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WEAK CONVERGENCE THEOREMS TO COMMON ATTRACTIVE POINTS OF NORMALLY 2-GENERALIZED HYBRID MAPPINGS WITH ERRORS

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Abstract. We prove weak convergence theorems for finding common attractive and fixed points of two normally 2-generalized hybrid mappings, which are not necessarily commutative. Basing on the ideas of mean convergence by Baillon [3], Shimizu and Takahashi [25, 26], and Atsushiba and Takahashi [2], we establish two alternative methods to approximate common attractive and fixed points. Moreover, we apply the method to a common null point problem for two maximal monotone multi-valued mappings. Our results are obtained under settings with finitely many error terms.

1. Introduction

Let H be a real Hilbert space. Its inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, where C is a nonempty subset of H. For nonexpansive mappings, many types of approximation methods for finding fixed points have been proposed. Reich [23] used Mann's type [21] iteration

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) T x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.1}$$

and showed that the sequence $\{x_n\}$ converges weakly to a fixed point of T in a setting of Banach spaces. In (1.1), $x_1 = x \in C$ is given, and $\mathbb{N}$ is the set of natural numbers, and $\{\lambda_n\}$ is a sequence of real numbers in the interval $[0, 1]$ that satisfies certain conditions. Atsushiba and Takahashi [2] employed the following iteration:

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.2}$$

where $x_1 \in C$ is given, and demonstrated weak convergence to common fixed points of S and T, where S and T are nonexpansive mappings such that $ST = TS$. The idea of mean convergence as (1.2) based on Baillon [3] and Shimizu and Takahashi [25, 26]. For further developments of iteration (1.2), see Kurokawa and Takahashi [20], Kohsaka [14], and Hojo and Takahashi [8].

Successive studies have demonstrated that conditions on the mapping T can be partly discarded to prove convergence theorems. Kocourek et al., in their 2010'

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paper [13], defined a wide class of mappings that contains nonexpansive mappings as special cases, and presented methods to approximate its fixed points. A mapping $T : C \to H$ is called *generalized hybrid* [13] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha) \|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta) \|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. The class of generalized hybrid mappings simultaneously includes nonexpansive mappings, *nonspreading mappings* [15], *hybrid mappings* [29], and λ-*hybrid mappings* [1] as special cases. Note that nonspreading mappings and hybrid mappings are not necessarily continuous; see [10]. The type of nonspreading mappings is deduced from optimization problems.

The class of generalized hybrid mappings has been further extended. A mapping $T : C \to C$ is called *normally* 2-*generalized hybrid* [16] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that $\sum_{n=0}^{2} (\alpha_n + \beta_n) \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$, and

$$\begin{aligned} \alpha_2 \left\|T^2x - Ty\right\|^2 &+ \alpha_1 \|Tx - Ty\|^2 + \alpha_0 \|x - Ty\|^2 \\ &+ \beta_2 \left\|T^2x - y\right\|^2 + \beta_1 \|Tx - y\|^2 + \beta_0 \|x - y\|^2 \leq 0 \end{aligned}$$

for all $x, y \in C$. This class of mappings contains generalized hybrid mappings and other classes of nonlinear mappings, e.g., *normally generalized hybrid mappings* [33] and 2-*generalized hybrid mappings* [22]. Hojo et al. [9] gave examples that were 2-generalized hybrid but not generalized hybrid. Unlike the case of 2-generalized hybrid mappings, it can be shown that a normally 2-generalized hybrid mapping has at most one fixed point if $\sum_{n=0}^{2} (\alpha_n + \beta_n) > 0$; see Theorem 4.3 in [18].

For a normally 2-generalized hybrid mapping T, Kondo and Takahashi [16] defined a sequence $\{x_n\}$ as follows:

$$x_{n+1} = a_n x_n + b_n T x_n + c_n T^2 x_n \in C, \tag{1.3}$$

where $x_1 \in C$ is given, and $\{a_n, b_n, c_n\}$ is a set of coefficients for a convex combination. They showed a weak convergence of $\{x_n\}$ to an *attractive point* [31] of T. An attractive point is defined in the next section. Let S and T be normally 2-generalized hybrid mappings from C into itself that are not necessarily commutative. Kondo and Takahashi [19] proved a weak convergence to common attractive and fixed points of S and T by using the following iteration:

$$x_{n+1} = a_n x_n + b_n S x_n + c_n S^2 x_n + d_n T x_n + e_n T^2 x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.4}$$

where $x_1 \in C$ is given, and $\{a_n, b_n, c_n, d_n, e_n\}$ is a set of coefficients for a convex combination. For a common attractive point problem of two noncommutative nonlinear mappings, see also [30] and [32].

In this paper, combining the ideas of iterations (1.2) and (1.4), we consider two types of iterations formulated by

$$x_{n+1} = a_n x_n + b_n \frac{1}{n} \sum_{k=1}^{n} S^k x_n + c_n \frac{1}{n} \sum_{k=1}^{n} T^k x_n \text{ and} \tag{1.5}$$

$$x_{n+1} = a_n x_n + b_n S x_n + c_n S^2 x_n + d_n \frac{1}{n} \sum_{k=1}^{n} T^k x_n, \tag{1.6}$$

where S and T are normally 2-generalized hybrid mappings, which are not necessarily commutative. Using these iterations, we show that the sequence $\{x_n\}$ converges weakly to common attractive and fixed points of S and T. Moreover, we apply the method to a common null point problem for two maximal monotone multi-valued mappings. Our results are obtained under settings with finitely many error terms. Each error term will vanish rapidly as Kamimura and Takahashi [12] and Takahashi [28].

2. Preliminaries

This section briefly presents preliminary information and results. Throughout this paper, we denote a real Hilbert space by H. Let $\{x_n\}$ be a sequence in H, and let x be an element of H. We write the strong and weak convergence of $\{x_n\}$ to x by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. It is well-known that a closed and convex subset of H is weakly closed. A sequence $\{x_n\}$ converges weakly to x if and only if for every subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ that converges weakly to x.

Let $T : C \to H$ be a mapping from C into H, where C is a nonempty subset of H. We denote sets of fixed and attractive points by

$$\begin{aligned} F(T) &= \{u \in C : Tu = u\} \text{ and} \\ A(T) &= \{u \in H : \|Ty - u\| \leq \|y - u\| \text{ for all } y \in C\}, \end{aligned}$$

respectively. Takahashi and Takeuchi, in their 2011's paper [31], introduced the concept of attractive points, and showed that the set of attractive points $A(T)$ is closed and convex in a Hilbert space. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|Tx - u\| \leq \|x - u\|$ for all $x \in C$ and $u \in F(T)$. If a mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive, then $F(T) \subset A(T)$. We know from [16] that a normally 2-generalized hybrid mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive. We also know that the set of fixed points $F(T)$ of a quasi-nonexpansive mapping is closed and convex; see [11]. A mapping $T : C \to H$ is called *firmly nonexpansive* if $\|Tx - Ty\|^2 \leq \langle x - y,\ Tx - Ty\rangle$ for all $x, y \in C$. It is well-known that a firmly nonexpansive mapping is nonexpansive. For this type of mappings, see, e.g., Browder [4] and Goebel and Kirk [5].

Let A be a nonempty, closed, and convex subset of H, and let P_A be the *metric projection* from H onto A, that is, $\|x - P_A x\| \leq \|x - z\|$ for all $x \in H$ and $z \in A$. It is known that the metric projection is firmly nonexpansive, and thus, it is nonexpansive. Furthermore, it holds that $\langle x - P_A x,\ P_A x - z\rangle \geq 0$ for all $x \in H$ and $z \in A$.

Let $B : H \to 2^H$ be a multi-valued mapping defined on H. We write it as $B \subset H \times H$. Its effective domain is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping $B \subset H \times H$ is called *monotone* if $\langle x - y,\ u - v\rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. For a monotone multi-valued mapping B on H and $r > 0$, define $J_r \equiv (I + rB)^{-1}$, where I is the identity mapping on $D(B)$. It is called the *resolvent* of B for $r > 0$. It is well-known that J_r is single-valued and firmly nonexpansive. Also, it is known that $F(J_r) = B^{-1}0$ for all $r > 0$, where $B^{-1}0 = \{x \in H : 0 \in Bx\}$. For a monotone multi-valued mapping B on H and $r > 0$, *Yoshida approximation* is defined as $A_r \equiv \frac{1}{r}(I - J_r)$, which is also a single valued-mapping from $D(B)$ into H. It is known that $(J_r x, A_r x) \in A$ for all $x \in H$ and $r > 0$.

A monotone mapping is said to be *maximal* if its graph is not properly contained by any other monotone mappings on H. For a maximal monotone multi-valued mapping $B \subset H \times H$, its null point set $B^{-1}0$ is a closed and convex subset of its effective domain $D(B)$. It is also known that if a multi-valued mapping B is maximal monotone, its resolvent $J_r \equiv (I + rB)^{-1}$ and Yoshida approximation $A_r \equiv \frac{1}{r}(I - J_r)$ are defined on the whole area of H. In other words, J_r is a mapping from H into $D(B)$, and A_r is a mapping from H into itself. Let $B \subset H \times H$ be a maximal monotone multi-valued mapping, and $v, w \in H$. Then, the following holds: if $\langle a - v,\ b - w\rangle \geq 0$ for all $(a, b) \in B$, then $(v, w) \in B$. For more details, see Takahashi [27] and [28].

Hojo [6] proved the following theorem, which clarifies a set of assumptions that guarantees that there exist common attractive and fixed points of two normally 2-generalized hybrid mappings.

Theorem 2.1 ([6])**.** *Let C be a nonempty subset of H, and let S and T be commutative normally 2-generalized hybrid mappings from C into itself. Suppose that there exists an element $z \in C$ such that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then, $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty.*

In the next lemma, the part (a) was proved by Takahashi [28], while (b) was established by Maruyama et al. [22] to deal with 2-generalized hybrid mappings. For a proof of (c), see [19].

Lemma 2.2 ([22, 28])**.** *Let $x, y, z, w \in H$ and $a, b, c, d \in \mathbb{R}$. Then, the following hold:*

(a) *If $a + b = 1$, then $\|ax + by\|^2 = a\|x\|^2 + b\|y\|^2 - ab\|x - y\|^2$.*

(b) *If $a + b + c = 1$, then*

$$\begin{aligned}\|ax + by + cz\|^2 &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 \\ &\quad - ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2.\end{aligned}$$

(c) *If $a + b + c + d = 1$, then*

$$\begin{aligned}\|ax + by + cz + dw\|^2 &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 \\ &\quad - ab\|x - y\|^2 - ac\|x - z\|^2 - ad\|x - w\|^2 \\ &\quad - bc\|y - z\|^2 - bd\|y - w\|^2 - cd\|z - w\|^2.\end{aligned}$$

The following lemma was proved by Takahashi [28] as Problem 8.2.1. For completeness, we prove it here.

Lemma 2.3 ([28])**.** *Let $\{x_n\}$ be a sequence of real numbers that is bounded from below, and let $\{\eta_n\}$ be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} \eta_n < \infty$. Supposed that $x_{n+1} \leq x_n + \eta_n$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is convergent.*

Proof. We prove that $\limsup_{m\to\infty} x_m \leq \liminf_{n\to\infty} x_n$ and $\limsup_{m\to\infty} x_m < \infty$. Let $m, n \in \mathbb{N}$ with $m \geq n$. It holds from an assumption that

$$x_{n+1} \leq x_n + \eta_n,$$

$$\begin{aligned} x_{n+2} &\leq x_{n+1} + \eta_{n+1}, \\ &\vdots \\ x_{m+1} &\leq x_m + \eta_m. \end{aligned}$$

Summing these inequalities, we obtain

$$x_{m+1} \leq x_n + \sum_{k=n}^{m} \eta_k \leq x_n + \sum_{k=n}^{\infty} \eta_k$$

for all $m \in \mathbb{N}$ with $m \geq n$. This means that $\limsup_{m\to\infty} x_m < \infty$. Taking $\limsup_{m\to\infty}$, we have that

$$\limsup_{m\to\infty} x_m \leq x_n + \sum_{k=n}^{\infty} \eta_k$$

for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} \eta_k < \infty$, it holds that $\sum_{k=n}^{\infty} \eta_k \to 0$ as $n \to \infty$. Therefore, taking $\liminf_{n\to\infty}$, we obtain

$$\limsup_{m\to\infty} x_m \leq \liminf_{n\to\infty} x_n.$$

This completes the proof. □

Note that if $\eta_n = 0$ for all $n \in \mathbb{N}$ in Lemma 2.3, the lemma simply asserts that a sequence of real numbers that is monotone decreasing and bounded from below is convergent. The next lemma, together with Lemma 2.3, will be utilized in the proofs of our main theorems. Basing on the proofs of Lemma 2 in Kamimura and Takahashi [12] and Lemma 8.2.1 in Takahashi [28], we extend their results to a case with finitely many error terms.

Lemma 2.4. *Let C be a nonempty and convex subset of H, let A be a nonempty, closed and convex subset of H, and let P_A be the metric projection from H onto A. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n + d_n = 1$ for all $n \in \mathbb{N}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, and $\sum_{n=1}^{\infty} \delta_n < \infty$. Let $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ be sequences in C. Given $x_1 \in C$, define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = a_n X_n + b_n Y_n + c_n Z_n + d_n W_n \, (\in C),$$

where

$$X_n \in C \quad \text{such that } \|X_n - x_n\| \leq \alpha_n, \tag{2.1}$$

$$Y_n \in C \quad \text{such that } \|Y_n - y_n\| \leq \beta_n, \tag{2.2}$$

$$Z_n \in C \quad \text{such that } \|Z_n - z_n\| \leq \gamma_n, \tag{2.3}$$

$$W_n \in C \quad \text{such that } \|W_n - w_n\| \leq \delta_n. \tag{2.4}$$

Suppose that

$$\|y_n - u\| \leq \|x_n - u\|, \quad \|z_n - u\| \leq \|x_n - u\|, \tag{2.5}$$

$$\|w_n - u\| \leq \|x_n - u\| \tag{2.6}$$

for all $u \in A$ and $n \in \mathbb{N}$. Then, $\{P_A x_n\}$ converges in A, in other words, there exists an element $\overline{x} \in A$ and $P_A x_n \to \overline{x}$.

Proof. Define $\eta_n \equiv \alpha_n + \beta_n + \gamma_n + \delta_n \in \mathbb{R}$. Then, it holds that $\eta_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \eta_n < \infty$. First, we verify that

$$\|x_{n+1} - u\| \leq \|x_n - u\| + \eta_n \tag{2.7}$$

for all $u \in A$ and $n \in \mathbb{N}$. Indeed, it holds from (2.1)–(2.4) that

$$\begin{aligned} \|x_{n+1} - u\| &= \|a_n (X_n - u) + b_n (Y_n - u) + c_n (Z_n - u) + d_n (W_n - u)\| \\ &\leq a_n \|X_n - u\| + b_n \|Y_n - u\| + c_n \|Z_n - u\| + d_n \|W_n - u\| \\ &\leq a_n (\|X_n - x_n\| + \|x_n - u\|) + b_n (\|Y_n - y_n\| + \|y_n - u\|) \\ &\quad + c_n (\|Z_n - z_n\| + \|z_n - u\|) + d_n (\|W_n - w_n\| + \|w_n - u\|) \\ &\leq a_n (\alpha_n + \|x_n - u\|) + b_n (\beta_n + \|y_n - u\|) \\ &\quad + c_n (\gamma_n + \|z_n - u\|) + d_n (\delta_n + \|w_n - u\|). \end{aligned}$$

Using (2.5) and (2.6), we have that

$$\begin{aligned} \|x_{n+1} - u\| &\leq a_n (\alpha_n + \|x_n - u\|) + b_n (\beta_n + \|x_n - u\|) \\ &\quad + c_n (\gamma_n + \|x_n - u\|) + d_n (\delta_n + \|x_n - u\|) \\ &\leq \|x_n - u\| + \alpha_n + \beta_n + \gamma_n + \delta_n \\ &= \|x_n - u\| + \eta_n. \end{aligned}$$

From Lemma 2.3, $\{\|x_n - u\|\}$ converges for all $u \in A$, and thus, $\{x_n\}$ is bounded. From (2.5)–(2.6), $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are bounded. Since the metric projection is nonexpansive, $\{P_A x_n\}$ is also bounded.

Define $g : A \to \mathbb{R}$ by

$$g(u) = \lim_{n\to\infty} \|x_n - u\| \quad \text{for each } u \in A.$$

Then, g is nonexpansive and convex. Additionally, it satisfies that if $\|u_m\| \to \infty$, then $g(u_m) \to \infty$, where $\{u_m\}$ is a sequence in A. Therefore, there exists a unique element $\overline{x} \in A$ such that

$$g(\overline{x}) = \inf_{u\in A} g(u).$$

(For these points, see Problem 5.3 in Takahashi [28].)

Define $l \equiv g(\overline{x})$ $(\in \mathbb{R})$, that is,

$$l \equiv g(\overline{x}) \equiv \lim_{n\to\infty} \|x_n - \overline{x}\| = \inf_{u\in A} \lim_{n\to\infty} \|x_n - u\|. \tag{2.8}$$

Since $P_A x_m, \overline{x} \in A$, we have that $\frac{1}{2}(P_A x_m + \overline{x}) \in A$ for all $m \in \mathbb{N}$. Therefore,

$$l \leq g\left(\frac{1}{2}(P_A x_m + \overline{x})\right) = \lim_{n\to\infty} \left\|x_n - \frac{1}{2}(P_A x_m + \overline{x})\right\| \tag{2.9}$$

for all $m \in \mathbb{N}$. Define

$$M \equiv 2 \sup_{n\in\mathbb{N}} \left\|x_n - \frac{1}{2}(P_A x_n + \overline{x})\right\| + \sum_{n=1}^{\infty} \eta_n \,(\in \mathbb{R}). \tag{2.10}$$

Since $\{x_n\}$ and $\{P_A x_n\}$ are bounded, M is a real number.

Next, we show that

$$\limsup_{n\to\infty} \|x_n - P_A x_n\| \le l. \tag{2.11}$$

Since $\overline{x} \in A$, it holds that $\|x_n - P_A x_n\| \le \|x_n - \overline{x}\|$ for all $n \in \mathbb{N}$. Taking the lim sup as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \|x_n - P_A x_n\| \le \lim_{n\to\infty} \|x_n - \overline{x}\| = l.$$

Thus, (2.11) holds.

Our aim is to show that $P_A x_n \to \overline{x}$. Suppose by way of contradiction that $P_A x_n \nrightarrow \overline{x}$. Then,

$$\begin{aligned} &\exists \varepsilon > 0 \text{ that satisfies the following:} \\ &\forall n \in \mathbb{N},\ \exists n_0 \ge n \text{ such that } \|P_A x_{n_0} - \overline{x}\| \ge \varepsilon. \end{aligned} \tag{2.12}$$

For $l \ge 0$ and $\varepsilon > 0$, choose $b > 0$ such that

$$0 < b < \sqrt{l^2 + \frac{\varepsilon^2}{8}} - l, \quad \text{i.e., } (l+b)^2 < l^2 + \frac{\varepsilon^2}{8}. \tag{2.13}$$

For $l \ge 0$ and $b > 0$, we have from (2.8), (2.11), and $\sum_{n=1}^\infty \eta_n < \infty$ that

$$\begin{aligned} &\exists n_1 \in \mathbb{N} \text{ such that } \forall n \ge n_1, \\ &\qquad \|x_n - \overline{x}\| \le l + b, \quad \|x_n - P_A x_n\| \le l + b, \quad M \sum_{i=n}^\infty \eta_i \le \frac{\varepsilon^2}{8}, \end{aligned}$$

where M is defined in (2.10). From (2.12), we can re-choose $n_0\,(\ge n_1)$ that satisfies the following:

$$\|P_A x_{n_0} - \overline{x}\| \ge \varepsilon, \tag{2.14}$$

$$\|x_{n_0} - \overline{x}\| \le l + b, \tag{2.15}$$

$$\|x_{n_0} - P_A x_{n_0}\| \le l + b, \tag{2.16}$$

$$M \sum_{i=n_0}^\infty \eta_i \le \frac{\varepsilon^2}{8}. \tag{2.17}$$

Since $\frac{1}{2}(P_A x_{n_0} + \overline{x}) \in A$, we have from (2.7) that

$$\begin{aligned} \left\| x_{n+n_0+1} - \frac{1}{2}(P_A x_{n_0} + \overline{x}) \right\| &\le \left\| x_{n+n_0} - \frac{1}{2}(P_A x_{n_0} + \overline{x}) \right\| + \eta_{n+n_0} \\ &\le \cdots \\ &\le \left\| x_{n_0} - \frac{1}{2}(P_A x_{n_0} + \overline{x}) \right\| + \sum_{i=n_0}^{n+n_0} \eta_i \\ &\le \left\| x_{n_0} - \frac{1}{2}(P_A x_{n_0} + \overline{x}) \right\| + \sum_{i=n_0}^{\infty} \eta_i \end{aligned}$$

for all $n \in \mathbb{N}$. Thus, we obtain

$$\begin{aligned}\left\|x_{n+n_0+1} - \frac{1}{2}(P_A x_{n_0} + \overline{x})\right\|^2 &\leq \left(\left\|x_{n_0} - \frac{1}{2}(P_A x_{n_0} + \overline{x})\right\| + \sum_{i=n_0}^{\infty} \eta_i\right)^2 \\ &\leq \left\|x_{n_0} - \frac{1}{2}(P_A x_{n_0} + \overline{x})\right\|^2 + M \sum_{i=n_0}^{\infty} \eta_i \\ &= \left\|\frac{1}{2}(x_{n_0} - P_A x_{n_0}) + \frac{1}{2}(x_{n_0} - \overline{x})\right\|^2 + M \sum_{i=n_0}^{\infty} \eta_i.\end{aligned}$$

By using Lemma 2.2-(a), we have that

$$\begin{aligned}\left\|x_{n+n_0+1} - \frac{1}{2}(P_A x_{n_0} + \overline{x})\right\|^2 \leq\ & \frac{1}{2}\|x_{n_0} - P_A x_{n_0}\|^2 + \frac{1}{2}\|x_{n_0} - \overline{x}\|^2 \\ & -\frac{1}{4}\|P_A x_{n_0} - \overline{x}\|^2 + M \sum_{i=n_0}^{\infty} \eta_i.\end{aligned}$$

From (2.14)–(2.17), we obtain

$$\begin{aligned}\left\|x_{n+n_0+1} - \frac{1}{2}(P_A x_{n_0} + \overline{x})\right\|^2 &\leq (l+b)^2 - \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{8} \\ &= (l+b)^2 - \frac{\varepsilon^2}{8}\end{aligned} \tag{2.18}$$

for all $n \in \mathbb{N}$. Since $\frac{1}{2}(P_A x_{n_0} + \overline{x}) \in A$, we have from (2.7) and Lemma 2.3 that the sequence

$$\left\{\left\|x_{n+n_0+1} - \frac{1}{2}(P_A x_{n_0} + \overline{x})\right\|^2\right\}_{n\in\mathbb{N}}$$

is convergent. It follows from (2.9) and (2.18) that

$$\begin{aligned}l^2 &\leq \lim_{n\to\infty}\left\|x_{n+n_0+1} - \frac{1}{2}(P_A x_{n_0} + \overline{x})\right\|^2 \\ &\leq (l+b)^2 - \frac{\varepsilon^2}{8}.\end{aligned}$$

Thus, $l^2 + \frac{\varepsilon^2}{8} \leq (l+b)^2$, which contradicts (2.13). Thus, we obtain $P_A x_n \to \overline{x}$ as claimed. □

Letting $\alpha_n = \beta_n = \gamma_n = \delta_n = 0$ in Lemma 2.4, we obtain the following corollary.

Corollary 2.5. *Let C be a nonempty and convex subset of H, let A be a nonempty, closed and convex subset of H, and let P_A be the metric projection from H onto A. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n+b_n+c_n+d_n = 1$ for all $n \in \mathbb{N}$. Let $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ be sequences in C. Given $x_1 \in C$, define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = a_n x_n + b_n y_n + c_n z_n + d_n w_n \,(\in C)\,.$$

Suppose that $\|y_n - u\| \leq \|x_n - u\|$, $\|z_n - u\| \leq \|x_n - u\|$, *and* $\|w_n - u\| \leq \|x_n - u\|$ *for all* $u \in A$ *and* $n \in \mathbb{N}$. *Then,* $\{P_A x_n\}$ *converges in* A, *in other words, there exists an element* $\overline{x} \in A$ *and* $P_A x_n \to \overline{x}$.

The following two lemmas conclude that a weak limit of a sequence is an attractive point. The proof of Lemma 2.6 was developed in [26] and [20].

Lemma 2.6 ([17])**.** *Let* C *be a nonempty subset of* H, *and let* T *be a normally* 2-*generalized hybrid mapping from* C *into itself with* $A(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a bounded sequence in* H. *Define* $z_n \equiv \frac{1}{n}\sum_{k=1}^{n} T^k x_n \,(\in H)$, *and suppose that* $z_{n_i} \rightharpoonup v \,(\in H)$, *where* $\{z_{n_i}\}$ *is a subsequence of* $\{z_n\}$. *Then,* $v \in A(T)$.

For development of proofs of Lemma 2.7, see [13] and [22].

Lemma 2.7 ([16])**.** *Let* C *be a nonempty subset of* H, *let* S *be a normally* 2-*generalized hybrid mapping from* C *into itself, and let* $\{x_n\}$ *be a sequence in* C. *If* $\{x_n\}$ *satisfies* $Sx_n - x_n \to 0$, $S^2 x_n - x_n \to 0$ *and* $x_n \rightharpoonup v$, *then* $v \in A(S)$.

The next lemma was substantially included in Kamimura and Takahashi [12].

Lemma 2.8. *Let* C *be a nonempty subset of* H, *and let* $A \subset H \times H$ *be a maximal monotone multi-valued mapping on* H *such that its domain is included in* C. *Define* $J_r = (I + rA)^{-1}$ *for* $r > 0$, *where* I *is the identity mapping. Let* $\{r_n\}$ *be a sequence of positive real numbers such that* $r_n \to \infty$, *and let* $\{x_n\}$ *be a bounded sequence in* H. *Suppose that* $J_{r_{n_i}} x_{n_i} \rightharpoonup v \in C$, *where* $\{r_{n_i}\}$ *and* $\{x_{n_i}\}$ *are subsequences of* $\{r_n\}$ *and* $\{x_n\}$, *respectively. Then,* $v \in A^{-1}0$.

Proof. Define $A_r = \frac{1}{r}(I - J_r)$, which is the Yoshida approximation of A for $r > 0$. Since $A \subset H \times H$ is maximal monotone, the domains of J_r and A_r are H. Thus, for the sequence $\{x_n\}$ in H, $\{J_{r_n} x_n\}$ and $\{A_{r_n} x_n\} \,(\subset H)$ are well-defined.

First, we show that $A_{r_{n_i}} x_{n_i} \to 0$. Since $\{x_n\}$ is bounded and J_{r_n} is nonexpansive for all $n \in \mathbb{N}$, $\{J_{r_n} x_n\}$ is also bounded. Therefore, we have from $r_n \to \infty$ that

$$\left\|A_{r_{n_i}} x_{n_i}\right\| = \frac{1}{r_{n_i}} \left\|x_{n_i} - J_{r_{n_i}} x_{n_i}\right\| \to 0$$

as $i \to \infty$.

Our goal is to prove that $(v, 0) \in A$. Since A is maximal monotone, it suffices to demonstrate that $\langle a - v,\ b - 0\rangle \geq 0$ for all $(a, b) \in A$. Let $(a, b) \in A$. We know that $(J_{r_n} x_n, A_{r_n} x_n) \in A$ for all $n \in \mathbb{N}$. Since A is monotone, it holds that

$$\langle a - J_{r_n} x_n,\ b - A_{r_n} x_n\rangle \geq 0.$$

Since $J_{r_{n_i}} x_{n_i} \rightharpoonup v$ and $A_{r_{n_i}} x_{n_i} \to 0$, replacing n by n_i and taking the limit as $i \to \infty$, we obtain $\langle a - v,\ b - 0\rangle \geq 0$. This completes the proof. □

The following lemma was demonstrated by Takahashi and Takeuchi [31], which is useful to prove fixed point approximations.

Lemma 2.9 ([31])**.** *Let* C *be a nonempty subset of* H, *and let* T *be a mapping from* C *into* H. *Then,* $A(T) \cap C \subset F(T)$.

3. Weak Convergence for Nonlinear Mappings

This section presents two types of iterations (1.5) and (1.6) to approximate common attractive points of normally 2-generalized hybrid mappings. The results are obtained under settings with finitely many error terms. The proofs do not rely on the assumption that the domains of the mappings are closed. By additionally supposing that the domains are closed, we obtain approximation methods for finding fixed points. The fundamentals of the proofs were developed by many authors. For example, see [7, 12, 13, 19, 22, 28, 33].

Theorem 3.1. *Let C be a nonempty and convex subset of H, and let S and T be normally 2-generalized hybrid mappings from C into itself. Suppose that $A(S) \cap A(T)$ is nonempty. Let P_A be the metric projection from H onto $A(S) \cap A(T)$. Let $a, b \in (0,1)$ such that $a \leq b$, and let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers such that $a_n + b_n + c_n = 1$ and $0 < a \leq a_n, b_n, c_n \leq b < 1$ for all $n \in \mathbb{N}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C : \textit{given},$$

$$X_n \in C \quad \textit{such that} \quad \|X_n - x_n\| \leq \alpha_n, \tag{3.1}$$

$$Y_n \in C \quad \textit{such that} \quad \left\| Y_n - \frac{1}{n} \sum_{k=1}^{n} S^k x_n \right\| \leq \beta_n, \tag{3.2}$$

$$Z_n \in C \quad \textit{such that} \quad \left\| Z_n - \frac{1}{n} \sum_{k=1}^{n} T^k x_n \right\| \leq \gamma_n, \tag{3.3}$$

$$x_{n+1} = a_n X_n + b_n Y_n + c_n Z_n \in C \quad \textit{for all } n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to a common attractive point $\overline{x} \in A(S) \cap A(T)$, where $\overline{x} \equiv \lim_{n\to\infty} P_A x_n$. Additionally, if C is closed, then $\{x_n\}$ converges weakly to a common fixed point $\widehat{x} \equiv \lim_{n\to\infty} P_F x_n \in F(S) \cap F(T)$, where P_F is the metric projection form H onto $F(S) \cap F(T)$.

Proof. Note that from Takahashi and Takeuchi [31], $A(S) \cap A(T)$ is a closed and convex subset of H. Since $A(S) \cap A(T) \neq \emptyset$ is assumed, there exists the metric projection P_A from H onto $A(S) \cap A(T)$.

Define $y_n \equiv \frac{1}{n} \sum_{k=1}^{n} S^k x_n \,(\in C)$ and $z_n \equiv \frac{1}{n} \sum_{k=1}^{n} T^k x_n \,(\in C)$. It is easy to verify that

$$\|y_n - u\| \leq \|x_n - u\| \quad \text{and} \quad \|z_n - u\| \leq \|x_n - u\| \tag{3.4}$$

for all $u \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. Indeed, since $u \in A(S)$, we have that

$$\begin{aligned} \|y_n - u\| &= \left\| \frac{1}{n} \sum_{k=1}^{n} S^k x_n - u \right\| = \frac{1}{n} \left\| \sum_{k=1}^{n} S^k x_n - nu \right\| \\ &= \frac{1}{n} \left\| \sum_{k=1}^{n} \left(S^k x_n - u \right) \right\| \leq \frac{1}{n} \sum_{k=1}^{n} \left\| \left(S^k x_n - u \right) \right\| \end{aligned}$$

$$\leq \frac{1}{n} \sum_{k=1}^{n} \|x_n - u\| = \|x_n - u\| .$$

Similarly, $\|z_n - u\| \leq \|x_n - u\|$ can be proved since $u \in A(T)$.

We show that the sequence $\{P_A x_n\}$ is convergent in $A(S) \cap A(T)$. Consider the case of $d_n = 0$ for all $n \in \mathbb{N}$ in Lemma 2.4. Assumptions (3.1)–(3.3) imply (2.1)–(2.3), respectively. Also, from (3.4), the conditions in (2.5) are satisfied. Thus, from Lemma 2.4, there exists $\overline{x} \in A(S) \cap A(T)$ such that $P_A x_n \to \overline{x}$. Our first aim is to show that $x_n \rightharpoonup \overline{x}$.

Note that the following hold:

$$\begin{aligned} \|X_n - u\| &\leq \|x_n - u\| + \alpha_n, \\ \|Y_n - u\| &\leq \|x_n - u\| + \beta_n, \\ \|Z_n - u\| &\leq \|x_n - u\| + \gamma_n \end{aligned} \tag{3.5}$$

for all $u \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. Indeed, it holds from (3.1) that

$$\begin{aligned} \|X_n - u\| &\leq \|X_n - x_n\| + \|x_n - u\| \\ &\leq \alpha_n + \|x_n - u\| . \end{aligned}$$

It follows from (3.2) and (3.4) that

$$\begin{aligned} \|Y_n - u\| &\leq \|Y_n - y_n\| + \|y_n - u\| \\ &\leq \beta_n + \|x_n - u\| \end{aligned}$$

since $u \in A(S)$. Similarly, since $u \in A(T)$, we can obtain $\|Z_n - u\| \leq \|x_n - u\| + \gamma_n$ by using (3.3) and (3.4).

Next, we show that the sequence $\{\|x_n - u\|\}$ is convergent in $\mathbb{R}$ for all $u \in A(S) \cap A(T)$. Define $\eta_n \equiv \alpha_n + \beta_n + \gamma_n \, (\geq 0)$. Since $\sum_{n=1}^{\infty} \eta_n < \infty$, from Lemma 2.3, it suffices to demonstrate that

$$\|x_{n+1} - u\| \leq \|x_n - u\| + \eta_n \tag{3.6}$$

for all $u \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. The inequality (3.6) can be verified by using (3.5) as follows:

$$\begin{aligned} \|x_{n+1} - u\| &= \|a_n (X_n - u) + b_n (Y_n - u) + c_n (Z_n - u)\| \\ &\leq a_n \|X_n - u\| + b_n \|Y_n - u\| + c_n \|Z_n - u\| \\ &\leq a_n (\|x_n - u\| + \alpha_n) + b_n (\|x_n - u\| + \beta_n) + c_n (\|x_n - u\| + \gamma_n) \\ &\leq \|x_n - u\| + \eta_n. \end{aligned}$$

Thus, $\{\|x_n - u\|\}$ is convergent. Consequently, $\{x_n\}$ is bounded. Since P_A is nonexpansive, $\{P_A x_n\}$ is also bounded.

Let us show that

$$\begin{aligned} &a_n b_n \|X_n - Y_n\|^2 + b_n c_n \|Y_n - Z_n\|^2 + c_n a_n \|Z_n - X_n\|^2 \\ &\qquad \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + M \eta_n \end{aligned} \tag{3.7}$$

for all $u \in A(S) \cap A(T)$ and $n \in \mathbb{N}$, where

$$M \equiv \sup_{n \in \mathbb{N}} (2 \|x_n - u\| + \eta_n) .$$

Since $\{x_n\}$ and $\{\eta_n\}$ are bounded, M is a real number. The inequality (3.7) can be demonstrated as follows. By using Lemma 2.2-(b), we obtain

$$\begin{aligned}\|x_{n+1}-u\|^2 &= \|a_n(X_n-u)+b_n(Y_n-u)+c_n(Z_n-u)\|^2\\ &= a_n\|X_n-u\|^2+b_n\|Y_n-u\|^2+c_n\|Z_n-u\|^2\\ &\quad -a_nb_n\|X_n-Y_n\|^2-b_nc_n\|Y_n-Z_n\|^2-c_na_n\|Z_n-X_n\|^2.\end{aligned}$$

From (3.5), we have that

$$\begin{aligned}& \|x_{n+1}-u\|^2\\ \leq\ & a_n(\|x_n-u\|+\alpha_n)^2+b_n(\|x_n-u\|+\beta_n)^2+c_n(\|x_n-u\|+\gamma_n)^2\\ & -a_nb_n\|X_n-Y_n\|^2-b_nc_n\|Y_n-Z_n\|^2-c_na_n\|Z_n-X_n\|^2\\ =\ & a_n\|x_n-u\|^2+b_n\|x_n-u\|^2+c_n\|x_n-u\|^2\\ & +a_n\left(2\|x_n-u\|\alpha_n+\alpha_n^2\right)+b_n\left(2\|x_n-u\|\beta_n+\beta_n^2\right)\\ & +c_n\left(2\|x_n-u\|\gamma_n+\gamma_n^2\right)\\ & -a_nb_n\|X_n-Y_n\|^2-b_nc_n\|Y_n-Z_n\|^2-c_na_n\|Z_n-X_n\|^2\\ \leq\ & \|x_n-u\|^2+\left(2\|x_n-u\|\alpha_n+\alpha_n^2\right)+\left(2\|x_n-u\|\beta_n+\beta_n^2\right)\\ & +\left(2\|x_n-u\|\gamma_n+\gamma_n^2\right)\\ & -a_nb_n\|X_n-Y_n\|^2-b_nc_n\|Y_n-Z_n\|^2-c_na_n\|Z_n-X_n\|^2\\ \leq\ & \|x_n-u\|^2+\alpha_n(2\|x_n-u\|+\alpha_n)+\beta_n(2\|x_n-u\|+\beta_n)\\ & +\gamma_n(2\|x_n-u\|+\gamma_n)\\ & -a_nb_n\|X_n-Y_n\|^2-b_nc_n\|Y_n-Z_n\|^2-c_na_n\|Z_n-X_n\|^2\\ \leq\ & \|x_n-u\|^2+\alpha_n(2\|x_n-u\|+\eta_n)+\beta_n(2\|x_n-u\|+\eta_n)\\ & +\gamma_n(2\|x_n-u\|+\eta_n)\\ & -a_nb_n\|X_n-Y_n\|^2-b_nc_n\|Y_n-Z_n\|^2-c_na_n\|Z_n-X_n\|^2\\ \leq\ & \|x_n-u\|^2+M\eta_n\\ & -a_nb_n\|X_n-Y_n\|^2-b_nc_n\|Y_n-Z_n\|^2-c_na_n\|Z_n-X_n\|^2.\end{aligned}$$

This means that (3.7) holds.

Since $\{\|x_n-u\|\}$ is convergent for $u\in A(S)\cap A(T)$ and $\eta_n\to 0$, we have from (3.7) that

$$X_n-Y_n\to 0 \quad\text{and}\quad X_n-Z_n\to 0. \tag{3.8}$$

Furthermore, it holds that

$$x_n-y_n\to 0 \quad\text{and}\quad x_n-z_n\to 0. \tag{3.9}$$

Indeed, it follow from (3.1), (3.8) and (3.2) that

$$\begin{aligned}\|x_n-y_n\| &\leq \|x_n-X_n\|+\|X_n-Y_n\|+\|Y_n-y_n\|\\ &\leq \alpha_n+\|X_n-Y_n\|+\beta_n\to 0.\end{aligned}$$

Similarly, we can prove that $x_n-z_n\to 0$.

We show that $x_n \rightharpoonup \overline{x}$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ and $v \in H$ such that $x_{n_j} \rightharpoonup v$. From (3.9), it holds that

$$y_{n_j} \rightharpoonup v \text{ and } z_{n_j} \rightharpoonup v.$$

We have from Lemma 2.6 that $v \in A(S) \cap A(T)$. Therefore, it follows that

$$\langle x_n - P_A x_n,\ P_A x_n - v \rangle \geq 0$$

for all $n \in \mathbb{N}$. Since $\{x_n\}$ and $\{P_A x_n\}$ are bounded, we have

$$\begin{aligned} \langle x_n - P_A x_n,\ v - \overline{x} \rangle &\leq \langle x_n - P_A x_n,\ P_A x_n - \overline{x} \rangle \\ &\leq \|x_n - P_A x_n\| \|P_A x_n - \overline{x}\| \\ &\leq L \|P_A x_n - \overline{x}\| \end{aligned}$$

for all $n \in \mathbb{N}$, where $L \equiv \sup_{n\in\mathbb{N}} \|x_n - P_A x_n\| \in \mathbb{R}$. Replacing n by n_j, and taking the limit as $j \to \infty$, we obtain

$$\langle v - \overline{x},\ v - \overline{x} \rangle \leq 0.$$

Thus, $v = \overline{x}$. This means that $x_n \rightharpoonup \overline{x}$.

Assume, in addition to the other assumptions, that C is closed in H. Our goal is to prove that $x_n \rightharpoonup \widehat{x} \equiv \lim_{k\to\infty} P_F x_k$. Since C is weakly closed and $x_n \rightharpoonup \overline{x}$, it follows that $\overline{x} \in C$, where $\overline{x} \equiv \lim_{n\to\infty} P_A x_n$. This implies that $\overline{x} \in C \cap A(S) \cap A(T)$. From Lemma 2.9, we have that $\overline{x} \in F(S) \cap F(T)$. Thus, $F(S) \cap F(T)$ is nonempty. Since S and T are quasi-nonexpansive, $F(S) \cap F(T)$ is closed and convex. Hence, there exists the metric projection P_F from H onto $F(S) \cap F(T)$. In much the same way as the proof of (3.4), we obtain

$$\|y_n - u\| \leq \|x_n - u\| \quad \text{and} \quad \|z_n - u\| \leq \|x_n - u\|$$

for all $u \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ since S and T are quasi-nonexpansive. Thus, we have from Lemma 2.4 that $\{P_F x_n\}$ converges strongly to an element $\widehat{x}$ of $F(S) \cap F(T)$, that is, $\widehat{x} \equiv \lim_{n\to\infty} P_F x_n$. We show that

$$\overline{x}\left(\equiv \lim_{n\to\infty} P_A x_n\right) = \widehat{x}\left(\equiv \lim_{n\to\infty} P_F x_n\right).$$

Since $\overline{x} \in F(S) \cap F(T)$, we have that

$$\langle x_n - P_F x_n,\ P_F x_n - \overline{x} \rangle \geq 0$$

for all $n \in \mathbb{N}$. Since $x_n \rightharpoonup \overline{x}$ and $P_F x_n \to \widehat{x}$, we have that $\langle \overline{x} - \widehat{x},\ \widehat{x} - \overline{x} \rangle \geq 0$, which means that $\widehat{x} = \overline{x}$. Hence, $\{x_n\}$ converges weakly to $\widehat{x} = \lim_{n\to\infty} P_F x_n \in F(S) \cap F(T)$. This completes the proof. □

Letting $\alpha_n = \beta_n = \gamma_n = 0$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let C be a nonempty and convex subset of H, and let S and T be normally 2-generalized hybrid mappings from C into itself. Suppose that $A(S) \cap A(T)$ is nonempty. Let P_A be the metric projection from H onto $A(S) \cap A(T)$. Let $a, b \in (0,1)$ such that $a \leq b$, and let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers such that $a_n + b_n + c_n = 1$ and $0 < a \leq a_n, b_n, c_n \leq b < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:*

$x_1 \in C$: *given*

$$x_{n+1} = a_n x_n + b_n \frac{1}{n} \sum_{k=1}^{n} S^k x_n + c_n \frac{1}{n} \sum_{k=1}^{n} T^k x_n \in C \quad \text{for all } n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ *converges weakly to a common attractive point* $\overline{x} \in A(S) \cap A(T)$, *where* $\overline{x} \equiv \lim_{n\to\infty} P_A x_n$. *Additionally, if* C *is closed, then* $\{x_n\}$ *converges weakly to a common fixed point* $\widehat{x} \equiv \lim_{n\to\infty} P_F x_n \in F(S) \cap F(T)$, *where* P_F *is the metric projection form* H *onto* $F(S) \cap F(T)$.

The next theorem provides an alternative method to approximate common attractive and fixed points of normally 2-generalized hybrid mappings.

Theorem 3.3. *Let* C *be a nonempty and convex subset of* H, *and let* S *and* T *be normally 2-generalized hybrid mappings from* C *into itself. Suppose that* $A(S) \cap A(T)$ *is nonempty. Let* P_A *be the metric projection from* H *onto* $A(S) \cap A(T)$. *Let* $a, b \in (0,1)$ *such that* $a \leq b$, *and let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, *and* $\{d_n\}$ *be sequences of real numbers such that* $a_n + b_n + c_n + d_n = 1$ *and* $0 < a \leq a_n, b_n, c_n, d_n \leq b < 1$ *for all* $n \in \mathbb{N}$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers such that* $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, *and* $\sum_{n=1}^{\infty} \delta_n < \infty$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_1 \in C : \text{given}$$

$$X_n \in C \quad \text{such that } \|X_n - x_n\| \leq \alpha_n, \tag{3.10}$$

$$Y_n \in C \quad \text{such that } \|Y_n - Sx_n\| \leq \beta_n, \tag{3.11}$$

$$Z_n \in C \quad \text{such that } \left\|Z_n - S^2 x_n\right\| \leq \gamma_n, \tag{3.12}$$

$$W_n \in C \quad \text{such that } \left\|W_n - \frac{1}{n}\sum_{k=1}^{n} T^k x_n\right\| \leq \delta_n, \tag{3.13}$$

$$x_{n+1} = a_n X_n + b_n Y_n + c_n Z_n + d_n W_n \in C \quad \text{for all } n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ *converges weakly to a common attractive point* $\overline{x} \in A(S) \cap A(T)$, *where* $\overline{x} \equiv \lim_{n\to\infty} P_A x_n$. *Additionally, if* C *is closed, then* $\{x_n\}$ *converges weakly to a common fixed point* $\widehat{x} \equiv \lim_{n\to\infty} P_F x_n \in F(S) \cap F(T)$, *where* P_F *is the metric projection form* H *onto* $F(S) \cap F(T)$.

The proof is analogous to that of Theorem 3.1.

Proof. Note that there exists the metric projection P_A from H onto $A(S) \cap A(T)$. The following relationships can be easily verified:

$$\|Sx_n - u\| \leq \|x_n - u\|, \quad \left\|S^2 x_n - u\right\| \leq \|x_n - u\|, \tag{3.14}$$
$$\left\|\frac{1}{n}\sum_{k=1}^{n} T^k x_n - u\right\| \leq \|x_n - u\|$$

for all $u \in A(S) \cap A(T)$ and $n \in \mathbb{N}$.

From (3.14), we have that the sequence $\{P_A x_n\}$ is convergent in $A(S) \cap A(T)$. This fact can be ascertained as follows: Remind Lemma 2.4. Define $y_n \equiv Sx_n\,(\in C)$, $z_n \equiv S^2 x_n\,(\in C)$, and $w_n \equiv \frac{1}{n}\sum_{k=1}^{n} T^k x_n\,(\in C)$. Assumptions (3.10)–(3.13) imply

(2.1)–(2.4), respectively. Also, from (3.14), the conditions (2.5) and (2.6) are satisfied. Thus, from Lemma 2.4, there exists $\overline{x} \in A(S) \cap A(T)$ such that $P_A x_n \to \overline{x}$. Our first aim is to show that $x_n \rightharpoonup \overline{x}$.

By using (3.14), we can show that

$$\begin{aligned} \|X_n - u\| &\leq \|x_n - u\| + \alpha_n, \quad \|Y_n - u\| \leq \|x_n - u\| + \beta_n, \\ \|Z_n - u\| &\leq \|x_n - u\| + \gamma_n, \quad \|W_n - u\| \leq \|x_n - u\| + \delta_n \end{aligned} \tag{3.15}$$

for all $u \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. Indeed, it holds from (3.11) and (3.14) that

$$\begin{aligned} \|Y_n - u\| &\leq \|Y_n - Sx_n\| + \|Sx_n - u\| \\ &\leq \beta_n + \|x_n - u\| \end{aligned}$$

since $u \in A(S)$. Similarly, we can obtain the other parts of (3.15).

Next, we prove that the sequence $\{\|x_n - u\|\}$ is convergent in $\mathbb{R}$ for all $u \in A(S) \cap A(T)$. Define $\eta_n \equiv \alpha_n + \beta_n + \gamma_n + \delta_n \,(\in \mathbb{R})$. Since $\sum_{n=1}^{\infty} \eta_n < \infty$, it suffices to demonstrate that

$$\|x_{n+1} - u\| \leq \|x_n - u\| + \eta_n \tag{3.16}$$

for all $u \in A(S) \cap A(T)$. It holds from (3.15) that

$$\begin{aligned} \|x_{n+1} - u\| &= \|a_n (X_n - u) + b_n (Y_n - u) + c_n (Z_n - u) + d_n (W_n - u)\| \\ &\leq a_n \|X_n - u\| + b_n \|Y_n - u\| + c_n \|Z_n - u\| + d_n \|W_n - u\| \\ &\leq a_n (\|x_n - u\| + \alpha_n) + b_n (\|x_n - u\| + \beta_n) \\ &\quad + c_n (\|x_n - u\| + \gamma_n) + d_n (\|x_n - u\| + \delta_n) \\ &\leq \|x_n - u\| + \eta_n. \end{aligned}$$

We have from Lemma 2.3 and (3.16) that $\{\|x_n - u\|\}$ is convergent for all $u \in A(S) \cap A(T)$. Thus, the sequence $\{x_n\}$ is bounded. Since P_A is nonexpansive, $\{P_A x_n\}$ is also bounded.

Let us show that

$$\begin{aligned} & a_n b_n \|X_n - Y_n\|^2 + a_n c_n \|X_n - Z_n\|^2 + a_n d_n \|X_n - W_n\|^2 \\ & + b_n c_n \|Y_n - Z_n\|^2 + b_n d_n \|Y_n - W_n\|^2 + c_n d_n \|Z_n - W_n\|^2 \\ & \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + M \eta_n \end{aligned} \tag{3.17}$$

for all $u \in A(S) \cap A(T)$ and $n \in \mathbb{N}$, where

$$M \equiv \sup_{n \in \mathbb{N}} (2 \|x_n - u\| + \eta_n) \in \mathbb{R}.$$

Indeed, by using Lemma 2.2-(c), we obtain

$$\begin{aligned} \|x_{n+1} - u\|^2 &= \|a_n (X_n - u) + b_n (Y_n - u) + c_n (Z_n - u) + d_n (W_n - u)\|^2 \\ &= a_n \|X_n - u\|^2 + b_n \|Y_n - u\|^2 + c_n \|Z_n - u\|^2 + d_n \|W_n - u\|^2 \\ &\quad - a_n b_n \|X_n - Y_n\|^2 - a_n c_n \|X_n - Z_n\|^2 - a_n d_n \|X_n - W_n\|^2 \\ &\quad - b_n c_n \|Y_n - Z_n\|^2 - b_n d_n \|Y_n - W_n\|^2 - c_n d_n \|Z_n - W_n\|^2. \end{aligned}$$

We have from (3.15) that

$$\|x_{n+1} - u\|^2 \leq a_n (\|x_n - u\| + \alpha_n)^2 + b_n (\|x_n - u\| + \beta_n)^2$$

$$
\begin{aligned}
& +c_n \left(\|x_n-u\|+\gamma_n\right)^2+d_n \left(\|x_n-u\|+\delta_n\right)^2 \\
& -a_n b_n \|X_n-Y_n\|^2-a_n c_n \|X_n-Z_n\|^2-a_n d_n \|X_n-W_n\|^2 \\
& -b_n c_n \|Y_n-Z_n\|^2-b_n d_n \|Y_n-W_n\|^2-c_n d_n \|Z_n-W_n\|^2 \\
= \; & a_n \|x_n-u\|^2+b_n \|x_n-u\|^2+c_n \|x_n-u\|^2+d_n \|x_n-u\|^2 \\
& +a_n \left(2 \|x_n-u\| \alpha_n+\alpha_n^2\right)+b_n \left(2 \|x_n-u\| \beta_n+\beta_n^2\right) \\
& +c_n \left(2 \|x_n-u\| \gamma_n+\gamma_n^2\right)+d_n \left(2 \|x_n-u\| \delta_n+\delta_n^2\right) \\
& -a_n b_n \|X_n-Y_n\|^2-a_n c_n \|X_n-Z_n\|^2-a_n d_n \|X_n-W_n\|^2 \\
& -b_n c_n \|Y_n-Z_n\|^2-b_n d_n \|Y_n-W_n\|^2-c_n d_n \|Z_n-W_n\|^2 \\
\leq \; & \|x_n-u\|^2+\alpha_n \left(2 \|x_n-u\|+\alpha_n\right)+\beta_n \left(2 \|x_n-u\|+\beta_n\right) \\
& +\gamma_n \left(2 \|x_n-u\|+\gamma_n\right)+\delta_n \left(2 \|x_n-u\|+\delta_n\right) \\
& -a_n b_n \|X_n-Y_n\|^2-a_n c_n \|X_n-Z_n\|^2-a_n d_n \|X_n-W_n\|^2 \\
& -b_n c_n \|Y_n-Z_n\|^2-b_n d_n \|Y_n-W_n\|^2-c_n d_n \|Z_n-W_n\|^2 \\
\leq \; & \|x_n-u\|^2+\alpha_n \left(2 \|x_n-u\|+\eta_n\right)+\beta_n \left(2 \|x_n-u\|+\eta_n\right) \\
& +\gamma_n \left(2 \|x_n-u\|+\eta_n\right)+\delta_n \left(2 \|x_n-u\|+\eta_n\right) \\
& -a_n b_n \|X_n-Y_n\|^2-a_n c_n \|X_n-Z_n\|^2-a_n d_n \|X_n-W_n\|^2 \\
& -b_n c_n \|Y_n-Z_n\|^2-b_n d_n \|Y_n-W_n\|^2-c_n d_n \|Z_n-W_n\|^2 \\
\leq \; & \|x_n-u\|^2+M \left(\alpha_n+\beta_n+\gamma_n+\delta_n\right) \\
& -a_n b_n \|X_n-Y_n\|^2-a_n c_n \|X_n-Z_n\|^2-a_n d_n \|X_n-W_n\|^2 \\
& -b_n c_n \|Y_n-Z_n\|^2-b_n d_n \|Y_n-W_n\|^2-c_n d_n \|Z_n-W_n\|^2 .
\end{aligned}
$$

This means that (3.17) holds.

Since $\{\|x_n-u\|\}$ is convergent and $\eta_n \to 0$, we have from (3.17) that

$$X_n-Y_n \to 0, \quad X_n-Z_n \to 0, \quad X_n-W_n \to 0. \tag{3.18}$$

Furthermore, it holds that

$$x_n-Sx_n \to 0, \quad x_n-S^2 x_n \to 0, \tag{3.19}$$

$$x_n-\frac{1}{n}\sum_{k=1}^{n} T^k x_n \to 0. \tag{3.20}$$

Indeed, it follow from (3.10), (3.11) and (3.18) that

$$
\begin{aligned}
\|x_n-Sx_n\| & \leq \|x_n-X_n\|+\|X_n-Y_n\|+\|Y_n-Sx_n\| \\
& \leq \alpha_n+\|X_n-Y_n\|+\beta_n \to 0.
\end{aligned}
$$

Similarly, we can prove $x_n-S^2x_n \to 0$ and $x_n-\frac{1}{n}\sum_{k=1}^{n} T^k x_n \to 0$.

We show that $x_n \rightharpoonup \overline{x}$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ and $v \in H$ such that $x_{n_j} \rightharpoonup v$. We have from (3.19) and Lemma 2.7 that $v \in A(S)$. Furthermore, it follows from (3.20) that

$$\frac{1}{n_j}\sum_{k=1}^{n_j} T^k x_{n_j} \rightharpoonup v. \tag{3.21}$$

From (3.21) and Lemma 2.6, we have that $v \in A(T)$. Therefore, $v \in A(S) \cap A(T)$. As a result, it follows that

$$\langle x_n - P_A x_n,\ P_A x_n - v\rangle \geq 0$$

for all $n \in \mathbb{N}$. Since $\{x_n\}$ and $\{P_A x_n\}$ are bounded, we have

$$\begin{aligned}\langle x_n - P_A x_n,\ v - \overline{x}\rangle &\leq \langle x_n - P_A x_n,\ P_A x_n - \overline{x}\rangle \\ &\leq \|x_n - P_A x_n\| \, \|P_A x_n - \overline{x}\| \\ &\leq L \, \|P_A x_n - \overline{x}\|\end{aligned}$$

for all $n \in \mathbb{N}$, where $L \equiv \sup_{n\in\mathbb{N}} \|x_n - P_A x_n\| \in \mathbb{R}$. Replacing n by n_j, and taking the limit as $j \to \infty$, we obtain

$$\langle v - \overline{x},\ v - \overline{x}\rangle \leq 0.$$

Thus, $v = \overline{x}$. This means that $x_n \rightharpoonup \overline{x}$.

Next, we additionally assume that C is closed in H. We prove that $x_n \rightharpoonup \widehat{x} \equiv \lim_{k\to\infty} P_F x_k$. Since C is weakly closed, it follows that $\overline{x} \in C \cap A(S) \cap A(T)$, where $\overline{x} \equiv \lim_{n\to\infty} P_A x_n$. From Lemma 2.9, we have that $\overline{x} \in F(S) \cap F(T)$. Thus, $F(S) \cap F(T)$ is nonempty. Since S and T are quasi-nonexpansive, $F(S) \cap F(T)$ is closed and convex. Hence, there exists the metric projection P_F from H onto $F(S) \cap F(T)$. We can easily prove that

$$\|Sx_n - u\| \leq \|x_n - u\|, \quad \left\|S^2 x_n - u\right\| \leq \|x_n - u\|,$$
$$\left\|\frac{1}{n}\sum_{k=1}^{n} T^k x_n - u\right\| \leq \|x_n - u\|$$

for all $u \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ since S and T are quasi-nonexpansive. Thus, we have from Lemma 2.4 that $\{P_F x_n\}$ converges strongly to an element $\widehat{x}$ of $F(S) \cap F(T)$, that is, $\widehat{x} \equiv \lim_{n\to\infty} P_F x_n$. We show that

$$\overline{x}\left(\equiv \lim_{n\to\infty} P_A x_n\right) = \widehat{x}\left(\equiv \lim_{n\to\infty} P_F x_n\right).$$

Since $\overline{x} \in F(S) \cap F(T)$, we have from a property of the metric projection that

$$\langle x_n - P_F x_n,\ P_F x_n - \overline{x}\rangle \geq 0$$

for all $n \in \mathbb{N}$. Since $x_n \rightharpoonup \overline{x}$ and $P_F x_n \to \widehat{x}$, we have that $\langle \overline{x} - \widehat{x}, \widehat{x} - \overline{x}\rangle \geq 0$, which means that $\widehat{x} = \overline{x}$. This implies that $\{x_n\}$ converges weakly to $\widehat{x} = \lim_{n\to\infty} P_F x_n \in F(S) \cap F(T)$. This completes the proof. □

Letting $\alpha_n = \beta_n = \gamma_n = \delta_n = 0$ in Theorem 3.3, we obtain the following corollary, which is a hybrid of the mean convergence method (1.2) and Kondo and Takahashi's type (1.4).

Corollary 3.4. *Let C be a nonempty and convex subset of H, and let S and T be normally 2-generalized hybrid mappings from C into itself. Suppose that $A(S) \cap A(T)$ is nonempty. Let P_A be the metric projection from H onto $A(S) \cap A(T)$. Let $a, b \in (0,1)$ such that $a \leq b$, and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real* *numbers such that* $a_n + b_n + c_n + d_n = 1$ *and* $0 < a \leq a_n, b_n, c_n, d_n \leq b < 1$ *for all* $n \in \mathbb{N}$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_1 \in C : \textit{given}$$

$$x_{n+1} = a_n x_n + b_n S x_n + c_n S^2 x_n + d_n \frac{1}{n} \sum_{k=1}^{n} T^k x_n \in C \quad \textit{for all } n \in \mathbb{N}.$$

Then, $\{x_n\}$ *converges weakly to a common attractive point* $\overline{x} \in A(S) \cap A(T)$, *where* $\overline{x} \equiv \lim_{n\to\infty} P_A x_n$. *Additionally, if* C *is closed, then* $\{x_n\}$ *converges weakly to a common fixed point* $\widehat{x} \equiv \lim_{n\to\infty} P_F x_n \in F(S) \cap F(T)$, *where* P_F *is the metric projection form* H *onto* $F(S) \cap F(T)$.

4. Weak Convergence for Resolvent

In this section, we present a weak convergence theorem to a common null point of two maximal monotone multi-valued mappings by using resolvents.

Theorem 4.1. *Let* C *be a nonempty, closed and convex subset of* H. *Let* $A, B \subset H \times H$ *be maximal monotone multi-valued mappings on* H *such that their domains are included in* C. *Let* $J_r^A = (I + rA)^{-1}$ *be the resolvent of* A *for* $r > 0$, *and let* $J_s^B = (I + sB)^{-1}$ *be the resolvent of* B *for* $s > 0$. *Suppose that* $A^{-1}0 \cap B^{-1}0$ *is nonempty. Let* $P : H \to A^{-1}0 \cap B^{-1}0$ *be the metric projection from* H *onto* $A^{-1}0 \cap B^{-1}0$. *Let* $\{r_n\}$ *and* $\{s_n\}$ *be sequences of positive real numbers such that* $r_n \to \infty$ *and* $s_n \to \infty$. *Let* $a, b \in (0,1)$ *such that* $a \leq b$, *and let* $\{a_n\}$, $\{b_n\}$, *and* $\{c_n\}$ *be sequences of real numbers such that* $a_n + b_n + c_n = 1$ *and* $0 < a \leq a_n, b_n, c_n \leq b < 1$ *for all* $n \in \mathbb{N}$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *be sequences of nonnegative real numbers such that* $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_1 \in C : \textit{given}$$

$$X_n \in C \quad \textit{such that } \|X_n - x_n\| \leq \alpha_n, \tag{4.1}$$

$$Y_n \in C \quad \textit{such that } \left\|Y_n - J_{r_n}^A x_n\right\| \leq \beta_n, \tag{4.2}$$

$$Z_n \in C \quad \textit{such that } \left\|Z_n - J_{s_n}^B x_n\right\| \leq \gamma_n, \tag{4.3}$$

$$x_{n+1} = a_n X_n + b_n Y_n + c_n Z_n \in C \quad \textit{for all } n \in \mathbb{N}.$$

Then, $\{x_n\}$ *converges weakly to a common null point* $\overline{x} \in A^{-1}0 \cap B^{-1}0$, *where* $\overline{x} \equiv \lim_{n\to\infty} P x_n$.

The proof is analogous to that of Theorem 3.1.

Proof. Since A and B are maximal monotone, $A^{-1}0 \cap B^{-1}0$ is closed and convex in C. Since $A^{-1}0 \cap B^{-1}0 \neq \emptyset$ is assumed, there exists the metric projection P from H onto $A^{-1}0 \cap B^{-1}0$. Furthermore, note that $J_{r_n}^A$ and $J_{s_n}^B$ are single-valued mappings from H into C. Define $y_n \equiv J_{r_n}^A x_n \, (\in C)$ and $z_n \equiv J_{s_n}^B x_n \, (\in C)$. It holds that

$$\|y_n - u\| \leq \|x_n - u\| \quad \text{and} \quad \|z_n - u\| \leq \|x_n - u\| \tag{4.4}$$

for all $u \in A^{-1}0 \cap B^{-1}0$ and $n \in \mathbb{N}$ since $A^{-1}0 = F\left(J_{r_n}^A\right)$, $B^{-1}0 = F\left(J_{s_n}^B\right)$, and the resolvents are nonexpansive. From Lemma 2.4, we can prove that the sequence $\{Px_n\}$ is convergent in $A^{-1}0 \cap B^{-1}0$. Indeed, consider the case of $d_n = 0$ in that lemma. Assumptions (4.1)–(4.3) imply (2.1)–(2.3), respectively. Also, from (4.4), the conditions in (2.5) are satisfied. Thus, from Lemma 2.4, there exists an element $\overline{x} \in A^{-1}0 \cap B^{-1}0$ and $Px_n \to \overline{x}$. Our aim is to show that $x_n \rightharpoonup \overline{x}$.

By using (4.4), we can show that

$$\begin{aligned} \|X_n - u\| &\leq \|x_n - u\| + \alpha_n, \\ \|Y_n - u\| &\leq \|x_n - u\| + \beta_n, \\ \|Z_n - u\| &\leq \|x_n - u\| + \gamma_n \end{aligned} \tag{4.5}$$

for all $u \in A^{-1}0 \cap B^{-1}0$ and $n \in \mathbb{N}$. Indeed, it follows from (4.1) that

$$\begin{aligned} \|X_n - u\| &\leq \|X_n - x_n\| + \|x_n - u\| \\ &\leq \alpha_n + \|x_n - u\| . \end{aligned}$$

Furthermore, it holds from (4.2) and (4.4) that

$$\begin{aligned} \|Y_n - u\| &\leq \|Y_n - y_n\| + \|y_n - u\| \\ &\equiv \left\|Y_n - J_{r_n}^A x_n\right\| + \left\|J_{r_n}^A x_n - u\right\| \\ &\leq \beta_n + \|x_n - u\| \end{aligned}$$

since $J_{r_n}^A$ is nonexpansive and $u \in A^{-1}0 = F\left(J_{r_n}^A\right)$. Similarly, by using (4.3), (4.4) and $u \in B^{-1}0 = F\left(J_{s_n}^B\right)$, we obtain $\|Z_n - u\| \leq \|x_n - u\| + \gamma_n$.

Next, we verify that the sequence $\{\|x_n - u\|\}$ is convergent in $\mathbb{R}$ for all $u \in A^{-1}0 \cap B^{-1}0$. Define $\eta_n \equiv \alpha_n + \beta_n + \gamma_n \,(\in \mathbb{R})$. In much the same way as the proof of (3.6), we can demonstrate that

$$\|x_{n+1} - u\| \leq \|x_n - u\| + \eta_n \tag{4.6}$$

for all $u \in A^{-1}0 \cap B^{-1}0$ by using (4.5). Since $\sum_{n=1}^{\infty} \eta_n < \infty$, it holds from Lemma 2.3 that $\{\|x_n - u\|\}$ converges for all $u \in A^{-1}0 \cap B^{-1}0$. Consequently, $\{x_n\}$ is a bounded sequence. Since P is nonexpansive, $\{Px_n\}$ is also bounded.

As the proof of (3.7), we can demonstrate that

$$\begin{aligned} &a_n b_n \|X_n - Y_n\|^2 + b_n c_n \|Y_n - Z_n\|^2 + c_n a_n \|Z_n - X_n\|^2 \\ &\leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + M\eta_n \end{aligned} \tag{4.7}$$

for all $u \in A^{-1}0 \cap B^{-1}0$ and $n \in \mathbb{N}$, where

$$M \equiv \sup_{n\in\mathbb{N}} \left(2 \|x_n - u\| + \eta_n\right) \in \mathbb{R},$$

by using Lemma 2.2 and (4.5). Since $\{\|x_n - u\|\}$ is convergent and $\eta_n \to 0$, we have from (4.7) that

$$X_n - Y_n \to 0 \quad \text{and} \quad X_n - Z_n \to 0. \tag{4.8}$$

Furthermore, it follows from (4.8) that

$$x_n - y_n \to 0 \quad \text{and} \quad x_n - z_n \to 0. \tag{4.9}$$

Indeed, note that

$$\begin{aligned} \|x_n - y_n\| &\leq \|x_n - X_n\| + \|X_n - Y_n\| + \|Y_n - y_n\| \\ &\leq \alpha_n + \|X_n - Y_n\| + \beta_n. \end{aligned}$$

Since $\alpha_n \to 0$ and $\beta_n \to 0$, we have from (4.8) that $x_n - y_n \to 0$. Similarly, we can prove that $x_n - z_n \to 0$.

We show that $x_n \rightharpoonup \overline{x}$. For any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ and $v \in H$ such that $x_{n_j} \rightharpoonup v$. From (4.9),

$$y_{n_j} \equiv J^A_{r_{n_j}} x_{n_j} \rightharpoonup v \quad \text{and} \quad z_{n_j} \equiv J^B_{s_{n_j}} x_{n_j} \rightharpoonup v.$$

Since $r_{n_j} \to \infty$ and $s_{n_j} \to \infty$, we have from Lemma 2.8 that $v \in A^{-1}0 \cap B^{-1}0$. Therefore, it follows that

$$\langle x_n - Px_n,\ Px_n - v\rangle \geq 0$$

for all $n \in \mathbb{N}$. Since $\{x_n\}$ and $\{Px_n\}$ are bounded, we have

$$\begin{aligned} \langle x_n - Px_n,\ v - \overline{x}\rangle &\leq \langle x_n - Px_n,\ Px_n - \overline{x}\rangle \\ &\leq \|x_n - Px_n\| \, \|Px_n - \overline{x}\| \\ &\leq L \, \|Px_n - \overline{x}\| \end{aligned}$$

for all $n \in \mathbb{N}$, where $L \equiv \sup_{n\in\mathbb{N}} \|x_n - Px_n\| \in \mathbb{R}$. Replacing n by n_j, and taking the limit as $j \to \infty$, we obtain

$$\langle v - \overline{x},\ v - \overline{x}\rangle \leq 0.$$

Thus, $v = \overline{x}$. This means that $x_n \rightharpoonup \overline{x}$. The completes the proof. $\square$

Letting $\alpha_n = \beta_n = \gamma_n = 0$, we obtain the following corollary:

Corollary 4.2. *Let C be a nonempty, closed and convex subset of H. Let $A, B \subset H \times H$ be maximal monotone multi-valued mapping on H such that their domains are included in C. Let $J^A_r = (I + rA)^{-1}$ be the resolvent of A for $r > 0$, and let $J^B_s = (I + sB)^{-1}$ be the resolvent of B for $s > 0$. Suppose that $A^{-1}0 \cap B^{-1}0$ is nonempty. Let $P : H \to A^{-1}0 \cap B^{-1}0$ be the metric projection from H onto $A^{-1}0 \cap B^{-1}0$. Let $\{r_n\}$ and $\{s_n\}$ be sequences of positive real numbers such that $r_n \to \infty$ and $s_n \to \infty$. Let $a, b \in (0, 1)$ such that $a \leq b$, and let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers such that $a_n + b_n + c_n = 1$ and $0 < a \leq a_n, b_n, c_n \leq b < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} &x_1 \in C : \textit{given} \\ &x_{n+1} = a_n x_n + b_n J^A_{r_n} x_n + c_n J^B_{s_n} x_n \in C \quad \textit{for all } n \in \mathbb{N}. \end{aligned}$$

Then, $\{x_n\}$ converges weakly to a common null point $\overline{x} \in A^{-1}0 \cap B^{-1}0$, where $\overline{x} \equiv \lim_{n\to\infty} Px_n$.

As is well-known, the null point problems have direct links with optimization problems; see Rockafellar [24], Kamimura and Takahashi [12], and Takahashi [28].

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