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STRONG CONVERGENCE THEOREMS OF HALPERN'S TYPE FOR NORMALLY 2-GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. This paper establishes two types of strong convergence theorems of finding attractive points of nonlinear mappings in Hilbert spaces. Sequences that are constructed in the Halpern's type iteration converge strongly to attractive points of normally 2-generalized hybrid mappings. These theorems are proved without assuming that the domain of the mapping is closed. Same types of results regarding fixed points are also demonstrated by additionally supposing that the domain of the mapping is closed. Our results extend many existing results in the literature.

1. Introduction

Let H be a real Hilbert space equipped with inner product $\langle \cdot,\ \cdot \rangle$ and norm $\|\cdot\|$. We denote the sets of natural and real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. Let C be a nonempty subset of H, and let T be a mapping from C to H. The sets of fixed points and attractive points of T are denoted by

$$\begin{aligned} F(T) &= \{u \in H : Tu = u\} \text{ and} \\ A(T) &= \{u \in H : \|Ty - u\| \leq \|y - u\| \text{ for all } y \in C\}, \end{aligned}$$

respectively. The concept of attractive points was proposed by Takahashi and Takeuchi [15]. A mapping $T : C \to H$ is called

(i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

It is easily ascertained that a fixed point of a nonexpansive mapping is an attractive point. For a nonexpansive mapping, Wittmann [19] proved the following strong convergence theorem of Halpern's iteration [4] in Hilbert spaces:

Theorem 1.1 ([19]). *Let C be a nonempty, closed and convex subset of H, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = \lambda_n x + (1 - \lambda_n) T x_n$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence of real numbers in the interval $[0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

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Wittmann's theorem has been extended to various directions. Kocourek, Takahashi and Yao [8] introduced a wide class of mappings, which contains nonexpansive mappings as special cases, and proved a weak convergence theorem in Hilbert spaces. A mapping $T : C \to H$ is called

(ii) *generalized hybrid* [8] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha) \|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta) \|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. Such a mapping T is called (α, β)-generalized hybrid. If $(\alpha, \beta) = (1, 0)$, a generalized hybrid mapping T is nonexpansive. The generalized hybrid mappings are further extended. A mapping T is called

(iii) *normally generalized hybrid* [16] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 \leq 0 \tag{1.2}$$

for all $x, y \in C$, where (1) $\alpha + \beta + \gamma + \delta \geq 0$ and (2) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$;

(iv) 2-*generalized hybrid* [11] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} \alpha_2 \left\|T^2 x - Ty\right\|^2 &+ \alpha_1 \|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2) \|x - Ty\|^2 \\ &\leq \beta_2 \left\|T^2 x - y\right\|^2 + \beta_1 \|Tx - y\|^2 + (1 - \beta_1 - \beta_2) \|x - y\|^2 \end{aligned}$$

for all $x, y \in C$.

Putting $\beta = 1 - \alpha$ and $\delta = -(1 - \gamma)$ in (1.2), we can recognize that a normally generalized hybrid mapping is $(\alpha, -\gamma)$-generalized hybrid. A 2-generalized hybrid mapping is generalized hybrid if $\alpha_2 = \beta_2 = 0$. Hojo, Takahashi and Termwuttipong [7] proved the following strong convergence theorem for 2-generalized hybrid mappings in a Hilbert space. Important precursors of [7] are Kurokawa and Takahashi [10] and Hojo and Takahashi [5]. For other types of convergence theorems, see also Hojo and Takahashi [6], Termwuttipong, Pongsriiam and Yao [18] and Alizadeh and Moradlou [1, 2].

Theorem 1.2 ([7]). *Let C be a nonempty, closed and convex subset of H, and let $T : C \to C$ be a 2-generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Given $x_1, z \in C$, define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = \lambda_n z + (1 - \lambda_n) \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to $\overline{z} \equiv P_{F(T)} z$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

On the other hand, Takahashi, Wong and Yao [17] established the following strong convergence theorem for generalized hybrid mappings in a Hilbert space.

Theorem 1.3 ([17]). *Let C be a nonempty and convex subset of H, and let $T : C \to C$ be a generalized hybrid mapping with $A(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\alpha_n\}$ be sequences of real numbers in the interval $(0, 1)$ such that*

$$\lim_{n\to\infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty \quad \textit{and} \quad \liminf_{n\to\infty} \alpha_n (1 - \alpha_n) > 0.$$

Given $x_1, z \in C$, define a sequence $\{x_n\}$ in C as follows:

$$x_{n+1} = \lambda_n z + (1-\lambda_n)\left(\alpha_n x_n + (1-\alpha_n) T x_n\right)$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to $\overline{z} \equiv P_{A(T)} z$, where $P_{A(T)}$ is the metric projection from H onto $A(T)$.

Very recently, normally generalized hybrid mappings (iii) and 2-generalized hybrid mappings (iv) are unified. A mapping $T : C \to C$ is called

(v) *normally* 2*-generalized hybrid* [9] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_2 \left\|T^2 x - Ty\right\|^2 + \alpha_1 \|Tx - Ty\|^2 + \alpha_0 \|x - Ty\|^2 \\ + \beta_2 \left\|T^2 x - y\right\|^2 + \beta_1 \|Tx - y\|^2 + \beta_0 \|x - y\|^2 \le 0\end{aligned}$$

for all $x, y \in C$, where (1) $\sum_{n=0}^{2} (\alpha_n + \beta_n) \ge 0$ and (2) $\alpha_2 + \alpha_1 + \alpha_0 > 0$. It is also called an $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$*-normally* 2*-generalized hybrid* mapping. Although Kondo and Takahashi [9] proved weak convergence theorems for normally 2-generalized hybrid mappings, any strong convergence theorem has not yet been known.

The main purpose of this paper is to establish two types of strong convergence theorems (Theorem 3.2 and Theorem 3.4) of finding attractive points of normally 2-generalized hybrid mappings in Hilbert spaces. Theorem 3.2 and Theorem 3.4 extend Theorem 1.2 and Theorem 1.3, respectively. These theorems are proved without assuming that the domain of the mapping is closed. Same types of results (Theorem 4.1 and Theorem 4.2) regarding fixed points are also demonstrated by additionally supposing that the domain of the mapping is closed.

2. Preliminaries

This section briefly offers background information and results. For more details, see Takahashi [13, 14] and earlier studies. We know that

$$\|x+y\|^2 \le \|x\|^2 + 2\langle x+y,\ y\rangle \tag{2.1}$$

for all $x, y \in H$. We also know that

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$. In particular, if $\lambda \in [0,1]$,

$$\|\lambda x + (1-\lambda) y\|^2 \le \lambda \|x\|^2 + (1-\lambda)\|y\|^2. \tag{2.3}$$

Let C be a nonempty subset of H. For $T : C \to H$ and $v \in H$, it is easy to verify that

$$v \in A(T) \Longleftrightarrow \|Ty - y\|^2 + 2\langle Ty - y,\ y - v\rangle \le 0, \quad \forall y \in C. \tag{2.4}$$

A mapping $T : C \to H$ with $F(T) \ne \emptyset$ is called *quasi-nonexpansive* if

$$\|Tx - u\| \le \|x - u\|$$

for all $x \in C$ and $u \in F(T)$. It is known in the literature that all types of mappings (i)–(v) mentioned in Introduction are quasi-nonexpansive if $F(T) \ne \emptyset$ (see [9]).

Furthermore, we know that the set of fixed points $F(T)$ of a quasi-nonexpansive mapping is closed and convex (see [8]), which plays important roles in many studies.

Strong and weak convergence of a sequence $\{x_n\}$ in H to a point $x \in H$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let A be a nonempty, closed and convex subset of H. We know that for any $z \in H$, there exists a unique nearest point $\overline{z} \in A$, that is, $\|z - \overline{z}\| = \inf_{u \in A} \|z - u\|$. This correspondence is called the *metric projection* from H onto A, and is denoted by P_A. We know that if P_A is the metric projection from H onto A, then

$$\langle z - P_A z,\ P_A z - v \rangle \geq 0 \tag{2.5}$$

for all $z \in H$ and $v \in A$.

The following lemmas are utilized in the proofs of the main theorems of this paper.

Lemma 2.1 ([15])**.** *Let C be a nonempty subset of H, and let T be a mapping from C to H. Then, $A(T)$ is a closed and convex subset of H.*

Lemma 2.2 ([3]; see also [20])**.** *Let $\{X_n\}$ be a sequence of nonnegative real numbers, let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1)$ such that $\sum_{n=1}^{\infty} \lambda_n = \infty$, and let $\{Y_n\}$ be a sequence of real numbers such that $\limsup_{n\to\infty} Y_n \leq 0$. If $X_{n+1} \leq (1-\lambda_n) X_n + \lambda_n Y_n$ for all $n \in \mathbb{N}$, then $X_n \to 0$ as $n \to \infty$.*

Lemma 2.3 ([11])**.** *Let $x, y, z \in H$ and $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$. Then,*

$$\begin{aligned}\|ax + by + cz\|^2 = {} & a\|x\|^2 + b\|y\|^2 + c\|z\|^2 \\ & - ab\|x-y\|^2 - bc\|y-z\|^2 - ca\|z-x\|^2.\end{aligned}$$

Additionally, if $a, b, c \in [0,1]$, then

$$\|ax + by + cz\|^2 \leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2.$$

Lemma 2.4 ([9])**.** *Let C be a nonempty subset of H, let $T : C \to C$ be a normally 2-generalized hybrid mapping, and let $\{x_n\}$ be a sequence in C satisfying $x_n - Tx_n \to 0$, $T^2 x_n - x_n \to 0$ and $x_n \rightharpoonup v$. Then, $v \in A(T)$.*

Lemma 2.5 ([12])**.** *Let $\{X_n\}$ be a sequence of real numbers. Assume that $\{X_n\}$ is not monotone decreasing for sufficiently large $n \in \mathbb{N}$, in other words, there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : X_k < X_{k+1}\} \neq \emptyset$. Define a sequence $\{\tau(n)\}_{n \geq n_0}$ of natural numbers as follows:*

$$\tau(n) = \max\{k \leq n : X_k < X_{k+1}\}, \quad \forall n \geq n_0.$$

Then, the followings hold:

(i) $\tau(n) \to \infty$ *as* $n \to \infty$;

(ii) $X_n \leq X_{\tau(n)+1}$ *and* $X_{\tau(n)} < X_{\tau(n)+1}$, $\quad \forall n \geq n_0$.

Lemma 2.6 ([9])**.** *Let C be a nonempty subset of H, and let $T : C \to C$ be a normally 2-generalized hybrid mapping with $F(T) \neq \emptyset$. Then, T is quasi-nonexpansive.*

Lemma 2.7. *Let C be a nonempty subset of H, and let $T : C \to C$ be a normally 2-generalized hybrid mapping. Then, $F(T) \subset A(T)$.*

Proof. Since T is a normally 2-generalized hybrid, there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_2 \left\|T^2x - Ty\right\|^2 + \alpha_1 \|Tx - Ty\|^2 + \alpha_0 \|x - Ty\|^2 \\ + \beta_2 \left\|T^2x - y\right\|^2 + \beta_1 \|Tx - y\|^2 + \beta_0 \|x - y\|^2 \leq 0\end{aligned}$$

for all $x, y \in C$, where (1) $\sum_{n=0}^{2} (\alpha_n + \beta_n) \geq 0$ and (2) $\alpha_2 + \alpha_1 + \alpha_0 > 0$.

Let $u \in F(T)$. We will prove that $u \in A(T)$. For any $y \in C$, we have that

$$\begin{aligned}\alpha_2 \left\|T^2u - Ty\right\|^2 + \alpha_1 \|Tu - Ty\|^2 + \alpha_0 \|u - Ty\|^2 \\ + \beta_2 \left\|T^2u - y\right\|^2 + \beta_1 \|Tu - y\|^2 + \beta_0 \|u - y\|^2 \leq 0.\end{aligned}$$

Since $u = Tu = T^2u$,

$$(\alpha_2 + \alpha_1 + \alpha_0) \|u - Ty\|^2 + (\beta_2 + \beta_1 + \beta_0) \|u - y\|^2 \leq 0.$$

Since $\sum_{n=0}^{2} (\alpha_n + \beta_n) \geq 0$, we have that

$$\begin{aligned}(\alpha_2 + \alpha_1 + \alpha_0) \|u - Ty\|^2 &\leq -(\beta_2 + \beta_1 + \beta_0) \|u - y\|^2 \\ &\leq (\alpha_2 + \alpha_1 + \alpha_0) \|u - y\|^2 .\end{aligned}$$

From $\alpha_2 + \alpha_1 + \alpha_0 > 0$, we obtain that $\|u - Ty\|^2 \leq \|u - y\|^2$ for all $y \in C$, which means that $u \in A(T)$. □

Lemma 2.8 ([15])**.** *Let C be a nonempty subset of H, and let T be a mapping from C to H. Then, $A(T) \cap C \subset F(T)$.*

The following theorems (Theorem 2.9 and 2.10) reveal a sufficient and necessary conditions for a normally 2-generalized hybrid mapping to have an attractive/fixed point.

Theorem 2.9 ([9])**.** *Let C be a nonempty subset of H, and let $T : C \to C$ be a normally 2-generalized hybrid mapping. Then, the following three statements are equivalent:*

(a) *for any $x \in C$, $\{T^n x\}$ is a bounded sequence in C;*
(b) *there exists $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C;*
(c) *$A(T) \neq \emptyset$.*

Theorem 2.10 ([9])**.** *Let C be a nonempty, closed and convex subset of H, and let $T : C \to C$ be a normally 2-generalized hybrid mapping. Then, the following four statements are equivalent:*

(a) *for any $x \in C$, $\{T^n x\}$ is a bounded sequence in C;*
(b) *there exists $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C;*
(c) *$A(T) \neq \emptyset$;*
(d) *$F(T) \neq \emptyset$.*

3. Strong convergence to attractive points

This section presents two methods of finding attractive points of normally 2-generalized hybrid mappings in Hilbert spaces. The theorems (Theorems 3.2 and 3.4) in this section are proved without supposing that a domain of mappings is closed. We start with preparing the following lemma, which extends that of Hojo, Takahashi and Termwuttipong [7].

Lemma 3.1. *Let C be a nonempty subset of H, let $T : C \to C$ be a normally 2-generalized hybrid mapping with $A(T) \neq \emptyset$, and let $\{x_n\}$ be a bounded sequence in C. Define $z_n \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x_n \,(\in H)$ and assume that $z_{n_i} \rightharpoonup v$, where $\{z_{n_i}\}$ is a subsequence of $\{z_n\}$. Then, $v \in A(T)$.*

Proof. Let T be $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$ -normally 2-generalized hybrid. Note that the sequences $\left\{T^{n+1}x_n\right\}$, $\{T^n x_n\}$ and $\{Tx_n\}$ in C are bounded. Indeed, let $u \in A(T)$ and $M \equiv \max\{\|x_n - u\| : n \in \mathbb{N}\}$. Then,

$$\left\|T^{n+1}x_n - u\right\| \leq \|T^n x_n - u\| \leq \cdots \leq \|Tx_n - u\| \leq \|x_n - u\| \leq M$$

for all $n \in \mathbb{N}$, which means that $\left\{T^{n+1}x_n\right\}$, $\{T^n x_n\}$ and $\{Tx_n\}$ are bounded.

Let $y \in C$. From (2.4), it is sufficient to prove that

$$\|Ty - y\|^2 + 2\langle Ty - y,\ y - v\rangle \leq 0.$$

Since T is $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$-normally 2-generalized hybrid, it holds that

$$\begin{aligned}\alpha_2 \left\|T^{k+2}x_n - Ty\right\|^2 &+ \alpha_1 \left\|T^{k+1}x_n - Ty\right\|^2 + \alpha_0 \left\|T^k x_n - Ty\right\|^2 \\ &+ \beta_2 \left\|T^{k+2}x_n - y\right\|^2 + \beta_1 \left\|T^{k+1}x_n - y\right\|^2 + \beta_0 \left\|T^k x_n - y\right\|^2 \leq 0\end{aligned}$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$. We have that

$$\begin{aligned}\alpha_2 &\left(\left\|T^{k+2}x_n - y\right\|^2 + 2\left\langle T^{k+2}x_n - y,\ y - Ty\right\rangle + \|y - Ty\|^2\right) \\ &+ \alpha_1 \left(\left\|T^{k+1}x_n - y\right\|^2 + 2\left\langle T^{k+1}x_n - y,\ y - Ty\right\rangle + \|y - Ty\|^2\right) \\ &+ \alpha_0 \left(\left\|T^k x_n - y\right\|^2 + 2\left\langle T^k x_n - y,\ y - Ty\right\rangle + \|y - Ty\|^2\right) \\ &+ \beta_2 \left\|T^{k+2}x_n - y\right\|^2 + \beta_1 \left\|T^{k+1}x_n - y\right\|^2 + \beta_0 \left\|T^k x_n - y\right\|^2 \leq 0\end{aligned}$$

and hence

$$\begin{aligned}&(\alpha_2 + \alpha_1 + \alpha_0)\|y - Ty\|^2 + (\alpha_2 + \beta_2)\left\|T^{k+2}x_n - y\right\|^2 \\ &+ (\alpha_1 + \beta_1)\left\|T^{k+1}x_n - y\right\|^2 + (\alpha_0 + \beta_0)\left\|T^k x_n - y\right\|^2 \\ &+ 2\left\langle \alpha_2 T^{k+2}x_n + \alpha_1 T^{k+1}x_n + \alpha_0 T^k x_n - (\alpha_2 + \alpha_1 + \alpha_0)\, y,\ y - Ty\right\rangle \leq 0.\end{aligned}$$

Using the condition $\sum_{n=0}^{2}(\alpha_n+\beta_n)\geq 0$, we obtain that

$$\begin{aligned}&(\alpha_2+\alpha_1+\alpha_0)\|y-Ty\|^2+(\alpha_2+\beta_2)\left\|T^{k+2}x_n-y\right\|^2\\&\quad+(\alpha_1+\beta_1)\left\|T^{k+1}x_n-y\right\|^2-[(\alpha_2+\beta_2)+(\alpha_1+\beta_1)]\left\|T^kx_n-y\right\|^2\\&\qquad+2\Big\langle\alpha_2T^{k+2}x_n+\alpha_1T^{k+1}x_n+(\alpha_2+\alpha_1+\alpha_0)T^kx_n-(\alpha_2+\alpha_1)T^kx_n\\&\qquad\quad-(\alpha_2+\alpha_1+\alpha_0)y,\quad y-Ty\Big\rangle\leq 0.\end{aligned}$$

This yields that

$$\begin{aligned}&(\alpha_2+\alpha_1+\alpha_0)\|y-Ty\|^2+(\alpha_2+\beta_2)\left(\left\|T^{k+2}x_n-y\right\|^2-\left\|T^kx_n-y\right\|^2\right)\\&\quad+(\alpha_1+\beta_1)\left(\left\|T^{k+1}x_n-y\right\|^2-\left\|T^kx_n-y\right\|^2\right)\\&\qquad+2\Big\langle\alpha_2\left(T^{k+2}x_n-T^kx_n\right)+\alpha_1\left(T^{k+1}x_n-T^kx_n\right)+(\alpha_2+\alpha_1+\alpha_0)T^kx_n\\&\qquad\quad-(\alpha_2+\alpha_1+\alpha_0)y,\quad y-Ty\Big\rangle\leq 0.\end{aligned}$$

Summing these inequalities with respect to k from 0 to $n-1$ and dividing it by n, we obtain that

$$\begin{aligned}&(\alpha_2+\alpha_1+\alpha_0)\|y-Ty\|^2\\&\quad+(\alpha_2+\beta_2)\frac{1}{n}\left(\left\|T^{n+1}x_n-y\right\|^2+\|T^nx_n-y\|^2-\|Tx_n-y\|^2-\|x_n-y\|^2\right)\\&\qquad+(\alpha_1+\beta_1)\frac{1}{n}\left(\|T^nx_n-y\|^2-\|x_n-y\|^2\right)\\&\qquad\quad+2\Big\langle\alpha_2\frac{1}{n}\left(T^{n+1}x_n+T^nx_n-Tx_n-x_n\right)+\alpha_1\frac{1}{n}\left(T^nx_n-x_n\right)\\&\qquad\qquad+(\alpha_2+\alpha_1+\alpha_0)z_n-(\alpha_2+\alpha_1+\alpha_0)y,\quad y-Ty\Big\rangle\leq 0.\end{aligned}$$

Replacing n by n_i and taking the limit as $i\to\infty$, we have that

$$(\alpha_2+\alpha_1+\alpha_0)\|y-Ty\|^2+2(\alpha_2+\alpha_1+\alpha_0)\langle v-y,\ y-Ty\rangle\leq 0.$$

Since $\alpha_2+\alpha_1+\alpha_0>0$, we obtain that $\|y-Ty\|^2+2\langle v-y,\ y-Ty\rangle\leq 0$ for all $y\in C$. This means that $v\in A(T)$. □

Theorem 3.2. *Let C be a nonempty and convex subset of H, and let $T:C\to C$ be a normally 2-generalized hybrid mapping with $A(T)\neq\emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1)$ such that $\lim_{n\to\infty}\lambda_n=0$ and $\sum_{n=1}^{\infty}\lambda_n=\infty$. Given $x_1,z\in C$, define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1}=\lambda_nz+(1-\lambda_n)\frac{1}{n}\sum_{k=0}^{n-1}T^kx_n$$

for all $n\in\mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to $\overline{z}\equiv P_{A(T)}z$, where $P_{A(T)}$ is the metric projection from H onto $A(T)$.

Proof. First, note from Lemma 2.1 that $A(T)$ is a closed and convex subset of H. Since $A(T) \neq \emptyset$ is assumed, the metric projection $P_{A(T)}$ from H onto $A(T)$ is well-defined. Define $z_n \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x_n$. Since C is convex, $\{z_n\}$ is a sequence in C. We can ascertain that

$$\|z_n - u\| \leq \|x_n - u\| \tag{3.1}$$

for all $u \in A(T)$ and $n \in \mathbb{N}$. Indeed, since $u \in A(T)$, we have that

$$\begin{aligned} \|z_n - u\| &\equiv \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n - u \right\| \\ &= \left\| \frac{1}{n} \sum_{k=0}^{n-1} \left(T^k x_n - u \right) \right\| \\ &\leq \frac{1}{n} \sum_{k=0}^{n-1} \left\| T^k x_n - u \right\| \\ &\leq \|x_n - u\|. \end{aligned}$$

Next, we will show that the sequence $\{x_n\}$ is bounded by using the mathematical induction. Let $u \in A(T)$, and define $M \equiv \max\{\|z - u\|, \ \|x_1 - u\|\}$.

(i) For the case of $n = 1$, it holds form the definition of M that $\|x_1 - u\| \leq M$.

(ii) Suppose that $\|x_k - u\| \leq M$. From (3.1), we have that

$$\begin{aligned} \|x_{k+1} - u\| &\equiv \|\lambda_k z + (1 - \lambda_k) z_k - u\| \\ &= \|\lambda_k (z - u) + (1 - \lambda_k)(z_k - u)\| \\ &\leq \lambda_k \|z - u\| + (1 - \lambda_k) \|z_k - u\| \\ &\leq \lambda_k \|z - u\| + (1 - \lambda_k) \|x_k - u\| \\ &\leq \lambda_k M + (1 - \lambda_k) M = M. \end{aligned}$$

Thus, $\{x_n\}$ is bounded. From (3.1), $\{z_n\}$ is also bounded. For $u \in A(T)$, it holds that $\|Tx_n - u\| \leq \|x_n - u\|$, and thus $\{Tx_n\}$ is also bounded. Since $\{z_n\}$ is bounded and $\lambda_n \to 0$ is assumed, we obtain that $x_{n+1} - z_n \to 0$ from

$$\begin{aligned} \|x_{n+1} - z_n\| &\equiv \|\lambda_n z + (1 - \lambda_n) z_n - z_n\| \\ &= \lambda_n \|z - z_n\|. \end{aligned}$$

Define $X_n \equiv \|x_n - \overline{z}\|^2$, where $\overline{z} \equiv P_{A(T)} z$. Our aim is to show that $X_n \to 0$. From (2.1) and (3.1), we have that

$$\begin{aligned} X_{n+1} &\equiv \|x_{n+1} - \overline{z}\|^2 \\ &= \|\lambda_n (z - \overline{z}) + (1 - \lambda_n)(z_n - \overline{z})\|^2 \\ &\leq (1 - \lambda_n)^2 \|z_n - \overline{z}\|^2 + 2 \langle x_{n+1} - \overline{z}, \ \lambda_n (z - \overline{z}) \rangle \\ &\leq (1 - \lambda_n) \|z_n - \overline{z}\|^2 + 2\lambda_n \langle x_{n+1} - \overline{z}, \ z - \overline{z} \rangle \\ &\leq (1 - \lambda_n) \|x_n - \overline{z}\|^2 + 2\lambda_n \langle x_{n+1} - \overline{z}, \ z - \overline{z} \rangle \\ &\equiv (1 - \lambda_n) X_n + 2\lambda_n \langle x_{n+1} - \overline{z}, \ z - \overline{z} \rangle. \end{aligned}$$

From Lemma 2.2, it is sufficient to demonstrate that

$$\limsup_{n\to\infty} \langle x_{n+1} - \overline{z},\ z - \overline{z}\rangle \leq 0.$$

Without loss of gererality, there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n\to\infty} \langle x_{n+1} - \overline{z},\ z - \overline{z}\rangle = \lim_{i\to\infty} \langle x_{n_i+1} - \overline{z},\ z - \overline{z}\rangle$$

and $x_{n_i+1} \rightharpoonup v$ for some $v \in H$. Since $x_{n+1} - z_n \to 0$, it holds that $z_{n_i} \rightharpoonup v$. Since the sequence $\{x_n\}$ is bounded and T is normally 2-generalized hybrid, we have from Lemma 3.1 that $v \in A(T)$. Since $\overline{z} \equiv P_{A(T)}z$ and $v \in A(T)$, we obtain from (2.5) that

$$\begin{aligned}\limsup_{n\to\infty} \langle x_{n+1} - \overline{z},\ z - \overline{z}\rangle &= \lim_{i\to\infty} \langle x_{n_i+1} - \overline{z},\ z - \overline{z}\rangle \\ &= \langle v - \overline{z},\ z - \overline{z}\rangle \leq 0.\end{aligned}$$

This completes the proof. □

Remark 3.3. From (3.1) in the proof of Theorem 3.2, the averaged sequence

$$\left\{ z_n \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x_n \right\}$$

also converges strongly to the attractive point $\overline{z} \equiv P_{A(T)}z$.

The following theorem offers an alternative method to construct sequences that converge strongly to attractive points of normally 2-generalized hybrid mappings.

Theorem 3.4. *Let C be a nonempty and convex subset of H, and let $T : C \to C$ be a normally 2-generalized hybrid mapping with $A(T) \neq \emptyset$. Let $\{\lambda_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers in the interval $(0, 1)$ such that*

$$\lim_{n\to\infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty, \tag{3.2}$$

$$a_n + b_n + c_n = 1, \quad \forall n \in \mathbb{N}, \tag{3.3}$$

$$\liminf_{n\to\infty} a_n b_n > 0, \quad \liminf_{n\to\infty} b_n c_n > 0 \quad \textit{and} \quad \liminf_{n\to\infty} c_n a_n > 0. \tag{3.4}$$

Given $x_1, z \in C$, define a sequence $\{x_n\}$ in C as follows:

$$x_{n+1} = \lambda_n z + (1 - \lambda_n)\left(a_n x_n + b_n T x_n + c_n T^2 x_n\right)$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to $\overline{z} \equiv P_{A(T)}z$, where $P_{A(T)}$ is the metric projection from H onto $A(T)$.

Proof. Define $z_n \equiv a_n x_n + b_n T x_n + c_n T^2 x_n$. Since C is convex, from (3.3), $\{z_n\}$ is a sequence in C. It is easily ascertained that

$$\|z_n - u\| \leq \|x_n - u\| \tag{3.5}$$

for all $u \in A(T)$ and $n \in \mathbb{N}$. Indeed, since $u \in A(T)$, we have from (3.3) that

$$\begin{aligned} \|z_n - u\| &\equiv \left\| \left[a_n x_n + b_n T x_n + c_n T^2 x_n \right] - u \right\| \\ &\leq a_n \|x_n - u\| + b_n \|Tx_n - u\| + c_n \left\| T^2 x_n - u \right\| \\ &\leq a_n \|x_n - u\| + b_n \|x_n - u\| + c_n \|x_n - u\| \\ &= \|x_n - u\| . \end{aligned}$$

Furthermore, it holds that the sequences $\{x_n\}$, $\{z_n\}$ and $\{Tx_n\}$ are bounded. Since their proofs are just same as those in the proof of Theorem 3.2, we omit them here.

Next, note that

$$\begin{aligned} \|x_{n+1} - x_n\| \leq \lambda_n \|z - x_n\| \\ + (1 - \lambda_n) b_n \|Tx_n - x_n\| + (1 - \lambda_n) c_n \left\| T^2 x_n - x_n \right\| \end{aligned} \tag{3.6}$$

for all $n \in \mathbb{N}$. The relationship (3.6) can be verified as follows:

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &\quad \equiv \|\lambda_n z + (1 - \lambda_n) z_n - x_n\| \\ &\quad \leq \lambda_n \|z - x_n\| + (1 - \lambda_n) \|z_n - x_n\| \\ &\quad \equiv \lambda_n \|z - x_n\| + (1 - \lambda_n) \left\| a_n x_n + b_n T x_n + c_n T^2 x_n - x_n \right\| \\ &\quad = \lambda_n \|z - x_n\| + (1 - \lambda_n) \left\| a_n x_n + b_n T x_n + c_n T^2 x_n - (a_n + b_n + c_n) x_n \right\| \\ &\quad = \lambda_n \|z - x_n\| + (1 - \lambda_n) \left\| b_n (T x_n - x_n) + c_n \left(T^2 x_n - x_n \right) \right\| \\ &\quad \leq \lambda_n \|z - x_n\| + (1 - \lambda_n) b_n \|Tx_n - x_n\| + (1 - \lambda_n) c_n \left\| T^2 x_n - x_n \right\| . \end{aligned}$$

Furthermore, it holds that

$$\begin{aligned} a_n b_n \|x_n - Tx_n\|^2 + b_n c_n \left\| Tx_n - T^2 x_n \right\|^2 + c_n a_n \left\| T^2 x_n - x_n \right\|^2 \\ \leq \lambda_n \|z - u\|^2 + \|x_n - u\|^2 - \|x_{n+1} - u\|^2 \end{aligned} \tag{3.7}$$

for all $u \in A(T)$ and $n \in \mathbb{N}$. Indeed, from (2.3), (3.3) and Lemma 2.3,

$$\begin{aligned} &\|x_{n+1} - u\|^2 \\ &\quad = \|\lambda_n (z - u) + (1 - \lambda_n)(z_n - u)\|^2 \\ &\quad \leq \lambda_n \|z - u\|^2 + (1 - \lambda_n) \|z_n - u\|^2 \\ &\quad \leq \lambda_n \|z - u\|^2 + \left\| a_n (x_n - u) + b_n (Tx_n - u) + c_n \left(T^2 x_n - u \right) \right\|^2 \\ &\quad = \lambda_n \|z - u\|^2 + a_n \|x_n - u\|^2 + b_n \|Tx_n - u\|^2 + c_n \left\| T^2 x_n - u \right\|^2 \\ &\qquad - a_n b_n \|x_n - Tx_n\|^2 - b_n c_n \left\| Tx_n - T^2 x_n \right\|^2 - c_n a_n \left\| T^2 x_n - x_n \right\|^2 \\ &\quad \leq \lambda_n \|z - u\|^2 + a_n \|x_n - u\|^2 + b_n \|x_n - u\|^2 + c_n \|x_n - u\|^2 \\ &\qquad - a_n b_n \|x_n - Tx_n\|^2 - b_n c_n \left\| Tx_n - T^2 x_n \right\|^2 - c_n a_n \left\| T^2 x_n - x_n \right\|^2 \\ &\quad \leq \lambda_n \|z - u\|^2 + \|x_n - u\|^2 \\ &\qquad - a_n b_n \|x_n - Tx_n\|^2 - b_n c_n \left\| Tx_n - T^2 x_n \right\|^2 - c_n a_n \left\| T^2 x_n - x_n \right\|^2 , \end{aligned}$$

which implies that (3.7) is satisfied.

Define $X_n \equiv \|x_n - \overline{z}\|^2$, where $\overline{z} \equiv P_{A(T)}z$. Our aim is to show that $X_n \to 0$. The rest of the proof is divided into two cases.

Case (A). Suppose that there exists a natural number N such that $X_{n+1} \leq X_n$ for all $n \geq N$. In this case, the sequence $\{X_n\}$ is convergent. From (3.7), it holds that

$$\begin{aligned} &a_n b_n \|x_n - Tx_n\|^2 + b_n c_n \|Tx_n - T^2 x_n\|^2 + c_n a_n \|T^2 x_n - x_n\|^2 \\ &\quad \leq \lambda_n \|z - \overline{z}\|^2 + \|x_n - \overline{z}\|^2 - \|x_{n+1} - \overline{z}\|^2 \\ &\quad \equiv \lambda_n \|z - \overline{z}\|^2 + X_n - X_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. We have from (3.2) and (3.4) that

$$x_n - Tx_n \to 0, \quad Tx_n - T^2 x_n \to 0 \text{ and } T^2 x_n - x_n \to 0 \tag{3.8}$$

as $n \to \infty$. Since $\{x_n\}$ is bounded, it holds from (3.6) and (3.2) that $x_{n+1} - x_n \to 0$. By using (2.1) and (3.5), we have

$$\begin{aligned} X_{n+1} &\equiv \|x_{n+1} - \overline{z}\|^2 \\ &= \|\lambda_n (z - \overline{z}) + (1 - \lambda_n)(z_n - \overline{z})\|^2 \\ &\leq (1 - \lambda_n)^2 \|z_n - \overline{z}\|^2 + 2\langle x_{n+1} - \overline{z}, \ \lambda_n (z - \overline{z})\rangle \\ &\leq (1 - \lambda_n) \|z_n - \overline{z}\|^2 + 2\lambda_n \langle x_{n+1} - \overline{z}, \ z - \overline{z}\rangle \\ &\leq (1 - \lambda_n) \|x_n - \overline{z}\|^2 + 2\lambda_n \langle x_{n+1} - \overline{z}, \ z - \overline{z}\rangle \\ &= (1 - \lambda_n) X_n + 2\lambda_n \left(\langle x_{n+1} - x_n, \ z - \overline{z}\rangle + \langle x_n - \overline{z}, \ z - \overline{z}\rangle\right). \end{aligned}$$

Since $x_{n+1} - x_n \to 0$, from Lemma 2.2, it is sufficient to prove that

$$\limsup_{n\to\infty} \langle x_n - \overline{z}, \ z - \overline{z}\rangle \leq 0.$$

Since $\{x_n\}$ is bounded, without loss of generality, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - \overline{z}, \ z - \overline{z}\rangle = \lim_{i\to\infty} \langle x_{n_i} - \overline{z}, \ z - \overline{z}\rangle$$

and $x_{n_i} \rightharpoonup v$ for some $v \in H$. Since T is normally 2-generalized hybrid, we have from (3.8) and Lemma 2.4 that $v \in A(T)$. Since $\overline{z} \equiv P_{A(T)}z$ and $v \in A(T)$, we obtain from (2.5) that

$$\begin{aligned} \limsup_{n\to\infty} \langle x_n - \overline{z}, \ z - \overline{z}\rangle &= \lim_{i\to\infty} \langle x_{n_i} - \overline{z}, \ z - \overline{z}\rangle \\ &= \langle v - \overline{z}, \ z - \overline{z}\rangle \leq 0. \end{aligned}$$

This completes the proof for Case (A).

Case (B). Suppose that there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : X_k < X_{k+1}\} \neq \emptyset$. Define $\tau(n) = \max\{k \leq n : X_k < X_{k+1}\}$ for all $n \geq n_0$. As results from Lemma 2.5, the followings hold:

$$\tau(n) \to \infty \text{ as } n \to \infty; \tag{3.9}$$

$$X_n \leq X_{\tau(n)+1} \text{ for all } n \geq n_0; \tag{3.10}$$

$$X_{\tau(n)} < X_{\tau(n)+1} \text{ for all } n \geq n_0. \tag{3.11}$$

From (3.2) and (3.4), it holds that

$$\lambda_{\tau(n)} \to 0; \tag{3.12}$$

$$0 < \liminf_{n\to\infty} a_{\tau(n)} b_{\tau(n)}; \tag{3.13}$$

$$0 < \liminf_{n\to\infty} b_{\tau(n)} c_{\tau(n)}; \tag{3.14}$$

$$0 < \liminf_{n\to\infty} c_{\tau(n)} a_{\tau(n)}. \tag{3.15}$$

From (3.10), it is sufficient to prove that $X_{\tau(n)+1} \equiv \left\| x_{\tau(n)+1} - \overline{z} \right\|^2 \to 0$. From (3.7), we have

$$\begin{aligned} a_{\tau(n)} b_{\tau(n)} & \left\| x_{\tau(n)} - T x_{\tau(n)} \right\|^2 + b_{\tau(n)} c_{\tau(n)} \left\| T x_{\tau(n)} - T^2 x_{\tau(n)} \right\|^2 \\ & + c_{\tau(n)} a_{\tau(n)} \left\| T^2 x_{\tau(n)} - x_{\tau(n)} \right\|^2 \\ & \leq \lambda_{\tau(n)} \left\| z - \overline{z} \right\|^2 + \left\| x_{\tau(n)} - \overline{z} \right\|^2 - \left\| x_{\tau(n)+1} - \overline{z} \right\|^2 \\ & \equiv \lambda_{\tau(n)} \left\| z - \overline{z} \right\|^2 + X_{\tau(n)} - X_{\tau(n)+1} \end{aligned}$$

for all $n \geq n_0$. From (3.11),

$$\begin{aligned} a_{\tau(n)} b_{\tau(n)} \| x_{\tau(n)} - T x_{\tau(n)} \|^2 + b_{\tau(n)} c_{\tau(n)} \left\| T x_{\tau(n)} - T^2 x_{\tau(n)} \right\|^2 & \\ + c_{\tau(n)} a_{\tau(n)} \left\| T^2 x_{\tau(n)} - x_{\tau(n)} \right\|^2 & \leq \lambda_{\tau(n)} \left\| z - \overline{z} \right\|^2 . \end{aligned}$$

From (3.9), (3.12)–(3.15), we obtain that

$$x_{\tau(n)} - T x_{\tau(n)} \to 0, \quad T x_{\tau(n)} - T^2 x_{\tau(n)} \to 0 \text{ and } T^2 x_{\tau(n)} - x_{\tau(n)} \to 0 \tag{3.16}$$

as $n \to \infty$. From (3.6),

$$\begin{aligned} \left\| x_{\tau(n)+1} - x_{\tau(n)} \right\| \leq \lambda_{\tau(n)} \left\| z - x_{\tau(n)} \right\| + \left(1 - \lambda_{\tau(n)}\right) b_{\tau(n)} \left\| T x_{\tau(n)} - x_{\tau(n)} \right\| & \\ + \left(1 - \lambda_{\tau(n)}\right) c_{\tau(n)} \left\| T^2 x_{\tau(n)} - x_{\tau(n)} \right\| & \end{aligned}$$

for all $n \geq n_0$. Since $\left\{ x_{\tau(n)} \right\}$ is bounded, we have from (3.9), (3.12)–(3.16) that $x_{\tau(n)+1} - x_{\tau(n)} \to 0$, and thus, $X_{\tau(n)+1} - X_{\tau(n)} \to 0$. We will demonstrate that $X_{\tau(n)} \equiv \left\| x_{\tau(n)} - \overline{z} \right\|^2 \to 0$. From (2.1) and (3.5),

$$\begin{aligned} X_{\tau(n)+1} & \equiv \left\| x_{\tau(n)+1} - \overline{z} \right\|^2 \\ & = \left\| \lambda_{\tau(n)} \left(z - \overline{z}\right) + \left(1 - \lambda_{\tau(n)}\right) \left(z_{\tau(n)} - \overline{z}\right) \right\|^2 \\ & \leq \left(1 - \lambda_{\tau(n)}\right)^2 \left\| z_{\tau(n)} - \overline{z} \right\|^2 + 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{z},\ z - \overline{z} \right\rangle \\ & \leq \left(1 - \lambda_{\tau(n)}\right) \left\| x_{\tau(n)} - \overline{z} \right\|^2 + 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{z},\ z - \overline{z} \right\rangle \\ & \equiv \left(1 - \lambda_{\tau(n)}\right) X_{\tau(n)} + 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{z},\ z - \overline{z} \right\rangle . \end{aligned}$$

This yields

$$\lambda_{\tau(n)} X_{\tau(n)} \leq X_{\tau(n)} - X_{\tau(n)+1} + 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{z},\ z - \overline{z} \right\rangle .$$

From (3.11),

$$\lambda_{\tau(n)} X_{\tau(n)} \leq 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{z},\ z - \overline{z} \right\rangle .$$

Since $\lambda_{\tau(n)} > 0$, we have

$$\begin{aligned} X_{\tau(n)} &\leq 2\left\langle x_{\tau(n)+1} - \overline{z},\ z - \overline{z}\right\rangle \\ &= 2\left\langle x_{\tau(n)+1} - x_{\tau(n)},\ z - \overline{z}\right\rangle + 2\left\langle x_{\tau(n)} - \overline{z},\ z - \overline{z}\right\rangle. \end{aligned}$$

Since $x_{\tau(n)+1} - x_{\tau(n)} \to 0$, it is sufficient to prove that

$$\limsup_{n\to\infty} \left\langle x_{\tau(n)} - \overline{z},\ z - \overline{z}\right\rangle \leq 0.$$

Since $\left\{x_{\tau(n)}\right\}$ is bounded, without loss of generality, there exists a subsequence $\left\{x_{\tau(n_i)}\right\}$ of $\left\{x_{\tau(n)}\right\}$ such that

$$\limsup_{n\to\infty} \left\langle x_{\tau(n)} - \overline{z},\ z - \overline{z}\right\rangle = \lim_{i\to\infty} \left\langle x_{\tau(n_i)} - \overline{z},\ z - \overline{z}\right\rangle$$

and $x_{\tau(n_i)} \rightharpoonup v$ for some $v \in H$. Since T is normally 2-generalized hybrid, we have from (3.16) and Lemma 2.4 that $v \in A(T)$. Since $\overline{z} \equiv P_{A(T)}z$ and $v \in A(T)$, we obtain from (2.5) that

$$\begin{aligned} \limsup_{n\to\infty} \left\langle x_{\tau(n)} - \overline{z},\ z - \overline{z}\right\rangle &= \lim_{i\to\infty} \left\langle x_{\tau(n_i)} - \overline{z},\ z - \overline{z}\right\rangle \\ &= \left\langle v - \overline{z},\ z - \overline{z}\right\rangle \leq 0. \end{aligned}$$

This completes the proof. □

Remark 3.5. From (3.5) in the proof of Theorem 3.4, the "averaged sequence"

$$\left\{z_n \equiv a_n x_n + b_n T x_n + c_n T^2 x_n\right\}$$

also converges strongly to the attractive point $\overline{z} \equiv P_{A(T)}z$.

4. Strong convergence to fixed points

In this section, we add an assumption that C is closed, and obtain strong convergence theorems of finding fixed points for normally 2-generalized hybrid mappings in Hilbert spaces. From Theorem 3.2, we obtain the following theorem:

Theorem 4.1. *Let C be a nonempty, closed and convex subset of H, and let $T : C \to C$ be a normally 2-generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1)$ such that $\lim_{n\to\infty}\lambda_n = 0$ and $\sum_{n=1}^{\infty}\lambda_n = \infty$. Given $x_1, z \in C$, define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = \lambda_n z + (1-\lambda_n)\frac{1}{n}\sum_{k=0}^{n-1} T^k x_n$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to $\widehat{z} \equiv P_{F(T)}z$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Proof. We know from Lemma 2.6 that T is quasi-nonexpansive. As a consequence, $F(T)$ is closed and convex. Since $F(T) \neq \emptyset$ is assumed, the metric projection $P_{F(T)}$ from H onto $F(T)$ is well-defined. We have from Lemmas 2.7 and 2.8 that

$$F(T) \subset A(T) \quad\text{and}\quad A(T) \cap C \subset F(T), \tag{4.1}$$

respectively. Since $F(T) \neq \emptyset$, it holds from (4.1) that $A(T) \neq \emptyset$. From Lemma 2.1, the metric projection $P_{A(T)}$ from H onto $A(T)$ is well-defined.

Define $\overline{z} \equiv P_{A(T)}z$. From Theorem 3.2, we obtain that $x_n \to \overline{z}$. Thus, it is sufficient to prove that $(\widehat{z} \equiv) P_{F(T)}z = \overline{z} \left(\equiv P_{A(T)}z\right)$. First, note that $\overline{z} \in F(T)$. Indeed, since $\{x_n\}$ is a sequence in C that converges to $\overline{z}$ and C is closed in H, it holds that $\overline{z} \in C$. Since $\overline{z} \left(\equiv P_{A(T)}z\right) \in A(T)$, it holds from (4.1) that $\overline{z} \in F(T)$. Next, we will verify that $\|z - \overline{z}\| \le \|z - u\|$ for all $u \in F(T)$, in other words, $\overline{z}$ is the nearest point of $F(T)$ from z. Let $u \in F(T)$. Using (4.1), we have that

$$\begin{aligned} \|z - \overline{z}\| &= \inf\{\|z - q\| : q \in A(T)\} \\ &\le \inf\{\|z - q\| : q \in F(T)\} \\ &\le \|z - u\|, \end{aligned}$$

which means $\overline{z} = P_{F(T)}z \equiv \widehat{z}$. This completes the proof. $\square$

From Theorem 3.4, we obtain the following strong approximation method of finding fixed points of normally 2-generalized hybrid mappings.

Theorem 4.2. *Let C be a nonempty, closed and convex subset of H, and let $T : C \to C$ be a normally 2-generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers in the interval $(0,1)$ such that*

$$\lim_{n\to\infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty,$$
$$a_n + b_n + c_n = 1, \quad \forall n \in \mathbb{N},$$
$$\liminf_{n\to\infty} a_n b_n > 0, \quad \liminf_{n\to\infty} b_n c_n > 0 \quad \text{and} \quad \liminf_{n\to\infty} c_n a_n > 0.$$

Given $x_1, z \in C$, define a sequence $\{x_n\}$ in C as follows:

$$x_{n+1} = \lambda_n z + (1 - \lambda_n)\left(a_n x_n + b_n T x_n + c_n T^2 x_n\right)$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to $\widehat{z} \equiv P_{F(T)}z$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Proof. The proof is just same as that of Theorem 4.1. $\square$

As the final remark, notice that all results in this paper are extended to normally N-generalized hybrid mappings. A mapping $T : C \to C$ is called *normally N-generalized hybrid* [9] if there exist real numbers $\alpha_0, \beta_0, \ldots, \alpha_N, \beta_N \in \mathbb{R}$ such that

$$\sum_{n=0}^{N} \alpha_n \|T^n x - Ty\|^2 + \sum_{n=0}^{N} \beta_n \|T^n x - y\|^2 \le 0$$

for all $x, y \in C$, where $\sum_{n=0}^{N} (\alpha_n + \beta_n) \ge 0$ and $\sum_{n=0}^{N} \alpha_n > 0$.

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