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# EXISTENCE RESULTS FOR SYSTEMS OF FIRST-ORDER STIELTJES DIFFERENTIAL EQUATIONS VIA THE METHOD OF $g$-SOLUTION-REGIONS

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*Dedicated to Professor Sehie Park*

Abstract. In this article, we establish existence results for systems of first-order Stieltjes differential equations with an initial value condition or a periodic boundary condition. For this purpose, we introduce the method of $g$-solution-regions, that generalizes and is based on the method of solution-regions introduced by Frigon [6] in 2018. Examples present different sets that are $g$-solution-regions. An application to model the evolution of the voltage across an electrical circuit composed by a resonant tunnelling diode (RTD) driving a laser diode (LD) is presented.

## 1. Introduction

Let $g : \mathbb{R} \to \mathbb{R}$ be a monotone nondecreasing and left-continuous function. In this article, we establish existence results for systems of first-order Stieltjes differential equations of the form

$$u_g'(t) = f(t, u(t)) \quad \text{for } g\text{-almost all } t \in I = [0, T], \quad u \in \mathcal{B}, \tag{1.1}$$

where $u_g'$ denotes the derivative of the function $u$ with respect to $g$, $f : I \times \mathbb{R}^N \to \mathbb{R}^N$ is a $g$-Carathéodory function and $\mathcal{B}$ denotes the initial value condition or the periodic boundary condition

$$u(0) = r, \tag{1.2}$$

$$u(0) = u(T). \tag{1.3}$$

The Stieltjes derivative was introduced by López Pouso and Rodríguez in [17] to unify continuous, discrete and impulsive calculus, which is particularly useful to model situations with impulsions and stationary periods. In the last years, this theory has been thoroughly developed, and results using different methods have been found for the problems (1.1), (1.2) and (1.1), (1.3). Basic existence and uniqueness theory has first been developed for the initial value condition in [7]. Stieltjes versions of classical uniqueness results, such as Yosie-type criterion, Peano's uniqueness theorem, Osgood's uniqueness theorem, Montel-Tonelli uniqueness criterion and Perron's theorem, were proved in [21] for the initial value condition. The

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method of lower and upper solutions has also been very useful to find existence results. Indeed, existence results were found for the case $N = 1$ in [14,18,21], and for systems in [16,19,21]. In particular, [14,16,21] assumed a version of the following monotonicity condition

$$x \mapsto x + f(t,x)(g(t^+) - g(t)) \quad \text{is nondecreasing on } [\alpha(t), \beta(t)]$$

for every $t \in [0,T) \cap D_g$, where $D_g$ is the set of discontinuity points of $g$, and $\alpha, \beta$ are lower and upper solutions, respectively. This condition was avoided in [18,19]. Schaeffer's fixed point theorem was used in [29] to establish existence theorems for Stieltjes differential equations with a periodic boundary condition. Furthermore, results for Stieltjes equations using the fixed point index theory were recently found in [12] for a generalized periodic boundary condition.

Problems with systems of Stietljes differential equations with multiple derivators were also considered in [15,19,22]. López Pouso and Márquez Albés [15] first developed basic existence and uniqueness theory for initial value problems with different derivators in 2019. This was further expanded by Márquez Albés and Tojo [22] who obtained theorems based on Osgood or Montel-Tonelli conditions. Existence results for systems with different derivators were also found by Maia, El Khattabi and Frigon [19] using the method of lower and upper solutions.

Stieltjes differential inclusions with a periodic boundary condition were considered in [23, 28]. Satco and Smyrlis [28] used Bohnenblust-Karlin set-valued fixed point theorem to guarantee the existence of regulated solutions. Maraffa and Satco [23] found existence results when $F : [0,T] \times \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ is convex, compact-valued and upper semicontinuous everywhere except on a set that can be dense, using the notion of contingent $g$-derivative.

In [6], Frigon introduced the notion of solution-region to generalize the method of lower and upper solutions (see [1–4,9,11,13,24,26]) and the method of solution-tubes (see [5,8,25]) for existence and multiplicity results of systems of classical first-order differential equations, using the fixed point index. Recent developments on the method of solution-regions for problems with generalized boundary conditions have also been made in [10,31].

In this paper, we introduce the method of $g$-solution-regions to establish existence results for systems of the form (1.1). A $g$-solution-region will be a set $R \subset I \times \mathbb{R}^N$ that will guarantee the existence of a solution of (1.1) whose graph is contained in $R$. Our results will generalize those using the method of solution-regions, which already generalizes the method of lower and upper solutions and the method of solution-tubes for classical differential equations, as we can consider $g(t) = t$. It will also generalize the existence results from [16,18] and partially those from [14,19,21], as lower and upper solutions for Stieltjes differential equations are a particular case of $g$-solution-regions.

The rest of the paper is organized as follows. In Section 2, we present preliminary definitions and results. We also introduce the notion of $(g \times I_{\mathbb{R}^N})$-differentiability, related concepts and results that follow, which will be useful for our definition of $g$-admissible region. In Section 3, we recall an exponential function introduced in [7] and prove related results for systems of linear Stieltjes differential equations with a periodic boundary condition that will be useful for our existence result for the periodic boundary condition problem. Section 4 introduces the notion of $g$-admissible regions and presents examples of these regions. Section 5 introduces $g$-solution-regions for both the initial value condition and the periodic boundary condition problems, with examples for both cases. Section 6 is the main section of our paper, where we prove existence results relying on those new notions. We present multiple examples to see how $g$-solution-regions generalize usual methods. Finally, in Section 7, we present an application of our existence result for the periodic boundary condition problem to the evolution of the voltage across an electrical circuit composed by a resonant tunnelling diode (RTD) driving a laser diode (LD).

## 2. Preliminaries

Let $g : \mathbb{R} \to \mathbb{R}$ be a monotone nondecreasing function and continuous from the left everywhere. Let us define two sets that will be useful. Let $C_g$ be the set of points where $g$ is constant in a neighborhood denoted by

$$C_g = \{t \in \mathbb{R} : g \text{ is constant on } (t-\epsilon, t+\epsilon) \text{ for a certain } \epsilon > 0\}. \tag{2.1}$$

We also denote the set of discontinuities of $g$, which is a countable set since $g$ is monotone, by

$$D_g = \{t \in \mathbb{R} : g(t^+) - g(t) > 0\}, \tag{2.2}$$

where we note

$$g(t^+) = \lim_{s \to t^+} g(s).$$

A notion of $g$-continuity can be defined from this function $g$ (see [7]).

**Definition 2.1.** A function $f : A \subset \mathbb{R} \to \mathbb{R}^N$ is *$g$-continuous at the point $t_0 \in A$* (or *continuous with respect to $g$ at the point $t_0$*) if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$[t \in A, |g(t) - g(t_0)| < \delta] \Rightarrow \|f(t) - f(t_0)\| < \epsilon.$$

We say that $f$ is *$g$-continuous on $A$* if $f$ is $g$-continuous at every point $t_0 \in A$.

We recall the following proposition, shown in [7], that contains useful properties of $g$-continuous functions.

**Proposition 2.2.** *Let $a, b \in \mathbb{R}$ be such that $a < b$. If $f : [a, b] \to \mathbb{R}^N$ is $g$-continuous on $[a, b]$, then we have that*

(i) *$f$ is continuous from the left at every point $t_0 \in (a, b]$.*
(ii) *If $g$ is continuous at $t_0 \in [a, b)$, then $f$ is also continuous at $t_0$.*
(iii) *If $g$ is constant on a certain interval $[\alpha, \beta] \subset [a, b]$, then $f$ is also constant on that interval.*

*In particular, $g$-continuous functions on $[a, b]$ are continuous on $[a, b]$ when $g$ is continuous on $[a, b)$.*

Denote by $\mathcal{BC}_g([a, b], \mathbb{R}^N)$ the subset of $g$-continuous functions with values in $\mathbb{R}^N$ that are also bounded on $[a, b]$. The set $\mathcal{BC}_g([a, b], \mathbb{R}^N)$ is a Banach space when it is equipped with the following norm:

$$\|f\|_0 = \sup_{t \in [a,b]} \|f(t)\| \quad \text{for all } f \in \mathcal{BC}_g([a, b], \mathbb{R}^N).$$

The function $g$ also generates a unique Lebesgue-Stieltjes measure, denoted $\mu_g : \mathcal{M}_g \to [0,\infty]$, from the following base formula

$$\mu_g([a,b)) = g(b) - g(a) \quad \text{for all } a,b \in \mathbb{R}, a < b,$$

where $\mathcal{M}_g$ is the $\sigma$-algebra of subsets $A \subset \mathbb{R}$ that respect a Carathéodory condition. This $g$-measure $\mu_g$ shares many properties with the Lebesgue measure, but the main difference between the two is that every $t \in D_g$ is an atom. Indeed, let $t \in D_g$, we have

$$\begin{aligned}\mu_g(\{t\}) &= \mu_g\left(\bigcap_{n=1}^{\infty}[t, t+1/n)\right)\\ &= \lim_{n\to\infty} \mu_g([t,t+1/n))\\ &= \lim_{n\to\infty} g(t+1/n) - g(t)\\ &= g(t^+) - g(t) > 0.\end{aligned}$$

We will say that a null set with respect to $\mu_g$ is a *set of $g$-measure zero.* A property will be *true $g$-almost everywhere* if it is true outside a set of $g$-measure zero. Finally, a function $f : E \in \mathcal{M}_g \to \mathbb{R}$ is a *$g$-measurable function* if, for every open set $V \subset \mathbb{R}$, we have $f^{-1}(V) \in \mathcal{M}_g$. From this $g$-measure, the Lebesgue-Stieltjes integral can be defined, and we note

$$\mathcal{L}^1_g(E) = \left\{ f : E \to \mathbb{R} \mid f \text{ is } g\text{-measurable and } \int_E |f(t)|d\mu_g < \infty \right\}.$$

The notion of the derivative of a function with respect to $g$ was introduced in [17].

**Definition 2.3.** Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}^N$. The *derivative with respect to $g$* (or *$g$-derivative*, or *Stieltjes derivative*) *of $f$ at the point* $t_0 \in E\backslash C_g$ is given by

$$f'_g(t_0) = \lim_{t\to t_0} \frac{f(t)-f(t_0)}{g(t)-g(t_0)} \quad \text{if } t_0 \notin D_g,$$

$$f'_g(t_0) = \lim_{t\to t_0^+} \frac{f(t)-f(t_0)}{g(t)-g(t_0)} \quad \text{if } t_0 \in D_g,$$

if the limit exists, in which case we say that $f$ is *$g$-differentiable at $t_0$*. We say that $f$ is *$g$-differentiable $g$-almost everywhere on $E$* if $f$ is $g$-differentiable at every $t_0 \in E\backslash S$, where $\mu_g(S) = 0$.

It is not necessary to define the $g$-derivative at points in $C_g$, since by [17, Proposition 2.5], $\mu_g(C_g) = 0$, and we will have $C_g \subset S$.

We will look for solutions of (1.1) which are $g$-absolutely continuous. We recall the definition given in [17].

**Definition 2.4.** Let $a,b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$. The function $f$ is *absolutely continuous with respect to $g$* (or *$g$-absolutely continuous*) if, for every

$\epsilon > 0$, there exists $\delta > 0$ such that, for every family $\{(a_n, b_n)\}_{n=1}^m$ of pairwise disjoint open subintervals of $[a, b]$ satisfying

$$\sum_{n=1}^{m}(g(b_n) - g(a_n)) < \delta,$$

we have

$$\sum_{n=1}^{m}|f(b_n) - f(a_n)| < \epsilon.$$

We note $\mathcal{AC}_g([a, b])$ the set of $g$-absolutely continuous functions.

We can now state the fundamental theorem of calculus for the Lebesgue-Stieltjes integral proved in [17, Theorem 5.4].

**Theorem 2.5.** *A function $F : [a, b] \to \mathbb{R}$ is $g$-absolutely continuous on $[a, b]$ if and only if the following three conditions are respected:*

- *there exists $F_g'(t)$ for $g$-almost all $t \in [a, b]$;*
- *$F_g' \in \mathcal{L}_g^1([a, b))$;*
- *for all $t \in [a, b]$, we have*

$$F(t) = F(a) + \int_{[a,t)} F_g'(s)d\mu_g.$$

We say that a function $f : [a, b] \to \mathbb{R}^N$ is *$g$-absolutely continuous* on $[a, b]$ if each of its components is $g$-absolutely continuous. We note $\mathcal{AC}_g([a, b], \mathbb{R}^N)$ the set of $g$-absolutely continuous functions with values in $\mathbb{R}^N$.

The following two propositions are direct generalizations of [7, Propositions 5.5 and 5.6], by considering functions component by component.

**Proposition 2.6.** *The set $\mathcal{AC}_g([a, b], \mathbb{R}^N)$ is included in $\mathcal{BC}_g([a, b], \mathbb{R}^N)$.*

**Proposition 2.7.** *Let $\mathcal{S} \subset \mathcal{AC}_g([a, b], \mathbb{R}^N)$ be such that $\{F(a) : F \in \mathcal{S}\}$ is bounded. Suppose that there exists $h \in \mathcal{L}_g^1([a, b), [0, \infty))$ such that*

$$\|F_g'(t)\| \leq h(t) \quad \text{for } g\text{-almost all } t \in [a, b) \text{ and for all } F \in \mathcal{S}.$$

*Then, $\mathcal{S}$ is relatively compact in $\mathcal{BC}_g([a, b], \mathbb{R}^N)$.*

Let us recall the definition of a $g$-Carathéodory function and some related notions.

**Definition 2.8.** Let $X$ be a nonempty subset of $\mathbb{R}^N$. A map $f : [a, b] \times X \to \mathbb{R}^N$ is *$g$-Carathéodory* if it satisfies the following conditions:

(i) for all $x \in X$, $f(\cdot, x)$ is $g$-measurable;
(ii) for $g$-almost all $t \in [a, b]$, $f(t, \cdot)$ is continuous on $X$;
(iii) for all $r > 0$, there exists $h_r \in \mathcal{L}_g^1([a, b))$ such that

$$\|f(t, x)\| \leq h_r(t) \quad \text{for } g\text{-almost all } t \in [a, b) \text{ and for all } x \in X \text{ such that } \|x\| \leq r.$$

**Definition 2.9.** Let $X$ be a nonempty subset of $\mathbb{R}^N$. A map $f : [a, b] \times X \to \mathbb{R}^N$ is *$g$-integrably bounded* if there exists $h \in \mathcal{L}_g^1([a, b), [0, \infty))$ such that

$$\|f(t, x)\| \leq h(t) \quad \text{for } g\text{-almost all } t \in [a, b) \text{ and for all } x \in X.$$

A set $A \subset \mathcal{L}^1_g([a,b),\mathbb{R}^N)$ is *uniformly $g$-integrably bounded in $\mathcal{L}^1_g([a,b),\mathbb{R}^N)$* if there exists $h \in \mathcal{L}^1_g([a,b),[0,\infty))$ such that

$$\|u(t)\| \leq h(t) \quad \text{for } g\text{-almost all } t \in [a,b) \text{ and for all } u \in A.$$

The interested reader is referred to Theorem 7.5 of [7] or Lemma 2.12 of [18] for the proof of the following lemma.

**Lemma 2.10.** *Let $f : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$ be a $g$-Carathéodory function. Then, the operator $N_f : \mathcal{BC}_g([a,b],\mathbb{R}^N) \to \mathcal{BC}_g([a,b],\mathbb{R}^N)$ defined by*

$$N_f(x)(t) = \int_{[a,t)} f(s,x(s))d\mu_g$$

*is continuous and completely continuous. Also, if $f$ is $g$-integrably bounded, then $N_f$ is a compact operator.*

The following definition, introduced in [7], will be helpful to define the $g$-admissible regions.

**Definition 2.11.** Let $f : A \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^M$ and let $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$ such that $(t,x) \in A$. We say that $f$ is $(g \times I_{\mathbb{R}^N})$*-continuous at the point* $(t,x)$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$[(s,y) \in A, |g(s) - g(t)| < \delta, \|y - x\| < \delta] \Rightarrow \|f(s,y) - f(t,x)\| < \epsilon.$$

We introduce the notion of partial $g$-derivative.

**Definition 2.12.** Let $f : \mathbb{R}^N \to \mathbb{R}$. The *partial $g$-derivative of $f$ with respect to $x_i$ at the point* $x = (x_1, \ldots, x_N)$ is given by

$$\frac{\partial_g f}{\partial_g x_i}(x) = \begin{cases} \displaystyle\lim_{y_i \to x_i} \frac{f(x_1,\ldots,y_i,\ldots,x_N) - f(x_1,\ldots,x_i,\ldots,x_N)}{g(y_i) - g(x_i)} & \text{if } x_i \notin D_g, \\ \displaystyle\lim_{y_i \to x_i^+} \frac{f(x_1,\ldots,y_i,\ldots,x_N) - f(x_1,\ldots,x_i,\ldots,x_N)}{g(y_i) - g(x_i)} & \text{if } x_i \in D_g, \end{cases}$$

if the limit exists.

Let us introduce the definition of a $(g \times I_{\mathbb{R}^N})$-differentiable function at a point $(t,x)$ where $t \notin C_g$.

**Definition 2.13.** Let $A \subset \mathbb{R}$, $B \subset \mathbb{R}^N$ and $t \in A \backslash C_g$. A function $f : A \times B \to \mathbb{R}$ is $(g \times I_{\mathbb{R}^N})$*-differentiable at the point* $(t,x) \in A \times B$ if there exists a vector $J(t,x) \in \mathbb{R}^{N+1}$ and a function $r : A \times B \to \mathbb{R}$ such that, for every $s \in A$ and $y \in B$,

$$f(s,y) - f(t,x) = \langle J(t,x), (g(s) - g(t), y - x)\rangle + r(s,y),$$

where

$$\lim_{(s,y)\to(t,x)} \frac{r(s,y)}{\|(g(s) - g(t), y - x)\|} = 0 \quad \text{if } t \notin D_g,$$
$$\lim_{(s,y)\to(t^+,x)} \frac{r(s,y)}{\|(g(s) - g(t), y - x)\|} = 0 \quad \text{if } t \in D_g.$$

Theorems similar to the classical differentiability case can be obtained.

**Proposition 2.14.** *Let $A \subset \mathbb{R}, B \subset \mathbb{R}^N$ and $t \in A \backslash (D_g \cup C_g)$. If $f : A \times B \to \mathbb{R}$ is $(g \times I_{\mathbb{R}^N})$-differentiable at the point $(t, x) \in A \times B$, then the vector $J(t, x) \in \mathbb{R}^{N+1}$ from Definition 2.13 is given by*

$$J(t,x) = \left( \frac{\partial_g f}{\partial_g t}(t,x), \nabla_x f(t,x) \right),$$

*where*

$$\nabla_x f(t,x) = \left( \frac{\partial f}{\partial x_1}(t,x), \dots, \frac{\partial f}{\partial x_N}(t,x) \right).$$

**Theorem 2.15.** *Let $A \subset \mathbb{R}, B \subset \mathbb{R}^N$ and $t \in A \backslash (D_g \cup C_g)$. Let $h : A \times B \to \mathbb{R}$ and $u : E \subset \mathbb{R} \to \mathbb{R}^N$ such that the following conditions are satisfied:*

(i) *$u$ is $g$-differentiable at $t$;*
(ii) *$u(E) \subset B$;*
(iii) *$h$ is $(g \times I_{\mathbb{R}^N})$-differentiable at $(t, u(t))$.*

*Then, $h(\cdot, u(\cdot)) : \mathbb{R} \to \mathbb{R}$ is $g$-differentiable at $t$ and*

$$(h(t,u(t)))'_g = \frac{\partial_g h}{\partial_g t}(t,u(t)) + \langle \nabla_x h(t,u(t)), u'_g(t) \rangle.$$

*Proof.* Since $h$ is $(g \times I_{\mathbb{R}^N})$-differentiable at $(t, u(t))$, we have that, for $s \in A$ and $y \in B$,

$$h(s,y) - h(t,u(t)) = \langle J(t,u(t)), (g(s)-g(t), y-u(t)) \rangle + r(s,y), \tag{2.3}$$

where

$$\lim_{(s,y)\to(t,u(t))} \frac{r(s,y)}{\|(g(s)-g(t), y-u(t))\|} = 0. \tag{2.4}$$

Consider $s \neq t, y = u(s) \in B$ and divide the equation (2.3) by $g(s) - g(t)$, we have that

$$\begin{aligned}\frac{h(s,u(s)) - h(t,u(t))}{g(s)-g(t)} &= \frac{\langle J(t,u(t)), (g(s)-g(t), u(s)-u(t)) \rangle}{g(s)-g(t)} + \frac{r(s,u(s))}{g(s)-g(t)} \\ &= \frac{\partial_g h}{\partial_g t}(t,u(t)) + \left\langle \nabla_x h(t,u(t)), \frac{u(s)-u(t)}{g(s)-g(t)} \right\rangle + \frac{r(s,u(s))}{g(s)-g(t)}\end{aligned}$$

by Proposition 2.14. When $s \to t$, we have that $u(s) \to u(t)$ since $t \notin D_g$ and $u$ is $g$-differentiable at $t \notin (D_g \cup C_g)$, so $u$ is continuous by Proposition 2.2. Also, observe that

$$\begin{aligned}\lim_{s\to t} \frac{|r(s,u(s))|}{|g(s)-g(t)|} &= \lim_{s\to t} \frac{|r(s,u(s))|}{|g(s)-g(t)| \left\| \left(1, \frac{u(s)-u(t)}{g(s)-g(t)}\right) \right\|} \left\| \left(1, \frac{u(s)-u(t)}{g(s)-g(t)}\right) \right\| \\ &= \lim_{\substack{(s,y)\to(t,u(t)) \\ y=u(s)}} \frac{|r(s,y)|}{\|(g(s)-g(t), y-u(t))\|} \|(1, u'_g(t))\| \\ &= 0\end{aligned}$$

by (2.4) and the fact that $u$ is $g$-differentiable at $t$. We then obtain

$$\lim_{s\to t} \frac{h(s,u(s)) - h(t,u(t))}{g(s)-g(t)} = \lim_{s\to t} \frac{\partial_g h}{\partial_g t}(t,u(t)) + \left\langle \nabla_x h(t,u(t)), \frac{u(s)-u(t)}{g(s)-g(t)} \right\rangle$$

$$
\begin{aligned}
&+ \frac{r(s,u(s))}{g(s)-g(t)} \\
&= \frac{\partial_g h}{\partial_g t}(t,u(t)) + \langle \nabla_x h(t,u(t)), u_g'(t)\rangle.
\end{aligned}
$$

So $h(\cdot,u(\cdot))$ is $g$-differentiable at $t$ and

$$(h(t,u(t)))_g' = \frac{\partial_g h}{\partial_g t}(t,u(t)) + \langle \nabla_x h(t,u(t)), u_g'(t)\rangle.$$

$\square$

## 3. Linear Stieltjes differential equations

We recall an exponential function introduced in [7]. Let $c \in \mathcal{L}_g^1([a,b))$ be such that

$$1 + c(t)\mu_g(\{t\}) \neq 0 \quad \text{for all } t \in [a,b) \cap D_g. \tag{3.1}$$

Let $T_c^- = \{t \in [a,b) \cap D_g : 1 + c(t)\mu_g(\{t\}) < 0\}$. This set has finite cardinality by [7, Lemma 6.4]. If $T_c^- = \{t_1, \dots, t_m\}$ with $a \le t_1 < t_2 < \cdots < t_m$, we define $\hat{e}_c(\cdot,a) : [a,b] \to \mathbb{R}\backslash\{0\}$ by

$$\hat{e}_c(t,a) = \begin{cases} e^{\int_{[a,t)} \hat{c}(s)d\mu_g} & \text{if } a \le t \le t_1, \\ (-1)^i e^{\int_{[a,t)} \hat{c}(s)d\mu_g} & \text{if } t_i < t \le t_{i+1}, i = 1, \dots, m, \end{cases} \tag{3.2}$$

where $t_{m+1} = b$ and

$$\hat{c}(t) = \begin{cases} c(t) & \text{if } t \in [a,b]\backslash D_g, \\ \frac{\log\left|1+c(t)\mu_g(\{t\})\right|}{\mu_g(\{t\})} & \text{if } t \in [a,b) \cap D_g. \end{cases} \tag{3.3}$$

With [20, Lemma 3.1] and [7, Lemma 6.2], we see that $\hat{e}_c(\cdot,a)$ is well defined.

**Lemma 3.1.** *Let $c \in \mathcal{L}_g^1([a,b))$ be such that (3.1) is verified. Then, we have*

$$\sum_{t\in[a,b)\cap D_g} \big| \log \left|1 + c(t)\mu_g(\{t\})\right| \big| < \infty.$$

*In particular, $\hat{c} \in \mathcal{L}_g^1([a,b))$.*

Now, consider the following system of nonhomogeneous linear Stieltjes differential equations

$$\begin{aligned} u_g'(t) + d(t)u(t) &= k(t) \quad \text{for } g\text{-almost all } t \in [a,b), \\ u(a) &= u_0, \end{aligned} \tag{3.4}$$

where $u_0 \in \mathbb{R}^N$, $k \in \mathcal{L}_g^1([a,b),\mathbb{R}^N)$ and $d \in \mathcal{L}_g^1([a,b))$.

We can easily obtain its solution from [7, Proposition 6.8].

**Proposition 3.2.** *Let $u_0 \in \mathbb{R}^N, k \in \mathcal{L}_g^1([a,b),\mathbb{R}^N)$ and let $d \in \mathcal{L}_g^1([a,b))$ satisfying*

$$d(t)\mu_g(\{t\}) \neq 1 \quad \textit{for all } t \in [a,b) \cap D_g. \tag{3.5}$$

*Then,* (3.4) *has a unique solution* $u \in \mathcal{AC}_g([a,b],\mathbb{R}^N)$*. Also,* $u$ *verifies*

$$u(t) = \hat{e}_{\overline{d}}^{-1}(t,a)\left(u_0 + \int_{[a,t)} \hat{e}_{\overline{d}}(s,a)\overline{k}(s)d\mu_g\right) \quad \textit{for all } t \in [a,b],$$

*where*

$$\overline{d}(t) = \frac{d(t)}{1 - d(t)\mu_g(\{t\})} \tag{3.6}$$

*and*

$$\overline{k}(t) = \frac{k(t)}{1 - d(t)\mu_g(\{t\})}. \tag{3.7}$$

Consider the following system of nonhomogeneous linear Stieltjes differential equations with a periodic boundary condition:

$$\begin{aligned} u_g'(t) + d(t)u(t) = k(t) \quad &\text{for } g\text{-almost all } t \in [a,b), \\ u(a) = u(b), \end{aligned} \tag{3.8}$$

where $k \in \mathcal{L}_g^1([a,b),\mathbb{R}^N)$ and $d \in \mathcal{L}_g^1([a,b))$.

An explicit solution to this problem can be deduced from Proposition 3.2.

**Proposition 3.3.** *Let* $k \in \mathcal{L}_g^1([a,b),\mathbb{R}^N)$ *and* $d \in \mathcal{L}_g^1([a,b))$ *verifying* (3.5)*. If* $\hat{e}_{\overline{d}}(b,a) \neq 1$*, then* (3.8) *has a unique solution* $u \in \mathcal{AC}_g([a,b],\mathbb{R}^N)$ *given by*

$$u(t) = \hat{e}_{\overline{d}}^{-1}(t,a)\left(\frac{1}{\hat{e}_{\overline{d}}(b,a)-1}\int_{[a,b)} \hat{e}_{\overline{d}}(s,a)\overline{k}(s)d\mu_g + \int_{[a,t)} \hat{e}_{\overline{d}}(s,a)\overline{k}(s)d\mu_g\right) \quad \textit{for all } t \in [a,b], \tag{3.9}$$

*where* $\overline{d}$ *and* $\overline{k}$ *are defined in* (3.6) *and* (3.7) *respectively.*

*Proof.* We remark that the solution of (3.4) will be a solution of (3.8) for a good choice of $u_0$. Indeed, by Proposition 3.2, the solution of (3.4) is given by

$$u(t) = \hat{e}_{\overline{d}}^{-1}(t,a)\left(u_0 + \int_{[a,t)} \hat{e}_{\overline{d}}(s,a)\overline{k}(s)d\mu_g\right). \tag{3.10}$$

By taking $t = b$ in (3.10), we must have

$$u_0 = u(b) = \hat{e}_{\overline{d}}^{-1}(b,a)\left(u_0 + \int_{[a,b)} \hat{e}_{\overline{d}}(s,a)\overline{k}(s)d\mu_g\right).$$

So, we get

$$(\hat{e}_{\overline{d}}(b,a) - 1)u_0 = \int_{[a,b)} \hat{e}_{\overline{d}}(s,a)\overline{k}(s)d\mu_g.$$

The unique solution $u \in \mathcal{AC}_g([a,b],\mathbb{R}^N)$ of (3.8) is then given by

$$u(t) = \hat{e}_{\overline{d}}^{-1}(t,a)\left(\frac{1}{\hat{e}_{\overline{d}}(b,a)-1}\int_{[a,b)} \hat{e}_{\overline{d}}(s,a)\overline{k}(s)d\mu_g + \int_{[a,t)} \hat{e}_{\overline{d}}(s,a)\overline{k}(s)d\mu_g\right) \quad \text{for all } t \in [a,b],$$

since $\hat{e}_{\overline{d}}(b,a) \neq 1$ by hypothesis. $\square$

Let $L : \mathcal{L}^1_g([a,b),\mathbb{R}^N) \to \mathcal{BC}_g([a,b],\mathbb{R}^N)$ be an operator defined by

$$L(k) = u, \tag{3.11}$$

where $u$ verifies (3.8) and is given by (3.9) for a function $d(t) = d > 0$ constant that respects certain hypotheses. This operator verifies some nice properties, as shown in the next lemma.

**Lemma 3.4.** *Let $d > 0$ be such that*

$$d\mu_g(\{t\}) \neq 1 \quad \textit{for all } t \in [a,b) \cap D_g$$

*and such that $\hat{e}_{\overline{d}}(b,a) \neq 1$, where*

$$\overline{d}(t) = \frac{d}{1 - d\mu_g(\{t\})}.$$

*Let $L : \mathcal{L}^1_g([a,b),\mathbb{R}^N) \to \mathcal{BC}_g([a,b],\mathbb{R}^N)$ be defined in* (3.11). *Then, $L$ is linear and continuous. Also, $L(A)$ is relatively compact for all $A \subset \mathcal{L}^1_g([a,b),\mathbb{R}^N)$ uniformly $g$-integrably bounded in $\mathcal{L}^1_g([a,b),\mathbb{R}^N)$.*

*Proof.* The linearity of the integral implies directly that $L$ is linear. Let us show that $L$ is continuous. Since $L$ is linear, it suffices to show that there exists $c \geq 0$ such that

$$\|L(k)\|_0 \leq c\|k\|_{\mathcal{L}^1_g([a,b),\mathbb{R}^N)} \quad \text{for all } k \in \mathcal{L}^1_g([a,b),\mathbb{R}^N). \tag{3.12}$$

For $k \in \mathcal{L}^1_g([a,b),\mathbb{R}^N)$, we find that

$$\begin{aligned}
\|L(k)\|_0 &= \sup_{t\in[a,b]} \Bigg\| \hat{e}^{-1}_{\overline{d}}(t,a) \Bigg( \frac{1}{\hat{e}_{\overline{d}}(b,a) - 1} \int_{[a,b)} \hat{e}_{\overline{d}}(s,a)\overline{k}(s)\,d\mu_g \\
&\qquad + \int_{[a,t)} \hat{e}_{\overline{d}}(s,a)\overline{k}(s)\,d\mu_g \Bigg) \Bigg\| \\
&\leq \sup_{t\in[a,b]} |\hat{e}^{-1}_{\overline{d}}(t,a)| \Bigg( \frac{1}{|\hat{e}_{\overline{d}}(b,a) - 1|} \int_{[a,b)} |\hat{e}_{\overline{d}}(s,a)|\|\overline{k}(s)\|\,d\mu_g \\
&\qquad + \int_{[a,t)} |\hat{e}_{\overline{d}}(s,a)|\|\overline{k}(s)\|\,d\mu_g \Bigg) \\
&\leq \sup_{t\in[a,b]} e^{\|\hat{\overline{d}}\|_{\mathcal{L}^1_g([a,b))}} \Bigg( e^{\|\hat{\overline{d}}\|_{\mathcal{L}^1_g([a,b))}} \Bigg( \frac{1}{|\hat{e}_{\overline{d}}(b,a) - 1|} \int_{[a,b)} \|\overline{k}(s)\|\,d\mu_g \\
&\qquad + \int_{[a,t)} \|\overline{k}(s)\|\,d\mu_g \Bigg) \Bigg) \\
&\leq \left( e^{\|\hat{\overline{d}}\|_{\mathcal{L}^1_g([a,b))}} \right)^2 \left( \frac{1}{|\hat{e}_{\overline{d}}(b,a) - 1|} + 1 \right) \|\overline{k}\|_{\mathcal{L}^1_g([a,b),\mathbb{R}^N)} \\
&= m\|\overline{k}\|_{\mathcal{L}^1_g([a,b),\mathbb{R}^N)},
\end{aligned}$$

where

$$m = \left( e^{\|\hat{\overline{d}}\|_{\mathcal{L}^1_g([a,b))}} \right)^2 \left( \frac{1}{\left|\hat{e}_{\overline{d}}(b,a) - 1\right|} + 1 \right) > 0. \tag{3.13}$$

Let $A^* = \{t \in [a,b) : d\mu_g(\{t\}) > 1/2\}$. We see that $A^*$ has finite cardinality, since $d \in \mathcal{L}^1_g([a,b))$ and thus

$$\infty > \|d\|_{L^1_g} \geq \sum_{t \in A^*} |d\mu_g(\{t\})| > \sum_{t \in A^*} \frac{1}{2}.$$

We then have

$$\begin{aligned} \|\overline{k}\|_{\mathcal{L}^1_g([a,b),\mathbb{R}^N)} &= \int_{[a,b)} \|\overline{k}(s)\| d\mu_g \\ &\leq \int_{[a,b)\setminus A^*} 2\|k(s)\| d\mu_g + \sum_{s \in [a,b) \cap A^*} \frac{\|k(s)\|}{|1 - d\mu_g(\{s\})|} \mu_g(\{s\}) \\ &\leq 2\|k\|_{\mathcal{L}^1_g([a,b),\mathbb{R}^N)} + \sum_{s \in [a,b) \cap A^*} \|k\|_{\mathcal{L}^1_g([a,b),\mathbb{R}^N)} \frac{1}{|1 - d\mu_g(\{s\})|} \\ &= r\|k\|_{\mathcal{L}^1_g([a,b),\mathbb{R}^N)}, \end{aligned} \tag{3.14}$$

where

$$r = \left( 2 + \sum_{s \in [a,b) \cap A^*} \frac{1}{|1 - d\mu_g(\{s\})|} \right) > 0. \tag{3.15}$$

By taking $c = mr > 0$, we have that $L$ is continuous.

Now, we show that $L(A)$ is relatively compact for all $A \subset \mathcal{L}^1_g([a,b),\mathbb{R}^N)$ uniformly $g$-integrably bounded in $\mathcal{L}^1_g([a,b),\mathbb{R}^N)$ with the help of Proposition 2.7. Let $A \subset \mathcal{L}^1_g([a,b),\mathbb{R}^N)$ be a uniformly $g$-integrably bounded set in $\mathcal{L}^1_g([a,b),\mathbb{R}^N)$. First, we directly have that $L(A) \subset \mathcal{AC}_g([a,b],\mathbb{R}^N)$ by Proposition 3.2. Let us show that $\{L(k)(a) : L(k) \in L(A)\}$ is bounded. Let $k \in A$, we have that, for all $t \in [a,b]$,

$$\|L(k)(t)\| \leq \|L(k)\|_0 \leq c\|k\|_{\mathcal{L}^1_g([a,b),\mathbb{R}^N)} \leq c\|h\|_{\mathcal{L}^1_g([a,b))} \in \mathbb{R} \tag{3.16}$$

by (3.12), where $h \in \mathcal{L}^1([a,b),[0,\infty))$ comes from the definition of $A$ uniformly $g$-integrably bounded.

Finally, let us show that there exists $\tilde{h} \in \mathcal{L}^1_g([a,b),[0,\infty))$ such that

$$\|(L(k))'_g(t)\| \leq \tilde{h}(t) \quad \text{for } g\text{-almost all } t \in [a,b) \text{ and for all } L(k) \in L(A).$$

Let $k \in A$. Since $L(k)$ verifies (3.8), we find that, for $g$-almost all $t \in [a,b)$,

$$\begin{aligned} \|(L(k))'_g(t)\| &= \|k(t) - dL(k)(t)\| \\ &\leq h(t) + d\|L(k)(t)\| \\ &\leq h(t) + dc\|h\|_{\mathcal{L}^1_g([a,b))} \\ &= \tilde{h}(t) \in \mathcal{L}^1_g([a,b),[0,\infty)), \end{aligned}$$

since $A$ is uniformly $g$-integrably bounded and by (3.16). It follows from Proposition 2.7 that $L(A)$ is relatively compact. $\square$

## 4. $g$-admissible regions

Let $g : \mathbb{R} \to \mathbb{R}$ be continuous from the left and monotone nondecreasing. To this function, we associate the sets $C_g$ and $D_g$ as defined in (2.1), (2.2) respectively.

Consider $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ and

$$u'_g(t) = f(t, u(t)) \quad \text{for } g\text{-almost all } t \in I = [0,T], u \in \mathcal{B}, \tag{4.1}$$

where $\mathcal{B}$ denotes the initial value condition or the periodic boundary condition

$$u(0) = r, \tag{4.2}$$

$$u(0) = u(T). \tag{4.3}$$

We generalize the concept of admissible regions presented in [6].

**Definition 4.1.** We say that a set $R \subset I \times \mathbb{R}^N$ is a *$g$-admissible region* if there exist $h : I \times \mathbb{R}^N \to \mathbb{R}$ and $p = (p_1, p_2) : I \times \mathbb{R}^N \to I \times \mathbb{R}^N$ such that:

(i) $R = \{(t,x) \in I \times \mathbb{R}^N : h(t,x) \leq 0\}$ is bounded and, for all $t \in I$,

$$R_t = \{x \in \mathbb{R}^N : (t,x) \in R\} \neq \emptyset;$$

(ii) $h$ is $(g \times I_{\mathbb{R}^N})$-continuous;
(iii) $h$ is bounded on every bounded set $X \subset I \times \mathbb{R}^N$;
(iv) $h$ is $(g \times I_{\mathbb{R}^N})$-differentiable for $g$-almost all $t \in I \backslash D_g$ and for all $x \in \mathbb{R}^N$ such that $(t,x) \in R^c$;
(v) $p$ is a continuous and bounded function such that $p(t,x) = (t,x)$ for all $(t,x) \in R$, and

$$\big\langle \nabla_x h(t,x), p_2(t,x) - x \big\rangle < 0 \quad \text{for } g\text{-almost all } t \in I \backslash D_g \text{ and all } x \text{ such that } (t,x) \in R^c.$$

We call $(h,p)$ an *associated $g$-admissible pair to $R$.*

**Remark 4.2.** If $\mathcal{B}$ denotes the initial value condition, we can weaken the condition (v) in Definition 4.1 by taking the non strict inequality. If $\mathcal{B}$ denotes the periodic boundary condition, we can consider the inequality of the condition (v) in Definition 4.1 on a set of strictly positive measure. We use the condition (v) in Definition 4.1 to alleviate the text.

We can see that from our definition of $g$-admissible region, one obtains the definition of an admissible region by taking $g(t) = t$.

Let us present some examples of $g$-admissible regions.

**Example 4.3.** Let $M : I \to [0, \infty)$ and $v : I \to \mathbb{R}^N$ be functions such that $M \in \mathcal{AC}_g(I, [0,\infty))$ and $v \in \mathcal{AC}_g(I, \mathbb{R}^N)$. The region

$$R = \{(t,x) \in I \times \mathbb{R}^N : \|x - v(t)\| \leq M(t)\}$$

is a $g$-admissible region with the associated $g$-admissible pair $(h,p)$ given by

$$h(t,x) = \|x - v(t)\| - M(t)$$

and $p(t,x) = (t, p_2(t,x))$, where $p_2(t,x)$ is the projection of $x$ on $R_t$.

**Example 4.4.** Let $M : I \to [0,\infty)$, $v : I \to \mathbb{R}^N$ and $a : I \to (0,\infty)^N$ be functions such that $M \in \mathcal{AC}_g(I,[0,\infty))$, $v \in \mathcal{AC}_g(I,\mathbb{R}^N)$ and $a \in \mathcal{AC}_g(I,(0,\infty)^N)$. The region

$$R = \big\{(t,x) \in I \times \mathbb{R}^N : \big\langle a(t),(x - v(t))^{\odot 2}\big\rangle \le M^2(t)\big\},$$

where we denote $x^{\odot 2} = (x_1^2,\dots,x_N^2) \in \mathbb{R}^N$, is a $g$-admissible region with the associated $g$-admissible pair $(h,p)$ given by

$$h(t,x) = \big\langle a(t),(x - v(t))^{\odot 2}\big\rangle - M^2(t)$$

and $p(t,x) = (t,p_2(t,x))$, where $p_2(t,x)$ is the projection of $x$ on $R_t$.

**Example 4.5.** Let $\alpha_i,\beta_i : I \to \mathbb{R}$ be functions such that $\alpha_i,\beta_i \in \mathcal{AC}_g(I)$ and $\alpha_i(t) \le \beta_i(t)$ for all $t \in I$ and all $i \in \{1,\dots,N\}$. The region

$$R = \big\{(t,x) \in I \times \mathbb{R}^N : \alpha_i(t) \le x_i \le \beta_i(t) \text{ for all } i \in \{1,\dots,N\}\big\}$$

is a $g$-admissible region with the associated $g$-admissible pair $(h,p)$ given by

$$h(t,x) = \sum_{i=1}^N c_i(t,x),$$

where, for all $i \in \{1,\dots,N\}$,

$$c_i(t,x) = \begin{cases} 0 & \text{if } x_i \in [\alpha_i(t),\beta_i(t)], \\ (\alpha_i(t) - x_i)^2 & \text{if } x_i < \alpha_i(t), \\ (x_i - \beta_i(t))^2 & \text{if } x_i > \beta_i(t), \end{cases} \tag{4.4}$$

and $p(t,x) = (t,p_2(t,x))$, where $p_2(t,x)$ is the projection of $x$ on $R_t$.

## 5. $g$-SOLUTION-REGIONS

Let us now define the notion of $g$-solution-region that generalizes the concept of solution-region presented in [6]. These regions will guarantee the existence of a solution to the problem (4.1) whose graph will be in the region.

### 5.1. **Initial value condition.**

Let us first look at $g$-solution-regions for the problem (4.1) with the initial value condition (4.2).

**Definition 5.1.** We say that a set $R \subset I \times \mathbb{R}^N$ is a *$g$-solution-region of* (4.1)*,* (4.2) if it is a $g$-admissible region with an associated $g$-admissible pair $(h,p)$ satisfying the following conditions:

(i) the function $h(\cdot,u(\cdot)) \in \mathcal{AC}_g(J)$ for all $u \in \mathcal{AC}_g(I,\mathbb{R}^N)$, where $J = \{t \in I : h(t^+,u(t^+)) > 0\}$;

(ii) for $g$-almost all $t \in I \backslash D_g$ and all $x$ such that $(t,x) \in R^c$, we have

$$\frac{\partial_g h}{\partial_g t}(t,x) + \big\langle \nabla_x h(t,x), f(p(t,x))\big\rangle \le 0;$$

(iii) for all $t \in I \cap D_g$ and all $x$ such that

$$(t,x) \in K_I^+ = \big\{(t,x) \in (I \cap D_g) \times \mathbb{R}^N : h\big(t^+, x + f(p(t,x))\mu_g(\{t\})\big) > 0\big\},$$

we have

$$h\big(t^+, x + f(p(t,x))\mu_g(\{t\})\big) \le h(t,x);$$

(iv) $h(0,r) \leq 0$.

We present some examples of $g$-solution-regions of (4.1), (4.2).

**Example 5.2.** Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N, M \in \mathcal{AC}_g(I,[0,\infty))$ and $v \in \mathcal{AC}_g(I,\mathbb{R}^N)$. Let $(h,p)$ be the associated $g$-admissible pair to the $g$-admissible region

$$R = \{(t,x) \in I \times \mathbb{R}^N : \|x - v(t)\| \leq M(t)\}$$

given in Example 4.3. Suppose that $h(0,r) \leq 0$ and that the following conditions are verified:

(i) for $g$-almost all $t \in I \backslash D_g$ and all $x$ such that $(t,x) \in R^c$, we have

$$\left\langle f(p(t,x)) - v_g'(t), \frac{x - v(t)}{\|x - v(t)\|} \right\rangle - M_g'(t) \leq 0;$$

(ii) for all $t \in I \cap D_g$ and all $x$ such that $(t,x) \in K_I^+$, we have

$$\left\|x + f(p(t,x))\mu_g(\{t\}) - v(t^+)\right\| - M(t^+) \leq \|x - v(t)\| - M(t).$$

It is easy to verify that $R$ is a $g$-solution-region of (4.1), (4.2).

**Example 5.3.** Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N$, $M \in \mathcal{AC}_g(I,[0,\infty))$, $v \in \mathcal{AC}_g(I,\mathbb{R}^N)$ and $a \in \mathcal{AC}_g(I,(0,\infty)^N)$. Let $(h,p)$ be the associated $g$-admissible pair to the $g$-admissible region

$$R = \left\{(t,x) \in I \times \mathbb{R}^N : \left\langle a(t), (x - v(t))^{\odot 2}\right\rangle \leq M^2(t)\right\}$$

given in Example 4.4. Suppose that $h(0,r) \leq 0$ and the following conditions are verified:

(i) for $g$-almost all $t \in I \backslash D_g$ and all $x$ such that $(t,x) \in R^c$, we have

$$\left\langle a_g'(t), (x - v(t))^{\odot 2}\right\rangle + 2\left(\left\langle f(p(t,x)) - v_g'(t), a(t) \odot (x - v(t))\right\rangle - M(t)M_g'(t)\right) \leq 0,$$

where we denote $x \odot y = (x_1y_1, \ldots, x_Ny_N) \in \mathbb{R}^N$, called the *Hadamard product*;

(ii) for all $t \in I \cap D_g$ and all $x$ such that $(t,x) \in K_I^+$, we have

$$\left\langle a(t^+), \left(x + f(p(t,x))\mu_g(\{t\}) - v(t^+)\right)^{\odot 2}\right\rangle - M^2(t^+) \leq \left\langle a(t), (x - v(t))^{\odot 2}\right\rangle - M^2(t).$$

It is not difficult to show that $R$ is a $g$-solution-region of (4.1), (4.2).

**Example 5.4.** Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N, \alpha_i, \beta_i \in \mathcal{AC}_g(I)$ be functions such that $\alpha_i(t) \leq \beta_i(t)$ for all $t \in I$ and all $i \in \{1, \ldots, N\}$. Let $(h,p)$ be the associated $g$-admissible pair to the $g$-admissible region

$$R = \left\{(t,x) \in I \times \mathbb{R}^N : \alpha_i(t) \leq x_i \leq \beta_i(t) \text{ for all } i \in \{1, \ldots, N\}\right\}$$

given in Example 4.5. Suppose that $h(0,r) \leq 0$ and that the following conditions are verified:

(i) for $g$-almost all $t \in I \backslash D_g$ and all $x$ such that $(t,x) \in R^c$, we have

$$\sum_{i=1}^N d_i(t,x) + k_i(t,x)f_i(p(t,x)) \leq 0,$$

where

$$d_i(t,x) = \begin{cases} 0 & \text{if } x_i \in [\alpha_i(t), \beta_i(t)], \\ \alpha'_{i,g}(t)(\alpha_i(t) - x_i) & \text{if } x_i < \alpha_i(t), \\ -\beta'_{i,g}(t)(x_i - \beta_i(t)) & \text{if } x_i > \beta_i(t), \end{cases} \tag{5.1}$$

and

$$k_i(t,x) = \begin{cases} 0 & \text{if } x_i \in [\alpha_i(t), \beta_i(t)], \\ -(\alpha_i(t) - x_i) & \text{if } x_i < \alpha_i(t), \\ x_i - \beta_i(t) & \text{if } x_i > \beta_i(t); \end{cases} \tag{5.2}$$

(ii) for all $t \in I \cap D_g$ and all $x$ such that $(t,x) \in K_I^+$, we have

$$\sum_{i=1}^{N} c_i\big(t^+, x + f(p(t,x))\mu_g(\{t\})\big) - c_i(t,x) \leq 0,$$

where $c_i$ is defined in (4.4).

One can verify that $R$ is a $g$-solution-region of (4.1), (4.2).

5.2. **Periodic boundary condition.** We now are interested in the Stieltjes differential equation (4.1) with the periodic boundary condition (4.3).

**Definition 5.5.** We say that a set $R \subset I \times \mathbb{R}^N$ is a *$g$-solution-region of* (4.1), (4.3) if it is a $g$-admissible region with an associated $g$-admissible pair $(h,p)$ satisfying the following conditions:

(i) the function $h(\cdot, u(\cdot)) \in \mathcal{AC}_g(J)$ for all $u \in \mathcal{AC}_g(I, \mathbb{R}^N)$, where $J = \big\{t \in I : h(t^+, u(t^+)) > 0\big\}$;

(ii) for $g$-almost all $t \in I \backslash D_g$ and all $x$ such that $(t,x) \in R^c$, we have

$$\frac{\partial_g h}{\partial_g t}(t,x) + \big\langle \nabla_x h(t,x), f(p(t,x)) \big\rangle \leq 0;$$

(iii) there exists $d > 0$ such that

$$d\mu_g(\{t\}) \neq 1 \quad \text{for all } t \in I \cap D_g, \quad \hat{e}_{\overline{d}}(T,0) \neq 1,$$

where $\hat{e}_{\overline{d}}(\cdot,0)$ and $\overline{d}$ are defined in (3.2), (3.6) respectively, and for all $t \in I \cap D_g$ and all $x$ such that

$$(t,x) \in K_P^+ = \big\{(t,x) \in (I \cap D_g) \times \mathbb{R}^N : h\big(t^+, x + f^d(p(t,x))\mu_g(\{t\})\big) > 0\big\},$$

we have

$$h\big(t^+, x + f^d(p(t,x))\mu_g(\{t\})\big) \leq h(t,x),$$

where we note $f^d(p(t,x)) = d(p_2(t,x) - x) + f(p(t,x))$;

(iv) for all $x$ such that $(0,x) \in R^c$, we have $h(0,x) \leq h(T,x)$.

**Remark 5.6.** We can weaken the conditions of Definition 4.1 by taking the non strict inequality in the condition (v) of Definition 4.1 if we imposed one of the following stronger conditions in Definition 5.5:

(i) we add to the condition (iii) of Definition 5.5 that there exists $t \in I \cap D_g$, $x \in \mathbb{R}^N$ such that $(t,x) \in K_P^+$ and

$$h\big(t^+, x + f^d(p(t,x))\mu_g(\{t\})\big) < h(t,x);$$

(ii) we replace the inequality by the strict inequality in the condition (iv) of Definition 5.5.

We will use the condition (v) of Definition 4.1 as well as conditions (iii) and (iv) of Definition 5.5 to alleviate the text.

**Remark 5.7.** One can see that the condition (iii) of Definition 5.1 can be obtained from the condition (iii) of Definition 5.5 by taking $d = 0$ and omitting $\hat{e}_{\overline{d}}(T,0) \neq 1$.

We notice that, from our definition of $g$-solution-region, one obtains the definition of a solution-region by taking $g(t) = t$.

Let us present some examples of $g$-solution-regions of (4.1), (4.3).

**Example 5.8.** Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N$, $M \in \mathcal{AC}_g(I,[0,\infty))$ and $v \in \mathcal{AC}_g(I,\mathbb{R}^N)$. Let $(h,p)$ be the associated $g$-admissible pair to the $g$-admissible region

$$R = \{(t,x) \in I \times \mathbb{R}^N : \|x - v(t)\| \le M(t)\}$$

given in Example 4.3. Suppose that the following conditions are verified:

(i) for $g$-almost all $t \in I \backslash D_g$ and all $x$ such that $(t,x) \in R^c$, we have

$$\left\langle f(p(t,x)) - v_g'(t), \frac{x - v(t)}{\|x - v(t)\|} \right\rangle - M_g'(t) \le 0;$$

(ii) there exists $d > 0$ such that

$$d\mu_g(\{t\}) \neq 1 \quad \text{for all } t \in I \cap D_g, \quad \hat{e}_{\overline{d}}(T,0) \neq 1,$$

and for all $t \in I \cap D_g$ and all $x$ such that $(t,x) \in K_P^+$, we have

$$\big\|x + f^d(p(t,x))\mu_g(\{t\}) - v(t^+)\big\| - M(t^+) \le \|x - v(t)\| - M(t);$$

(iii) for all $x$ such that

$$\|x - v(0)\| - M(0) > 0,$$

we have

$$\|x - v(0)\| - M(0) \le \|x - v(T)\| - M(T).$$

It is easy to show that $R$ is a $g$-solution-region of (4.1), (4.3).

**Example 5.9.** Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N$, $M \in \mathcal{AC}_g(I,[0,\infty))$, $v \in \mathcal{AC}_g(I,\mathbb{R}^N)$ and $a \in \mathcal{AC}_g(I,(0,\infty)^N)$. Let $(h,p)$ be the associated $g$-admissible pair to the $g$-admissible region

$$R = \big\{(t,x) \in I \times \mathbb{R}^N : \langle a(t), (x - v(t))^{\odot 2}\rangle \le M^2(t)\big\}$$

given in Example 4.4. Suppose that the following conditions are verified:

(i) for $g$-almost all $t \in I \backslash D_g$ and all $x$ such that $(t,x) \in R^c$, we have

$$\langle a_g'(t), (x - v(t))^{\odot 2}\rangle + 2\big(\langle f(p(t,x)) - v_g'(t), a(t) \odot (x - v(t))\rangle - M(t)M_g'(t)\big) \le 0,$$

(ii) there exists $d > 0$ such that

$$d\mu_g(\{t\}) \neq 1 \quad \text{for all } t \in I \cap D_g, \quad \hat{e}_{\overline{d}}(T, 0) \neq 1,$$

and for all $t \in I \cap D_g$ and all $x$ such that $(t, x) \in K_P^+$, we have

$$\begin{aligned}&\left\langle a(t^+), \left(x + f^d(p(t,x))\mu_g(\{t\}) - v(t^+)\right)^{\odot 2}\right\rangle - M^2(t^+)\\ &\leq \left\langle a(t), (x - v(t))^{\odot 2}\right\rangle - M^2(t);\end{aligned}$$

(iii) for all $x$ such that

$$\left\langle a(0), (x - v(0))^{\odot 2}\right\rangle - M^2(0) > 0,$$

we have

$$\left\langle a(0), (x - v(0))^{\odot 2}\right\rangle - M^2(0) \leq \left\langle a(T), (x - v(T))^{\odot 2}\right\rangle - M^2(T).$$

One can verify that $R$ is a $g$-solution-region of (4.1), (4.3).

**Example 5.10.** Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N$, $\alpha_i, \beta_i \in \mathcal{AC}_g(I)$ be functions such that $\alpha_i(t) \leq \beta_i(t)$ for all $t \in I$ and all $i \in \{1, \ldots, N\}$. Let $(h, p)$ be the associated $g$-admissible pair to the $g$-admissible region

$$R = \left\{(t, x) \in I \times \mathbb{R}^N : \alpha_i(t) \leq x_i \leq \beta_i(t) \text{ for all } i \in \{1, \ldots, N\}\right\}$$

given in Example 4.5. Suppose that the following conditions are verified:

(i) for $g$-almost all $t \in I \backslash D_g$ and all $x$ such that $(t, x) \in R^c$, we have

$$\sum_{i=1}^{N} d_i(t, x) + k_i(t, x) f_i(p(t, x)) \leq 0,$$

where $d_i, k_i$ are defined in (5.1), (5.2) respectively;

(ii) there exists $d > 0$ such that

$$d\mu_g(\{t\}) \neq 1 \quad \text{for all } t \in I \cap D_g, \quad \hat{e}_{\overline{d}}(T, 0) \neq 1,$$

and for all $t \in I \cap D_g$ and all $x$ such that $(t, x) \in K_P^+$, we have

$$\sum_{i=1}^{N} c_i\left(t^+, x + f^d(p(t,x))\mu_g(\{t\})\right) - c_i(t, x) \leq 0,$$

where $c_i$ is defined in (4.4);

(iii) for all $x$ such that

$$\sum_{i=1}^{N} c_i(0, x) > 0,$$

we have

$$\sum_{i=1}^{N} c_i(0, x) \leq \sum_{i=1}^{N} c_i(T, x).$$

Then, $R$ is a $g$-solution-region of (4.1), (4.3).

## 6. Existence results

We start with a comparison lemma that is a direct generalization from Lemma 2.13 of [18] and that will be useful to prove our existence results.

**Lemma 6.1.** *Let $u, v \in \mathcal{BC}_g([a,b])$. Let $J = \{t \in [a,b] : v(t^+) < u(t^+)\}$. Suppose that $u, v \in \mathcal{AC}_g(J)$ and are such that*

(i) *$u'_g(t) \leq v'_g(t)$ $g$-almost everywhere on $J$,*
(ii) *$u(a) - v(a) \leq u(b) - v(b)$ or $u(a) \leq v(a)$.*

*Then, $u(t) \leq v(t)$ for all $t \in [a,b]$, or there exists $c > 0$ such that $u(t) = v(t) + c$ for all $t \in [a,b]$.*

### 6.1. **Initial value condition.**

Let us show the first main theorem. It establishes the existence of a solution to the system of Stieltjes differential equations with the initial value condition.

**Theorem 6.2.** *Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N$ be a $g$-Carathéodory function. Suppose that there exists a $g$-solution-region $R$ of* (4.1), (4.2) *such that $f \circ p$ is a $g$-Carathéodory function, where $(h, p)$ is an associated $g$-admissible pair to $R$. Then, the problem* (4.1), (4.2) *has a solution $u \in \mathcal{AC}_g(I, \mathbb{R}^N)$ such that $(t, u(t)) \in R$ for all $t \in I$.*

*Proof.* Consider the following problem:

$$\begin{aligned} u'_g(t) &= f(p(t, u(t))) \quad \text{for } g\text{-almost all } t \in I, \\ u(0) &= r. \end{aligned} \tag{6.1}$$

By the condition (iii) of a $g$-Carathéodory function for $f \circ p$ and by the fact that $p$ is a bounded function by the condition (v) of Definition 4.1, we deduce that $f \circ p$ is $g$-integrably bounded. Let $\mathcal{T} : \mathcal{BC}_g(I, \mathbb{R}^N) \to \mathcal{BC}_g(I, \mathbb{R}^N)$ be the operator defined by

$$\mathcal{T}(u)(t) = r + \int_{[0,t)} f(p(s, u(s))) d\mu_g.$$

We see that $\mathcal{T}$ is well defined, since $f \circ p$ is a $g$-Carathéodory function and $g$-integrably bounded. So, for all $u \in \mathcal{BC}_g(I, \mathbb{R}^N)$, we have

$$\begin{aligned} \|\mathcal{T}(u)\|_0 &= \sup_{t \in I} \left\| r + \int_{[0,t)} f(p(s, u(s))) d\mu_g \right\| \\ &\leq \sup_{t \in I} \|r\| + \int_{[0,t)} \|f(p(s, u(s)))\| d\mu_g \\ &\leq \|r\| + \int_{[0,T)} h(s) d\mu_g \\ &< \infty \end{aligned}$$

for a certain $h \in \mathcal{L}^1_g([0,T), [0,\infty))$. We also have that $\mathcal{T}$ is a continuous and compact operator by Lemma 2.10.

Let us show that the fixed points of $\mathcal{T}$ are solutions of the problem (6.1). Indeed, if $\mathcal{T}(u) = u$, then for all $t \in I$, we have

$$u(t) = r + \int_{[0,t)} f(p(s, u(s))) d\mu_g.$$

We directly obtain from Theorem 2.5 that $u \in \mathcal{AC}_g(I, \mathbb{R}^N)$ and

$$u'_g(t) = f(p(t, u(t))) \quad \text{for } g\text{-almost all } t \in I.$$

We also have that $u(0) = r$, so it is a solution of the problem (6.1). Schauder's fixed point theorem implies that $\mathcal{T}$ has at least one fixed point that is a solution of (6.1).

We now need to deduce that this solution $u \in \mathcal{AC}_g(I, \mathbb{R}^N)$ is also a solution of the problem (4.1), (4.2). In order to do this, since $p(t, x) = (t, x)$ for all $(t, x) \in R$, it suffices to show that $(t, u(t)) \in R$ for all $t \in I$. We then want to show that $h(t, u(t)) \leq 0$ for all $t \in I$. We will use Lemma 6.1 with the functions $h(\cdot, u(\cdot))$ and 0 to conclude.

Let $J = \{t \in I : 0 < h(t^+, u(t^+))\}$. First, we directly have that $0 \in \mathcal{AC}_g(I) \subset \mathcal{BC}_g(I) \cap \mathcal{AC}_g(J)$. Since $u \in \mathcal{BC}_g(I, \mathbb{R}^N)$, the conditions (ii) and (iii) of Definition 4.1 imply that $h(\cdot, u(\cdot))$ is $g$-continuous on $I$ and bounded. So, we have that $h(\cdot, u(\cdot)) \in \mathcal{BC}_g(I)$.

We also have that $h(\cdot, u(\cdot)) \in \mathcal{AC}_g(J)$ by the condition (i) of Definition 5.1, since $u \in \mathcal{AC}_g(I, \mathbb{R}^N)$.

Furthermore, condition (iv) of Definition 5.1 assures that

$$h(0, u(0)) = h(0, r) \leq 0.$$

Now, we want to show that

$$(h(t, u(t)))'_g \leq 0 \quad g\text{-almost everywhere on } J. \tag{6.2}$$

Let $t \in J$. We will separate the proof in two distinct cases.

Case 1: If $t \notin D_g$, we have that

$$\lim_{s \to t} u(s) = u(t), \tag{6.3}$$

since $u \in \mathcal{AC}_g(I, \mathbb{R}^N) \subset \mathcal{BC}_g(I, \mathbb{R}^N)$ and by Proposition 2.2. We also have that $(t, u(t)) \in R^c$, since $h(\cdot, u(\cdot))$ is $g$-continuous and $t \in J$, so

$$h(t, u(t)) = h(t^+, u(t^+)) > 0.$$

The condition (iv) of Definition 4.1 assures that $h$ is $(g \times I_{\mathbb{R}^N})$-differentiable at point $(t, u(t))$ for $g$-almost all $t \in J \backslash D_g \subset I \backslash D_g$. In addition, $u$ is $g$-differentiable at $t$ for $g$-almost all $t \in J \backslash D_g \subset I$. By Theorem 2.15, we find that

$$(h(t, u(t)))'_g = \frac{\partial_g h}{\partial_g t}(t, u(t)) + \left\langle \nabla_x h(t, u(t)), u'_g(t) \right\rangle$$

for $g$-almost all $t \in J \backslash D_g$. We conclude from condition (ii) of Definition 5.1 and from the fact that $u$ is a solution of (6.1) that

$$\begin{aligned}(h(t, u(t)))'_g &= \frac{\partial_g h}{\partial_g t}(t, u(t)) + \left\langle \nabla_x h(t, u(t)), f(p(t, u(t))) \right\rangle \\ &\leq 0\end{aligned}$$

for $g$-almost all $t \in J \backslash D_g \subset I \backslash D_g$, since $(t, u(t)) \in R^c$.

Case 2: If $t \in D_g$, we have that

$$(h(t, u(t)))'_g = \frac{h(t^+, u(t^+)) - h(t, u(t))}{g(t^+) - g(t)}$$

$$\begin{aligned}
&= \frac{h\left(t^+, u(t) + \frac{u(t^+)-u(t)}{\mu_g(\{t\})} \cdot \mu_g(\{t\})\right) - h(t,u(t))}{\mu_g(\{t\})} \\
&= \frac{h\big(t^+, u(t) + u_g'(t) \cdot \mu_g(\{t\})\big) - h(t,u(t))}{\mu_g(\{t\})} \\
&= \frac{h\big(t^+, u(t) + f(p(t,u(t))) \cdot \mu_g(\{t\})\big) - h(t,u(t))}{\mu_g(\{t\})} \\
&\le 0
\end{aligned}$$

by condition (iii) of Definition 5.1. Indeed, we have that $(t, u(t)) \in K_I^+$ since

$$u(t) + f(p(t,u(t)))\mu_g(\{t\}) = u(t^+) \text{ and } h(t^+, u(t^+)) > 0.$$

Furthermore, since $t \in J \cap D_g$, we deduce that

$$h(t,u(t)) = h(t^+, u(t^+)) - (h(t,u(t)))_g'\mu_g(\{t\}) \ge h(t^+, u(t^+)) > 0$$

and then $(t, u(t)) \in R^c$.

Therefore, (6.2) is satisfied. Lemma 6.1 assures that $h(t, u(t)) \le 0$ for all $t \in I$, or that there exists $c > 0$ such that $h(t, u(t)) = c$ for all $t \in I$. This last case is however impossible, since $h(0, u(0)) = h(0, r) \le 0$. So we have that $h(t, u(t)) \le 0$ for all $t \in I$, which is equivalent to have $(t, u(t)) \in R$ for all $t \in I$. The solution $u$ of the problem (6.1) is then also a solution to the problem (4.1), (4.2). $\square$

Here is an example of an application of Theorem 6.2 for a system of two $g$-differential equations with an initial value condition.

**Example 6.3.** Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} t & \text{if } t \le 1, \\ t+1 & \text{if } t > 1. \end{cases}$$

The function is continuous from the left and nondecreasing. We also have $D_g = \{1\}$ and $\mu_g(\{1\}) = 1$.

Consider the following initial value problem

$$\begin{aligned} u_g'(t) &= f(t, u(t)) \quad \text{for } g\text{-almost all } t \in [0,2], \\ u(0) &= (4, -4), \end{aligned} \tag{6.4}$$

where $f = (f_1, f_2) : [0, 2] \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f_1(t, x_1, x_2) = \begin{cases} -\left((x_1-3)^3 + \frac{1}{2}\right)\left(\cos^2(t^2 x_1^5 x_2^3) + 1\right) & \text{if } t \in [0,1), \\ -1 & \text{if } t = 1, \\ -(x_1 - 2)e^{2x_2}\log(1 + t^3 x_1^2) + \frac{t}{2} & \text{if } t \in (1, 2]. \end{cases}$$

and

$$f_2(t, x_1, x_2) = \begin{cases} -\big((x_2+5)^5 + 1\big)\big(\sin^2(t^5 e^{x_1^3}) + 1\big) & \text{if } t \in [0,1), \\ 1 & \text{if } t = 1, \\ -(x_2+4)^3 t^4 e^{6x_1x_2}|x_1 - x_2| + 1 & \text{if } t \in (1, 2]. \end{cases}$$

Let us define $v : [0, 2] \to \mathbb{R}^2$ by

$$v(t) = \begin{cases} (3, -5) & \text{if } t \in [0, 1], \\ (2, -4) & \text{if } t \in (1, 2], \end{cases}$$

and $M : [0, 2] \to [0, \infty)$ by

$$M(t) = \begin{cases} 3t + 2 & \text{if } t \in [0, 1], \\ 2t + 3 & \text{if } t \in (1, 2]. \end{cases}$$

Let

$$R = \{(t, x) \in [0, 2] \times \mathbb{R}^2 : \|x - v(t)\| \le M(t)\},$$

$h : [0, 2] \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$h(t, x) = \|x - v(t)\| - M(t),$$

and $p : [0, 2] \times \mathbb{R}^2 \to [0, 2] \times \mathbb{R}^2$ defined by

$$\begin{aligned} p(t, x) &= (t, p_2(t, x)) \\ &= \left(t, M(t)\frac{x - v(t)}{\|x - v(t)\|} + v(t)\right), \end{aligned}$$

where $p_2(t, x)$ is the projection of $x$ on $R_t$. We directly see that $v \in \mathcal{AC}_g([0, 2], \mathbb{R}^2)$ and $M \in \mathcal{AC}_g([0, 2], [0, \infty))$. We also have that $h(0, 4, -4) \le 0$.

Let $t \in (0, 1)\backslash D_g$ and $(x_1, x_2) \in \mathbb{R}^2$ such that $(t, x_1, x_2) \in R^c$. We see that $v'_g(t) = (0, 0)$ and $M'_g(t) = 3$. We find that

$$\begin{aligned} &\left\langle f(p(t, x)) - v'_g(t), \frac{x - v(t)}{\|x - v(t)\|} \right\rangle - M'_g(t) \\ &= -\Big(((p_2(t, x))_1 - 3)^3 + \frac{1}{2}\Big)\Big(\cos^2(t^2((p_2(t, x))_1)^5((p_2(t, x))_2)^3) + 1\Big)\Big(\frac{x_1 - 3}{\|x - v(t)\|}\Big) \\ &\quad - \Big(((p_2(t, x))_2 + 5)^5 + 1\Big)\Big(\sin^2(t^5 e^{((p_2(t,x))_1)^3}) + 1\Big)\Big(\frac{x_2 + 5}{\|x - v(t)\|}\Big) - 3 \\ &\le -M^3(t)\left(\frac{x_1 - 3}{\|x - v(t)\|}\right)^4 (0 + 1) - M^5(t)\left(\frac{x_2 + 5}{\|x - v(t)\|}\right)^6 (0 + 1) \\ &\quad + \frac{1}{2}(1 + 1)\left(\frac{|x_1 - 3|}{\|x - v(t)\|}\right) + (1 + 1)\left(\frac{|x_2 + 5|}{\|x - v(t)\|}\right) - 3 \\ &\le 0, \end{aligned}$$

since $M(t) \ge 0$ for any $t \in [0, 2]$, where $(p_2(t, x))_i$ is the component $i$ of the vector $p_2(t, x)$ for $i \in \{1, 2\}$. A similar argument shows that the same inequality holds for $t \in (1, 2)\backslash D_g$ and $(x_1, x_2) \in \mathbb{R}^2$ such that $(t, x_1, x_2) \in R^c$. Since $\mu_g(\{0, 2\}) = 0$, the condition (i) in Example 5.2 is respected.

Let $t = 1 \in [0, 2] \cap D_g = \{1\}$ and $x \in \mathbb{R}^2$ such that $(t, x) \in K_I^+$, we have that

$$\begin{aligned} \big\|x + f(p(1, x))\mu_g(\{1\}) - v(1^+)\big\| - M(1^+) &= \|x + (-1, 1) - (2, -4)\| - 5 \\ &= \|x - (3, -5)\| - 5 \\ &= \|x - v(1)\| - M(1). \end{aligned}$$

The condition (ii) in Example 5.2 holds, thus $R$ is a $g$-solution-region of (6.4). We conclude by remarking that $f$ and $f \circ p$ are $g$-Carathéodory functions. Then, by Theorem 6.2, there exists a solution $u \in \mathcal{AC}_g([0,2],\mathbb{R}^2)$ to the problem (6.4) such that $(t,u(t)) \in R$ for all $t \in [0,2]$.

6.2. **Periodic boundary condition.** We can now prove an existence theorem for the problem with the periodic boundary condition.

**Theorem 6.4.** *Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N$ be a $g$-Carathéodory function. Suppose there exists a $g$-solution-region $R$ of* (4.1), (4.3) *such that $f \circ p$ is a $g$-Carathéodory function, where $(h,p)$ is an associated $g$-admissible pair to $R$. Then, the problem* (4.1), (4.3) *has a solution $u \in \mathcal{AC}_g(I,\mathbb{R}^N)$ such that $(t,u(t)) \in R$ for all $t \in I$.*

*Proof.* Consider the following problem:

$$\begin{aligned} u'_g(t) + du(t) &= dp_2(t,u(t)) + f(p(t,u(t))) \quad \text{for } g\text{-almost all } t \in I, \\ u(0) &= u(T), \end{aligned} \tag{6.5}$$

where $d > 0$ satisfies condition (iii) in Definition 5.5.

First, for the same reason as in the proof of Theorem 6.2, we have that $f \circ p$ is $g$-integrably bounded. Consider the operator $\mathcal{S} : \mathcal{BC}_g(I,\mathbb{R}^N) \to \mathcal{L}^1_g([0,T),\mathbb{R}^N)$ defined by

$$S(u)(t) = dp_2(t,u(t)) + f(p(t,u(t))).$$

Note that $S$ is well defined, since for all $u \in \mathcal{BC}_g(I,\mathbb{R}^N)$, we have

$$\begin{aligned} \|S(u)\|_{\mathcal{L}^1_g([0,T),\mathbb{R}^N)} &= \int_{[0,T)} \big\| dp_2(s,u(s)) + f(p(s,u(s))) \big\| d\mu_g \\ &\le \int_{[0,T)} d\|p_2(s,u(s))\| + \|f(p(s,u(s)))\| d\mu_g \\ &\le \int_{[0,T)} dQ + h(s) d\mu_g \\ &= dQ\mu_g([0,T)) + \|h\|_{\mathcal{L}^1_g([0,T))} \\ &< \infty, \end{aligned}$$

where $Q \ge 0$ is a bound of $\|p_2(t,x)\|$ by Definition 4.1 and $h \in \mathcal{L}^1_g([0,T),[0,\infty))$ comes from the fact that $f \circ p$ is a $g$-integrably bounded function. Also, consider the operator $L : \mathcal{L}^1_g([0,T),\mathbb{R}^N) \to \mathcal{BC}_g(I,\mathbb{R}^N)$ defined in (3.11), which is well defined by Lemma 3.4.

We remark that the fixed points of $L \circ S$ are solutions to the problem (6.5) by definition of $L$ and $S$. Let us show that $L \circ S : \mathcal{BC}_g(I,\mathbb{R}^N) \to \mathcal{BC}_g(I,\mathbb{R}^N)$ has a fixed point insured by Schauder's fixed point theorem. We need to show that $L \circ S$ is compact.

First, we show that $L \circ S$ is continuous. By Lemma 3.4, it suffices to show that $S$ is a continuous operator to conclude, since the composition of continuous operators is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $\mathcal{BC}_g(I,\mathbb{R}^N)$. Since $p$ is continuous, we have that

$$p_2(s,u_n(s)) \to p_2(s,u(s)) \quad \text{for all } s \in I.$$

Furthermore, since

$$\|p_2(s,u_n(s))\| \le Q \quad \text{for all } s \in I \text{ and for all } n \in \mathbb{N},$$

the Lebesgue's dominated convergence theorem implies that

$$\int_{[0,T)} \big\|p_2(s,u_n(s)) - p_2(s,u(s))\big\| d\mu_g \to 0. \tag{6.6}$$

Since $f \circ p$ is a $g$-Carathéodory function, we find by the same argument as in [18, Lemma 2.12] that

$$\int_{[0,T)} \big\|f(p(s,u_n(s))) - f(p(s,u(s)))\big\| d\mu_g \to 0. \tag{6.7}$$

By combining (6.6) and (6.7), we find that

$$\begin{aligned}
&\big\|S(u_n) - S(u)\big\|_{\mathcal{L}^1_g([0,T),\mathbb{R}^N)} \\
&= \int_{[0,T)} \big\|dp_2(s,u_n(s)) + f(p(s,u_n(s))) - (dp_2(s,u(s)) + f(p(s,u(s))))\big\| d\mu_g \\
&\le \int_{[0,T)} d\big\|p_2(s,u_n(s)) - p_2(s,u(s))\big\| + \big\|f(p(s,u_n(s))) - f(p(s,u(s)))\big\| d\mu_g \\
&\to 0.
\end{aligned}$$

So $S$ is continuous.

We now need to show that $L\big(S(\mathcal{BC}_g(I,\mathbb{R}^N))\big)$ is relatively compact. By Lemma 3.4, it suffices to show that $S(\mathcal{BC}_g(I,\mathbb{R}^N))$ is uniformly $g$-integrably bounded. For all $u \in \mathcal{BC}_g(I,\mathbb{R}^N)$, we have that

$$\begin{aligned}
\|S(u)(t)\| &= \big\|dp_2(t,u(t)) + f(p(t,u(t)))\big\| \\
&\le Qd + h(t) \\
&= \tilde{h}(t) \in \mathcal{L}^1_g([0,T),[0,\infty)).
\end{aligned}$$

So, $L(S(\mathcal{BC}_g(I,\mathbb{R}^N)))$ is relatively compact and $L \circ S$ is a compact operator. By Schauder's fixed point theorem, there exists a function $u \in \mathcal{BC}_g(I,\mathbb{R}^N)$ such that $u$ is a fixed point of $L \circ S$, which is a solution of the problem (6.5). We remark that $u \in \mathcal{AC}_g(I,\mathbb{R}^N)$ by Proposition 3.3.

We now have to show that this solution $u \in \mathcal{AC}_g(I,\mathbb{R}^N)$ is a solution of the problem (4.1), (4.3). Since $p(t,x) = (t,x)$ on $R$, it suffices to deduce that $(t,u(t)) \in R$ for all $t \in I$ to conclude. To do so, let us show that $h(t,u(t)) \le 0$ for all $t \in I$ with the help of Lemma 6.1.

Let $J = \{t \in I : 0 < h(t^+,u(t^+))\}$. By the same argument as in Theorem 6.2, we have that $0, h(\cdot,u(\cdot)) \in \mathcal{AC}_g(J) \cap \mathcal{BC}_g(I)$. Let us show that

$$(h(t,u(t)))'_g \le 0 \quad g\text{-almost everywhere on } J.$$

Let $t \in J$. We will separate the proof in two distinct cases.

Case 1: If $t \notin D_g$, then by the same argument as in Theorem 6.2, we have that $h(t,u(t)) = h(t^+,u(t^+)) > 0$ and

$$(h(t,u(t)))'_g = \frac{\partial_g h}{\partial_g t}(t,u(t)) + \big\langle \nabla_x h(t,u(t)), u'_g(t)\big\rangle$$

for $g$-almost all $t \in J \backslash D_g$. Since $u$ is a solution of (6.5), we conclude from the condition (v) of Definition 4.1, the condition (ii) of Definition 5.5 and from the hypothesis that $d > 0$ that

$$
\begin{aligned}
(h(t,u(t)))'_g &= \frac{\partial_g h}{\partial_g t}(t,u(t)) \\
&\quad + \big\langle \nabla_x h(t,u(t)), d\big(p_2(t,u(t)) - u(t)\big) + f(p(t,u(t)))\big\rangle \\
&= \frac{\partial_g h}{\partial_g t}(t,u(t)) + \big\langle \nabla_x h(t,u(t)), f(p(t,u(t)))\big\rangle \\
&\quad + d\big\langle \nabla_x h(t,u(t)), p_2(t,u(t)) - u(t)\big\rangle \\
&< 0
\end{aligned}
\tag{6.8}
$$

for $g$-almost all $t \in J \backslash D_g$, since $(t,u(t)) \in R^c$.

Case 2: If $t \in D_g$, we have that

$$
\begin{aligned}
(h(t,u(t)))'_g &= \frac{h\big(t^+, u(t^+)\big) - h(t,u(t))}{\mu_g(\{t\})} \\
&= \frac{h\Big(t^+, u(t) + \frac{u(t^+)-u(t)}{\mu_g(\{t\})}\cdot \mu_g(\{t\})\Big) - h(t,u(t))}{\mu_g(\{t\})} \\
&= \frac{h\big(t^+, u(t) + u'_g(t)\mu_g(\{t\})\big) - h(t,u(t))}{\mu_g(\{t\})} \\
&= \frac{h\big(t^+, u(t) + f^d(p(t,u(t)))\mu_g(\{t\})\big) - h(t,u(t))}{\mu_g(\{t\})} \\
&\le 0,
\end{aligned}
$$

since $u$ is a solution of (6.5) and by condition (iii) of Definition 5.5. Indeed, we have that $(t,u(t)) \in K_P^+$, since

$$u(t) + f^d(p(t,u(t)))\mu_g(\{t\}) = u(t^+) \quad \text{and} \quad h\big(t^+, u(t^+)\big) > 0.$$

Furthermore, since $t \in J \cap D_g$, we deduce by the same argument as in the proof of Theorem 6.2 that $(t,u(t)) \in R^c$. In every case, we have that $(h(t,u(t)))'_g \le 0$ for $g$-almost all $t \in J$.

Let us show that $h(0,u(0)) \le h(T,u(T))$ to conclude. Let

$$A = \{t \in I : h(t,u(t)) > 0\}.$$

If $0 \in A$, then, by condition (iv) of Definition 5.5 and the fact that $u$ respects the periodic boundary condition $u(0) = u(T)$, we have that

$$0 < h(0,u(0)) \le h(T,u(0)) = h(T,u(T)).$$

If $0 \notin A$, then $h(0,u(0)) \le 0$. In every case, the condition (ii) of Lemma 6.1 is satisfied.

Lemma 6.1 then implies that $h(t,u(t)) \le 0$ for all $t \in I$ or that there exists $c > 0$ such that $h(t,u(t)) = c$ for all $t \in I$. This last case is impossible by (6.8). We then have that $(t,u(t)) \in R$ for all $t \in I$ and our solution $u$ of the problem (6.5) is also a solution of the problem (4.1), (4.3). □

Let us present an example of an application of Theorem 6.4 to a system of $g$-differential equations with the periodic boundary condition.

**Example 6.5.** Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} t^3 + t^2 + t & \text{if } t \leq 1, \\ t^3 + t^2 + t + 1 & \text{if } t > 1. \end{cases}$$

We see that $g$ is continuous from the left and nondecreasing. We also have $D_g = \{1\}$ and $\mu_g(\{1\}) = 1$.

Consider the following problem with the periodic boundary condition

$$\begin{aligned} &u_g'(t) = f(t, u(t)) \quad \text{for } g\text{-almost all } t \in [0, 2], \\ &u(0) = u(2), \end{aligned} \tag{6.9}$$

where $f : [0, 2] \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f(t, x_1, x_2) = \begin{cases} -\frac{1}{2}(x_1, x_2) & \text{if } t = 1, \\ \left(-2x_1^3 - 3x_1 - x_2 + \frac{t}{2}, -2x_2^3 + x_1 - 3x_2 + \frac{t}{2}\right) & \text{otherwise.} \end{cases}$$

Define $v : [0, 2] \to \mathbb{R}^2$ by

$$v(t) = (0, 0) \quad \text{for all } t \in [0, 2]$$

and $M : [0, 2] \to [0, \infty)$ par

$$M(t) = \begin{cases} 4 & \text{if } t \in [0, 1], \\ 2 & \text{if } t \in (1, 2]. \end{cases}$$

We see that $v \in \mathcal{AC}_g([0, 2], \mathbb{R}^2)$ and $M \in \mathcal{AC}_g([0, 2], [0, \infty))$. Let

$$R = \{(t, x_1, x_2) \in [0, 2] \times \mathbb{R}^2 : \|x - v(t)\| \leq M(t)\}.$$

We can also directly calculate that the conditions of Example 5.8 are verified with our choice of $v$ and $M$ by choosing $d = 1/2 > 0$ for the second condition. Thus $R$ is a $g$-solution-region of (6.9). Since $f$ and $f \circ p$ are $g$-Carathéodory functions, Theorem 6.4 guarantees that there exists a solution $u \in \mathcal{AC}_g([0, 2], \mathbb{R}^2)$ such that $(t, u(t)) \in R$ for every $t \in [0, 2]$.

## 7. Application

A resonant tunnelling diode (RTD) is an electrical component that lets current go in one direction only and that has the negative differential resistance property, which corresponds to a decrease of the current going through the diode when the voltage at its terminals increases. It can be connected to a laser diode (LD) to manage the emission of the laser. It is an optoelectronic integrated circuit, and it is anticipated that this circuit will lead to new applications in optical communications, notably in data encryption. Interested readers are referred to [27, 30] for more information about this RTD-LD circuit.

The voltage output $V(t)$ across a RTD-LD circuit can be described with the help of a Liénard equation of the following form:

$$V''(t) + H(V)V'(t) + G(V) = V_{AC} \sin(2\pi f_{in} t), \tag{7.1}$$

where $H(V)$ and $G(V)$ are given by

$$H(V)=\frac{R}{L}+\frac{1}{C}\frac{d\tilde{f}(V)}{dV}$$

and

$$G(V)=\frac{V(t)}{LC}+\frac{R}{LC}\tilde{f}(V)-\frac{V_{DC}}{LC}$$

for $\tilde{f}:\mathbb{R}\to\mathbb{R}$ a function of class $C^1$, and constants $R, V_{AC}, V_{DC}, f_{in}\geq 0$ and $L, C>0$, where $R$ is the equivalent resistance of the RTD and LD, $V_{AC}$ is the driving alternating current voltage, $V_{DC}$ is a direct current bias voltage, $f_{in}$ is the excitation frequency, $L$ is the inductance and $C$ is the RTD capacitance. Remark that this differential equation of order two can be transformed in a system of two differential equations of order one. Indeed, let $y=V$ and $z=V'$, we find that (7.1) can be expressed by

$$\begin{aligned}&y'(t)=z(t)\\&z'(t)=V_{AC}\sin\big(2\pi f_{in}t\big)-G(y(t))-H(y(t))z(t).\end{aligned}$$

Let us consider the voltage output across a RTD-LD circuit over a period of 10 ns, where, to simplify the situation, the constants will be such that $f_{in}\geq 0$, $V_{AC}=0$, $R=C=L=1$ and $V_{DC}=3$. Furthermore, suppose that $\tilde{f}:\mathbb{R}\to\mathbb{R}$ is such that

$$0\leq\tilde{f}(w)\leq\frac{1}{4}\quad\forall w\in\mathbb{R},$$

and

$$\Psi(w):=\frac{d\tilde{f}}{dV}(w)\geq -1\quad\forall w\in\mathbb{R}.$$

A voltage regulator will also be connected to the circuit to maintain a constant voltage between 4 and 6 ns. At 8 ns, the voltage regulator will adjust the voltage in a proportional and opposite manner to the difference between the voltage and 3. Similarly, the derivative of the voltage will be impacted proportionally in the opposite direction of $V'(t)$ by the voltage regulator. The voltage regulator will act rapidly enough to consider these adjustments as instantaneous.

Let $y(t)$ be the voltage in volts across the RTD-LD circuit at time $t$ ns, and let $z(t)$ be the derivative of the voltage across the RTD-LD circuit at time $t$ ns, where $t\in[0,10]$. We can describe this situation with the following system of two Stieltjes differential equations

$$\big(y_g'(t),z_g'(t)\big)=\big(f_1(t,y(t),z(t)),f_2(t,y(t),z(t))\big)\text{ for }g\text{-almost all }t\in[0,10],\tag{7.2}$$

where $f=(f_1,f_2):[0,10]\times\mathbb{R}^2\to\mathbb{R}$ is defined by

$$f_1(t,x_1,x_2)=\begin{cases}x_2 & \text{if } t\in[0,10]\backslash\{8\},\\ -\frac{4}{5}(x_1-3) & \text{if } t=8,\end{cases}$$

and

$$f_2(t,x_1,x_2)=\begin{cases}V_{AC}\sin\big(2\pi f_{in}t\big)-G(x_1)-H(x_1)x_2 & \text{if } t\in[0,10]\backslash\{8\},\\ -\frac{4x_2}{5} & \text{if } t=8,\end{cases}$$

$$= \begin{cases} -\big(x_1 + \tilde{f}(x_1) - 3\big) - \big(1 + \Psi(x_1)\big)x_2 & \text{if } t \in [0,10]\backslash\{8\}, \\ -\frac{4x_2}{5} & \text{if } t = 8, \end{cases}$$

and where $g : \mathbb{R} \to \mathbb{R}$ is defined by

$$g(t) = \begin{cases} t & \text{if } t \in [0,4], \\ 4 & \text{if } t \in (4,6], \\ t-2 & \text{if } t \in (6,8], \\ t-1 & \text{if } t \in (8,10), \\ 9n + g(t-10n) & \text{if } t - 10n \in [0,10), n \in \mathbb{Z}\backslash\{0\}, \end{cases}$$

see Figure 7. We first see that this system is not piecewise linear when $\Psi$ is not constant, and that there is no trivial solution if $\tilde{f}(3) \neq 0$. Furthermore, we have that

$$C_g = \bigcup_{n\in\mathbb{Z}} (4+10n, 6+10n)$$

and

$$D_g = \bigcup_{n\in\mathbb{Z}} \{8+10n\},$$

where $\mu_g(\{t\}) = 1$ for all $t \in D_g$.

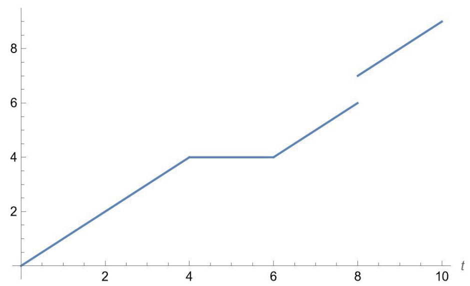


FIGURE 7.1. Graph of the function $g$

We will also consider the periodic boundary condition,

$$(y(0), z(0)) = (y(10), z(10)) \tag{7.3}$$

to ensure the long-term stability of the circuit.

Let us show that the problem (7.2), (7.3) has a solution insured by Theorem 6.4. Let $v : [0,10] \to \mathbb{R}^2$ and $M : [0,10] \to [0,\infty)$ be defined by

$$v(t) = (3,0)$$

and

$$M(t) = \begin{cases} \frac{t}{4} + 1 & \text{if } t \in [0,4], \\ 2 & \text{if } t \in (4,6], \\ \frac{t}{4} + \frac{1}{2} & \text{if } t \in (6,8], \\ \frac{t}{4} - \frac{3}{2} & \text{if } t \in (8,10]. \end{cases}$$

Observe that $v \in \mathcal{AC}_g([0,10],\mathbb{R}^2)$ and that $M \in \mathcal{AC}_g([0,10],[0,\infty))$. By Example 4.3, the set

$$R = \{(t,x) \in [0,10] \times \mathbb{R}^2 : \|x - v(t)\| \le M(t)\}$$

is a $g$-admissible region with the associated $g$-admissible pair $(h,p)$, where $h : [0,10] \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$h(t,x) = \|x - v(t)\| - M(t)$$

and $p : [0,10] \times \mathbb{R}^2 \to [0,10] \times \mathbb{R}^2$ is defined by

$$p(t,x) = (t, p_2(t,x)) = \left(t, M(t)\frac{x - v(t)}{\|x - v(t)\|} + v(t)\right),$$

where $p_2(t,x)$ is the projection of $x$ on $R_t$, see Figure 7.

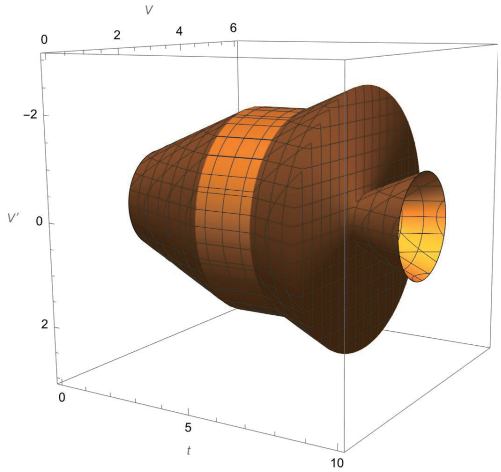


FIGURE 7.2. Region $R$

Now, let us show that $R$ is a $g$-solution-region of (7.2), (7.3) by verifying the conditions of Example 5.8. First, let us verify that, for $g$-almost all $t \in [0,10]\backslash D_g$ and all $x$ such that $(t,x) \in R^c$, we have

$$\left\langle f(p(t,x)) - v_g'(t), \frac{x - v(t)}{\|x - v(t)\|} \right\rangle - M_g'(t) \le 0.$$

Let $t \in (0,10)\backslash(D_g \cup C_g \cup \{4,6\})$ and $x$ be such that $\|x - v(t)\| > M(t)$. Then, since $M(t) \ge 0$, $\Psi(w) \ge -1$ and $0 \le \tilde{f}(w) \le \frac{1}{4}$ for all $w \in \mathbb{R}$ and $t \in [0,10]$, we have that

$$\begin{aligned}
&\left\langle f(p(t,x)) - v_g'(t), \frac{x - v(t)}{\|x - v(t)\|} \right\rangle - M_g'(t) \\
&= M(t)\frac{x_2}{\|x - v(t)\|}\frac{(x_1 - 3)}{\|x - v(t)\|} \\
&\quad + \left(-M(t)\frac{x_1 - 3}{\|x - v(t)\|} - 3 - \tilde{f}((p_2(t,x))_1) + 3\right)\frac{x_2}{\|x - v(t)\|} \\
&\quad - \left(1 + \Psi\left((p_2(t,x))_1\right)\right) M(t)\left(\frac{x_2}{\|x - v(t)\|}\right)^2 - \frac{1}{4}
\end{aligned}$$

$$\leq \frac{|x_2|}{4\|x - v(t)\|} - \frac{1}{4}$$
$$\leq 0,$$

where $(p_2(t,x))_1$ is the first component of the vector $p_2(t,x)$. The condition (i) of Example 5.8 is verified, since $\mu_g(C_g \cup \{0,4,6,10\}) = 0$.

Let us show the condition (ii), which asks for the existence of a constant $d > 0$ such that

$$d\mu_g(\{t\}) \neq 1 \quad \text{for all } t \in I \cap D_g, \quad \hat{e}_{\overline{d}}(10,0) \neq 1,$$

and for all $t \in I \cap D_g$ and all $x$ such that $(t,x) \in K_P^+$, we have that

$$\left\|x + f^d(p(t,x))\mu_g(\{t\}) - v(t^+)\right\| - M(t^+) \leq \|x - v(t)\| - M(t).$$

Let $d = \frac{4}{5} > 0$. We see that $d\mu_g(\{8\}) = \frac{4}{5} \neq 1$ and

$$\overline{d}(t) = \frac{4}{5 - 4\mu_g(\{t\})} > 0 \quad \text{for all } t \in [0,10],$$

which implies that $\hat{e}_{\overline{d}}(10,0) > 1$. Finally, let $t = 8 \in [0,10] \cap D_g = \{8\}$ and $x$ such that $(t,x) \in K_P^+ = \{(t,x) \in (I \cap D_g) \times \mathbb{R}^N : h(t^+, x + f^d(p(t,x))\mu_g(\{t\})) > 0\}$. We have that

$$\begin{aligned}
0 &< h\big(t^+, x + f^d(p(t,x))\mu_g(\{t\})\big) \\
&= \left\|x + \big(d(p_2(t,x) - x) + f(p(t,x))\big)\mu_g(\{t\}) - v(t^+)\right\| - M(t^+) \\
&= \left\|x + \frac{4}{5}\left(p_2(t,x) - x\right) - \frac{4}{5}p_2(t,x) + \frac{4}{5}(3,0) - (3,0)\right\| - \frac{1}{2} \\
&= \frac{1}{5}\left(\|x - (3,0)\| - \frac{5}{2}\right).
\end{aligned}$$

We also have that

$$h(t,x) = \|x - v(t)\| - M(t) = \|x - (3,0)\| - \frac{5}{2}.$$

So,

$$0 < \frac{1}{5}\left(\|x - (3,0)\| - \frac{1}{2}\right) \leq \|x - (3,0)\| - \frac{1}{2}.$$

The condition (ii) is then verified. The last condition is directly verified, since $v(0) = v(10) = (3,0)$ and $M(0) = M(10) = 1$. Thus, we have, for any $x \in \mathbb{R}^2$,

$$\|x - v(0)\| - M(0) \leq \|x - v(10)\| - M(10).$$

By Example 5.8, $R$ is a $g$-solution-region of (7.2), (7.3).

We see that $f$ and $f \circ p$ are $g$-Carathéodory functions. Thus, Theorem 6.4 guarantees the existence of a solution $u = (y,z) \in \mathcal{AC}_g([0,10], \mathbb{R}^2)$ to the problem (7.2), (7.3) such that $(t, y(t), z(t)) \in R$ for all $t \in [0,10]$.

Figure 7 shows the $g$-solution-region $R$ and the graph of a numerical approximation of a solution to the problem for $\tilde{f}$ given in [30],

$$\tilde{f}(V) = A \log\left(\frac{1 + e^{q(B - C + n_1 V(t))/k_B T}}{1 + e^{q(B - C - n_1 V(t))/k_B T}}\right)\left(\frac{\pi}{2} + \arctan\left(\frac{C - n_1 V(t)}{D}\right)\right)$$

$$+ H(e^{n_2 qV(t)/k_B T} - 1),$$

where $A = 6.48 \cdot 10^{-3}$, $B = 0.0875$, $C = 0.1449$, $D = 0.02132$, $H = 7.901 \cdot 10^{-4}$, $n_1 = 0.1902$, $n_2 = 0.0284$, $T = 300$K, $q = 1.602 \cdot 10^{-19}$ coulombs, the electric charge, and $k_B = 1.381 \cdot 10^{-23}$ J$\cdot$K$^{-1}$, the Boltzmann constant. We see that the graph of the approximate solution is in the $g$-solution-region $R$ for all $t \in [0, 10]$.

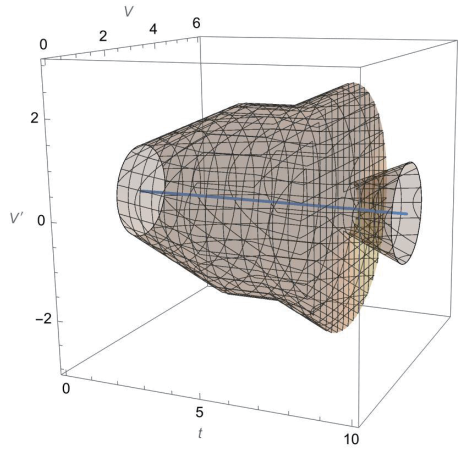


FIGURE 7.3. The $g$-solution-region $R$ and graph of a numerical approximation of a solution

We also see in Figure 7 that the solution is not trivial, is constant over $(4, 6)$, which is the orange segment, and has a discontinuity at $t = 8$.

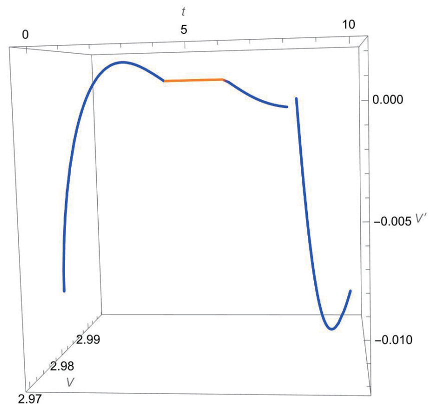


FIGURE 7.4. Graph of a numerical approximation of a solution

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