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VIEWING ATTRACTIVE POINT SETS THROUGH THE KIRSZBRAUN-VALENTINE THEOREM

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday with admiration and respect.

Abstract. The concept of attractive points introduced by Takahashi et. al. [14] is considered via the Kirszbraun-Valentine theorem.

1. Introduction

Let C be a nonempty subset of a Hilbert space H and let $T : C \to H$ be a mapping. If $\|Tx - Ty\| \leq \|x - y\|$ for $x, y \in C$, then T is said to be *nonexpansive*. We denote by $F(T)$ the set of *fixed points* of T, i.e.,

$$F(T) = \{z \in C : Tz = z\}.$$

Takahashi et. al. [14] introduce the set of *attractive points* $A(T)$ of T as follow:

$$A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, x \in C\}.$$

For a family of mappings $\mathcal{S}$, let $F(\mathcal{S})$ and $A(\mathcal{S})$ stand for the set of common fixed points and the set of common attractive points of elements in $\mathcal{S}$, respectively. That is,

$$F(\mathcal{S}) = \bigcap_{T \in \mathcal{S}} F(T), \quad A(\mathcal{S}) = \bigcap_{T \in \mathcal{S}} A(T).$$

For each semigroup S, let $B(S)$ be the Banach space of all bounded real-valued mappings on S with supremum norm. A continuous linear functional $\mu \in B(S)^*$ (the dual space of $B(S)$) is called a *mean* on $B(S)$ if $\|\mu\| = \mu(1) = 1$. For any $f \in B(S)$, we use the following notation:

$$\mu(f) = \mu_s(f(s)).$$

A mean μ on $B(S)$ is said to be *left invariant* [respectively, *right invariant*] if $\mu_s(f(ts)) = \mu_s(f(s))$ [respectively, $\mu_s(f(st)) = \mu_s(f(s))$] for all $f \in B(S)$ and for all $t \in S$. We will say that μ is an *invariant mean* if it is both left and right invariants. If $B(S)$ has an invariant mean, we call S an *amenable semigroup*. It is well known that every commutative semigroup is amenable [6]. For each $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ by $(l_s f)(t) = f(st)$ and

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$(r_sf)(t) = f(ts)$ for any $t \in S$, respectively. A net $\{\mu_\alpha\}$ of means on $B(S)$ is said to be *asymptotically invariant* if for each $f \in B(S)$,

$$\lim_\alpha(\mu_\alpha(l_sf) - \mu_\alpha(f)) = 0 = \lim_\alpha(\mu_\alpha(r_sf) - \mu_\alpha(f)).$$

From now on, let S be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous.

Let C be a nonempty subset of a Hilbert space H. A family $\mathcal{S} = \{T_s : s \in S\}$ of mappings on C is said to be a *nonexpansive semigroup* on C into itself if it satisfies the following conditions:

(i) for each $s \in S$, T_s is nonexpansive,
(ii) $T_{ts} = T_tT_s$ for each $t, s \in S$,
(iii) for each $x \in C$, $s \mapsto T_sx$ is continuous.

Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C. Assume $\{T_sx : s \in S\}$ is bounded for each $x \in C$. Then, for any mean μ on $C(S)$, the space of bounded real valued continuous functions on S under supremun norm, and $x \in C$, there exists a unique $x_0 \in C$ such that

$$\mu_s \langle T_sx, y\rangle = \langle x_0, y\rangle$$

for all $y \in H$. Putting $T_\mu x = x_0$ for all $x \in C$.

The following facts [(I), (II)] suggest that $A(T)$ can play the role of $F(T)$. The aim of this paper is then to show that these two concepts are closely related. First we collect some facts on the two concepts.

(I) *Some facts on $F(T)$.*

Theorem 1.1 ([5, Theorem 8]). *Let C be a nonempty subset of a Hilbert space H. Let S be a semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C. Then $F(\mathcal{S})$ is a closed convex subset of H.*

Theorem 1.2 ([8, Theorem 2]). *Let C be a closed convex subset of a Hilbert space H and S a semitopological semigroup such that $C(S)$ has a left invariant mean. Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C. Then the followings are equivalent:*

(i) *$\{T_sx : s \in S\}$ is bounded for some $x \in C$;*
(ii) *$\{T_sx : s \in S\}$ is bounded for all $x \in C$;*
(iii) *$F(\mathcal{S}) = \bigcap_{s\in S} F(T_s) \neq \emptyset$.*

Theorem 1.3 ([12]). *Let S be an amenable semigroup, C a closed convex subset of a Hilbert space H, $\mathcal{S}$ a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{\mu_\alpha\}$ be an asymptotically invariant net of means. Then for each $u \in C$, $\{T_{\mu_\alpha}u\}$ converges weakly to u_0 in $F(\mathcal{S})$ where $u_0 = \lim_s \pi_{F(\mathcal{S})}T_su$ and $\pi_{F(\mathcal{S})}$ is the metric projection from H onto $F(\mathcal{S})$.*

Theorem 1.3 is extendable to CAT(0) spaces:

Theorem 1.4 ([1, Theorem 3.9]). *Let X be a complete* CAT(0) *space that has* Property (N)*, C be a closed convex subset of X, S a commutative semigroup, and*

$\mathcal{S} = \{T_s : s \in S\}$ *a nonexpansive semigroup on* C *with* $F(\mathcal{S}) \neq \emptyset$*. Suppose* $\{\mu_\alpha\}$ *is an asymptotically invariant net of means on* $B(S)$ *satisfying condition:*

$$\mu_{\alpha_s}(d^2(T_s x, y)) - \mu_{\alpha_s}(d^2(T_{st} x, y)) \to 0 \text{ uniformly for } y \in C. \tag{1.1}$$

Then $\{T_{\mu_\alpha x}\}$ Δ*-converges to* $x_0 = \lim_s \pi_{F(\mathcal{S})} T_s x$ *in* $F(\mathcal{S})$ *for all* $x \in C$.

In the Hilbert space setting, Property (N) and condition (1.1) are always satisfied.

Theorem 1.5 ([13, Theorem 1])**.** *Let* C *be a closed convex subset of a Hilbert space* H *and* S *a semitopological semigroup such that* $C(S)$ *has an invariant mean. Let* $\mathcal{S} = \{T_s : s \in S\}$ *be a nonexpansive semigroup on* C *with* $F(\mathcal{S}) \neq \emptyset$*. Then* T_μ *satisfies the followings:*

(i) $T_\mu T_s = T_s T_\mu = T_\mu$ *for all* $s \in S$,
(ii) T_μ *is a nonexpansive retraction of* C *onto* $F(\mathcal{S})$,
(iii) $T_\mu x \in \overline{co}\,\{T_s x : s \in S\}$ *for all* $x \in C$.

Theorem 1.6. [10, Lemma 3.2] *Let* C *be a closed convex subset of a Hilbert space* H *and* S *a left reversible semitopological semigroup, i.e., any two closed right ideals of* S *have nonvoid intersection. Let* $\mathcal{S} = \{T_s : s \in S\}$ *be a nonexpansive semigroup on* C*. Suppose that* $\{T_s x : s \in S\}$ *is bounded for some* $x \in C$*. Then* $F(\mathcal{S}) \neq \emptyset$.

(II) *Some facts on* $A(T)$.

Theorem 1.7 ([14, Lemma 2.3])**.** *Let* H *be a Hilbert space, let* C *be a nonempty subset of* H*, and let* T *be a mapping from* C *into* H*. Then* $A(T)$ *is a closed and convex subset of* H.

Theorem 1.8 ([15, Lemma 2.4])**.** *Let* H *be a Hilbert space, let* C *be a nonempty subset of* H*, and let* T *be a quasi-nonexpansive mapping from* C *into* H*. Then* $A(T) \cap C = F(T)$.

Theorem 1.9 ([2, Lemma 3.1])**.** *Let* C *be a nonempty closed convex subset of Hilbert space* H*,* S *a commutative semigroup and* $\mathcal{S} = \{T_s : s \in S\}$ *a nonexpansive semigroup on* C*. Then, if* $A(\mathcal{S}) \neq \emptyset$*,* $F(\mathcal{S}) \neq \emptyset$.

Theorem 1.10 ([2, Theorem 4.1])**.** *Let* H *be a Hilbert space, let* C *be a nonempty subset of* H*. Let* S *be a commutative semigroup and let* $\mathcal{S} = \{T_s : s \in S\}$ *be a nonexpansive semigroup on* C *into itself such that* $\{T_s x : s \in S\}$ *is bounded for some* $x \in C$*. Let* $\{\mu_\alpha\}$ *be a strongly asymptotically invariant net of means on* $C(S)$*, i.e.,*

$$\lim_\alpha \|\mu_\alpha - l_s^* \mu_\alpha\| = 0,$$

where l_s^* *is the adjoint operator of* l_s*. Then, the followings hold:*

(i) $A(\mathcal{S})$ *is nonempty, closed and convex,*
(ii) *for any* $u \in C$*,* $\{T_{\mu_\alpha} u\}$ *converges weakly to* u_0 *in* $A(\mathcal{S})$*, where* $u_0 = \lim_s \pi_{A(\mathcal{S})} T_s u$.

We ask if we can extend the given mapping T to a closed and convex domain having $A(T)$ as its fixed point set and results from (I) can be applied to (II) using only assumptions for T on C. Fortunately, the answer is "affirmative", thanks to the Kirszbraun-Valentine Theorem.

Theorem 1.11 ([16, Kirszbraun-Valentine Theorem (KV)]). *Let C be an arbitrary nonempty subset of a Hilbert space H and let $T : C \to H$ be nonexpansive. Then there exists a nonexpansive extension $\overline{\overline{T}}$ of T for which $\overline{\overline{T}} : H \to \overline{co}\, T(C)$.*

2. Preliminaries

A short proof of (KV) can be found in [3] using the Minty's surjectivity theorem. See [7] for a simpler proof. Several attempt aimed on extending (KV) to more general spaces. For example, [9] obtained (KV) on CAT(k) spaces under some conditions.

In general, however, (KV) may fail in CAT(0) spaces:

Example 2.1. *Let (X, ρ) be the gluing of two CAT(0) spaces $\mathbb{R}^2 \times \{0\}$ and $\{(0,0)\} \times \mathbb{R}$ at the point $(0,0,0)$. Since gluing of CAT(0) spaces alongs a convex subset yields a CAT(0) space by the Basic Gluing Theorem in* [4, page 347], *(X, ρ) is a CAT(0) space. By Lemma 5.24 in* [4, page 67], *the distance ρ is given by*

$$\rho(x,y) = \begin{cases} |x-y| & x, y \in \mathbb{R}, \\ |x-y| & x, y \in \{(0,0)\} \times \mathbb{R}, \\ |x| + |y| & \text{otherwise.} \end{cases}$$

Next consider points $x = (0,0,1)$, $y = (0,0,-1)$, and $z = (1,0,0)$. It is easy to see that $\rho(x,y) = \rho(x,z) = \rho(y,z) = 2$. Moreover, the intersection of closed balls

$$\overline{B}(x,1) \cap \overline{B}(y,1) \cap \overline{B}(z,1) = \{(0,0,0)\} \neq \emptyset.$$

If we consider points $u = (0,0,0), v = (2,0,0), w = (1, \sqrt{3}, 0)$, we see that $\rho(u,v) = \rho(u,w) = \rho(v,w) = 2$ since the restriction of ρ on $\mathbb{R}^2 \times \{0\}$ is simply the Euclidean distance. However, the intersection

$$\overline{B}(u,1) \cap \overline{B}(v,1) \cap \overline{B}(w,1) = \emptyset.$$

Let $T : \{x, y, z\} \to \{u, v, w\}$ be a bijection. Then T is actually an isometry. Clearly, there is no way to extend T to a nonexpensive map $\overline{\overline{T}} : co(x,y,z) \to X$ because if such $\overline{\overline{T}}$ were to exist, we would have $\rho(T(a), \overline{\overline{T}}(0,0,0)) \leq \rho(a,(0,0,0)) = 1$ for all $a \in \{x,y,z\}$ and hence $\overline{\overline{T}}(0,0,0) \in \overline{B}(u,1) \cap \overline{B}(v,1) \cap \overline{B}(w,1)$ which leads to a contradiction.

In the above example, we can verify directly that (X, ρ) is a CAT(0) space, i.e., it satisfies the CN-inequality: For points $x, y, w \in X$, if $z = \frac{x \oplus y}{2}$ is the mid-point of x and y, there holds

$$\rho^2(w,z) \leq \frac{1}{2}\rho^2(w,x) + \frac{1}{2}\rho^2(w,y) - \frac{1}{4}\rho^2(x,y). \tag{2.1}$$

We only consider only the following cases.

Case 1 [$x = (a,b,0)$, $y = (c,d,0)$, $w = (0,0,u)$]: We know that $|z|^2 = \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 - \frac{1}{4}|x-y|^2$ and $2|z| \leq |x| + |y|$, thus

$$\begin{aligned} \rho^2(w,z) &= (|w| + |z|)^2 \\ &= |w|^2 + |z|^2 + 2|w|\,|z| \end{aligned}$$

$$
\begin{aligned}
&= |w| + \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 - \frac{1}{4}|x-y|^2 + 2|w||z| \\
&\leq |w|^2 + \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 - \frac{1}{4}|x-y|^2 + |w|(|x|+|y|) \\
&= \frac{1}{2}(|w|+|x|)^2 + \frac{1}{2}(|w|+|y|)^2 - \frac{1}{4}|x-y|^2 .
\end{aligned}
$$

Case 2 $[x=(0,0,a),\ y=(b,c,0)]$: Since the segment $[x,y]$ is an R-tree on which the CN-inequality holds, we only consider for the case when w is of the form $(d,e,0)$. *Case 2.1* $z \in [0,x]$: Under this case, we consider the point w' on the ray starting from the origin in the direction of y, and $|w'| = |w|$. Clearly, $|y-w'| \leq |w-y|$. Now, as being an R-tree comprising of the segment $[x,y] \cup [x,w']$, (2.1) implies

$$
\begin{aligned}
\rho^2(w,z) &= (|w|+|z|)^2 \\
&= (|w'|+|z|)^2 \\
&= \rho^2(w',z) \\
&\leq \frac{1}{2}\rho^2(w',x) + \frac{1}{2}\rho^2(w',y) - \frac{1}{4}\rho^2(x,y) \\
&= \frac{1}{2}(|w'|+|x|)^2 + \frac{1}{2}|w'-y|^2 - \frac{1}{4}\rho^2(x,y) \\
&\leq \frac{1}{2}\rho^2(w,x) + \frac{1}{2}\rho^2(w,y) - \frac{1}{4}\rho^2(x,y).
\end{aligned}
$$

Case 2.2 $z \in [0,y]$: For this case we consider the point x' on the ray starting from the origin in the direction of $-y$ having $|x'| = |x|$. Being a CAT(0) space of $\mathbb{R}^2 \times \{0\}$, a similar computation as in *Case 2.1* giving us an inequality :

$$
\rho^2(w,z) \leq \frac{1}{2}\rho^2(w,x) + \frac{1}{2}\rho^2(w,y) - \frac{1}{4}\rho^2(x,y).
$$

Thus, the CN-inequality holds for both subcases.

Observe that, in the above example, the space (X,ρ) comprises of two spaces, one is an R-tree, the other is a Hilbert space, and (KV) holds on both spaces. This space (X,ρ) is an example of the following gluing spaces: For two metric spaces (Y,d) and (Z,η) where $d=\eta$ on $Y \cap Z$. Suppose that $A = Y \cap Z$ is a nonempty closed subset of (Y,d) and (Z,η). Let $X = Y \cup Z$ and define a metric ρ on X by $\rho = d$ on Y, $\rho = \eta$ on Z, and for $y \in Y$ and $z \in Z$, let $\rho(y,z) = \inf\{d(y,w) + \eta(w,z)\}$ where the infimum is taken over all $w \in A$. Since A is closed, ρ is indeed a metric on X. Call X a gluing space of Y and Z.

Proposition 2.2. *Let (X,ρ) be a gluing space of (Y,d) and (Z,η) and let $A = Y \cap Z$ be nonempty and closed in Y and Z. Let $T : E \to X$ be a nonexpansive mapping on a subset E of X and $A \subset E$. If* (KV) *holds on (Y,d) and (Z,η), $T(E \cap Y) \subset Y$, and $T(E \cap Z) \subset Z$, then T can be extended to a nonexpansive mapping on X.*

Proof. Extend T to the domain X by extending T on Y and Z respectively, and call the extension also as T. We show that T is nonexpansive. Let $y \in Y$ and $z \in Z$. Take any $w \in A$ and consider an estimate :

$$
\rho(Ty,Tz) \leq \rho(Ty,Tw) + \rho(Tw,Tz)
$$

$$\leq d(y, w) + \eta(w, z),$$

to conclude that $\rho(Ty, Tz) \leq \rho(y, z)$, and this completes the proof. □

3. Main results

Let C be a nonempty subset of a Hilbert space H and $T : C \to H$ be nonexpansive. Let K be the closed convex hull of the image $T(C)$. In a proof of (KV), we first obtain an extension $\overline{T} : H \to H$ and then put $\overline{\overline{T}} = \pi_K \overline{T} : H \to K$ where π_K is the metric projection onto K. Thus

$$A(T) = \{z \in H : \|z - Tx\| \, \|z - x\|, \, x \in C\},$$
$$A(\overline{T}) = \left\{z \in H : \left\|z - \overline{T}x\right\| \|z - x\|, \, x \in H\right\},$$
$$A(\overline{\overline{T}}) = \left\{z \in H : \left\|z - \overline{\overline{T}}x\right\| \|z - x\|, \, x \in H\right\}.$$

The attractive point sets and the Kirszbraun-Valentine theorem are related. In fact we have

$$A(T) = \bigcup F(\overline{T}) \tag{3.1}$$

where the union is taken over all extensions $\overline{T} : E \to X$ of T over a subset E of X. Obviously, one of $\overline{T}$ in (3.1) is the mapping $U : C \cup A(T) \to H$ defined by putting $U = T$ on C and U to be the identity mapping on $A(T)$. Clearly, by the definition of the attractive point set $A(T)$, U is a nonexpansive extension of T having $A(T) = F(U)$. By the proof of (KV) outlined above, we can extend U to obtain a nonexpansive extension $\overline{T} : H \to H$ with $F(\overline{T}) = A(T)$. Thus we have:

Theorem 3.1. *For any nonexpansive mapping $T : C \to C$, there exists a nonexpansive extension $\overline{T} : H \to H$ of T having $A(T) = A(\overline{T}) = F(\overline{T}) \supset F(\overline{\overline{T}}) = A(\overline{\overline{T}}) = A(T) \cap K = \pi_K(A(T))$, and all sets are closed and convex. Here $\overline{\overline{T}} = \pi_K \overline{T}$.*

Proof. We need only to prove that (i) $F(\overline{T}) \supset F(\overline{\overline{T}})$ and (ii) $A(T) \cap K = \pi_K(A(T))$.

(i): Let $z = \overline{\overline{T}}z$ and note that $z = \pi_K \overline{T}(z) \in K$ and for every $v \in C$, $Tv = \overline{T}v = \overline{\overline{T}}v$ which implies $\left\|Tv - \overline{\overline{T}}z\right\| = \left\|\overline{\overline{T}}v - \overline{\overline{T}}z\right\| \leq \|v - z\|$. If $\overline{T}z \in K$, then $z = \overline{\overline{T}}(z) = \overline{T}(z)$ and thus $z \in F(\overline{T})$. Next we show that this is the only possibility. Otherwise we choose $v \in C$ so that $\|z - v\|^2 < d(z, C)^2 + \left\|z - \overline{T}z\right\|^2 \leq \|Tv - z\|^2 + \left\|z - \overline{T}z\right\|^2$, where $d(z, C)$ is the distance from z to C. Therefore $\left\|Tv - \overline{T}z\right\|^2 \leq \|v - z\|^2 < \|Tv - z\|^2 + \left\|z - \overline{T}z\right\|^2$ a contradiction.

(ii): It suffices to show that $\pi_K(z) \in F(\overline{T})$ for $z \in F(\overline{T})$. For this we take $z \in F(\overline{T})$, and observe that $\left\|\overline{T}\pi_K(z) - z\right\| = \left\|\overline{T}\pi_{K_0}(z) - \overline{T}z\right\| \leq \|\pi_K(z) - z\|$ and thus $\overline{T}\pi_K(z) = \pi_K(z)$. □

In the following, for a given T, the mappings $\overline{T}$ and $\overline{\overline{T}}$ will stand for the corresponding mappings obtained from T in Theorem 3.1.

By Theorem 1.9, we have

Corollary 3.2. $A(T) \neq \emptyset \implies A(\overline{T}) \neq \emptyset \implies A(\overline{\overline{T}}) \neq \emptyset$.

Corollary 3.3. *If C is a nonempty subset of a Hilbert space H, $T : C \to C$ is nonexpansive, and $T(C)$ is bounded, then $A(T) \cap K$ is nonempty.*

Proof. Just consider the restriction $T : T(C) \to T(C)$ to obtain a fixed point of $\overline{\overline{T}}$. □

The following results are some examples which are consequences of our main result. It is noted that the conditions given on the mappings $T's$ and their domains in each theorem are sufficient for their extensions $\overline{T}'s$ and $\overline{\overline{T}}'s$.

Theorem 3.4 ([2, Lemma 3.2])**.** *Let C be a nonempty subset of a Hilbert space H. Let S be a commutative semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C. Then $A(\mathcal{S})$ is a closed convex subset of H.*

Theorem 3.5. *Let C be a nonempty subset of a Hilbert space H and S a semitopological semigroup such that $C(S)$ has a left invariant mean. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C. Then the followings are equivalent:*

(i) *$\{T_s x : s \in S\}$ is bounded for some $x \in C$;*
(ii) *$\{T_s x : s \in S\}$ is bounded for all $x \in C$;*
(iii) *$A(\mathcal{S}) \neq \emptyset$.*

Theorem 3.6. *Let C be a nonempty subset of a Hilbert space H and S a left reversible semitopological semigroup. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C into itself. Suppose that $\{T_t x : t \in S\}$ is bounded for some $x \in C$. Then $A(\overline{\overline{\mathcal{S}}}) \neq \emptyset$. Here $\overline{\overline{\mathcal{S}}} = \left\{\overline{\overline{T}} : T \in \mathcal{S}\right\}$.*

Theorem 3.7 ([2, Theorem 4.1])**.** *Let C be a nonempty subset of a Hilbert space H. Let S be a commutative semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\{T_t x : t \in S\}$ is bounded for some $x \in C$. Let $\{\mu_\alpha\}$ be a strongly asymptotically invariant net of means on $C(S)$, i.e., a net of means on $C(S)$ such that*

$$\lim_{\alpha} \|\mu_\alpha - l_s^* \mu_\alpha\| = 0,$$

Then, for any $u \in C$, $\{T_{\mu_\alpha} u\}$ converges weakly to $u_0 \in A(\overline{\mathcal{S}})$ (or $A(\overline{\overline{\mathcal{S}}})$, where $u_0 = \lim_t \pi_{A(\overline{\mathcal{S}})} T_t u = \lim_t \pi_{A(\overline{\overline{\mathcal{S}}})} T_t u$. Here $\overline{\mathcal{S}} = \left\{\overline{T} : T \in \mathcal{S}\right\}$.

4. Open problems

1. Observe that the following result is not a consequence of our main theorem (Theorem 3.1):

Theorem 4.1 ([14])**.** *Let C be a nonempty subset of a Hilbert space H and let $T : C \to C$ be a generalized hybrid mapping. Then, $A(T) \neq \emptyset$ if and only if there exists an $x_0 \in C$ such that $\{T^n x_0\}$ is bounded.*

Thus it is natural to ask if the Kirszbraun-Valentine Theorem can be extended to a more general class of mappings.

2. It is also interesting to extend the Kirszbraun-Valentine Theorem to a more general space.

3. Extend Theorem 1.4 and Theorem 1.5 to cover the case where C is not necessary convex.

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