Journal of Nonlinear and Convex Analysis
Volume 16, Number 10, 2015, 1965–1983

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

HYBRID EXTRAGRADIENT METHODS FOR FINDING MINIMUM-NORM SOLUTIONS OF SPLIT FEASIBILITY PROBLEMS

LU-CHUAN CENG, NGAI-CHING WONG*, AND JEN-CHIH YAO

Abstract. In this paper, we consider the split feasibility problem (SFP) on a nonempty closed convex subset C of a Hilbert space of arbitrary dimension. When C is given as the common fixed point set of nonexpansive mappings, combining Mann's iterative method, Korpelevich's extragradient method and the hybrid steepest-descent method, we develop an iterative algorithm. This algorithm provides the strong convergence to the minimum-norm solution of the SFP. On the other hand, we study the hybrid extragradient methods for finding a common element of the solution set Γ of the SFP and the set $\mathrm{Fix}(S)$ of fixed points of a strictly pseudocontractive mapping S. We propose an iterative algorithm which generates sequences converging weakly to an element of $\mathrm{Fix}(S) \cap \Gamma$.

1. Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of arbitrary dimensions, respectively. The split feasibility problem (SFP) is to find a point x with the property:

$$x \in C \quad \text{and} \quad Ax \in Q,$$

where A is a bounded linear operator from $\mathcal{H}_1$ into $\mathcal{H}_2$.

In 1994, the SFP was first introduced by Censor and Elfving [11], in the finite-dimensional setting, for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. A number of image reconstruction problems can be formulated as the SFP; see, e.g., [2]. Recently, it is found that the SFP can also be applied to study intensity-modulated radiation therapy (IMRT) [10, 12, 14, 15]. A wide variety of iterative methods solving the SFP has been used in signal processing and image reconstruction; see, e.g., [2, 3, 10–14, 17, 31, 32, 36, 37, 45, 46] and the references therein.

A special case of the SFP is the following convex constrained linear inverse problem of finding an element x such that

$$x \in C \quad \text{and} \quad Ax = b.$$

2010 *Mathematics Subject Classification.* 49J40, 47J20, 47H10, 65K05, 47H09.

Key words and phrases. Split feasibility problems, fixed point problems, hybrid extragradient methods, strictly pseudocontractive mappings, nonexpansive mappings, minimum-norm solutions, demiclosedness principle, algorithms.

*Corresponding author. This research was partially supported by grants from the National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133), Leading Academic Discipline Project of Shanghai Normal University (DZL707), the NSYSU-KMU joint ventures (01A06802, P009), and the Taiwan NSC (102-2115-M-110-002-MY2 and 102-2111-E-037-004-MY3).

It has been extensively investigated in the literature using the projected Landweber iterative method [20, 24]. Comparatively, the general SFP has received much less attention so far, due to the complexity arising from the set Q. Therefore, whether various versions of the projected Landweber iterative method [24] can be extended to solve the SFP remains an interesting hot topic.

The original algorithm given in [11] involves the computation of the inverse A^{-1} (assuming the existence of the inverse of A), and thus, did not become very popular. A seemingly more popular algorithm that solves the SFP is the CQ algorithm of Byrne [2,3] which is found to be a gradient-projection method in convex minimization. It is also a special case of the proximal forward-backward splitting method [17]. The CQ algorithm involves only the computation of the projections P_C and P_Q onto the sets C and Q, respectively, and is therefore implementable in the case where P_C and P_Q have closed-form expressions; for example, C and Q are closed balls or half-spaces. However, it remains a challenge how to implement the CQ algorithm in the case where the projections P_C and/or P_Q fail to have closed-form expressions, though theoretically we can prove the (weak) convergence of the algorithm. Recently, Xu [37] applied Mann's algorithm to the SFP and purposed an averaged CQ algorithm which was proved to be weakly convergent to a solution of the SFP. He also established a strong convergence result, showing that the minimum-norm solution can be obtained.

Throughout this paper, assume that the SFP is consistent, that is, the solution set Γ of the SFP is nonempty. Set

$$\nabla f = A^*(I - P_Q)A,$$

where A^* is the adjoint of A.

Proposition 1.1 (see [37]). *For any x^* in $\mathcal{H}_1$, the following statements are equivalent.*

(i) *x^* solves the SFP.*

(ii) *x^* solves, for any $\lambda > 0$, the fixed point equation*

$$P_C(I - \lambda \nabla f)x^* = x^*.$$

(iii) *x^* solves the variational inequality problem (VIP) of finding x^* in C such that*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

It follows from Proposition 1.1 that

$$\Gamma = \mathrm{Fix}(P_C(I - \lambda \nabla f)) = \mathrm{VI}(C, \nabla f)$$

for all $\lambda > 0$, where $\mathrm{Fix}(P_C(I - \lambda \nabla f))$ and $\mathrm{VI}(C, \nabla f)$ denote the set of fixed points of $P_C(I - \lambda \nabla f)$ and the solution set of VIP (1.1), respectively.

Very recently, Ceng, Ansari and Yao [4] studied relaxed extragradient methods for finding a common element of the solution set Γ of the SFP and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S. Combining Mann's iterative method and Korpelevich's extragradient method [23, 27, 28, 33], they proposed two iterative algorithms for finding an element of $\mathrm{Fix}(S) \cap \Gamma$.

Assuming all (real) Hilbert spaces are of arbitrary dimension in this paper. When $C = \bigcap_{i=1}^N \mathrm{Fix}(S_i)$ is the set of common fixed points of nonexpansive mappings, we propose in Theorem 3.3 a hybrid steepest-descent extragradient algorithm ensuring the norm convergence to the minimum-norm solution of the SFP. On the other hand, if S is a strictly pseudo-contractive mapping, combining Mann's iterative method, Korpelevich's extragradient method and the hybrid steepest-descent method [44], we propose in Theorem 3.6 an algorithm ensuring the weak convergence to an element of $\mathrm{Fix}(S) \cap \Gamma$.

Results in this paper are new and novel. Our results supplement, improve, extend and develop corresponding results of Xu [37, Theorems 5.5 and 5.7], and Ceng, Ansari and Yao [4, Theorems 3.1 and 3.2]. Since the iterative scheme provided in Theorem 3.3 can solve two problems simultaneously: the SFP and the problem of finding a common fixed point of finitely many nonexpansive mappings, it should be more useful and more applicable than other established methods in literature. See Remark 3.8 for more details. We also remark that the recent extensive study of algorithms for split feasibility problems provides us a strong motivation of continuing investigation in this manuscript. See, e.g., [5,6,8,9,18,19,30,34,35,40,42,43,47,48].

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We write $x_n \to x$ to indicate that a sequence $\{x_n\}$ converges strongly to x, and $x_n \rightharpoonup x$ to indicate that $\{x_n\}$ converges weakly to x in $\mathcal{H}$, respectively. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Let K be a nonempty closed convex subset of $\mathcal{H}$. Now we present some known definitions and results which will be used in the sequel. Recall that the metric (or nearest point) projection from $\mathcal{H}$ onto K is the mapping $P_K : \mathcal{H} \to K$ which assigns to each point x in $\mathcal{H}$ the unique point $P_K x$ in K satisfying the property

$$\|x - P_K x\| = \inf_{y \in K} \|x - y\| =: d(x, K).$$

Proposition 2.1 ([21])**.** *For given x in $\mathcal{H}$ and z in K, we have*

(i) $z = P_K x \;\Leftrightarrow\; \langle x - z, y - z \rangle \leq 0,\ \forall y \in K$;

(ii) $z = P_K x \;\Leftrightarrow\; \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2,\ \forall y \in K$;

(iii) $\langle x - y, P_K x - P_K y \rangle \geq \|P_K x - P_K y\|^2,\ \forall y \in \mathcal{H}$.

Let C be a nonempty subset of $\mathcal{H}$ in the following.

Definition 2.2. A mapping $T : C \to \mathcal{H}$ is said to be:

(a) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(b) *firmly nonexpansive* if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

Alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : C \to \mathcal{H}$ is nonexpansive.

Definition 2.3. Let $T : C \to \mathcal{H}$ be a mapping.

(a) T is said to be *monotone* if

$$\langle x - y, Tx - Ty\rangle \geq 0, \quad \forall x, y \in C.$$

(b) T is said to be β*-strongly monotone* for some $\beta > 0$ if

$$\langle x - y, Tx - Ty\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in C.$$

(c) T is said to be ν*-inverse strongly monotone* for some $\nu > 0$ if

$$\langle x - y, Tx - Ty\rangle \geq \nu\|Tx - Ty\|^2, \quad \forall x, y \in C.$$

It it easy to see that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_K is firmly nonexpansive, monotone, and 1-inverse strongly monotone. Inverse strongly monotone (also referred to as *co-coercive*) operators have been applied widely in solving practical problems in various fields, for instance, in traffic assignment problems; see, e.g., [1, 22].

Definition 2.4. A mapping $T : C \to \mathcal{H}$ is said to be an *averaged mapping* if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T \equiv (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$ and $S : C \to \mathcal{H}$ is nonexpansive. We say that T is α*-averaged* in this case. Firmly nonexpansive mappings (in particular, projections) are $\frac{1}{2}$-averaged maps.

Proposition 2.5 ([3])**.** *Let $T : C \to \mathcal{H}$ be a mapping.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-inverse strongly monotone.*

(ii) *T is averaged if and only if the complement $I - T$ is ν-inverse strongly monotone for some $\nu > 1/2$. Indeed, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-inverse strongly monotone for α in $(0, 1)$.*

(iii) *If T is ν-inverse strongly monotone, then γT is $\frac{\nu}{\gamma}$-inverse strongly monotone for any $\gamma > 0$.*

Proposition 2.6 ([3, 16])**.** *Let $S, T, V : C \to \mathcal{H}$ be mappings.*

(i) *If $T = (1 - \alpha)S + \alpha V$ for some α in $(0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1 - \alpha)S + \alpha V$ for some α in $(0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composition of finitely many averaged mappings is averaged. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composition $T_1 \circ T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

(v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_N).$$

The following result is useful when we verify the weak convergence of a sequence. Its proof is an immediate consequence of Opial's property [29] of a Hilbert space; see Xu [38, Lemma 4.1].

Proposition 2.7. *Let K be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $\{x_n\}$ be a bounded sequence which satisfies the following properties:*

(i) *every weak limit point of $\{x_n\}$ lies in K;*
(ii) $\lim_{n\to\infty}\|x_n - x\|$ *exists for every x in K.*

Then $\{x_n\}$ converges weakly to a point in K.

Let C be a subset of a normed space, and let k be a constant in $[0, 1)$. A mapping $S : C \to C$ is called *k-strictly pseudocontractive* if

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C;$$

see [26]. In case C is a subset of a real Hilbert space, $S : C \to C$ is k-strictly pseudo-contractive if and only if there holds the following inequality:

$$\langle Sx - Sy, x - y\rangle \le \|x - y\|^2 - \frac{1-k}{2}\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \quad (2.1)$$

This immediately implies that if S is a k-strictly pseudo-contractive mapping, then $I - S$ is $\frac{1-k}{2}$-inverse strongly monotone; for further detail, we refer to [26] and the references therein. Nonexpansive mappings are strict pseudo-contractions.

The so-called demiclosedness principle for strict pseudo-contractive mappings in the following lemma will often be used.

Lemma 2.8 ([26, Proposition 2.1]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and let $S : C \to C$ be a k-strictly pseudo-contractive mapping.*

(i) *S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \le \frac{1+k}{1-k}\|x - y\|, \quad \forall x, y \in C.$$

(ii) *The mapping $I - S$ is demiclosed at 0, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - S)x_n \to 0$, then $(I - S)\tilde{x} = 0$.*
(iii) *The fixed point set $\mathrm{Fix}(S)$ of S is closed and convex so that the projection $P_{\mathrm{Fix}(S)}$ is well defined.*

Lemma 2.9 ([21]). *Let $\mathcal{H}$ be a real Hilbert space. Then, for all x, y in $\mathcal{H}$ and λ in $\mathbb{R}$,*

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Lemma 2.10 ([41]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $S : C \to C$ be a k-strictly pseudo-contractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma + \delta)k \le \gamma$. Then*

$$\|\gamma(x - y) + \delta(Sx - Sy)\| \le (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C.$$

The following elementary result is quite well-known.

Lemma 2.11 ([39, Lemma 2.1]). *Let $\{\gamma_n\}, \{\delta_n\}$ be sequences of real numbers such that*

(i) $\{\gamma_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \gamma_n = +\infty$, *or equivalently,*

$$\prod_{n=0}^{\infty}(1-\gamma_n) = \lim_{n\to\infty} \prod_{k=0}^{n}(1-\gamma_k) = 0;$$

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$, *or*

(ii)′ $\sum_{n=0}^{\infty} \gamma_n \delta_n$ *is convergent.*

Let $\{a_n\}$ *be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \quad \forall n \geq 0.$$

Then, $\lim_{n\to\infty} a_n = 0$.

3. Hybrid extragradient methods

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $F : C \to \mathcal{H}$ be κ-Lipschitz continuous and η-strongly monotone on C for some constants $\kappa, \eta > 0$; that is,

$$\|Fx - Fy\| \leq \kappa\|x-y\| \quad \text{and} \quad \langle Fx - Fy, x-y\rangle \geq \eta\|x-y\|^2, \quad \forall x, y \in C.$$

In this case, $0 < \eta < \kappa$. Let σ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to C$, we define the mapping $T^\sigma : C \to \mathcal{H}$ by

$$T^\sigma x := Tx - \sigma\mu F(Tx), \quad \forall x \in C.$$

Lemma 3.1 (see [39, Lemma 3.1])**.** T^σ *is a contraction provided* $0 < \mu < \frac{2\eta}{\kappa^2}$. *More precisely,*

$$\|T^\sigma x - T^\sigma y\| \leq (1-\sigma\tau)\|x-y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)} \in (0,1)$. *In particular, if* $T = I$ *the identity mapping, then*

$$\|(I - \sigma\mu F)x - (I-\sigma\mu F)y\| \leq (1-\sigma\tau)\|x-y\|, \quad \forall x, y \in C.$$

We remark that, whenever $0 < \mu < \frac{2\eta}{\kappa^2}$, the relation $0 < \eta < \kappa$ obviously implies

$$0 \leq \sqrt{(1-\mu\eta)^2} < \sqrt{1-\mu(2\eta-\mu\kappa^2)} < \sqrt{1 - 2\mu\eta + \frac{2\eta}{\kappa^2}\mu\kappa^2} = 1,$$

which hence leads to $\tau \in (0,1]$.

Let C and Q be nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of arbitrary dimensions, respectively. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Throughout the paper, we assume that the SFP:

$$x^* \in C \quad \text{and} \quad Ax^* \in Q,$$

is consistent; that is, the solution set Γ of the SFP is nonempty (and closed and convex).

Let $f, f_\alpha : \mathcal{H}_1 \to \mathbb{R}$ be the continuously differentiable functions defined by

$$f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2,$$

and

$$f_\alpha(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \quad \forall \alpha > 0.$$

As the minimization problem

$$\min_{x\in C} f(x)$$

is ill-posed, Xu [37] considered the following Tikhonov's regularization problem:

$$\min_{x\in C} f_\alpha(x), \tag{3.1}$$

where $\alpha > 0$ is the regularization parameter. The regularized minimization (3.1) has a unique solution x_α.

Proposition 3.2 ([37])**.** *If the SFP is consistent, then x_α converges strongly to the minimum-norm solution $x_{\min}$ of the SFP as $\alpha \to 0^+$; namely, $x_{\min}$ in Γ satisfies*

$$\|x_{\min}\| = \min\{\|x^*\| : x^* \in \Gamma\}.$$

The point $x_{\min}$ can be obtained in two steps. First, observing that the gradient

$$\nabla f_\alpha(x) = \nabla f(x) + \alpha I = A^*(I - P_Q)A + \alpha I$$

is $(\alpha + \|A\|^2)$-Lipschitz continuous and α-strongly monotone, we know that the mapping $P_C(I - \sigma\lambda\nabla f_\alpha)$ is a contraction with the (Lipschitz) coefficient

$$1 - \sigma\tau \le 1 - \sigma(1 - \sqrt{1 - \epsilon\alpha\lambda}) \le 1 - \frac{1}{2}\epsilon\alpha\sigma\lambda.$$

Here, $0 < \sigma \le 1,\ \tau = 1 - \sqrt{1 - \lambda(2\alpha - \lambda(\|A\|^2 + \alpha)^2)}$ and

$$0 < \lambda \le \frac{(2-\epsilon)\alpha}{(\|A\|^2 + \alpha)^2} < \frac{2\alpha}{(\|A\|^2 + \alpha)^2}, \tag{3.2}$$

for some ϵ in $(0, 1]$. Indeed, putting $F = \nabla f_\alpha$ and $T = I$ in Lemma 3.1, we have

$$\begin{aligned}
&\|P_C(I - \sigma\lambda\nabla f_\alpha)x - P_C(I - \sigma\lambda\nabla f_\alpha)y\| \\
\le&\|(I - \sigma\lambda\nabla f_\alpha)x - (I - \sigma\lambda\nabla f_\alpha)y\| \\
\le&\Big[1 - \sigma(1 - \sqrt{1 - \lambda(2\alpha - \lambda(\|A\|^2 + \alpha)^2)})\Big]\|x - y\| \\
\le&\Big[1 - \sigma\Big(1 - \sqrt{1 - 2\alpha\lambda + \lambda\cdot\frac{(2-\epsilon)\alpha}{(\|A\|^2+\alpha)^2}(\|A\|^2 + \alpha)^2)}\Big)\Big]\|x - y\| \\
=&[1 - \sigma(1 - \sqrt{1 - \epsilon\alpha\lambda})]\|x - y\| \\
\le&\Big[1 - \sigma\Big(1 - \Big(1 - \frac{1}{2}\epsilon\alpha\lambda\Big)\Big)\Big]\|x - y\| \\
=&\Big(1 - \frac{1}{2}\epsilon\alpha\sigma\lambda\Big)\|x - y\|.
\end{aligned}$$

It is worth noting that x_α is a fixed point of the mapping $P_C(I - \sigma\lambda\nabla f_\alpha)$ for any σ in $(0, 1]$ and $\lambda > 0$ satisfying (3.2), and can be obtained through the limit as $n \to \infty$ of the sequence of Picard iterates

$$x_{n+1}^\alpha = P_C(I - \sigma\lambda\nabla f_\alpha)x_n^\alpha.$$

Second, letting $\alpha \to 0$ yields $x_\alpha \to x_{\min}$ in norm.

It is interesting to know whether these two steps can be combined to get $x_{\min}$ in a hybrid steepest-descent extragradient algorithm. The following result shows that we can achieve the goals for suitable choices of σ, λ and α.

Theorem 3.3. *Let K be a nonempty closed convex subset of $\mathcal{H}_1$. Assume that $\{S_i : 1 \le i \le N\}$ be a pool of N nonexpansive mappings from K into K such that*

$$C = \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \neq \emptyset.$$

Suppose the non-negative sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}$ and $\{\lambda_n\}$, and the constant ϵ in $(0,1]$ satisfy the following conditions.

(i) $0 < \sigma_n \le 1$ *and* $0 < \lambda_n \le \frac{(2-\epsilon)\alpha_n}{(\|A\|^2+\alpha_n)^2}$ *for all large enough n.*

(ii) $\alpha_n \to 0$ *and* $\lambda_n \to 0$.

(iii) $\sum_{n=0}^{\infty} \alpha_n^2 \sigma_n \lambda_n \delta_n = +\infty$.

(iv) $\frac{|\sigma_n\lambda_n - \sigma_{n-1}\lambda_{n-1}| + \sigma_{n-1}\lambda_{n-1}|\alpha_n - \alpha_{n-1}|}{\alpha_n^3 \sigma_n^2 \lambda_n^2 \delta_n} \to 0$.

(v) $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ *and* $\beta_n + \gamma_n + \delta_n = 1$ *for all* $n \ge 0$.

(vi) $\frac{2(2-\epsilon)\delta_n}{\alpha_n + \|A\|^2} \le \epsilon \gamma_n \sigma_n \lambda_n$ *for all large enough n.*

Define a sequence $\{x_n\}$ through the following Mann type steepest-descent extragradient algorithm:

$$\text{(3.3)} \quad \begin{cases} x_1 = x \in K, \ \text{chosen arbitrarily}, \\ y_n = P_C(S_{[n]}x_n - \sigma_n\lambda_n \nabla f_{\alpha_n}(S_{[n]}x_n)), \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n P_C(S_{[n]}x_n - \sigma_n\lambda_n \nabla f_{\alpha_n}(S_{[n]}y_n)), \quad \forall n \ge 1, \end{cases}$$

Here, $S_{[n]} = S_{n \bmod N}$ and $\nabla f_{\alpha_n} = \alpha_n I + A^*(I - P_Q)A$ for each $n \ge 1$.

Then both the sequences $\{x_n\}$ and $\{y_n\}$ converge in norm to the minimum-norm solution $x_{\min}$ of the SFP.

Proof. For each $n \ge 0$, let z_n be the unique fixed point of the contraction

$$P_C(I - \sigma_n\lambda_n \nabla f_{\alpha_n}) : C \to C.$$

Then, $z_n \to x_{\min}$ in norm. We shall prove that

$$\|x_{n+1} - z_n\| \to 0.$$

Define

$$T_n := P_C(I - \sigma_n\lambda_n \nabla f_{\alpha_n})S_{[n]}.$$

From $z_n \in C = \bigcap_{i=1}^{N} \mathrm{Fix}(S_i)$ it follows that

$$T_n z_n = P_C(S_{[n]}z_n - \sigma_n\lambda_n \nabla f_{\alpha_n}(S_{[n]}z_n)) = P_C(z_n - \sigma_n\lambda_n \nabla f_{\alpha_n}(z_n)) = z_n.$$

Noting that $P_C(I - \sigma_n\lambda_n \nabla f_{\alpha_n})$ has a contraction ratio $1 - \frac{1}{2}\epsilon\alpha_n\sigma_n\lambda_n$, we have

$$\begin{aligned} \|y_n - z_n\| &= \|T_n x_n - T_n z_n\| \\ &= \|P_C(I - \sigma_n\lambda_n \nabla f_{\alpha_n})S_{[n]}x_n - P_C(I - \sigma_n\lambda_n \nabla f_{\alpha_n})S_{[n]}z_n\| \\ &\le \Big(1 - \frac{1}{2}\epsilon\alpha_n\sigma_n\lambda_n\Big)\|S_{[n]}x_n - S_{[n]}z_n\| \end{aligned} \tag{3.4}$$

$$\leq \Big(1 - \frac{1}{2}\epsilon\alpha_n\sigma_n\lambda_n\Big)\|x_n - z_n\|.$$

We now estimate

$$\begin{aligned}\|z_n - z_{n-1}\| &= \|T_n z_n - T_{n-1}z_{n-1}\| \\ &\leq \|T_n z_n - T_n z_{n-1}\| + \|T_n z_{n-1} - T_{n-1}z_{n-1}\| \\ &\leq \Big(1 - \frac{1}{2}\epsilon\alpha_n\sigma_n\lambda_n\Big)\|z_n - z_{n-1}\| + \|T_n z_{n-1} - T_{n-1}z_{n-1}\|.\end{aligned}$$

This implies that

$$\|z_n - z_{n-1}\| \leq \frac{2}{\epsilon\alpha_n\sigma_n\lambda_n}\|T_n z_{n-1} - T_{n-1}z_{n-1}\|. \tag{3.5}$$

However, since ∇f is Lipschitz continuous and $\{z_n\}$ is bounded, we have

$$\begin{aligned}&\|T_n z_{n-1} - T_{n-1}z_{n-1}\| \\ =&\|P_C(I - \sigma_n\lambda_n\nabla f_{\alpha_n})S_{[n]}z_{n-1} - P_C(I - \sigma_{n-1}\lambda_{n-1}\nabla f_{\alpha_{n-1}})S_{[n-1]}z_{n-1}\| \\ \leq&\|(I - \sigma_n\lambda_n\nabla f_{\alpha_n})S_{[n]}z_{n-1} - (I - \sigma_{n-1}\lambda_{n-1}\nabla f_{\alpha_{n-1}})S_{[n-1]}z_{n-1}\| \\ =&\|(I - \sigma_n\lambda_n\nabla f_{\alpha_n})z_{n-1} - (I - \sigma_{n-1}\lambda_{n-1}\nabla f_{\alpha_{n-1}})z_{n-1}\| \\ \leq&\|\sigma_n\lambda_n\nabla f_{\alpha_n}(z_{n-1}) - \sigma_{n-1}\lambda_{n-1}\nabla f_{\alpha_{n-1}}(z_{n-1})\| \\ =&\|(\sigma_n\lambda_n - \sigma_{n-1}\lambda_{n-1})\nabla f_{\alpha_n}(z_{n-1}) \\ &+ \sigma_{n-1}\lambda_{n-1}(\nabla f_{\alpha_n}(z_{n-1}) - \nabla f_{\alpha_{n-1}}(z_{n-1}))\| \\ \leq&|\sigma_n\lambda_n - \sigma_{n-1}\lambda_{n-1}|\|\nabla f(z_{n-1}) + \alpha_n z_{n-1}\| + \sigma_{n-1}\lambda_{n-1}|\alpha_n - \alpha_{n-1}|\|z_{n-1}\| \\ \leq&(|\sigma_n\lambda_n - \sigma_{n-1}\lambda_{n-1}| + \sigma_{n-1}\lambda_{n-1}|\alpha_n - \alpha_{n-1}|)M,\end{aligned} \tag{3.6}$$

where $M = \sup_{n\geq 1}\max\{\|\nabla f(z_{n-1}) + \alpha_n z_{n-1}\|, \|z_{n-1}\|\} < +\infty$. Utilizing conditions (i), (vi), and inequalities (3.3), (3.4), (3.5), we obtain for large n that

$$\begin{aligned}&\|x_{n+1} - z_n\| \\ =&\|\beta_n(x_n - z_n) + \gamma_n(y_n - z_n) + \delta_n[P_C(S_{[n]}x_n - \sigma_n\lambda_n\nabla f_{\alpha_n}(S_{[n]}y_n)) - z_n]\| \\ \leq&\beta_n\|x_n - z_n\| + \gamma_n\|y_n - z_n\| \\ &+ \delta_n\|P_C(S_{[n]}x_n - \sigma_n\lambda_n\nabla f_{\alpha_n}(S_{[n]}y_n)) - P_C(S_{[n]}z_n - \sigma_n\lambda_n\nabla f_{\alpha_n}(S_{[n]}z_n))\| \\ \leq&\beta_n\|x_n - z_n\| + \gamma_n\|y_n - z_n\| \\ &+ \delta_n\|(S_{[n]}x_n - \sigma_n\lambda_n\nabla f_{\alpha_n}(S_{[n]}y_n)) - (S_{[n]}z_n - \sigma_n\lambda_n\nabla f_{\alpha_n}(S_{[n]}z_n))\| \\ \leq&\beta_n\|x_n - z_n\| + \gamma_n\|y_n - z_n\| \\ &+ \delta_n[\|S_{[n]}x_n - S_{[n]}z_n\| + \sigma_n\lambda_n\|\nabla f_{\alpha_n}(S_{[n]}y_n) - \nabla f_{\alpha_n}(S_{[n]}z_n)\|] \\ \leq&\beta_n\|x_n - z_n\| + \gamma_n\|y_n - z_n\| \\ &+ \delta_n[\|x_n - z_n\| + \sigma_n\lambda_n(\alpha_n + \|A\|^2)\|S_{[n]}y_n - S_{[n]}z_n\|] \\ \leq&\beta_n\|x_n - z_n\| + \gamma_n\|y_n - z_n\| \\ &+ \delta_n[\|x_n - z_n\| + \sigma_n\lambda_n(\alpha_n + \|A\|^2)\|y_n - z_n\|] \qquad (3.7) \\ =&(\beta_n + \delta_n)\|x_n - z_n\| + (\gamma_n + \delta_n\sigma_n\lambda_n(\alpha_n + \|A\|^2))\|y_n - z_n\| \\ \leq&(1 - \gamma_n)\|x_n - z_n\| + \Big(\gamma_n + \delta_n\cdot\frac{(2-\epsilon)\alpha_n}{\alpha_n + \|A\|^2}\Big)\|y_n - z_n\|\end{aligned}$$

$$
\begin{aligned}
&\le (1-\gamma_n)\|x_n - z_n\| + \Big(\gamma_n + \delta_n \cdot \frac{(2-\epsilon)\alpha_n}{\alpha_n + \|A\|^2}\Big)\Big(1 - \frac{1}{2}\epsilon\alpha_n\sigma_n\lambda_n\Big)\|x_n - z_n\| \\
&= \Big[1 - \gamma_n + \gamma_n + \frac{(2-\epsilon)\alpha_n\delta_n}{\alpha_n + \|A\|^2} - \frac{1}{2}\epsilon\alpha_n\gamma_n\sigma_n\lambda_n - \frac{(2-\epsilon)\epsilon\alpha_n^2\sigma_n\lambda_n\delta_n}{2(\alpha_n + \|A\|^2)}\Big]\|x_n - z_n\| \\
&= \Big[1 + \alpha_n\Big(\frac{(2-\epsilon)\delta_n}{\alpha_n + \|A\|^2} - \frac{1}{2}\epsilon\gamma_n\sigma_n\lambda_n\Big) - \frac{(2-\epsilon)\epsilon\alpha_n^2\sigma_n\lambda_n\delta_n}{2(\alpha_n + \|A\|^2)}\Big]\|x_n - z_n\| \\
&\le \Big[1 - \frac{(2-\epsilon)\epsilon\alpha_n^2\sigma_n\lambda_n\delta_n}{2(\alpha_n + \|A\|^2)}\Big]\|x_n - z_n\| \\
&\le \Big[1 - \frac{(2-\epsilon)\epsilon\alpha_n^2\sigma_n\lambda_n\delta_n}{2(\alpha_n + \|A\|^2)}\Big]\|x_n - z_{n-1}\| + \|z_n - z_{n-1}\| \\
&\le \Big[1 - \frac{(2-\epsilon)\epsilon\alpha_n^2\sigma_n\lambda_n\delta_n}{2(\alpha_n + \|A\|^2)}\Big]\|x_n - z_{n-1}\| + \frac{2}{\epsilon\alpha_n\sigma_n\lambda_n}\|T_n z_{n-1} - T_{n-1}z_{n-1}\| \\
&\le \Big[1 - \frac{(2-\epsilon)\epsilon\alpha_n^2\sigma_n\lambda_n\delta_n}{2(\alpha_n + \|A\|^2)}\Big]\|x_n - z_{n-1}\| \\
&\quad + \frac{2M(|\sigma_n\lambda_n - \sigma_{n-1}\lambda_{n-1}| + \sigma_{n-1}\lambda_{n-1}|\alpha_n - \alpha_{n-1}|)}{\epsilon\alpha_n\sigma_n\lambda_n} \\
&= \Big[1 - \frac{(2-\epsilon)\epsilon\alpha_n^2\sigma_n\lambda_n\delta_n}{2(\alpha_n + \|A\|^2)}\Big]\|x_n - z_{n-1}\| + \frac{(2-\epsilon)\epsilon\alpha_n^2\sigma_n\lambda_n\delta_n}{2(\alpha_n + \|A\|^2)} \\
&\quad \times \frac{4M(\alpha_n + \|A\|^2)(|\sigma_n\lambda_n - \sigma_{n-1}\lambda_{n-1}| + \sigma_{n-1}\lambda_{n-1}|\alpha_n - \alpha_{n-1}|)}{(2-\epsilon)\epsilon^2\alpha_n^3\sigma_n^2\lambda_n^2\delta_n} \\
&= \Big[1 - \frac{(2-\epsilon)\epsilon\alpha_n^2\sigma_n\lambda_n\delta_n}{2(\alpha_n + \|A\|^2)}\Big]\|x_n - z_{n-1}\| + \frac{(2-\epsilon)\epsilon\alpha_n^2\sigma_n\lambda_n\delta_n}{2(\alpha_n + \|A\|^2)} \cdot \mu_n,
\end{aligned}
$$

where

$$
\mu_n = \frac{4M(\alpha_n + \|A\|^2)(|\sigma_n\lambda_n - \sigma_{n-1}\lambda_{n-1}| + \sigma_{n-1}\lambda_{n-1}|\alpha_n - \alpha_{n-1}|)}{(2-\epsilon)\epsilon^2\alpha_n^3\sigma_n^2\lambda_n^2\delta_n} \to 0
$$

due to conditions (ii) and (iv). Applying Lemma 2.11 to (3.6), we conclude $\|x_{n+1} - z_n\| \to 0$. Hence, $x_n \to x_{\min}$ in norm. Taking into consideration the strong convergence of both $\{x_n\}$ and $\{z_n\}$ to $x_{\min}$, we deduce from (3.4) that

$$
\|y_n - z_n\| \le \|x_n - z_n\| \to 0.
$$

Therefore, $y_n \to x_{\min}$ in norm. □

Corollary 3.4. *Let C be a nonempty closed convex subset of $\mathcal{H}_1$. Suppose the nonnegative sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions.*

(i) $0 < \lambda_n \le \frac{\alpha_n}{(\|A\|^2 + \alpha_n)^2}$ *for all large enough n.*

(ii) $\alpha_n \to 0$ *and* $\lambda_n \to 0$.

(iii) $\sum_{n=1}^{\infty} \alpha_n^2\lambda_n\delta_n = +\infty$.

(iv) $\frac{|\lambda_{n+1} - \lambda_n| + \lambda_n|\alpha_{n+1} - \alpha_n|}{\alpha_{n+1}^3\lambda_{n+1}^2\delta_{n+1}} \to 0$.

(v) $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ *and* $\beta_n + \gamma_n + \delta_n = 1$ *for all* $n \geq 1$.

(vi) $\frac{2\delta_n}{\alpha_n + \|A\|^2} \leq \gamma_n \lambda_n$ *for all large enough* n.

Define a sequence $\{x_n\}$ through the following Mann type extragradient algorithm:

$$\begin{cases} x_1 = x \in C, \text{ chosen arbitrarily,} \\ y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(x_n)), \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 1. \end{cases}$$

Here, $\nabla f_{\alpha_n} = \alpha_n I + A^*(I - P_Q)A$. Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge in norm to the minimum-norm solution of the SFP.

Proof. In Theorem 3.3, putting $K = C$, $S_i = I$ the identity mapping for each $i = 1, 2, \ldots, N$, and $\sigma_n = \epsilon = 1$ for all $n \geq 0$, we obtain that $\bigcap_{i=1}^N \mathrm{Fix}(S_i) = C \neq \emptyset$ and

$$\begin{cases} x_1 = x \in C, \text{ chosen arbitrarily,} \\ y_n = P_C(S_{[n]}x_n - \sigma_n \lambda_n \nabla f_{\alpha_n}(S_{[n]}x_n)) = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(x_n)), \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n P_C(S_{[n]}x_n - \sigma_n \lambda_n \nabla f_{\alpha_n}(S_{[n]}y_n)) \\ \quad = \beta_n x_n + \gamma_n y_n + \delta_n P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 1. \end{cases}$$

In this case, the assumptions of Theorem 3.3 are all satisfied. Therefore, Theorem 3.3 applies. ☐

Remark 3.5. (a) Theorem 3.3 includes [4, Theorem 3.1] as a special case.

(b) In Theorem 3.3, put $\sigma_n = \epsilon = 1$, $\alpha_n = n^{-\delta}$, $\lambda_n = n^{-\sigma}$ and $\delta_n = n^{-\theta}$, where $\delta = \frac{1}{10}$, $\sigma = \frac{1}{5}$ and $\theta = \frac{1}{4}$. It is easy to see that conditions (i)-(iv) in Theorem 3.3 are satisfied. If $\liminf_{n\to\infty} \gamma_n > 0$, condition (vi) is also satisfied.

(c) It is worth emphasizing that the Mann type steepest-descent extragradient algorithm employed in Theorem 3.3 is essentially the predictor-corrector algorithm. Indeed, the first iterative step $y_n = P_C(S_{[n]}x_n - \sigma_n \lambda_n \nabla f_{\alpha_n}(S_{[n]}x_n))$ is a predictor, and the second iterative step $x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n P_C(S_{[n]}x_n - \sigma_n \lambda_n \nabla f_{\alpha_n}(S_{[n]}y_n))$ is a corrector. Note also that we extend the iterative algorithm in [37, Theorem 5.5] to develop the Mann type steepest-descent extragradient algorithm in Theorem 3.3.

Theorem 3.6. *Let* $S : C \to C$ *be a* k*-strictly pseudocontractive mapping such that* $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. *Suppose the sequences of parameters* $\{\alpha_n\}$ *in* $(0, \infty)$ *and* $\{\beta_n\}, \{\sigma_n\}, \{\gamma_n\}, \{\delta_n\}$ *in* $[0,1]$ *satisfy the following conditions.*

(i) $\sum_{n=0}^{\infty} \alpha_n < +\infty$.

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

(iii) $\sigma_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$ *for all* $n \geq 0$.

(iv) $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$.

Assume that $0 < \lambda < \frac{2}{\|A\|^2}$, and let $\{x_n\}$ and $\{y_n\}$ be the sequences in C generated by the following Mann type hybrid extragradient-like algorithm:

(3.8)
$$\begin{cases} x_0 = x \in C, \text{ chosen arbitrarily,} \\ y_n = (1-\beta_n)x_n + \beta_n P_C(x_n - \lambda\nabla f_{\alpha_n}(x_n)), \\ x_{n+1} = \sigma_n x_n + \gamma_n P_C(y_n - \lambda\nabla f_{\alpha_n}(y_n)) + \delta_n S P_C(y_n - \lambda\nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0. \end{cases}$$

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element z in $\mathrm{Fix}(S)\cap \Gamma$.

Proof. First, we observe that $P_C(I-\mu\nabla f_\alpha)$ is ζ-averaged for each μ in $(0, \frac{2}{2\alpha+\|A\|^2})$, where

$$\zeta = \frac{2+\mu(2\alpha+\|A\|^2)}{4} \in (0,1).$$

See, e.g., [7] and from which it follows that $P_C(I-\lambda\nabla f_{\alpha_n})$ is nonexpansive for all λ in $(0, \frac{2}{\|A\|^2})$ and for all sufficiently large $n \geq 0$.

We divide the remainder of the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, take a fixed point p from $\mathrm{Fix}(S)\cap\Gamma$ arbitrarily. We get $Sp = p$ and $P_C(I-\lambda\nabla f)p = p$ for λ in $(0, \frac{2}{\|A\|^2})$. Hence,

$$\begin{aligned} \|y_n - p\| &= \|(1-\beta_n)(x_n - p) + \beta_n[P_C(I-\lambda\nabla f_{\alpha_n})x_n - p]\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|P_C(I-\lambda\nabla f_{\alpha_n})x_n - p\| \\ &= (1-\beta_n)\|x_n - p\| + \beta_n\|P_C(I-\lambda\nabla f_{\alpha_n})x_n - P_C(I-\lambda\nabla f)p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[\|P_C(I-\lambda\nabla f_{\alpha_n})x_n - P_C(I-\lambda\nabla f_{\alpha_n})p\| \\ &\quad + \|P_C(I-\lambda\nabla f_{\alpha_n})p - P_C(I-\lambda\nabla f)p\|] \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[\|x_n - p\| + \|P_C(I-\lambda\nabla f_{\alpha_n})p - P_C(I-\lambda\nabla f)p\|] \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[\|x_n - p\| + \lambda\alpha_n\|p\|] \\ &= \|x_n - p\| + \lambda\alpha_n\beta_n\|p\|. \end{aligned} \tag{3.9}$$

Since $(\gamma_n+\delta_n)k \leq \gamma_n$ for all $n \geq 0$, utilizing Lemma 2.10 we derive from (3.9) that

$$\begin{aligned} &\|x_{n+1} - p\| \\ =&\|\sigma_n(x_n - p) + \gamma_n(P_C(I-\lambda\nabla f_{\alpha_n})y_n - p) + \delta_n(SP_C(I-\lambda\nabla f_{\alpha_n})y_n - p)\| \\ \leq&\sigma_n\|x_n - p\| + \|\gamma_n(P_C(I-\lambda\nabla f_{\alpha_n})y_n - p) + \delta_n(SP_C(I-\lambda\nabla f_{\alpha_n})y_n - Sp)\| \\ \leq&\sigma_n\|x_n - p\| + (\gamma_n+\delta_n)\|P_C(I-\lambda\nabla f_{\alpha_n})y_n - p\| \\ \leq&\sigma_n\|x_n - p\| + (\gamma_n+\delta_n)[\|P_C(I-\lambda\nabla f_{\alpha_n})y_n - P_C(I-\lambda\nabla f_{\alpha_n})p\| \\ &\quad + \|P_C(I-\lambda\nabla f_{\alpha_n})p - P_C(I-\lambda\nabla f)p\|] \\ \leq&\sigma_n\|x_n - p\| + (\gamma_n+\delta_n)[\|y_n - p\| + \|(I-\lambda\nabla f_{\alpha_n})p - (I-\lambda\nabla f)p\|] \\ \leq&\sigma_n\|x_n - p\| + (\gamma_n+\delta_n)[\|y_n - p\| + \lambda\alpha_n\|p\|] \\ \leq&\sigma_n\|x_n - p\| + (\gamma_n+\delta_n)[\|x_n - p\| + \lambda\alpha_n\beta_n\|p\| + \lambda\alpha_n\|p\|] \\ \leq&\sigma_n\|x_n - p\| + (\gamma_n+\delta_n)[\|x_n - p\| + 2\lambda\alpha_n\|p\|] \\ \leq&\|x_n - p\| + 2\lambda\alpha_n\|p\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n < +\infty$, we conclude that

(3.10) $$\lim_{n\to\infty}\|x_n - p\| \text{ exists for each } p \text{ in } \mathrm{Fix}(S)\cap\Gamma.$$

Therefore, $\{x_n\}$ is bounded and so are $\{y_n\}, \{\nabla f(x_n)\}$ and $\{\nabla f(y_n)\}$.

Step 2. Let $u_n = P_C(I-\lambda\nabla f_{\alpha_n})x_n$ and $v_n = P_C(I-\lambda\nabla f_{\alpha_n})y_n$. We have

$$\lim_{n\to\infty}\|u_n - x_n\| = \lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|v_n - Sv_n\| = \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Indeed, by Lemma 2.9 we have

$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(u_n - p)\|^2\\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|u_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - u_n\|^2\\ &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n[\|x_n - p\| + \lambda\alpha_n\|p\|]^2 - \beta_n(1-\beta_n)\|x_n - u_n\|^2\\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \alpha_n\beta_n(2\lambda\|p\|\|x_n - p\| + \alpha_n\lambda^2\|p\|^2)\\ &\quad - \beta_n(1-\beta_n)\|x_n - u_n\|^2\\ &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \alpha_n M_1 - \beta_n(1-\beta_n)\|x_n - u_n\|^2\\ &= \|x_n - p\|^2 + \alpha_n M_1 - \beta_n(1-\beta_n)\|x_n - u_n\|^2.\end{aligned}$$

Here, $M_1 = \sup_{n\ge 0}\{\beta_n(2\lambda\|p\|\|x_n - p\| + \alpha_n\lambda^2\|p\|^2)\} < +\infty$.

Note that $\sigma_n + \gamma_n + \delta_n = 1$ and hence

$$x_{n+1} - x_n = \sigma_n x_n + \gamma_n v_n + \delta_n S v_n - x_n = \gamma_n(v_n - x_n) + \delta_n(Sv_n - x_n).$$

Since $(\gamma_n+\delta_n)k \le \gamma_n$ for all $n \ge 0$, utilizing Lemmas 2.9 and 2.10 we get from (3.9) that

$$\begin{aligned}&\|x_{n+1} - p\|^2\\ =&\|\sigma_n(x_n - p) + \gamma_n(v_n - p) + \delta_n(Sv_n - p)\|^2\\ =&\|\sigma_n(x_n - p) + (\gamma_n+\delta_n)\frac{1}{\gamma_n+\delta_n}(\gamma_n(v_n - p) + \delta_n(Sv_n - p))\|^2\\ =&\sigma_n\|x_n - p\|^2 + (\gamma_n+\delta_n)\|\frac{1}{\gamma_n+\delta_n}(\gamma_n(v_n - p) + \delta_n(Sv_n - p))\|^2\\ &- \sigma_n(\gamma_n+\delta_n)\|\frac{1}{\gamma_n+\delta_n}(\gamma_n(v_n - x_n) + \delta_n(Sv_n - x_n))\|^2\\ \le&\sigma_n\|x_n - p\|^2 + (\gamma_n+\delta_n)\|v_n - p\|^2 - \sigma_n(\gamma_n+\delta_n)\|\frac{1}{\gamma_n+\delta_n}(x_{n+1} - x_n)\|^2\\ =&\sigma_n\|x_n - p\|^2 + (\gamma_n+\delta_n)\|v_n - p\|^2 - \frac{\sigma_n}{1-\sigma_n}\|x_{n+1} - x_n\|^2\\ \le&\sigma_n\|x_n - p\|^2 + (1-\sigma_n)[\|y_n - p\| + \lambda\alpha_n\|p\|]^2 - \frac{\sigma_n}{1-\sigma_n}\|x_{n+1} - x_n\|^2\\ =&\sigma_n\|x_n - p\|^2 + (1-\sigma_n)\|y_n - p\|^2 + \alpha_n(1-\sigma_n)(2\lambda\|p\|\|y_n - p\| + \alpha_n\lambda^2\|p\|^2)\\ &- \frac{\sigma_n}{1-\sigma_n}\|x_{n+1} - x_n\|^2\\ \le&\sigma_n\|x_n - p\|^2 + (1-\sigma_n)\|y_n - p\|^2 + \alpha_n M_2 - \frac{\sigma_n}{1-\sigma_n}\|x_{n+1} - x_n\|^2\\ \le&\sigma_n\|x_n - p\|^2 + (1-\sigma_n)[\|x_n - p\|^2 + \alpha_n M_1 - \beta_n(1-\beta_n)\|x_n - u_n\|^2] + \alpha_n M_2\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{\sigma_n}{1-\sigma_n}\|x_{n+1}-x_n\|^2 \\
&\le \|x_n - p\|^2 + \alpha_n(M_1+M_2) - (1-\sigma_n)\beta_n(1-\beta_n)\|x_n-u_n\|^2 \\
&\quad - \frac{\sigma_n}{1-\sigma_n}\|x_{n+1}-x_n\|^2.
\end{aligned}
$$

Here, $M_2 = \sup_{n\ge 0}\{(1-\sigma_n)(2\lambda\|p\|\|y_n-p\| + \alpha_n\lambda^2\|p\|^2)\} < +\infty$. It follows that

$$
\begin{aligned}
(1-\sigma_n)\beta_n(1-\beta_n)\|x_n-u_n\|^2 + \frac{\sigma_n}{1-\sigma_n}\|x_{n+1}-x_n\|^2& \\
\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n(M_1+M_2)&.
\end{aligned}
$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, and $0 < \liminf_{n\to\infty}\sigma_n \le \limsup_{n\to\infty}\sigma_n < 1$, we deduce from the existence of $\lim_{n\to\infty}\|x_n - p\|$ that

$$
\lim_{n\to\infty}\|x_n-u_n\| = \lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.
$$

Utilizing (3.8) we get

$$
\lim_{n\to\infty}\|y_n-x_n\| = \lim_{n\to\infty}\beta_n\|u_n-x_n\| = 0.
$$

Taking into account the nonexpansiveness of $P_C(I-\lambda\nabla f_{\alpha_n})$, we have

$$
\begin{aligned}
\|v_n-x_n\| &\le \|v_n-u_n\| + \|u_n-x_n\| \\
&= \|P_C(I-\lambda\nabla f_{\alpha_n})y_n - P_C(I-\lambda\nabla f_{\alpha_n})x_n\| + \|u_n-x_n\| \\
&\le \|y_n-x_n\| + \|u_n-x_n\|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|\delta_n(Sv_n-x_n)\| &\le \|x_{n+1}-x_n\| + \gamma_n\|v_n-x_n\| \\
&\le \|x_{n+1}-x_n\| + \|v_n-x_n\| \\
&\le \|x_{n+1}-x_n\| + \|y_n-x_n\| + \|u_n-x_n\|.
\end{aligned}
$$

Since $\|x_{n+1}-x_n\|\to 0$, $\|y_n-x_n\|\to 0$, $\|u_n-x_n\|\to 0$ and $\liminf_{n\to\infty}\delta_n > 0$, we immediately obtain that

$$
\lim_{n\to\infty}\|v_n-x_n\| = \lim_{n\to\infty}\|Sv_n-x_n\| = 0.
$$

This implies

$$
\lim_{n\to\infty}\|Sv_n-v_n\| = 0.
$$

Step 3. $\omega_w(x_n)\subset \mathrm{Fix}(S)\cap\varGamma$.

Suppose that $\hat{x}\in\omega_w(x_n)$ and $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_j}\rightharpoonup\hat{x}$. Set $T = P_C(I-\lambda\nabla f)$. Then T is nonexpansive for each λ in $(0,\frac{2}{\|A\|^2})$. We have seen that $\nabla f = A^*(I-P_Q)A$ is $\frac{1}{\|A\|^2}$-inverse strongly monotone and $\lambda\nabla f = \lambda A^*(I-P_Q)A$ is $\frac{1}{\lambda\|A\|^2}$-inverse strongly monotone. Hence, by Proposition 2.5(ii) the complement $I-\lambda\nabla f$ is $\frac{\lambda\|A\|^2}{2}$-averaged for each λ in $(0,\frac{2}{\|A\|^2})$. Therefore, noting that P_C is

$\frac{1}{2}$-averaged and applying Proposition 2.6(iv), we know that $T = P_C(I - \lambda\nabla f)$ is α-averaged for each λ in $(0, \frac{2}{\|A\|^2})$, with

$$\alpha = \frac{1}{2} + \frac{\lambda\|A\|^2}{2} - \frac{1}{2}\cdot\frac{\lambda\|A\|^2}{2} = \frac{2+\lambda\|A\|^2}{4} \in (0,1).$$

Consequently, T is nonexpansive.

Now observe that

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - u_n\| + \|u_n - Tx_n\| \\ &= \|x_n - u_n\| + \|P_C(I-\lambda\nabla f_{\alpha_n})x_n - P_C(I-\lambda\nabla f)x_n\| \\ &\le \|x_n - u_n\| + \|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)x_n\| \\ &= \|x_n - u_n\| + \lambda\alpha_n\|x_n\|.\end{aligned}$$

From $\|x_n - u_n\| \to 0,\ \alpha_n \to 0$, and the boundedness of $\{x_n\}$ it follows that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Taking into account $x_{n_j} \rightharpoonup \hat{x}$ and utilizing Lemma 2.8(ii), we obtain $\hat{x} \in \mathrm{Fix}(T)$. But $\mathrm{Fix}(T) = \Gamma$; we therefore have $\hat{x} \in \Gamma$. Furthermore, since $x_{n_j} \rightharpoonup \hat{x}$ and

$$\lim_{n\to\infty}\|u_n - x_n\| = \lim_{n\to\infty}\|u_n - v_n\| = \lim_{n\to\infty}\|Sv_n - v_n\| = 0,$$

we have

$$v_{n_j} \rightharpoonup \hat{x} \quad \text{and} \quad \lim_{j\to\infty}\|Sv_{n_j} - v_{n_j}\| = 0.$$

Thus, from Lemma 2.8(ii) we get $\hat{x} \in \mathrm{Fix}(S)$. Therefore, we have $\hat{x} \in \mathrm{Fix}(S)\cap\Gamma$. This shows

$$\omega_w(x_n) \subset \mathrm{Fix}(S)\cap\Gamma. \tag{3.11}$$

Step 4. Both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element z in $\mathrm{Fix}(S)\cap\Gamma$.

Indeed, according to (3.10) and (3.11) we apply Proposition 2.7 to $\mathrm{Fix}(S)\cap\Gamma$ to show that $\{x_n\}$ converges weakly to a point z in $\mathrm{Fix}(S)\cap\Gamma$. Moreover, from $\|x_n - y_n\| \to 0$ it follows that $y_n \rightharpoonup z$. □

Corollary 3.7. *Let $S : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S)\cap\Gamma \ne \emptyset$. *Suppose the sequences of parameters $\{\alpha_n\}$ in $(0,\infty)$ and $\{\beta_n\}, \{\sigma_n\}$ in $[0,1]$ satisfy the following conditions.*

(i) $\sum_{n=0}^{\infty}\alpha_n < +\infty$.

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

(iii) $0 < \liminf_{n\to\infty}\sigma_n \le \limsup_{n\to\infty}\sigma_n < 1$.

Assume that $0 < \lambda < \frac{2}{\|A\|^2}$, and let $\{x_n\}$ and $\{y_n\}$ be the sequences in C generated by the following Mann type extragradient-like algorithm:

$$\begin{cases} x_0 = x \in C, \ \text{chosen arbitrarily}, \\ y_n = (1-\beta_n)x_n + \beta_n P_C(x_n - \lambda\nabla f_{\alpha_n}(x_n)), \\ x_{n+1} = \sigma_n x_n + (1-\sigma_n)SP_C(y_n - \lambda\nabla f_{\alpha_n}(y_n)), \quad \forall n \ge 0. \end{cases}$$

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element z in $\mathrm{Fix}(S) \cap \Gamma$.

Proof. In Theorem 3.6, putting $\gamma_n = 0$ for all $n \geq 0$, we conclude that $\sigma_n + \delta_n = \sigma_n + \gamma_n + \delta_n = 1$ and

$$\begin{cases} x_0 = x \in C, \text{ chosen arbitrarily,} \\ y_n = (1-\beta_n)x_n + \beta_n P_C(x_n - \lambda \nabla f_{\alpha_n}(x_n)), \\ x_{n+1} = \sigma_n x_n + \gamma_n P_C(y_n - \lambda \nabla f_{\alpha_n}(y_n)) + \delta_n S P_C(y_n - \lambda \nabla f_{\alpha_n}(y_n)) \\ \ = \sigma_n x_n + \delta_n S P_C(y_n - \lambda \nabla f_{\alpha_n}(y_n)) \\ \ = \sigma_n x_n + (1-\sigma_n) S P_C(y_n - \lambda \nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0. \end{cases}$$

Being nonexpansive, S is a k-strictly pseudocontractive mapping with constant $k = 0$. Moreover, we have $(\gamma_n + \delta_n)k \leq \gamma_n$ for all $n \geq 0$ and $\liminf_{n\to\infty} \delta_n = 1 - \limsup_{n\to\infty} \sigma_n > 0$. All conditions of Theorem 3.6 are satisfied. Therefore, Theorem 3.6 applies. □

Remarks 3.8. We would like to compare our theorems in this paper with other established results in recent literature.

(a) Theorem 3.6 includes [4, Theorem 3.2] as a special case.

(b) The corresponding iterative algorithms in [37, Theorem 5.7] and [4, Theorem 3.2] are extended to develop the Mann type hybrid extragradient-like algorithm in Theorem 3.6. However, the technique of proving weak convergence in Theorem 3.2 is very different from [37, Theorem 5.7] and [4, Theorem 3.2] because our technique depends on the demiclosedness principle for strictly contractive mappings in Hilbert spaces.

(c) Theorem 3.3 improves, extends, supplements and develops [4, Theorem 3.1] in the following aspects.

  (i) The problem considered in Theorem 3.3 is more general and more subtle than that considered in [4, Theorem 3.1]. We consider in Theorem 3.3 the problem of finding the minimum-norm solution of the SFP defined on the intersection $C = \bigcap_{i=1}^N \mathrm{Fix}(S_i)$ of the fixed point sets of finitely many nonexpansive mappings $\{S_i : 1 \leq i \leq N\}$. It reduces to the problem in [4, Theorem 3.1] in the special case that $K = C$ and $S_i = I$, the identity mapping of K, for each $i = 1, \ldots, N$.

  (ii) The iterative scheme in [4, Theorem 3.1] is extended to develop the iterative scheme in Theorem 3.3 by virtue of a hybrid steepest-descent method developed in [30, Theorem 3.2]. The proof of Theorem 3.3 makes heavy use of the argument technique given in [30, Lemma 3.1 and Theorem 3.2].

  (iii) The iterative scheme in Theorem 3.3 is more advantageous and more flexible than that in [4, Theorem 3.1]. More precisely, our iterative scheme involves solving two problems: the SFP and the problem of finding a common fixed point of finitely many nonexpansive mappings. It involves a hybrid steepest-descent method (namely, we add a Lipschitz continuous and strongly monotone operator F in our iterative scheme) such that the common fixed point problem of finitely many nonexpansive mappings $\{S_i : 1 \leq i \leq N\}$ can be solved as well. Thus it should be more useful and more applicable.

References

[1] D. P. Bertsekas and E. M. Gafni, *Projection methods for variational inequalities with applications to the traffic assignment problem*, Math. Program. Study **17** (1982), 139–159.

[2] C. Byrne, *Iterative oblique projection onto convex subsets and the split feasibility problem*, Inverse Problems **18** (2002), 441–453.

[3] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Problems **20** (2004), 103–120.

[4] L. C. Ceng, Q. H. Ansari and J. C. Yao, *Relaxed extragradient methods for finding minimum-norm solutions of the split feasibility problem*, Nonlinear Anal. **75** (2012), 2116–2125.

[5] L. C. Ceng, Q. H. Ansari and J. C. Yao, *An extragradient method for solving split feasibility and fixed point problems*, Comput. Math. Appl. **64** (2012), 633–642.

[6] L. C. Ceng, Q. H. Ansari and J. C. Yao, *Mann type iterative methods for finding a common solution of split feasibility and fixed point problems*, Positivity **16** (2012), 471–495.

[7] L. C. Ceng, A. Petruşel and J. C. Yao, *Relaxed extragradient methods with regularization for general system of variational inequalities with constraints of split feasibility and fixed point problems*, Abstr. Appl. Anal. 2013 (2013), Article ID 891232, 25pp.

[8] L. C. Ceng, M. M. Wong and J. C. Yao, *A hybrid extragradient-like approximation method with regularization for solving split feasibility and fixed point problems*, J. Nonlinear Convex Anal. **14** (2013), 163–182.

[9] L. C. Ceng and J. C. Yao, *Relaxed and hybrid viscosity methods for general system of variational inequalities with split feasibility problem constraint*, Fixed Point Theory Appl. 2013, 2013:43, 50pp.

[10] Y. Censor, T. Bortfeld, B. Martin and A. Trofimov, *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol. **51** (2006), 2353–2365.

[11] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms **8** (1994), 221–239.

[12] Y. Censor, T. Elfving, N. Kopf and T. Bortfeld, *The multiple-sets split feasibility problem and its applications for inverse problems*, Inverse Problems **21** (2005), 2071–2084.

[13] Y. Censor, A. Gibali and S. Reich, *Algorithms for the split variational inequality problem*, Numer. Algorithms **59** (2012), 301–323.

[14] Y. Censor, A. Motova and A. Segal, *Perturbed projections and subgradient projections for the multiple-sets split feasibility problem*, J. Math. Anal. Appl. **327** (2007), 1244–1256.

[15] Y. Censor and A. Segal, *Iterative projection methods in biomedical inverse problems*, in: Mathematical Methods in Biomedical Imaging and Intensity-Modulated Therapy, Y. Censor, M. Jiang, A. K. Louis (eds.), IMRT, Edizioni della Norale, Pisa, Italy, 2008, pp. 65–96.

[16] P. L. Combettes, *Solving monotone inclusions via compositions of nonexpansive averaged operators*, Optimization **53** (2004), 475–504.

[17] P. L. Combettes and V. Wajs, *Signal recovery by proximal forward-backward splitting*, Multiscale Model. Simul. **4** (2005), 1168–1200.

[18] Y. Dang and Y. Gao, *The strong convergence of a KM-CQ-like algorithm for a split feasibility problem*, Inverse Problem 27 (2011) 015007, 9pp.

[19] Y. Dang and Y. Gao, *The strong convergence of a three-step algorithm for the split feasibility problem*, Optimization Letters **7** (2013), 1325–1339.

[20] B. Eicke, *Iteration methods for convexly constrained ill-posed problems in Hilbert spaces*, Numer. Funct. Anal. Optim. **13** (1992), 413–429.

[21] K. Geobel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, 1990.

[22] D. Han and H. K. Lo, *Solving non-additive traffic assignment problems: A descent method for co-coercive variational inequalities*, European J. Operational Research **159** (2004), 529–544.

[23] G. M. Korpelevich, *An extragradient method for finding saddle points and for other problems*, Ekonomika Mat. Metody **12** (1976), 747–756.

[24] L. Landweber, *An iterative formula for Fredholm integral equations of the first kind*, Amer. J. Math. **73** (1951), 615–624.

[25] J. L. Lions, *Quelques Methodes de Resolution des Problems aux Limites Non Lineaires*, Dunod, Paris, 1969.

[26] G. Marino and H. K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl. **329** (2007), 336–346.

[27] N. Nadezhkina and W. Takahashi, *Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **128** (2006), 191–201.

[28] N. Nadezhkina and W. Takahashi, *Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings*, SIAM J. Optim. **16** (2006), 1230–1241.

[29] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[30] C. Popîrlan, *Projection algorithm for split feasibility problem*, Annals of the University of Craiova, Mathematics and Computer Science Series **37** (2010), 52–58.

[31] B. Qu and N. Xiu, *A note on the CQ algorithm for the split feasibility problem*, Inverse Problems **21** (2005), 1655–1665.

[32] M. I. Sezan and H. Stark, *Applications of convex projection theory to image recovery in tomography and related areas*, in: Image Recovery Theory and Applications, H. Stark (ed.), Academic, Orlando, 1987, pp. 415-462.

[33] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[34] F. Wang and H. K. Xu, *Approximating curve and strong convergence of the CQ algorithm for the split feasibility problem*, J. Inequal. Appl. **2010** (2010), Article ID 102085, 13pp.

[35] F. Wang and H. K. Xu, *Cyclic algorithms for split feasibility problems in Hilbert spaces*, Nonlinear Anal. **74** (2011), 4105–4111.

[36] H. K. Xu, *A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem*, Inverse Problems **22** (2006), 2021–2034.

[37] H. K. Xu, *Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces*, Inverse Problems **26** (2010), 105018.

[38] H. K. Xu, *Averaged mappings and the gradient-projection algorithm*, J. Optim. Theory Appl. **150** (2011), 360–378.

[39] H. K. Xu and T. H. Kim, *Convergence of hybrid steepest-descent methods for variational inequalities*, J. Optim. Theory Appl. **119** (2003), 185–201.

[40] Y. Yao, T. H. Kim, S. Chebbi and H. K. Xu, *A modified extragradient method for the split feasibility and fixed point problems*, J. Nonlinear Convex Anal. **13** (2012), 383–396.

[41] Y. Yao, Y. C. Liou and S. M. Kang, *Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method*, Comput. Math. Appl. **59** (2010), 3472–3480.

[42] Y. Yao, M. Postolache and Y. C. Liou, *Strong convergence of a self-adaptive method for the split feasibility problem*, Fixed Point Theory Appl. 2013, 2013:201, 12pp.

[43] Y. Yao, P. X. Yang and S. M. Kang, *Composite projection algorithms for the split feasibility problem*, Math. Comput. Model. **57** (2013), 693–700.

[44] I. Yamada, *The hybrid steepest-descent method for the variational inequality problem over the intersection of fixed-point sets of nonexpansive mappings, in Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, D. Butnariu, Y. Censor and S. Reich (eds.), Kluwer Academic Publishers, Dordrecht, Netherlands, 2001, pp. 473–504.

[45] Q. Yang, *The relaxed CQ algorithm for solving the split feasibility problem*, Inverse Problems **20** (2004), 1261–1266.

[46] J. Zhao and Q. Yang, *Several solution methods for the split feasibility problem*, Inverse Problems **21** (2005), 1791–1799.

[47] J. Zhao and Q. Yang, *Self-adaptive projection methods for the multiple-sets split feasibility problem*, Inverse Problems **27** (2011), Article ID 035009.

[48] J. Zhao, Y. Zhang and Q. Yang, *Modified projection methods for the split feasibility problem and the multiple-sets split feasibility problem*, Applied Mathematics and Computation **219** (2012), 1644–1653.

Manuscript received January 10, 2014
revised July 10, 2014

Lu-Chuan Ceng
Department of Mathematics, Shanghai Normal University, and Scientific Computing Key Laboratory of Shanghai Universities, Shanghai 200234, China
E-mail address: zenglc@hotmail.com

Ngai-Ching Wong
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 804, Taiwan
E-mail address: wong@math.nsysu.edu.tw

Jen-Chih Yao
Center for General Education, China Medical University, Taichung 40402, Taiwan
E-mail address: yaojc@mail.cmu.edu.tw