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WEAK CONVERGENCE OF ITERATIVE METHODS FOR TWO ACCRETIVE OPERATORS IN BANACH SPACES

YUAN-HENG WANG*, SOUHAIL CHEBBI†, AND HONG-KUN XU‡

This paper is dedicated to Professor Wataru Takahashi on the occasion of his 70th Birthday

Abstract. New iterative schemes are introduced to find a common zero of two m-accretive operators A and B in the setting of Banach spaces which either satisfy Opial's property or are uniformly convex with a Fréchet differentiable norm. Under certain mild conditions on the sequences of parameters, we prove the weak convergence of the schemes to a common zero of A and B. One feature of our methods is that we use distinct indexes of the resolvent of A and B as opposed to the equal index in the existing literature.

1. Introduction

Let E be a real Banach space and E^* be its dual. The normalized duality mapping J from E into the family of nonempty (by the Hahn-Banach theorem) weak-star compact subsets of E^* is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

for each $x \in E$.

Assume that C is a nonempty closed convex subset of E. Recall that a mapping $T : C \to E$ is nonexpansive if for all $x, y \in C$,

$$\|Tx - Ty\| \le \|x - y\|.$$

We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$.

Recall that a (possibly multivalued) operator $A \subset E \times E$ with domain $D(A)$ and range $R(A)$ in E is accretive if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists a $j \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j\rangle \ge 0.$$

An accretive operator A is said to satisfy the range condition if $cl(D(A)) \subseteq R(I+rA)$ for all $r > 0$, and to be m-accretive if $R(I + rA) = E$ for all $r > 0$. If A is

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an accretive operator satisfying the range condition, then we can define, for each $r > 0$, the resolvent J_r^A and the Moreau-Yosida approximation A_r by

$$J_r^A = (I + rA)^{-1}, \quad A_r = \frac{1}{r}(I - J_r^A).$$

It is known that J_r^A is nonexpansive and the fixed point set $F(J_r^A) = A^{-1}(0)$ for all $r > 0$. Here

$$A^{-1}(0) = \{x \in D(A) : 0 \in Ax\}$$

is the set of zeros of A.

It is well known that if A is an accretive operator, then the solutions of the problem $0 \in Ax$ correspond to the equilibrium points of the evolution equation

$$\frac{du(t)}{dt} + Au(t) \ni 0.$$

Therefore, the problem of solving the inclusion $0 \in Ax$ has been paid much attention for over 50 years. Rockafellar [20] is the first to introduce the proximal point algorithm (PPA) in a general Hilbert space H: Initializing with $x_0 \in H$, he defines the $(n + 1)$th iterate by

$$x_{n+1} = J_{r_n}^A x_n, \tag{1.1}$$

where $\{r_n\} \subset (0, \infty)$ satisfies the condition $\liminf_{n\to\infty} r_n > 0$. Rockafellar proves that the sequence $\{x_n\}$ converges weakly to an element of $A^{-1}(0)$ (if any). The weak and strong convergence of the sequence $\{x_n\}$ defined by (1.1) have been extensively discussed in Hilbert Banach spaces (see [3, 6, 11, 12, 19] and the references therein).

In 2000, Kamimura and Takahashi [13] showed a strong convergence theorem in a Hilbert space: For a maximal monotone operator A with $A^{-1}(0) \neq \emptyset$ and $u \in C$, let the sequence $\{x_n\}$ be defined by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n}^A x_n, \ n \geq 0. \tag{1.2}$$

Under certain conditions, they proved the iterative sequence $\{x_n\}$ converges strongly to a zero of A. There are several extensions of the above result to the setting of certain classes of Banach spaces such as uniformly smooth Banach spaces. For instance, in [5] the algorithm (1.2) is proved to be strongly convergent in a uniformly smooth Banach space with a weakly continuous duality map. In [14] and [26], the algorithm (1.2) is extended to a uniformly smooth Banach space or a reflexive Banach space having a weakly sequentially continuous duality mapping.

The iterative scheme (1.2) and its modified variants have been investigated by many researchers (see [4, 9, 21, 22, 24, 27, 28, 30] and the references therein). However, most of the work has been carried out for maximal monotone operators in Hilbert spaces, and much less attention has been paid to m-accretive operators in Banach spaces.

In some situations, common solutions are sought. In our framework of this paper, we consider the problem of finding a common zero of a finite family of m-accretive operators in a Banach space. Suppose $\{A_i\}_{i=1}^k$ are a family of k m-accretive operators with common zeros. It is then an interesting problem of finding a common zero via iteration. One can adopt this approach: make a convex combination of the resolvents with the identity map I and then apply the algorithm (1.2), that is, apply (1.2) to the case where the resovent $J_{r_n}^A$ is replaced with the following averaged resolvent map:

$$S_k := a_0 I + a_1 J_1^{A_1} + a_2 J_1^{A_2} + \cdots + a_k J_1^{A_k}$$

where $a_j > 0$ for all $0 \le j \le k$ and $\sum_{j=0}^k a_j = 1$. In other words, the sequence $\{x_n\}$ is generated by the iteration process ([29]).

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S_k x_n, \ n \ge 0. \tag{1.3}$$

This algorithm is however not innovative since the fixed point set of S_k is precisely the common fixed point set of the resolvents $\{J_1^{A_j}\}$ (hence the common zero set $\cap_j A_j^{-1}(0)$) ([29]) under the mild assumption of strict convexity of the underly Banach space E and under standard assumptions on the sequence of parameters, $\{\alpha_n\}$, for the strong convergence of the sequence $\{x_n\}$.

Another similar modification was recently introduced in [7] where the purpose is again to find a common zero of the family $\{A_j\}_{j=1}^k$. But the essence of this approach is to make a finite family of the resolvents of the accretive operators into a single nonexpansive averaged operator S_k without exploring possible features of each individual resolvent.

In this paper we will continue the same line of research, that is, finding a common zero of finitely many m-accretive operators. However, for the sake of simplicity, we confine our study to the case of two m-accretive operators A and B, and always assume $A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. We will take each individual resolvent of A and B into consideration with the hope that the implementation of the proposed algorithm would be relatively easily realized. In our algorithm we will consider an average idea instead of shrinking like the algorithm (1.2). In other words, we will adopt the idea of Mann's algorithm [15]. More precisely, our algorithm generates a sequence according to the following recursion:

$$\begin{cases} y_n = \beta_n J_{r_n}^A x_n + (1 - \beta_n) J_{s_n}^B x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \ n \ge 0, \end{cases} \tag{1.4}$$

where the initialization x_0 is chosen arbitrarily, and where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1)$ and $\{r_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$.

Notice that in (1.4), the resolvent of A and B occurs individually and plays equal roles. Notice also that if we take $\alpha_n = 0$, then the algorithm (1.4) is reduced to the algorithm:

$$x_{n+1} = \beta_n J_{r_n}^A x_n + (1 - \beta_n) J_{s_n}^B x_n, \ n \ge 0.$$

Another feature is that we allow distinct indexes for the resolvent of A and B.

The aim of this paper is to investigate properties of the algorithm (1.4), in particular, we will prove the weak convergence of the sequence $\{x_n\}$ to a common zero of the operators A and B under some mild assumptions on the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ and $\{s_n\}$, in a uniformly convex Banach space with a Fréchet differentiable norm.

The paper is structured as follows. In the next section we introduce the preliminaries, including uniform convexity of Banach spaces, differentiability of norms, Opial's property, the resolvent identity, and the demiclosedness principle for nonexpansive mappings. In the last section we will prove the weak convergence of our algorithm (1.4).

2. Preliminaries

Throughout this paper, we always use E to stand for a real Banach space and the notation: $x_n \to x$ (respectively, $x_n \rightharpoonup x$) for the strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x. We also use $\omega_w(x_n)$ to denote the set of weak cluster points of the sequence $\{x_n\}$, that is,

$$\omega_w(x_n) = \{x \in E : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{x_{n_j}\} \subset \{x_n\}\}.$$

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $\|x\| = \|y\| = 1, x \neq y$. The modulus of convexity of E is defined as the function $\delta_E : [0, 2] \to [0, 1]$ by

$$\delta_E(\varepsilon) = \inf\{1 - \|x - y\|/2 : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}.$$

E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for each $\varepsilon > 0$. Let $S(E) = \{x \in E : \|x\| = 1\}$ be the unit sphere of E and consider the

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}. \tag{2.1}$$

We have various differentiability concepts for the norm of E described by the limit (2.1) as follows.

- E is said to be Gâteaux differentiable (or smooth) if the limit (2.1) exists for each $x, y \in S(E)$.
- E is said to be uniformly Gâteaux differentiable if it is smooth and the limit (2.1) is attained uniformly over $x \in S(E)$, for each fixed $y \in S(E)$.
- E is said to be Fréchet differentiable if it is smooth and the limit (2.1) is attained uniformly over $y \in S(E)$, for each fixed $x \in S(E)$.
- E is said to be a uniformly Fréchet differentiable norm (or uniformly smooth) if it is smooth and the limit (2.1) is attained uniformly over both $x, y \in S(E)$.

Recall also that E is said to satisfy Opial's property if there holds the implication:

$$\forall \{x_n\} \subset E,\ x_n \rightharpoonup x \quad \Rightarrow \quad \liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,\ \forall y \neq x.$$

It is known [17] that for every $1 < p < \infty$, ℓ^p satisfies Opial's property [17], however, L^p fails to satisfy it unless $p = 2$. More details on Banach space can be found in [16].

For the proof of the weak convergence of our algorithm (1.4), we need the tools stated below.

Lemma 2.1 ([1])**.** (The Resolvent Identity) *For* $\lambda, \mu > 0,\ x \in E$,

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x\right).$$

Lemma 2.2 ([25])**.** *Suppose that E is a uniformly convex Banach space and C is a bounded subset of E. Then there exists a convex continuous and strictly increasing function $g : [0, +\infty) \to [0, \infty)$ satisfying $g(0) = 0$ and*

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g(\|x - y\|)$$

for any $t \in [0,1]$, $x, y \in C$.

Lemma 2.3 ([2]). (The Demiclosedness Principle) *Let C be a closed convex subset of a uniformly convex Banach space and let $T : C \to C$ be a nonexpansive mapping with fixed points. Then $I - T$ is demiclosed in the sense that*

$$\{z_n\} \subset C,\ z_n \rightharpoonup z \text{ and } (I-T)z_n \to 0 \quad \Longrightarrow \quad (I-T)z = 0,\ i.e., z \in F(T).$$

3. The algorithm and its convergence

Lemma 3.1. *Let E be a uniformly convex Banach space and C a nonempty closed convex subset of X. Suppose $T_j : C \to X$, $j = 1, 2$, be nonexpansive mappings such that $F(T_1) \cap F(T_2) \neq \emptyset$. Then for any $\beta \in (0,1)$,*

$$F(\beta T_1 + (1-\beta)T_2) = F(T_1) \cap F(T_2). \tag{3.1}$$

In particular, if A and B are m-accretive operators in X such that $A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$, then for $\beta \in (0,1)$ and $r, s > 0$, we have

$$F(\beta J_r^A + (1-\beta)J_s^B) = F(J_r^A) \cap F(J_s^B). \tag{3.2}$$

Proof. It suffices to show that $F(\beta T_1 + (1-\beta)T_2) \subset F(T_1) \cap F(T_2)$. To see this, we take some $p \in F(T_1) \cap F(T_2)$ and let $q \in F(\beta T_1 + (1-\beta)T_2)$. Then we have $T_1 p = T_2 p = p$ and $\beta T_1 q + (1-\beta)T_2 q = q$. It follows from Lemma 2.2 that

$$\begin{aligned} \|q-p\|^2 &= \|\beta(T_1 q - p) + (1-\beta)(T_2 q - p)\|^2 \\ &\leq \beta\|T_1 q - p\|^2 + (1-\beta)\|T_2 q - p\|^2 - \beta(1-\beta)g(\|T_1 q - T_2 q\|) \\ &\leq \beta\|q-p\|^2 + (1-\beta)\|q-p\|^2 - \beta(1-\beta)g(\|T_1 q - T_2 q\|) \\ &= \|q-p\|^2 - \beta(1-\beta)g(\|T_1 q - T_2 q\|). \end{aligned}$$

Consequently, $g(\|T_1 q - T_2 q\|) = 0$. Namely, $T_1 q = T_2 q$. This then implies that $T_1 q = T_2 q = q$. □

We now provide a machinery for proving weak convergence of a sequence in a uniformly convex Banach space.

Lemma 3.2. *Let X be a Banach space and $\{x_n\}$ be a bounded sequence in X such that there exists a nonempty closed convex subset K of X with the properties*

- (W1) *for each $x \in K$, the $\lim_{n\to\infty} \|x_n - x\|$ exists, and*
- (W2) *if $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$ weakly converging to $\hat{x}$, then $\hat{x} \in K$, namely, $\omega_w(x_n) \subset K$.*

Suppose in addition that either X satisfies Opial's property or X is uniformly convex with a Fréchet differentiable norm and there exists a sequence of nonexpansive mappings, $\{T_n\}$, such that

- (W3) *$F(T_n) \supset K$ for each n, and*
- (W4) *$x_{n+1} = T_n x_n$ for all n.*

Then $\{x_n\}$ weakly converges to a point in K.

Proof. First consider the case that X satisfies Opial's property. This case is quite trivial. Indeed, if $p, q \in \omega_w(x_n)$ and assume $x_{n_j} \rightharpoonup p$ and $x_{m_l} \rightharpoonup q$, then it follows from (W1) that, upon assuming $p \neq q$,

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{j\to\infty} \|x_{n_j} - p\| < \lim_{j\to\infty} \|x_{n_j} - q\| = \lim_{n\to\infty} \|x_n - q\|$$

and

$$\lim_{n\to\infty} \|x_n - q\| = \lim_{l\to\infty} \|x_{m_l} - q\| < \lim_{l\to\infty} \|x_{m_l} - p\| = \lim_{n\to\infty} \|x_n - p\|.$$

An evident contradiction is reached.

We next turn to the case where X is uniformly convex with a Fréchet differentiable norm. In this case, by a result of Reich [18] (see also [23] and [5, Lemma 3.4]), we have $\lim_{n\to\infty}\langle x_n, J(f_1 - f_2)\rangle$ exists, where $f_1, f_2 \in \cap_n F(T_n)$ and thus

$$\langle p_1 - p_2, J(f_1 - f_2)\rangle = 0, \quad p_1, p_2 \in \omega_w(x_n),\ f_1, f_2 \in \cap_n F(T_n).$$

This together with the conditions (W2) and (W3) guarantees that $\omega_w(x_n)$ consists of at most one point and consequently, the sequence $\{x_n\}$ must converge weakly to a point in K. □

Theorem 3.3. *Let E be a uniformly convex Banach space. Suppose in addition E either satisfies Opial's property or is Fréchet differentiable. Let A and B be m-accretive operators in E such that $A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Generate a sequence $\{x_n\}$ by the algorithm:*

initialization: $x_0 \in X$ is chosen arbitrarily,

iteration:

$$y_n = \beta_n J^A_{r_n} x_n + (1-\beta_n) J^B_{s_n} x_n, \qquad (3.3)$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\}$, $\{s_n\}$ sequences in $(0,\infty)$ satisfy the conditions:

(α) $0 < \underline{\alpha} \le \alpha_n \le \overline{\alpha} < 1$;

(β) $0 < \underline{\beta} \le \beta_n \le \overline{\beta} < 1$;

(rs) *either* $0 < \underline{c} \le r_n,\ s_n \le \overline{c} < \infty$ *or* $\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = \infty$.

Then the sequence $\{x_n\}$ converges weakly to a point $z \in A^{-1}(0) \cap B^{-1}(0)$.

Proof. Let $K = A^{-1}(0) \cap B^{-1}(0)$. Notice that K is a nonempty closed convex subset of X. To prove the weak convergence of $\{x_n\}$ to a point in K, it suffices to verify that the four conditions (W1)-(W4) in Lemma 3.2 are satisfied.

We proceed as follows. Let $p \in K$ be arbitrarily given. First of all, we claim that $\{x_n\}$ is bounded. As a matter of fact, using the nonexpansiveness of the resolvent, we get

$$\begin{aligned} \|y_n - p\| &= \|\beta_n (J^A_{r_n} x_n - p) + (1-\beta_n)(J^B_{s_n} x_n - p)\| \\ &\le \beta_n \|J^A_{r_n} x_n - p\| + (1-\beta_n)\|J^B_{s_n} x_n - p\| \\ &\le \beta_n \|x_n - p\| + (1-\beta_n)\|x_n - p\| = \|x_n - p\| \end{aligned}$$

and

$$\|x_{n+1} - p\| = \|\alpha_n (x_n - p) + (1-\alpha_n)(y_n - p)\|$$

$$\begin{aligned}&\le \alpha_n\|x_n-p\|+(1-\alpha_n)\|y_n-p\|\\&\le \|x_n-p\|.\end{aligned}$$

This verifies (W1) which particularly implies that $\{x_n\}$ is bounded.

However, to verify (W2) we need much more delicate analysis on the sequence $\{x_n\}$. Apply Lemma 2.2 to further derive that

$$\begin{aligned}\|y_n-p\|^2 &= \|\beta_n(J_{r_n}^A x_n-p)+(1-\beta_n)(J_{s_n}^B x_n-p)\|^2\\&\le \beta_n\|J_{r_n}^A x_n-p\|^2+(1-\beta_n)\|J_{s_n}^B x_n-p\|^2\\&\quad -\beta_n(1-\beta_n)g(\|J_{r_n}^A x_n-J_{s_n}^B x_n\|)\\&\le \|x_n-p\|^2-\beta_n(1-\beta_n)g(\|J_{r_n}^A x_n-J_{s_n}^B x_n\|)\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(x_n-p)+(1-\alpha_n)(y_n-p)\|^2\\&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2-\alpha_n(1-\alpha_n)g(\|x_n-y_n\|)\\&\le \|x_n-p\|^2-\alpha_n(1-\alpha_n)g(\|x_n-y_n\|)\\&\quad -(1-\alpha_n)\beta_n(1-\beta_n)g(\|J_{r_n}^A x_n-J_{s_n}^B x_n\|).\end{aligned}\tag{3.5}$$

This implies that

$$\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n)g(\|x_n-y_n\|)<\infty\tag{3.6}$$

and

$$\sum_{n=0}^{\infty}(1-\alpha_n)\beta_n(1-\beta_n)g(\|J_{r_n}^A x_n-J_{s_n}^B x_n\|)<\infty.\tag{3.7}$$

By the assumptions (α) and (β), we immediately derive from (3.6) and (3.7) the following

(a) $\lim_{n\to\infty}\|x_n-y_n\|=0$, and
(b) $\lim_{n\to\infty}\|J_{r_n}^A x_n-J_{s_n}^B x_n\|=0$.

Now by (3.3),

$$\begin{aligned}&\|x_{n+1}-x_n\|=(1-\alpha_n)\|x_n-y_n\|,\\&\|x_{n+1}-y_n\|=\alpha_n\|x_n-y_n\|,\\&\|y_{n+1}-y_n\|\le\|y_{n+1}-x_{n+2}\|+\|x_{n+2}-x_{n+1}\|+\|x_{n+1}-y_n\|,\end{aligned}$$

and combining with (a), we get

(c) $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$,
(d) $\lim_{n\to\infty}\|x_{n+1}-y_n\|=0$, and
(e) $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$.

We are now in a position to prove (W2). Assuming $x_{n_j}\rightharpoonup z$, we have by (a), (c) and (e) that $x_{n_j+1}\rightharpoonup z$ and $y_{n_j+1}\rightharpoonup z$. In addition we may assume without loss of generality (by taking a further subsequence if necessary) that

$$r_{n_j}\to\hat r>0,\quad s_{n_j}\to\hat s>0,\quad \alpha_{n_j}\to\hat\alpha\in(0,1),\quad \beta_{n_j}\to\hat\beta\in(0,1).$$

By the condition (rs), we have either $\hat{r}, \hat{s} \in [\underline{c}, \overline{c}]$ or $\hat{r} = \hat{s} = \infty$.

In the case of $\hat{r}, \hat{s} \in [\underline{c}, \overline{c}]$, we set

$$\widehat{T} = \hat{\beta} J^A_{\hat{r}} + (1 - \hat{\beta}) J^B_{\hat{s}}.$$

Note that by Lemma 3.1, we see that $F(\widehat{T}) = A^{-1}(0) \cap B^{-1}(0) = K$. We also set

$$T_{n_j} = \beta_{n_j} J^A_{r_{n_j}} + (1 - \beta_{n_j}) J^B_{s_{n_j}}.$$

Now since $\{x_n\}$ is bounded, an application of the resolvent identity (Lemma 2.1) implies that

$$\|J^A_{\hat{r}} x_{n_j} - J^A_{r_{n_j}} x_{n_j}\| \to 0, \quad \|J^B_{\hat{s}} x_{n_j} - J^B_{s_{n_j}} x_{n_j}\| \to 0$$

so that $\|\widehat{T} x_{n_j} - T_{n_j} x_{n_j}\| \to 0$, from which it follows that

$$\begin{aligned}\|x_{n_j} - \widehat{T} x_{n_j}\| &\le \|x_{n_j} - x_{n_j+1}\| + \|x_{n_j+1} - y_{n_j+1}\| + \|y_{n_j+1} - \widehat{T} x_{n_j}\| \\ &\le \|x_{n_j} - x_{n_j+1}\| + \|x_{n_j+1} - y_{n_j+1}\| + \|T_{n_j} x_{n_j} - \widehat{T} x_{n_j}\| \\ &\to 0 \quad (j \to \infty).\end{aligned}$$

By Lemma 2.3, we immediately get $z \in F(\widehat{T}) = K$ and (W2) is proven.

Next consider the case where $r_{n_j} \to \infty$ and $s_{n_j} \to \infty$. In this case, if we denote by A_λ the Moreau-Yoshida of A of index $\lambda > 0$, that is, $A_\lambda = (I - J^A_\lambda)/\lambda$, then we derive that

$$\begin{aligned}\|(I - J^A_1) J^A_{r_n} x_n\| &= \|A_1 J^A_{r_n} x_n\| \\ &\le |A J^A_{r_n} x_n| := \inf\{\|z\| : z \in A J^A_{r_n} x_n\} \\ &\le \|A_{r_n} x_n\| = \frac{1}{r_n} \|x_n - J^A_{r_n} x_n\|.\end{aligned}$$

It turns out that

$$\|(I - J^A_1) J^A_{r_{n_j}} x_{n_j}\| \le \frac{1}{r_{n_j}} \|x_{n_j} - J^A_{r_n} x_{n_j}\| \to 0 \quad (\text{as } r_{n_j} \to \infty). \tag{3.8}$$

We rewrite x_{n+1} in the form

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J^A_{r_n} x_n + (1 - \alpha_n)(1 - \beta_n)(J^B_{s_n} x_n - J^A_{r_n} x_n). \tag{3.9}$$

Noting the facts that $\alpha_{n_j} \to \hat{\alpha} < 1$, (b) and $x_{n_j+1} \rightharpoonup z$, we immediately get from (3.9) that

$$J^A_{r_{n_j}} x_{n_j} \rightharpoonup z. \tag{3.10}$$

By virtue of this and (3.8), we can apply Lemma 2.3 to obtain $z \in F(J^A_1) = A^{-1}(0)$.

By (b), we also have

$$J^B_{s_{n_j}} x_{n_j} \rightharpoonup z. \tag{3.11}$$

Note that (3.8) also holds for B, that is,

$$\|(I - J^B_1) J^B_{s_{n_j}} x_{n_j}\| \le \frac{1}{s_{n_j}} \|x_{n_j} - J^B_{s_n} x_{n_j}\| \to 0 \quad (\text{as } s_{n_j} \to \infty). \tag{3.12}$$

By (3.11)and (3.12), we find that Lemma 2.3 is applicable to $(I - J_1^B)$, hence, $z \in F(J_1^B) = B^{-1}(0)$, and we have again verified that $z \in K$.

In any case we have verified (W2).

The conditions (W3) and (W4) are satisfied with T_n defined by

$$T_n = \alpha_n I + (1 - \alpha_n)[\beta_n J_{r_n}^A + (1 - \beta_n) J_{s_n}^B]. \tag{3.13}$$

We obviously have that T_n is nonexpansive and $x_{n+1} = T_n x_n$. Moreover,

$$F(T_n) = F(\beta_n J_{r_n}^A + (1 - \beta_n) J_{s_n}^B) = F(J_{r_n}^A) \cap F(J_{s_n}^B) = A^{-1}(0) \cap B^{-1}(0) = K.$$

Therefore, by Lemma 3.2, we conclude that the sequence $\{x_n\}$ converges weakly to a point in K. □

Remark 3.4. In the proof of Theorem 3.3, we find that in the case where both $r_n \to \infty$ and $s_n \to \infty$, the sequence $\{\alpha_n\}$ is not required to be bounded below away from zero since we still have (3.10) from (3.9) (and hence (3.11)) as long as $\alpha_{n_j} \to \hat{\alpha} < 1$. We therefore have the following result.

Theorem 3.5. *Let E be a uniformly convex Banach space with Fréchet differentiable norm and let A, B be m-accretive operators such that $A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be generated by the algorithm (3.3), where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ and $\{s_n\}$ satisfy the conditions*

(i) $0 \le \alpha_n \le \overline{\alpha} < 1$;
(ii) $0 < \underline{\beta} \le \beta_n \le \overline{\beta} < 1$;
(iii) $\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = \infty$.

Then $\{x_n\}$ converges weakly to a point $z \in A^{-1}(0) \cap B^{-1}(0)$.

Taking $\alpha_n = 0$ for all n, we get the result below.

Corollary 3.6. *Let E be a uniformly convex Banach space with Fréchet differentiable norm and let A, B be m-accretive operators such that $A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Generate a sequence $\{x_n\}$ by the algorithm:*

$$x_{n+1} = \beta_n J_{r_n}^A x_n + (1 - \beta_n) J_{r_n}^B x_n, \ n \ge 0,$$

where we assume that $\{\beta_n\}$ and $\{r_n\}$ satisfy the conditions (ii) *and* (iii) *in Theorem 3.5. Then $\{x_n\}$ converges weakly to a point $z \in A^{-1}(0) \cap B^{-1}(0)$.*

If we take $B \equiv 0$ in Corollary 3.6, then we get the following

Corollary 3.7. *Let E be a uniformly convex Banach space with Fréchet differentiable norm and let A be an m-accretive operator such that $A^{-1}(0) \neq \emptyset$. Generate a sequence $\{x_n\}$ by the algorithm:*

$$x_{n+1} = (1 - \beta_n) x_n + \beta_n J_{r_n}^A x_n, \ n \ge 0,$$

where we assume that $\{\beta_n\}$ satisfy the condition (ii) *in Theorem 3.5, and $\lim_{n\to\infty} r_n = \infty$. Then $\{x_n\}$ converges weakly to a point $z \in A^{-1}(0)$.*

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Y. H. Wang
Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
E-mail address: wangyuanhengmath@163.com

S. Chebbi
Department of Mathematics, College of Science, King Saud University, Box 2455, Riyadh 11451, Saudi Arabia
E-mail address: schebbi@ksu.edu.sa

H. K. Xu
Department of Mathematics, School of Science, Hangzhou Dianzi University, Hangzhou 310018, Zhejiang, China;
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan
E-mail address: xuhk@hdu.edu.cn; xuhk@math.nsysu.edu.tw