that $\|K\| = \|\tilde K\| = 1$, and thus $\tilde K : W \to Y = X$ is an isometric, linear operator. Next we are going to show that for the extension $\tilde K$ the following holds true

$$(\tilde K u)(x) = \hat u(x) = u\Big(\frac{x}{|x|^2}\Big) = (Ku)(x), \quad \forall\, u \in W. \tag{3.11}$$

To this end let $u \in W$ be given, then there is $(\varphi_n) \subset C_c^\infty(B)$ with $\varphi_n \to u$ in $W$. Since $(\varphi_n)$ is a Cauchy sequence in $W$, from (3.10) it follows that the sequence of the Kelvin transforms $\hat\varphi_n = K\varphi_n = \tilde K\varphi_n$ is a Cauchy sequence in $Y = X$, and thus

$$\hat\varphi_n \to v = \tilde K u \ \text{ in } X, \tag{3.12}$$

which by using Lemma 3.4 yields

$$\lim_{n\to\infty}\int_\Omega |\nabla(\hat\varphi_n - v)|^N\,dy = 0, \quad \lim_{n\to\infty}\int_{\Omega\cap B_R} |\hat\varphi_n - v|^N\,dy = 0, \ \ \forall\, R > 1. \tag{3.13}$$

From (3.9) with $\varphi$ replaced by $\varphi_n - u$ and $\hat\varphi$ replaced by $\hat\varphi_n - \hat u$, respectively, we deduce for any $R > 1$

$$\begin{aligned}\int_{\Omega\cap B_R} |\hat\varphi_n(y) - \hat u(y)|^N\,dy &\le \int_{B\cap\{|x|>\frac1R\}} |\varphi_n(x) - u(x)|^N \frac{1}{|x|^{2N}}\,dx\\ &\le R^{2N}\int_B |\varphi_n(x) - u(x)|^N\,dx,\end{aligned}$$

which implies

$$\lim_{n\to\infty}\int_{\Omega\cap B_R} |\hat\varphi_n(y) - \hat u(y)|^N\,dy = 0, \ \ \forall\, R > 1,$$

and thus by (3.13) we obtain $\hat u(y) = v(y)$ for a.e. $y \in \Omega$, which proves (3.11). Therefore, $\tilde K u = Ku = \hat u$.

So far we have shown that the Kelvin transform $K : W \to Y = X$ is a linear, bounded, isometric and injective operator. To complete the proof we need to show that $K$ is surjective, i.e., $K(W) = Y = X$. Let $v \in X$, then there is a sequence $\psi_n \in C_c^\infty(\Omega)$ such that $\psi_n \to v$ in $X$, that is

$$\lim_{n\to\infty}\int_\Omega |\nabla(\psi_n(y) - v(y))|^N\,dy = 0, \quad \lim_{n\to\infty}\int_{\Omega\cap B_R} |\psi_n(y) - v(y)|^2\,dy = 0 \tag{3.14}$$

for all $R > 1$, where the second limit is due to Lemma 3.4. Clearly $\hat\psi_n \in C_c^\infty(B)$, and in view of (3.10) it follows that $(\hat\psi_n)$ is a Cauchy sequence in $W$, and thus $\hat\psi_n \to u$ in $W$ for some $u \in W$. Now we shall see that in fact $Ku = v$ holds true, which gives the surjectivity. From $\hat{\psi}_n \to u$ in $W$ and by using (3.9) we get for the corresponding Kelvin transforms $K\hat{\psi}_n - Ku = \psi_n - \hat{u}$

$$\lim_{n\to\infty} \int_{\Omega\cap B_R} |\psi_n(y) - \hat{u}(y)|^N \, dy = \lim_{n\to\infty} \int_{B\cap\{|x|>\frac{1}{R}\}} |\hat{\psi}_n(x) - u(x)|^N \frac{1}{|x|^4} \, dx = 0,$$

for any $R > 1$, which together (3.14) yields $\hat{u}(x) = v(x)$ for a.e. $x \in \Omega$, and hence the surjectivity of $K$. Clearly, $K$ is order preserving with respect to the natural partial ordering of functions. Finally, we readily verify that $K(Ku) = u$ for all $u \in H$, and thus $K = K^{-1}$. This completes the proof of the theorem. □

**Theorem 3.6.** *The Kelvin transform $\hat{u}(x) = (Ku)(x) = u\Big(\frac{x}{|x|^2}\Big)$ satisfies the following formula*

$$\int_B |\nabla u(x)|^{N-2} \nabla u(x) \nabla v(x) \, dx = \int_\Omega |\nabla \hat{u}(y)|^{N-2} \nabla \hat{u}(y) \nabla \hat{v}(y) \, dy. \tag{3.15}$$

*Proof.* Applying Lemma 3.3 and using Theorem 3.5 as well as the change of variable $x(y) = \frac{y}{|y|^2}$, a straightforward calculation yields

$$\begin{aligned}
&\int_\Omega |\nabla \hat{u}(y)|^{N-2} \nabla \hat{u}(y) \nabla \hat{v}(y) \, dy \\
&= \int_\Omega \frac{1}{|y|^{2(N-2)}} |\nabla u(x(y))|^{N-2} \Big( \frac{1}{|y|^2} \nabla u(x(y)) - \frac{2}{|y|^4} \langle \nabla u(x(y)), y \rangle y \Big) \times \\
&\qquad \Big( \frac{1}{|y|^2} \nabla v(x(y)) - \frac{2}{|y|^4} \langle \nabla v(x(y)), y \rangle y \Big) \, dy \\
&= \int_\Omega \frac{1}{|y|^{2(N-2)}} |\nabla u(x(y))|^{N-2} \frac{1}{|y|^4} \nabla u(x(y)) \nabla v(x(y)) \, dy \\
&= \int_B |x|^{2(N-2)} |\nabla u(x)|^{N-2} |x|^4 \nabla u(x) \nabla v(x) \frac{1}{|x|^{2N}} \, dx \\
&= \int_B |\nabla u(x)|^{N-2} \nabla u(x) \nabla v(x) \, dx
\end{aligned}$$

which completes the proof. □

We conclude this section with the following equivalence result.

**Theorem 3.7.** *Assume hypotheses (Ha) and (Hg)(i). Then the following holds:*

(i) *$u \in X$ is a solution of (1.1), that is*

$$-\Delta_N u = a(x) g(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega = \partial B,$$

*if and only if its Kelvin transform $Ku = \hat{u} \in W$ is a solution of (1.5), that is*

$$-\Delta_N \hat{u} = b(y) g(\hat{u}) \quad \text{in } B, \quad \hat{u} = 0 \quad \text{on } \partial B,$$

*where $b$ is given by (2.2) which fulfills (Hb).*

(ii) *Any solution of (1.5) is a critical point of the energy functional $E : W \to \mathbb{R}$ given by*

$$E(\hat{u}) = \frac{1}{N} \int_B |\nabla \hat{u}|^N \, dy - \int_B b(y) G(\hat{u}) \, dy, \tag{3.16}$$

*where $G(s) = \int_0^s g(t)\,dt$ is the primitive of $g$. The functional $E$ is well defined , $C^1$, and weakly lower semicontinuous.*

(iii) *For $u \in X$ and its Kelvin transform $Ku = \hat{u} \in W$ we have the equality*

$$
\begin{aligned}
E(\hat{u}) &= \frac{1}{N}\int_B |\nabla \hat{u}|^N\,dy - \int_B b(y)G(\hat{u})\,dy \\
&= \frac{1}{N}\int_\Omega |\nabla u|^N\,dx - \int_\Omega a(x)G(u)\,dx =: J(u),
\end{aligned}
\tag{3.17}
$$

*where $J : X \to \mathbb{R}$ is the energy functional related to (1.1). Moreover, $\hat{u} \in W$ is a critical point of $E$ if and only if its Kelvin transform $u = K\hat{u}$ is a critical point of $J$.*

*Proof.* Ad (i): Let $u \in X$ be a solution of (1.1), that is

$$\int_\Omega |\nabla u|^{N-2}\nabla u\,\nabla\varphi\,dx = \int_\Omega a(x)g(u)\,\varphi\,dx, \quad \forall\ \varphi \in \mathcal{D}\ (\text{resp. } X).$$

Consider the Kelvin transform $\hat{u} = Ku$ of the solution $u$ of (1.1) (note: $K = K^{-1}$). By applying Theorem 3.5 and Theorem 3.6, we immediately get with $x(y) = \frac{y}{|y|^2}$

$$
\begin{aligned}
\int_\Omega |\nabla u|^{N-2}\nabla u\,\nabla\varphi\,dx &= \int_B |\nabla \hat{u}|^{N-2}\nabla \hat{u}\,\nabla\hat{\varphi}\,dy, \quad \forall\ \hat{\varphi} = K\varphi \in W \\
\int_\Omega a(x)g(u)\,\varphi\,dx &= \int_B \frac{1}{|y|^{2N}}a\Big(\frac{y}{|y|^2}\Big)g(\hat{u})\hat{\varphi}\,dy \\
&= \int_B b(y)g(\hat{u})\hat{\varphi}\,dy, \quad \forall\ \hat{\varphi} = K\varphi \in W,
\end{aligned}
$$

which proves (i).

Ad (ii): As $b$ fulfills (Hb) and $W = W_0^{1,N}(B) \hookrightarrow\hookrightarrow L^q(B)$ is compactly embedded for any $q$ with $1 \le q < \infty$, by using (Hg)(i) one readily verifies that $E : W \to \mathbb{R}$ is well defined, and by standard arguments it is easily seen that $E$ is $C^1$ and weakly lower semicontinuous. Moreover,

$$\langle E'(\hat{u}), \hat{\varphi}\rangle = \int_B |\nabla \hat{u}|^{N-2}\nabla\hat{u}\nabla\hat{\varphi}\,dy - \int_B \hat{b}(y)g(\hat{u})\hat{\varphi}\,dy, \quad \forall\ \hat{\varphi} \in W,$$

which proves (ii).

Ad (iii): With (2.2)and (3.5) and applying the change of variable we obtain

$$
\begin{aligned}
E(\hat{u}) &= \frac{1}{N}\int_B |\nabla \hat{u}|^N\,dy - \int_B b(y)G(\hat{u})\,dy \\
&= \frac{1}{N}\int_B \frac{1}{|y|^{2N}}\Big|\nabla u\Big(\frac{y}{|y|^2}\Big)\Big|^N\,dy - \int_B \frac{1}{|y|^{2N}}a\big(\frac{y}{|y|^2}\big)G\Big(u\big(\frac{y}{|y|^2}\big)\Big)\,dy \\
&= \frac{1}{N}\int_\Omega |\nabla u|^N\,dx - \int_\Omega a(x)G(u)\,dx =: J(u)
\end{aligned}
$$

Let $u \in X$ be a critical point of $J$, then again by applying Theorem 3.6 we get

$$0 \;=\; \langle J'(u), \varphi\rangle = \int_\Omega |\nabla u|^{N-2}\nabla u\nabla\varphi\,dx - \int_\Omega a(x)g(u)\varphi\,dx$$

$$= \int_B |\nabla \hat{u}|^{N-2} \nabla \hat{u} \nabla \hat{\varphi}\, dy - \int_B \frac{1}{|y|^{2N}} a\Big(\frac{y}{|y|^2}\Big) g(\hat{u}) \hat{\varphi}\, dy$$
$$= \int_B |\nabla \hat{u}|^{N-2} \nabla \hat{u} \nabla \hat{\varphi}\, dy - \int_B b(y) g(\hat{u}) \hat{\varphi}\, dy = \langle E'(\hat{u}), \hat{\varphi} \rangle,$$

where $\hat{\varphi} = K\varphi$, which completes the proof. $\square$

**Corollary 3.8.** *Assume hypotheses (Ha) and (Hg)(i). Then the functional $J : X \to \mathbb{R}$ is well defined, $C^1$, weakly lower semicontinuous, and any solution of (1.1) is a critical point of $J$.*

*Proof.* By Theorem 3.5, the Kelvin transform $K : W \to X$ provides an isometric isomorphism. Due to Theorem 3.7, the properties of the functional $E$ transfer to $J$. $\square$

**Remark 3.9.** Theorem 3.5 and Theorem 3.7 allow us to study the exterior problem (1.1) via the elliptic boundary value problem (1.5) on the ball $B$.

## 4. A Brezis-Nirenberg type result

Let $V$ be the subspace of $W = W_0^{1,N}(B)$ introduced in (2.6), that is

$$V = \{v \in W : v \in C(\overline{B}) \cap C^1(\overline{A_{R_0}})\}, \quad \text{where } A_{R_0} = B \cap \Big\{x \in \mathbb{R}^N : |x| > \frac{1}{R_0}\Big\},$$

which is a Banach space under the norm $\|\cdot\|_V$ given by

$$\|v\|_V = \|v\|_{C(\overline{B})} + \|v\|_{C^1(\overline{A_{R_0}})} + \|v\|_W.$$

In this section we are going to prove the following Brezis-Nirenberg type result, which shows that a local minimizer of $E$ in the $V$-topology is also a local minimizer in the $W$-topology.

Let us first prove some regularity results for solutions of (1.5), that is,

$$-\Delta_N u = b(x) g(u) \text{ in } B, \quad u = 0 \text{ on } \partial B. \tag{4.1}$$

**Lemma 4.1.** *Assume hypothesis (Ha) (respectively, (Hb)) and (Hg)(i), and let $u \in W$ be a solution of (4.1). Then $u \in C^\alpha(\overline{B}) \cap C^{1,\alpha}(\overline{A_{R_0}})$ and there exists a constant $C = C(N, R_0, r, \tilde{q}, \alpha, \|b\|_r, \|u\|_W)$ such that*

$$\|u\|_{C^\alpha(\overline{B})} + \|u\|_{C^{1,\alpha}(\overline{A_{R_0}})} \le C(N, R_0, r, \tilde{q}, \alpha, \|b\|_r, \|u\|_W). \tag{4.2}$$

*Proof.* Hypothesis (Hg)(i) along with the embedding $W \hookrightarrow\hookrightarrow L^q(B)$ for any $q$ with $1 \le q < \infty$ and $b \in L^r(B) \cap L^\infty_{\mathrm{loc}}(\overline{B} \setminus \{0\})$ implies that the right-hand side function $\tilde{b}$ of (4.1)given by $\tilde{b}(x) = b(x) g(u(x))$ belongs to $L^\gamma(B)$ for some $\gamma > 1$ with $1 < \gamma < r$. Applying [13, Theorem 6.1.5], we infer that $u$ is bounded and there exists some constant $c = c(N, r, \tilde{q})$ such that

$$\|u\|_\infty \le c(N, r, \tilde{q}) \|\tilde{b}\|_\gamma. \tag{4.3}$$

Further, there is some constant $c = c(N, r)$ such that

$$\|\tilde{b}\|_\gamma \le c(N, r) \|b\|_r \Big(1 + \|u\|_W^{\tilde{q}}\Big), \tag{4.4}$$

which in view of (4.3) yields

$$\|u\|_\infty \le C = C(N, r, \tilde{q}, \|b\|_r, \|u\|_W). \tag{4.5}$$

Using [7, Corollary 7.1] we conclude that $u \in C^\alpha(\overline{B})$ with

$$\|u\|_{C^\alpha(\overline{B})} \le C(N, r, \tilde{q}, \alpha, \|b\|_r, \|u\|_W). \tag{4.6}$$

Since $b \in L^\infty_{\text{loc}}(\overline{B} \setminus \{0\})$ and $u \in C^\alpha(\overline{B})$, it follows that $\tilde{b} \in L^\infty_{\text{loc}}(\overline{B} \setminus \{0\})$, and thus $\tilde{b} \in L^q_{\text{loc}}(\overline{B} \setminus \{0\})$ for any $q$ with $1 \le q < \infty$, in particular $\tilde{b} \in L^\infty(A_{2R_0})$. Applying regularity results due to [6, Theorem 2, Remark and Corollary] and [10, Theorem 1], we obtain $u \in C^{1,\alpha}(\overline{A_{R_0}})$ with

$$\|u\|_{C^{1,\alpha}(\overline{A_{R_0}})} \le C(N, R_0, r, \tilde{q}, \alpha, \|b\|_r, \|u\|_W), \tag{4.7}$$

which completes the proof. □

The Brezis-Nirenberg type result reads as follows.

**Theorem 4.2.** *Assume hypotheses (Ha) (respectively, (Hb)) and (Hg)(i), and let $u_0 \in W$ be a weak solution of (4.1). If $u_0$ is a local minimizer of $E$ in the $V$-topology, i.e., $\exists\ \varepsilon > 0$ such that*

$$E(u_0) \le E(u_0 + w), \quad \forall\ w \in V : \|w\|_V \le \varepsilon,$$

*then $u_0$ is a local minimizer of $E$ in the $W$-topology, i.e., $\exists\ \delta > 0$ such that*

$$E(u_0) \le E(u_0 + v), \quad \forall\ v \in W : \|v\|_W \le \delta,$$

*Proof.* First, in view of Lemma 4.1, we have $u_0 \in C^\alpha(\overline{B}) \cap C^{1,\alpha}(\overline{A_{R_0}})$, and thus, in particular $u_0 \in V$.

Consider next the minimization problem

$$\beta_n = \inf_{u \in M_n} E(u) \quad \text{with } M_n = \Big\{u \in W : \|u - u_0\|_W \le \frac{1}{n}\Big\}. \tag{4.8}$$

By Theorem 3.7, $E : W \to \mathbb{R}$ is $C^1$ and weakly lower semicontinuous, so $E$ achieves its minimum at some $u_n \in M_n$, since $M_n$ is weakly compact. By applying Lagrange multiplier's rule there exists a Lagrange multiplier $\mu_n \in \mathbb{R}$ that can be shown to satisfy $\mu_n \le 0$ such that $u_n \in M_n$ satisfies the equality

$$\begin{aligned} \langle E'(u_n, \varphi\rangle &= \int_B \big(|\nabla u_n|^{N-2}\nabla u_n \nabla\varphi - b(x)g(u_n)\varphi\big)\,dx \\ &= \mu_n \int_B |\nabla(u_n - u_0)|^{N-2}\nabla(u_n - u_0)\nabla\varphi\,dx,\ \forall\ \varphi \in W, \end{aligned}$$

that is $u_n$ satisfies (in the distributional sense)

$$-\Delta_N u_n - b(x)g(u_n) = -\mu_n\,\Delta_N(u_n - u_0) \ \text{ in } B, \quad u_n = 0 \ \text{ on } \partial B. \tag{4.9}$$

Since $u_0$ is a weak solution of (4.1), that is

$$-\Delta_N u_0 = b(x)g(u_0) \ \text{ in } B, \quad u_0 = 0 \ \text{ on } \partial B \tag{4.10}$$

we get by subtracting (4.10) from (4.9)

$$-\big(\Delta_N u_n - \Delta_N u_0\big) + \mu_n\,\Delta_N(u_n - u_0) = b(x)\big(g(u_n) - g(u_0)\big). \tag{4.11}$$

Thus $w_n = u_n - u_0$ is a weak solution of the problem (note: $\mu_n \le 0$):

$$-\big(\Delta_N(u_0+w_n) - \Delta_N u_0\big) + \mu_n\,\Delta_N w_n = b(x)\big(g(u_0+w_n) - g(u_0)\big), \tag{4.12}$$

where $w_n = 0$ on $\partial B$. Let us introduce the quasilinear operator $A$ defined by the left-hand side of (4.12)

$$Au := -\big(\Delta_N(u_0+u) - \Delta_N u_0\big) + \mu_n\,\Delta_N u, \quad u \in W.$$

Since $u_0 \in V$, the operator $A : W \to W^*$ given by

$$\begin{aligned}\langle Au, \varphi\rangle &= \int_B \Big(|\nabla(u_0+u)|^{N-2}\nabla(u_0+u) - |\nabla u_0|^{N-2}\nabla u_0\Big)\nabla\varphi \\ &\quad -\mu_n\,|\nabla u|^{N-2}\nabla u \nabla\varphi\,dx, \quad u, \varphi \in W \end{aligned} \tag{4.13}$$

is well defined and continuous. Moreover, as $N \ge 2$ there is a constant $\theta > 0$ such that (note: $\mu_n \le 0$)

$$\begin{aligned}\langle Au, u\rangle &= \int_B \Big(|\nabla(u_0+u)|^{N-2}\nabla(u_0+u) - |\nabla u_0|^{N-2}\nabla u_0\Big)\nabla u \\ &\quad -\mu_n\,|\nabla u|^{N-2}\nabla u \nabla u\,dx \\ &\ge (\theta - \mu_n)\int_B |\nabla u|^N\,dx \ge \theta\int_B |\nabla u|^N\,dx, \end{aligned} \tag{4.14}$$

and thus $A$ has a positive ellipticity constant $\theta$ independent of $n$. In a similar way one readily verifies for all $u, v \in W$ the inequality

$$\langle Au - Av, u - v\rangle \ge \theta \int_B |\nabla(u-v)|^N\,dx. \tag{4.15}$$

From (4.13)-(4.15) it follows that $A : W \to W^*$ is a continuous and strongly monotone operator, which qualitatively behaves like $-\Delta_N$. Setting

$$\tilde g(x,s) = g(u_0(x) + s) - g(u_0(x)),$$

then the right hand side of (4.12) can be rewritten as $b(x)\tilde g(x, w_n)$, and equation (4.12) can equivalently be reformulated using the operator $A$ as

$$Aw_n = b(x)\tilde g(x, w_n). \tag{4.16}$$

Clearly, $\tilde g : B \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which due to (Hg) (i) and $u_0 \in V$ satisfies for some positive constant $c$ depending on $u_0$ the growth condition

$$|\tilde g(x,s)| \le c\,(1 + |s|^{\tilde q}), \quad \text{with } \tilde q > 1 \text{ as in (Hg) (i)}, \tag{4.17}$$

that is, $\tilde g$ is qualitatively equal to $g$. Now we may apply Lemma 4.1 with $-\Delta_N$ replaced by $A$, because the proof follows exactly the same line, which yields the existence of a constant $C = C(N, R_0, r, \tilde q, \alpha, \|b\|_r, \|w_n\|_W)$ such that

$$\|w_n\|_{C^\alpha(\overline{B})} + \|w_n\|_{C^{1,\alpha}(\overline{A_{R_0}})} \le C(N, R_0, r, \tilde q, \alpha, \|b\|_r, \|w_n\|_W) \le \hat C, \tag{4.18}$$

where $C(N, R_0, r, \tilde q, \alpha, \|b\|_r, \|w_n\|_W) \le \hat C$ with $\hat C$ independent of $n$ holds true because $\|w_n\|_W \to 0$ as $n \to \infty$. By Arzelá-Ascoli theorem there is a subsequence of $(w_n)$ (again denoted by $(w_n)$) with $w_n \to w$ in $V$. Since $\|w_n\|_W \to 0$, it follows

$w = 0$. By assumption $u_0$ is a local minimizer of $E$ in the $V$-topology, thus from $w_n \to 0$ in $V$ and $n$ large we get

$$E(u_0) \le E(u_0 + w_n) = E(u_n) = \inf_{\|u-u_0\|_W \le \frac{1}{n}} E(u),$$

which proves that $u_0$ must be a local minimizer of $E$ in the $W$-topology. □

**Remark 4.3.** Theorem 4.2 extends the by now classical Brezis-Nirenberg result (see [2]) in two ways. First, unlike in [2], the leading elliptic operator is quasilinear. Second, due to the low regularity of the right-hand side of (4.1), in particular of the coefficient $b$, for $u_0$ being a local minimizer in the $W$-topology it is sufficient that $u_0$ is only a local minimizer in the $V$-topology which is the $C(\overline{B}) \cap C^1(\overline{A_{R_0}})$-topology.

## 5. Proof of the main result

The proof of our main result, Theorem 2.7, is based on the one-to-one correspondence between solutions u of the exterior problem (1.1) and the bounded domain problem (1.5) (respectively (4.1)) via Kelvin transform as well as the invariance of the associated energy functionals under Kelvin transform, see Theorem 3.7. Since the Kelvin transform $K : W = W_0^{1,N}(B) \to X$ is an order-preserving, isometric isomorphism with $K = K^{-1}$ (see Theorem 3.5), Theorem 2.7 follows from the following result.

**Theorem 5.1.** *Assume (Ha) and (Hg) with $\lambda_1 < \mu < \infty$ and $-\infty \le \nu < \lambda_{1,R}$ where $\lambda_1$ and $\lambda_{1,R}$ are the first eigenvalues of (2.4) and (2.5) with corresponding positive eigenfunctions $\varphi_1$ and $\varphi_{1,R}$, respectively. Then the following holds true.*

(i) *The problem (1.5) (respectively (4.1)) has a positive solution $v_+ \in \operatorname{int}(V_+)$ and a negative solution $v_-$ with $-v_- \in \operatorname{int}(V_+)$.*

(ii) *If, in addition, $s \mapsto g(s)$ is nondecreasing, then there is a positive solution $\tilde{v}_+ \in \operatorname{int}(V_+)$ and a negative solution $\tilde{v}_-$ with $-\tilde{v}_- \in \operatorname{int}(V_+)$ that can be characterized as local minima of the associated energy functional $E : W \to \mathbb{R}$ given by*

$$E(u) = \frac{1}{N}\int_B |\nabla u|^N\,dx - \int_B b(x)G(u)\,dx, \tag{5.1}$$

*where $G(s) = \int_0^s g(t)\,dt$ is the primitive of $g$.*

*Proof.* As hypothesis (Ha) implies (Hb), we assume in what follows that (Hb) is fulfilled.

Ad (i): Let us first show that $\overline{v} = M\varphi_{1,R}$ is a supersolution of (1.5) for $M > 0$ large.

We use Corollary 2.3 and Lemma 2.5 and take into account Remark 2.6 of Section 2, from which we see that $\varphi_{1,R} : B_R \to \mathbb{R}$, $R > 1$, satisfies $\varphi_{1,R}|_{\overline{B}}(x) \ge c_m > 0$. In $B$ we have $b = b_R$, and thus $\overline{v} = M\varphi_{1,R}$ satisfies in $B$ (in the distributional sense)

$$\begin{aligned} -\Delta_N(M\varphi_{1,R}) - b(x)g(M\varphi_{1,R}) &= \lambda_{1,R} b(x)(M\varphi_{1,R})^{N-1} - b(x)g(M\varphi_{1,R}) \\ &= b(x)(M\varphi_{1,R})^{N-1}\Big(\lambda_{1,R} - \frac{g(M\varphi_{1,R})}{(M\varphi_{1,R})^{N-1}}\Big). \end{aligned}$$

Since $\varphi_{1,R}(x) \geq c_m > 0$ for all $x \in \overline{B}$, we get $\lim_{M\to\infty}(M\varphi_{1,R})^{N-1} = \infty$ uniformly with respect to $x \in \overline{B}$. By (Hg) (iii) and due to $\nu < \lambda_{1,R}$, we get for $M > 0$ large enough

$$\lambda_{1,R} - \frac{g(M\varphi_{1,R})}{(M\varphi_{1,R})^{N-1}} \geq 0,$$

which proves that $\overline{v} = M\varphi_{1,R}$ is a supersolution.

Next we are going to show that $\underline{v} = \varepsilon\,\varphi_1$ is a subsolution.

$$\begin{aligned} -\Delta_N(\varepsilon\,\varphi_1) - b(x)g(\varepsilon\,\varphi_1) &= \lambda_1 b(x)(\varepsilon\,\varphi_1)^{N-1} - b(x)g(\varepsilon\,\varphi_1) \\ &= b(x)(\varepsilon\,\varphi_1)^{N-1}\Big(\lambda_1 - \frac{g(\varepsilon\,\varphi_1}{(\varepsilon\,\varphi_1)^{N-1}}\Big). \end{aligned}$$

As $\varphi_1$ is, in particular, in $C(\overline{B})$ we get $\varepsilon\,\varphi_1(x) \to 0$ uniformly as $\varepsilon \to 0$. By (Hg) (ii) and due to $\mu > \lambda_1$, for $\varepsilon > 0$ small enough we obtain

$$\lambda_1 - \frac{g(\varepsilon\,\varphi_1}{(\varepsilon\,\varphi_1)^{N-1}} \leq 0,$$

which proves that $\underline{v} = \varepsilon\,\varphi_1$ is a subsolution. Moreover, as $\varphi_{1,R}(x) \geq c_m > 0$ for all $x \in \overline{B}$ and $\varphi_1 \in C(\overline{B})$, by choosing either $M$ larger or $\varepsilon$ smaller if necessary, one can always achieve $\underline{v} = \varepsilon\,\varphi_1 \leq \overline{v} = M\varphi_{1,R}$ in $\overline{B}$. Applying the sub-supersolution principle (see e.g. [5]), there exists a solution $v_+ \in W$ of (1.5) such that $\underline{v} \leq v_+ \leq \overline{v}$. Thus, $v_+(x) > 0$ for $x \in B$, and by Lemma 4.1, $v_+ \in C^\alpha(\overline{B}) \cap C^{1,\alpha}(\overline{A_{R_0}})$. Following the same arguments used in the proof of Lemma 2.5, we see that $v_+ \in \operatorname{int}(V_+)$.

In a similar way one shows that $\underline{v} = -M\varphi_{1,R}$ is a negative subsolution for $M > 0$ large, and $\overline{v} = -\varepsilon\,\varphi_1$ is a negative supersolution for $\varepsilon > 0$ small enough, such that $-M\varphi_{1,R} \leq -\varepsilon\,\varphi_1$, which yields the existence of a negative solution $v_-$ with $-M\varphi_{1,R} \leq v_- \leq -\varepsilon\,\varphi_1$. Clearly, we have $-v_- \in \operatorname{int}(V_+)$.

Ad (ii): Let us introduce the truncation function $\tau_+ : B \times \mathbb{R} \to \mathbb{R}$ related to the positive supersolution $\overline{v} = M\varphi_{1,R}$ defined by

$$\tau_+(x,s) = \begin{cases} 0 & \text{if } s \leq 0 \\ s & \text{if } 0 < s < \overline{v}(x) \\ \overline{v}(x) & \text{if } s \geq \overline{v}(x) \end{cases}$$

which is easily seen to be a uniformly bounded Carathéodory function. Define the corresponding 'truncated' energy functional

$$E_+(u) = \frac{1}{N}\,\|\nabla u\|_N^N - \int_B \int_0^{u(x)} b(x)g(\tau_+(x,s))\,ds\,dx.$$

Due to the compact embedding $W \hookrightarrow\hookrightarrow L^q(B)$ for any $q : 1 \leq q < \infty$, one readily verifies (similar as for $E$) that $E_+$ is $C^1$ and weakly lower semicontinuous. Let us check that $E_+$ is coercive. Since $(x,s) \to \tau_+(x,s)$ is uniformly bounded for all $(x,s) \in B \times \mathbb{R}$, for some positive constant $C$ we have $|g(\tau_+(x,s))| \leq C$ for all $(x,s) \in B \times \mathbb{R}$, and thus

$$\Big|\int_B \int_0^{u(x)} b(x)g(\tau_+(x,s))\,ds\,dx\Big| \leq C\int_B b(x)|u|\,dx \leq C\|b\|_r\|u\|_{r'} \leq \tilde{C}\|u\|_W,$$

which yields

$$E_+(u) \geq \frac{1}{N}\|u\|_W^N - \tilde{C}\|u\|_W \to \infty \quad \text{as } \|u\|_W \to \infty,$$

that is its coercivity. Hence, there exists a global minimizer $v$ of $E_+$, i.e., $\langle E'_+(v), \varphi\rangle = 0$ which means

$$\int_B |\nabla v|^{N-2}\nabla v \nabla\varphi \, dx = \int_B b(x)g(\tau_+(x, v(x))\varphi \, dx, \quad \forall\ \varphi \in W. \tag{5.2}$$

Moreover, the global minimizer $v$ of $E_+$ is nontrivial, which is seen as follows: For $t > 0$ small we have $0 \leq t\varphi_1(x) \leq \overline{v}(x)$, and thus $g(\tau_+(x, s)) = g(s)$ for $0 \leq s \leq \overline{v}(x)$ by the definition of $\tau_+$, which yields

$$E_+(t\varphi_1) = \frac{1}{N}t^N\lambda_1 \int_B b(x)\varphi_1^N \, dx - \int_B \int_0^{t\varphi_1(x)} b(x)g(s) \, ds \, dx.$$

By hypothesis (Hg) (ii) and $\mu > \lambda_1$ we get, in particular, $\mu > \lambda_1 + \varepsilon$ for $\varepsilon > 0$ small, and for $0 \leq s \leq \delta$ with $\delta$ small it follows $g(s) \geq s^{N-1}(\lambda_1 + \varepsilon)$, which yields for $t > 0$ sufficiently small

$$E_+(t\varphi_1) = \frac{1}{N}t^N\lambda_1 \int_B b(x)\varphi_1^N \, dx - \frac{1}{N}t^N(\lambda_1 + \varepsilon) \int_B b(x)\varphi_1^N \, dx < 0,$$

and therefore $E_+(v) < 0$, since $v$ is the global minimizer of $E_+$, that is $v \neq 0$. With the special test function $\varphi = (v - \overline{v})^+ = \max\{(v - \overline{v}), 0\}$ in (5.2) and in the relation for the supersolution $\overline{v}$, which is

$$\int_B |\nabla\overline{v}|^{N-2}\nabla\overline{v}\nabla(v - \overline{v})^+ \, dx \geq \int_B b(x)g(\overline{v}(x))(v - \overline{v})^+ \, dx, \tag{5.3}$$

we get by subtracting (5.3) from (5.2) ($\varphi = (v - \overline{v})^+$)

$$\begin{aligned} &\int_B \big(|\nabla v|^{N-2}\nabla v - |\nabla\overline{v}|^{N-2}\nabla\overline{v}\big)\nabla(v - \overline{v})^+ \, dx \\ &\leq \int_B b(x)\big(g(\tau_+(x, v(x)) - g(\overline{v}(x))\big)(v - \overline{v})^+ \, dx. \end{aligned} \tag{5.4}$$

The right-hand side of (5.4) is readily seen to be zero, and for the left-hand side we get

$$\begin{aligned} &\int_B \big(|\nabla v|^{N-2}\nabla v - |\nabla\overline{v}|^{N-2}\nabla\overline{v}\big)\nabla(v - \overline{v})^+ \, dx \\ &= \int_{\{x \in B: v(x) > \overline{v}(x)\}} \big(|\nabla v|^{N-2}\nabla v - |\nabla\overline{v}|^{N-2}\nabla\overline{v}\big)\nabla(v - \overline{v}) \, dx \\ &\geq c \int_{\{x \in B: v(x) > \overline{v}(x)\}} |\nabla(v - \overline{v})|^N \, dx = c \int_B |\nabla(v - \overline{v})^+|^N \, dx, \end{aligned}$$

for some positive constant $c$, which yields

$$\int_B |\nabla(v - \overline{v})^+|^N \, dx = 0,$$

and hence it follows $(v - \overline{v})^+ = 0$, that is $v \leq \overline{v}$. Testing (5.2) with the special test function $\varphi = v^- = \max\{-v, 0\}$, we get $\|v^-\|_W = 0$, and thus $v^- = 0$ which yields $v \geq 0$. Therefore, the global minimizer $v \neq 0$ of $E_+$ satisfies the inequality $0 \leq v \leq \overline{v}$, which implies that $v$ is a nontrivial, nonnegative weak solution of problem (1.5) (respectively (4.1)). To complete the proof we need to show that the global minimizer $v$ of $E_+$ even belongs to $\operatorname{int}(V_+)$ and is in fact a local minimizer of the functional $E$ which is related to (1.5) and given by (5.1). Since $0 \leq v \leq \overline{v}$ and $\overline{v}$ is bounded, we see that $v$ is a nonnegative and bounded solution of (1.5)(respectively (4.1)), which due to Lemma 4.1 and by following the proof of Lemma 2.5 yields $v \in \operatorname{int}(V_+)$. To show that the global minimizer $v$ of $E_+$ is a local minimizer of the functional $E$, we are going to prove that $v$ is a local minimizer of $E$ with respect to the $V$-topology, because then $v$ must be a local minimizer of $E$ with respect to the $W$-topology due to Theorem 4.2. To this end we have to show that there is a $\varepsilon$-ball in $V$ centered at $v$, i.e., $B(v,\varepsilon) \subset V$ such that $B(v,\varepsilon) \subset [0,\overline{v}]$ where $\overline{v} = M\varphi_{1,R}$. We note that $\varphi_{1,R}$ is a smooth positive supersolution of (1.5) which, in particular, is continuous in $\overline{B}$ and $\varphi_{1,R}(x) \geq c_m > 0$ for all $x \in \overline{B}$, and we have $0 \leq v \leq M\varphi_{1,R}$. Since $v \in \operatorname{int}(V_+)$, the proof of $B(v,\varepsilon) \subset [0,\overline{v}]$ for some positive $\varepsilon$ is accomplished provided there is some $\delta > 0$ such that

$$v(x) + \delta \leq \overline{v}(x) = M\varphi_{1,R}(x), \quad \forall\, x \in \overline{B}. \tag{5.5}$$

We recall that $v$ and $\overline{v}$ satisfy (in the distributional sense) the following equation and inequality, respectively,

$$-\Delta_N v = b(x)g(v) \quad \text{in } B, \quad v = 0 \ \text{ on } \partial B, \tag{5.6}$$

$$-\Delta_N \overline{v} \geq b(x)g(\overline{v}) \quad \text{in } B, \quad \overline{v}(x) \geq Mc_m =: \tilde{c} > 0 \ \text{ for } x \in \partial B. \tag{5.7}$$

Since $0 \leq v \leq \overline{v}$ and $s \mapsto g(s)$ is nondecreasing, we get

$$-\Delta_N v = b(x)g(v) \leq b(x)g(\overline{v}) \leq -\Delta_N \overline{v},$$

that is, (in the distributional sense)

$$-\Delta_N v \leq -\Delta_N \overline{v}. \tag{5.8}$$

Consider the set $\mathcal{N} = \{x \in B : v(x) = \overline{v}(x)\}$. We are going to show that $\mathcal{N}$ is a compact set. Since $v = 0$ on $\partial B$ and $\overline{v}(x) \geq \tilde{c} > 0$ for $x \in \partial B$ and both $v, \overline{v} \in C(\overline{B})$, we get by continuity arguments for some $\varrho$ with $0 < \varrho < 1$

$$(\overline{v} - v)(x) > 0 \quad \text{for all } x \in \overline{B} \setminus B(0,\varrho),$$

which implies that

$$\mathcal{N} = \{x \in B : v(x) = \overline{v}(x)\} = \{x \in \overline{B(0,\varrho)} : v(x) = \overline{v}(x)\}.$$

By continuity arguments $\{x \in \overline{B(0,\varrho)} : v(x) = \overline{v}(x)\}$ is compact, and thus $\mathcal{N}$ is a compact set. Now we are able to apply [12, Corollary 8.23] or [12, Corollary 8.25] which states that $\mathcal{N}$ must be empty, which along with $(\overline{v} - v)(x) \geq \tilde{c} > 0$ for $x \in \partial B$ shows that $\overline{v}(x) - v(x) > 0$ for all $x \in \overline{B}$. Thus again by continuity arguments there is a positive $\delta$ such that (5.5) is satisfied. Therefore, there is some $\varepsilon > 0$ sufficiently small with $\varepsilon < \delta$ such that $B(v,\varepsilon) \subset [0,\overline{v}]$, which completes the proof for $v$ being a local minimizer of $E$, i.e. $v = \tilde{v}_+$.

The proof for the existence of a negative local minimizer $\tilde{v}_-$ can be shown in a similar way. To this end introduce the truncation function $\tau_- : B \times \mathbb{R} \to \mathbb{R}$ related to the negative subsolution $\underline{v} = -M\varphi_{1,R}$ defined by

$$\tau_-(x,s) = \begin{cases} \underline{v}(x) & \text{if } s \le \underline{v}(x) \\ s & \text{if } \underline{v}(x) < s < 0 \\ 0 & \text{if } s \ge 0 \end{cases},$$

and the associated 'truncated' energy functional $E_-$ defined by

$$E_-(u) = \frac{1}{N}\|\nabla u\|_N^N - \int_B \int_0^{u(x)} b(x) g(\tau_-(x,s))\, ds\, dx.$$

Using similar arguments as above, one can show that $E_- : W \to \mathbb{R}$ has a global negative minimizer $\tilde{v}_-$ which can be shown to be a local minimizer of $E$. This completes the proof. ☐

Some remarks are in order.

**Remark 5.2.** If we drop the additional assumption of $s \mapsto g(s)$ being nondecreasing in Theorem 5.1 (ii), then one can still show the following results.

(i) The global minimizers $\tilde{v}_+$ and $\tilde{v}_-$ of $E_+$ and $E_-$, respectively, are local minimizers of the following 'truncated' functional $E_0 : W \to \mathbb{R}$ which is defined by

$$E_0(u) = \frac{1}{N}\|\nabla u\|_N^N - \int_B \int_0^{u(x)} b(x) g(\tau_0(x,s))\, ds\, dx,$$

where the truncation function $\tau_0 : B \times \mathbb{R} \to \mathbb{R}$ is given by

$$\tau_0(x,s) = \begin{cases} \underline{v}(x) & \text{if } s \le \underline{v}(x) \\ s & \text{if } \underline{v}(x) < s < \overline{v}(x) \\ \overline{v}(x) & \text{if } s \ge \overline{v}(x), \end{cases}$$

(ii) In a similar way as e.g. in [4] or in [3] (for the Laplacian) one can show that problem (1.5) (respectively (4.1)) has the smallest positive solution $v_+ \in \mathrm{int}(V_+)$ in $[0, \overline{v}]$ and the greatest negative solution $v_-$ with $-v_- \in \mathrm{int}(V_+)$, which are local minimizers of $E_0$ instead of $E$ when using $\tau_0$ as follows:

$$\tau_0(x,s) = \begin{cases} v_-(x) & \text{if } s \le v_-(x) \\ s & \text{if } v_-(x) < s < v_+(x) \\ v_+(x) & \text{if } s \ge v_+(x), \end{cases} \tag{5.9}$$

However, unlike in [4] the analysis here is much more involved, because the coefficient $b$ is only supposed to be locally bounded and in $L^r(B)$ $(r > 1)$ which requires to consider solutions in the function space $V$.

(iii) Following the idea of [3], a third sign-changing solution can be shown to exist as a Mountain Pass critical point of $E_0$ with $\tau_0$ given by (5.9).

**Proof of Theorem 2.7:** The proof follows readily from Theorem 5.1 by applying Theorem 3.5 and Theorem 3.7, according to which $u_+ = Kv_+ \in X$ is a positive solution of (1.1) and $u_- = Kv_- \in X$ is a negative solution of (1.1). Further, $\tilde{u}_+ = K\tilde{v}_+ \in X$ and $\tilde{u}_- = K\tilde{v}_- \in X$ are positive and negative local minimizer of the energy functional $J(u) = \frac{1}{N}\int_\Omega |\nabla u|^N\, dx - \int_\Omega a(x)G(u)\, dx$, that is related to (1.1) due to the isometric, order-preserving isomorphism provided by the Kelvin transform $K : W \to X$ and the invariance of the energy functional under Kelvin transform. $\square$

**Remark 5.3.** (i) In view of Remark 5.2 (ii), the smallest positive and greatest negative solution of (1.5) (respectively (4.1)) transfer via Kelvin transform to smallest positive and greatest negative solutions of (1.1), since $K : W \to X$ is, in particular, order-preserving.

(ii) The regularity of the solutions $v_+ \in \operatorname{int}(V_+)$, and $\tilde{v}_+ \in \operatorname{int}(V_+)$, transfers via Kelvin transform to $u_+ = Kv_+$, $\tilde{u}_+ = K\tilde{v}_+ \in \operatorname{int}(\hat{V}_+)$, where $\hat{V}$, $\hat{V}_+$, and $\operatorname{int}(\hat{V}_+)$ are as follows: For $R_0 > 1$ fixed, define the subspace $\hat{V}$ of $X$ by

$$\hat{V} = \{v \in X : v \in C(\overline{\Omega}) \cap L^\infty(\Omega) \cap C^1(\overline{\hat{A}_{R_0}})\},$$

where $\hat{A}_{R_0} = \Omega \cap \{x \in \mathbb{R}^N : |x| < R_0\}$, then $(\hat{V}, \|\cdot\|_{\hat{V}})$ is a Banach space with the norm given by

$$\|v\|_{\hat{V}} = \sup_{x \in \overline{\Omega}} |v(x)| + \|v\|_{C^1(\overline{\hat{A}_{R_0}})} + \|v\|_X.$$

Let $\hat{V}_+$ be the positve cone, i.e.,

$$\hat{V}_+ = \{v \in \hat{V} : v(x) \geq 0, \quad x \in \overline{\Omega}\},$$

then the interior of $\hat{V}_+$ is nonempty and can be characterized by

$$\begin{aligned}&\operatorname{int}(\hat{V}_+)\\ &= \{v \in \hat{V}_+ : v(x) > 0, \quad x \in \Omega, \ \frac{\partial v(x)}{\partial n} < 0 \text{ for all } x \in \partial\Omega = \partial B\}.\end{aligned}$$

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