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ATTRACTIVE POINT THEOREM FOR HARDY AND ROGERS TYPE CONTRACTION MAPPINGS IN COMPLETE METRIC SPACES

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Dedicated to Professor Sehie Park on the occasion of his 90th birthday

Abstract. For Hardy and Rogers type contraction mappings, we prove an attractive point theorem in complete metric spaces. The attractive point and fixed point theorems for contraction mappings and Kannan mappings are derived from our main theorem. This seems to be the first attempt to prove an attractive point theorem in metric spaces. Examples of attractive points are provided to illustrate the difference from fixed points. Fundamental properties regarding attractive points are also established in the framework of metric spaces, which includes some new results.

1. Introduction

Let (X, d) be a metric space. For a mapping $T : C \to X$, we denote by $F(T) = \{x \in C : Tx = x\}$ a set of fixed points, where C is a non-empty subset of X. A mapping $T : C \to X$ is called an *α-contraction* if

$$d(Tx, Ty) \leq \alpha d(x, y) \quad \text{for all } x, y \in X, \tag{1.1}$$

where $\alpha \in [0, 1)$. According to the Banach fixed point theorem [4], a contraction mapping has a unique fixed point in a complete metric space. Kannan [12] defined a class of mappings using the following condition:

$$d(Tx, Ty) \leq \beta (d(x, Tx) + d(y, Ty)) \quad \text{for all } x, y \in X, \tag{1.2}$$

where $\beta \in \left[0, \frac{1}{2}\right)$, and also proved the existence and uniqueness of a fixed point. We call a mapping with the condition (1.2) a *β-Kannan mapping*. The same conclusions hold true for mappings that satisfy

$$d(Tx, Ty) \leq \gamma (d(x, Ty) + d(Tx, y)) \quad \text{for all } x, y \in X, \tag{1.3}$$

where $\gamma \in \left[0, \frac{1}{2}\right)$; see Chatterjea [7]. We call this type of mapping a *γ-Chatterjea mapping*. The class of *Ćirić–Reich–Rus type contraction mappings* [8,23,26] contains both contraction mappings (1.1) and Kannan mappings (1.2) simultaneously. The condition for Ćirić–Reich–Rus type contraction mappings is as follows:

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \quad \text{for all } x, y \in X, \tag{1.4}$$

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where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma < 1$. Generalizing the condition (1.4), Hardy and Rogers [10] introduced the condition

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + \varepsilon d(Tx, y) \tag{1.5}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta, \varepsilon \geq 0$ and $\alpha + \beta + \gamma + \delta + \varepsilon < 1$. See also Karapınar *et al.* [16] and Roldan Lopez de Hierro *et al.* [25]. For relationships between these classes of mappings (1.1)–(1.5), see Rhoades [24]. For recent developments, see Agarwal *et al.* [1], Berinde [6], and Karapınar and Agarwal [15].

Let H be a real Hilbert space with the norm $\|\cdot\|$ induced from an inner product and let T be a mapping from C into H, where C is a non-empty subset of H. In 2011, Takahashi and Takeuchi [27] introduced a concept called *attractive points*. The set of attractive points is denoted by

$$A(T) = \{x \in H : \|Ty - x\| \leq \|y - x\| \text{ for all } y \in C\}.$$

They proved a nonlinear ergodic theorem without supposing that C is closed or convex. The following is a slightly modified version:

Theorem 1.1 ([27]). *Let C be a non-empty subset of H and let $T : C \to C$ be a generalized hybrid mapping, that is,*

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha) \|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta) \|x - y\|^2 \tag{1.6}$$

for all $x, y \in C$, where $\alpha, \beta \in \mathbb{R}$. Suppose that $A(T) \neq \emptyset$. Let $x \in C$ and define $S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x\, (\in H)$ for all $n \in \mathbb{N} = \{1, 2, \cdots\}$. Then, $\{S_n x\}$ converges weakly to $\widehat{x} \equiv \lim_{n\to\infty} P_{A(T)} T^n x \in A(T)$, where $P_{A(T)}$ is the metric projection from H onto $A(T)$.

The class of generalized hybrid mappings (1.6) was introduced by Kocourek *et al.* [18]. In the case of $\alpha = 1$ and $\beta = 0$ in (1.6), the mapping T is nonexpansive. Thus, Theorem 1.1 is applicable in cases of nonexpansive mappings. If C is closed and convex in Theorem 1.1, then the sequence $\{S_n x\}$ converges weakly to a fixed point of T. In this sense, Theorem 1.1 is an extension of the classical result by Baillon [3]. For various results regarding attractive points, see [2, 11, 20–22, 28] for instance. For various types of mappings included in the class of generalized hybrid mappings, see Appendix in Kondo [19].

In this research, we study a type of mappings characterized by the condition

$$\begin{aligned} d(Tx, Ty) \leq{}& a d(x, y) + b \left[\frac{1}{2} d(x, Tx) + \frac{1}{2} d(y, Ty)\right] \\ &+ c \left[\frac{1}{2} d(x, Ty) + \frac{1}{2} d(Tx, y)\right] \end{aligned} \tag{1.7}$$

for all $x, y \in C$, where $a, b, c \geq 0$ with $a + b + c < 1$. Although mappings in this class are special cases of Hardy and Rogers mappings (1.5), they include contraction mappings (1.1), Kannan mappings (1.2), and Chatterjea mappings (1.3) simultaneously. We prove an attractive point theorem for this class of mappings in complete metric spaces without assuming that the domain of the mapping is closed. If the domain is closed, it is guaranteed that there exists a fixed point. In Section 2, to explain the difference from fixed points, we provide examples of attractive points. Basic properties regarding attractive points are also revealed in the framework of metric spaces. In Section 3, the attractive point theorem is established. From this result, various fixed point theorems are deduced if the domain of the mapping is closed. In Section 4, closing comments are concisely presented.

2. Attractive points

In this section, we present examples and basic properties of attractive points. Let (X, d) be a metric space and C be a non-empty subset of X. The set of attractive points of a mapping $T : C \to X$ is defined as

$$A(T) = \{x \in X : d(Ty, x) \leq d(y, x) \text{ for all } y \in C\}.$$

The following examples illustrate the difference between attractive points and fixed points. For Example 2.3 below, the author referred to Example 1.1 in Berinde [5].

Example 2.1. (i) Let $C = (0, \infty)$ be an interval in $\mathbb{R}$. Consider the usual metric $d(x, y) = |x - y|$ in $\mathbb{R}$. Define $S : C \to C$ by $Sx = \frac{1}{2}x$ for all $x \in C$. Then, $A(S) = (-\infty, 0]$ while $F(S) = \emptyset$.

(ii) Let $C' = [0, \infty)$ be an interval in $\mathbb{R}$. Define $S : C' \to C'$ by $Sx = \frac{1}{2}x$ for all $x \in C'$. Then, $A(S) = (-\infty, 0]$ while $F(S) = \{0\}$.

Example 2.2. (i) Let $D = \mathbb{R}\backslash\{0\}$. Define $S' : D \to D$ by $S'x = -\frac{1}{2}x$ for all $x \in D$. Then, $A(S') = \{0\}$ while $F(S) = \emptyset$.

(ii) Let $D' = \mathbb{R}$. Define $S' : D' \to D'$ by $S'x = -\frac{1}{2}x$ for all $x \in D'$. Then, $A(S') = F(S') = \{0\}$.

Example 2.3. (i) Let $C = (0, \infty)$ be an interval in $\mathbb{R}$. Define $T : C \to \mathbb{R}$ as follows:

$$Tx = \begin{cases} 0 & \text{if } 0 < x \leq 4; \\ -1 & \text{if } x > 4. \end{cases}$$

Then, $A(T) = (-\infty, 0]$ while $F(T) = \emptyset$.

(ii) Let $C' = [0, \infty)$ be an interval in $\mathbb{R}$. Define $T : C' \to \mathbb{R}$ as follows:

$$Tx = \begin{cases} 0 & \text{if } 0 \leq x \leq 4; \\ -1 & \text{if } x > 4. \end{cases}$$

Then, $A(T) = (-\infty, 0]$ while $F(T) = \{0\}$.

The mappings S and S' in Examples 2.1 and 2.2 are $\frac{1}{2}$-contraction mappings while T in Example 2.3 is $\frac{1}{5}$-Kannan. As shown in Examples 2.1 and 2.3, an attractive point is not necessarily unique.

Assertions (1), (3), and (4) in the following proposition were proved in Takahashi and Takeuchi [27] while (2) was demonstrated in Kondo and Takahashi [20] in the setting of real Hilbert spaces:

Proposition 2.4. *Let C be a non-empty subset of a metric space X and let T be a mapping from C into X. Then, the following assertions hold:*

(1) *$A(T)$ is closed in X;*
(2) *When T is a self-mapping on C, it holds that $A(T) \subset A(T^M)$, where $M \in \mathbb{N}$;*
(3) *$A(T) \cap C \subset F(T)$;*

(4) *Assume that T is quasi-nonexpansive with $F(T) \neq \emptyset$, that is,*

$$d(Ty, q) \leq d(y, q) \quad \text{for all } y \in C \text{ and } q \in F(T). \tag{2.1}$$

Then, $F(T) \subset A(T)$.

Proof. (1) Let $\{x_n\}$ be a sequence in $A(T)$ such that $x_n \to x (\in X)$. We show that $x \in A(T)$, in other words, $d(Ty, x) \leq d(y, x)$ for any $y \in C$. Select $y \in C$ arbitrarily. As $x_n \in A(T)$, it follows that $d(Ty, x_n) \leq d(y, x_n)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we have the desired result.

(2) Let $x \in A(T)$ and $M \in \mathbb{N}$. We show that $x \in A(T^M)$. Let $y \in C$. As $x \in A(T)$ and $T^{M-1}y \in C$, it holds that

$$d(T^M y, x) = d(T(T^{M-1}y), x) \leq d(T^{M-1}y, x).$$

Similarly,

$$d(T^M y, x) \leq d(T^{M-1}y, x) \leq \cdots \leq d(y, x)$$

for arbitrarily chosen $y \in C$. This means that $x \in A(T^M)$ as claimed.

(3) Let $x \in A(T) \cap C$. As $x \in A(T)$, we have $d(Ty, x) \leq d(y, x)$ for all $y \in C$. As $x \in C$, set $y = x \in C$ in this expression. Then, we obtain $d(Tx, x) \leq d(x, x) = 0$. This indicates that $x \in F(T)$. Thus, we can conclude that $A(T) \cap C \subset F(T)$.

(4) Let $x \in F(T)$. We verify that $x \in A(T)$, that is, $d(Ty, x) \leq d(y, x)$ for all $y \in C$. Select $y \in C$ arbitrarily. From the definition of quasi-nonexpansive mappings (2.1), the desired result follows. □

To the author's best knowledge, the following Propositions 2.5 and 2.6 are new results in the literature:

Proposition 2.5. *Let T be a mapping from C into a metric space X, where C is a non-empty subset of X. Suppose that there exist $a, b, c \geq 0$ that satisfy $a + b + c \leq 1$ and the condition* (1.7) *for all $x, y \in C$. Then, $F(T) \subset A(T)$ holds true.*

Proof. Let $x \in F(T)$. Our aim is to show that $x \in A(T)$, in other words, $d(Ty, x) \leq d(y, x)$ for all $y \in C$. As $x = Tx$, using the condition (1.7), we have

$$\begin{aligned} d(Ty, x) &= d(Tx, Ty) \\ &\leq ad(x, y) + b\left[\frac{1}{2}d(x, Tx) + \frac{1}{2}d(y, Ty)\right] + c\left[\frac{1}{2}d(x, Ty) + \frac{1}{2}d(Tx, y)\right] \\ &= ad(x, y) + b\left[\frac{1}{2}d(y, Ty)\right] + c\left[\frac{1}{2}d(x, Ty) + \frac{1}{2}d(x, y)\right] \\ &\leq ad(x, y) + b\left[\frac{1}{2}d(y, x) + \frac{1}{2}d(x, Ty)\right] + c\left[\frac{1}{2}d(x, Ty) + \frac{1}{2}d(x, y)\right] \end{aligned}$$

for all $y \in C$. This yields

$$\left(1 - \frac{b}{2} - \frac{c}{2}\right) d(Ty, x) \leq \left(a + \frac{b}{2} + \frac{c}{2}\right) d(x, y). \tag{2.2}$$

Note that $1 - \frac{b}{2} - \frac{c}{2} > 0$. Define

$$\rho = \frac{a + \frac{b}{2} + \frac{c}{2}}{1 - \frac{b}{2} - \frac{c}{2}}.$$

From the hypotheses about the parameters a, b, c, it holds that $0 \leq \rho \leq 1$. Therefore, from (2.2), we obtain

$$d(Ty, x) \leq \rho d(x, y) \leq d(x, y).$$

This completes the proof. $\square$

Proposition 2.6. *Let C be a non-empty subset of a metric space X. Let T_n and T be mappings from C into itself, where $n \in \mathbb{N}$. Suppose that the sequence $\{T_n\}$ of mappings converges to T pointwisely, that is, $T_n y \to Ty$ for all $y \in C$. Let $x_n^* \in A(T_n)$ for all $n \in \mathbb{N}$ and assume that $x_n^* \to x^* \in X$ as $n \to \infty$. Then, $x^* \in A(T)$.*

Proof. As $x_n^* \in A(T_n)$, we have

$$d(T_n y, x_n^*) \leq d(y, x_n^*) \quad \text{for all } y \in C \text{ and } n \in \mathbb{N}. \tag{2.3}$$

Our goal is to show that $x^* \in A(T)$, in other words, $d(Ty, x^*) \leq d(y, x^*)$ for arbitrarily chosen $y \in C$. From (2.3), it follows that

$$\begin{aligned} d(Ty, x^*) &\leq d(Ty, T_n y) + d(T_n y, x_n^*) + d(x_n^*, x^*) \\ &\leq d(Ty, T_n y) + d(y, x_n^*) + d(x_n^*, x^*). \end{aligned}$$

As $T_n y \to Ty$ and $x_n^* \to x^*$, we have in the limit as $n \to \infty$ that $d(Ty, x^*) \leq d(y, x^*)$. This completes the proof. $\square$

3. Main result

In this section, we prove the main theorem of this study, which asserts that a mapping with the condition (1.7) has an attractive point. Attractive point and fixed point theorems for contraction mappings, Kannan mappings, and Chatterjea mappings are derived from our main theorem.

Theorem 3.1. *Let C be a non-empty subset of a complete metric space X and let T be a self-mapping defined on C. Suppose that there exist $a, b, c \geq 0$ that satisfy $a + b + c < 1$ and the condition (1.7), that is,*

$$d(Tx, Ty) \leq ad(x, y) + b\left[\frac{1}{2}d(x, Tx) + \frac{1}{2}d(y, Ty)\right] + c\left[\frac{1}{2}d(x, Ty) + \frac{1}{2}d(Tx, y)\right]$$

for all $x, y \in C$. Then, the set $A(T)$ of attractive points of T is not empty and for any $x \in C$, $\{T^n x\}$ converges to an attractive point $x^* \in A(T)\,(\subset X)$.

Proof. Let $x \in C$ and define $x_n = T^n x\,(\in C)$ for all $n \in \mathbb{N} \cup \{0\}$, where $x_0 = x$. We show that $\{x_n\}\,(\subset C \subset X)$ is a Cauchy sequence. It holds from (1.7) that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq ad(x_{n-1}, x_n) + b\left[\frac{1}{2}d(x_{n-1}, Tx_{n-1}) + \frac{1}{2}d(x_n, Tx_n)\right] \\ &\quad + c\left[\frac{1}{2}d(x_{n-1}, Tx_n) + \frac{1}{2}d(Tx_{n-1}, x_n)\right] \\ &= ad(x_{n-1}, x_n) + b\left[\frac{1}{2}d(x_{n-1}, x_n) + \frac{1}{2}d(x_n, x_{n+1})\right] \end{aligned}$$

$$
\begin{aligned}
&+ c\left[\frac{1}{2}d\left(x_{n-1},x_{n+1}\right)\right]\\
&\leq ad\left(x_{n-1},x_n\right)+b\left[\frac{1}{2}d\left(x_{n-1},x_n\right)+\frac{1}{2}d\left(x_n,x_{n+1}\right)\right]\\
&\quad + c\left[\frac{1}{2}d\left(x_{n-1},x_n\right)+\frac{1}{2}d\left(x_n,x_{n+1}\right)\right]
\end{aligned}
$$

for all $n\in\mathbb{N}$. From this, it follows that

$$\left(1-\frac{b}{2}-\frac{c}{2}\right)d\left(x_n,x_{n+1}\right)\leq\left(a+\frac{b}{2}+\frac{c}{2}\right)d\left(x_{n-1},x_n\right).$$

Defining

$$\rho=\frac{a+\frac{b}{2}+\frac{c}{2}}{1-\frac{b}{2}-\frac{c}{2}}\in[0,1), \tag{3.1}$$

we obtain

$$
\begin{aligned}
d\left(x_n,x_{n+1}\right)&\leq\rho d\left(x_{n-1},x_n\right)\\
&\leq\rho^2d\left(x_{n-2},x_{n-1}\right)\leq\cdots\leq\rho^n d\left(x_0,x_1\right).
\end{aligned}\tag{3.2}
$$

for all $n\in\mathbb{N}$. Let $m,n\in\mathbb{N}$ with $m\geq n$. Using (3.2), we have

$$
\begin{aligned}
d\left(x_n,x_m\right)&\leq d\left(x_n,x_{n+1}\right)+\cdots+d\left(x_{m-1},x_m\right)\\
&\leq\rho^n d\left(x_0,x_1\right)+\cdots+\rho^{m-1}d\left(x_0,x_1\right)\\
&\leq\rho^n d\left(x_0,x_1\right)\left(1+\rho+\rho^2+\cdots\right)\\
&=\frac{\rho^n}{1-\rho}d\left(x_0,x_1\right).
\end{aligned}
$$

It follows that $d\left(x_n,x_m\right)\to 0$ as $m,n\to\infty$. This indicates that $\{x_n\}\,(\subset C\subset X)$ is a Cauchy sequence as claimed.

As X is complete, there exists $x^*\in X$ such that $x_n\to x^*$. Note that $x^*\in C$ is not necessarily guaranteed and consequently, Tx^* cannot be considered. We verify that $x^*\in A\left(T\right)$, in other words, $d\left(Ty,x^*\right)\leq d\left(y,x^*\right)$ for all $y\in C$. Select $y\in C$ arbitrarily. Using (1.7), we have

$$
\begin{aligned}
d\left(Ty,x^*\right)&\leq d\left(Ty,x_{n+1}\right)+d\left(x_{n+1},x^*\right)\\
&=d\left(Ty,Tx_n\right)+d\left(x_{n+1},x^*\right)\\
&\leq ad\left(y,x_n\right)+b\left[\frac{1}{2}d\left(y,Ty\right)+\frac{1}{2}d\left(x_n,x_{n+1}\right)\right]\\
&\quad+c\left[\frac{1}{2}d\left(y,x_{n+1}\right)+\frac{1}{2}d\left(Ty,x_n\right)\right]+d\left(x_{n+1},x^*\right).
\end{aligned}
$$

We have in the limit as $n\to\infty$ that

$$
\begin{aligned}
d\left(Ty,x^*\right)&\leq ad\left(y,x^*\right)+b\frac{1}{2}d\left(y,Ty\right)+c\left[\frac{1}{2}d\left(y,x^*\right)+\frac{1}{2}d\left(Ty,x^*\right)\right]\\
&\leq ad\left(y,x^*\right)+b\left[\frac{1}{2}d\left(y,x^*\right)+\frac{1}{2}d\left(x^*,Ty\right)\right]+c\left[\frac{1}{2}d\left(y,x^*\right)+\frac{1}{2}d\left(Ty,x^*\right)\right].
\end{aligned}
$$

Thus, we obtain

$$\left(1-\frac{b}{2}-\frac{c}{2}\right)d\left(Ty,x^*\right)\le\left(a+\frac{b}{2}+\frac{c}{2}\right)d\left(y,x^*\right),$$

which indicates that

$$d\left(Ty,x^*\right)\le\rho d\left(y,x^*\right)\le d\left(y,x^*\right),$$

where $\rho\in[0,1)$ is defined in (3.1) and y is arbitrarily chosen as an element of C. This means that $x^*\in A(T)\,(\subset X)$. The proof is completed. $\square$

Remind that $A(T)\cap C\subset F(T)$ from (3) in Proposition 2.4. Using this, we obtain the following fixed point theorem as a corollary, which is a particular case of the Hardy and Rogers fixed point theorem:

Corollary 3.2 ([10]). *Let T be a self-mapping defined on X, where X is a complete metric space. Suppose that there exist $a,b,c\ge 0$ that satisfy $a+b+c<1$ and the condition* (1.7) *for all $x,y\in C$. Then, T has a unique fixed point $x^*\in F(T)$ and for any $x\in C$, $\{T^n x\}$ converges to the fixed point $x^*\in F(T)$.*

Proof. Set $C=X$ in Theorem 3.1. Then, $x^*\in A(T)\cap C\subset F(T)$, where x^* is the element in the proof of Theorem 3.1. Therefore, T has a fixed point x^* and $\{T^n x\}$ converges to x^* for all $x\in X$.

We show the uniqueness. Let $u,v\in F(T)$. Using (1.7), we have

$$d\left(Tu,Tv\right)\le ad\left(u,v\right)+b\left[\frac{1}{2}d\left(u,Tu\right)+\frac{1}{2}d\left(v,Tv\right)\right]+c\left[\frac{1}{2}d\left(u,Tv\right)+\frac{1}{2}d\left(Tu,v\right)\right].$$

As $u=Tu$ and $v=Tv$, it follows that

$$\begin{aligned}d\left(u,v\right)&=d\left(Tu,Tv\right)\\&\le ad\left(u,v\right)+c\left[\frac{1}{2}d\left(u,v\right)+\frac{1}{2}d\left(u,v\right)\right]\\&=\left(a+c\right)d\left(u,v\right).\end{aligned}$$

Thus, it holds that $(1-a-c)\,d(u,v)\le 0$. As $1-a-c>0$, we obtain $d(u,v)\le 0$, which implies that $u=v$. This ends the proof. $\square$

Three remarks are provided below. First, Theorem 3.1 does not imply the uniqueness of fixed points, as an attractive point is not guaranteed to be unique. Second, Theorem 3.1 and Corollary 3.2 yield the attractive point theorems and fixed point theorems for contraction mappings (1.1), Kannan mappings (1.2), and Chatterjea mappings (1.3), respectively. Third, if $C\,(\subset X)$ is closed in Theorem 3.1, then C itself is a complete metric space, as X is complete. In such a case, we assume $C=X$ and obtain Corollary 3.2.

4. Concluding remarks

In this study, we investigated Hardy and Rogers type contraction mappings (1.7) and proved an attractive point theorem without assuming that the domain of the mappings is closed. The existence of an attractive point is guaranteed while the uniqueness does not necessarily hold. From the main theorem, we obtained attractive point theorems for well-known types of contraction mappings. Furthermore, fixed point theorems are also derived if the domains of the mappings are closed.

However, the class of mappings for which an attractive point theorem has been established is still limited. It may be difficult to prove attractive point theorems for Hardy and Rogers contraction mappings (1.5), interpolative contraction mappings [13, 14], and quasi-contraction mappings [9]. Proving attractive point theorems for cyclic type contraction mappings [17] is also a topic of future research.

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