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HIGHER-ORDER DAEHEE OF THE SECOND KIND AND POLY-CAUCHY OF THE SECOND KIND MIXED-TYPE POLYNOMIALS

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Dedicated to Professor W. Takahashi on the occasion of his 70th birthday

ABSTRACT. In this paper, we consider higher-order Daehee of the second kind and poly-Cauchy of the second kind mixed-type polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

1. Introduction

In this paper, we consider the polynomials $\widehat{D}_n^{(r,k)}(x)$ whose generating function is given by

$$\left(\frac{(1+t)\ln(1+t)}{t}\right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^x=\sum_{n=0}^{\infty}\widehat{D}_n^{(r,k)}(x)\frac{t^n}{n!} \quad (k,r\in\mathbb{Z}_{>0})\,. \tag{1.1}$$

Recall that the Daehee polynomials of the second kind of order r, denoted by $\widehat{D}_n^{(r)}(x)$, is given by the generating function to be

$$\left(\frac{(1+t)\ln(1+t)}{t}\right)^r (1+t)^x=\sum_{n=0}^{\infty}\widehat{D}_n^{(r)}(x)\frac{t^n}{n!}\,.$$

Dahee polynomials were defined by the second author [5] and have been investigated in [3, 4, 6]. Also, the poly-Cauchy polynomials of the second kind $\widehat{c}_n^{(k)}(x)$ (of index k) are defined by the generating function as

$$\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^x=\sum_{n=0}^{\infty}\widehat{c}_n^{(k)}(x)\frac{t^n}{n!}\,,$$

where $\mathrm{Lif}_k(x)$ ($k\in\mathbb{Z}_{>0}$) is the polyfactorial function given by

$$\mathrm{Lif}_k(x)=\sum_{n=0}^{\infty}\frac{x^m}{m!(m+1)^k}\,,$$

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Poly-Cauchy polynomials of the first kind and of the second kind were defined by the third author [1] and have been investigated in [1].

In this paper, we consider higher-order Daehee of the second kind and poly-Cauchy of the second kind mixed-type polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2. Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F}=\left\{ f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k \middle| a_k\in\mathbb{C}\right\}. \tag{2.1}$$

Let $\mathbb{P}=\mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle=\langle L|p(x)\rangle+\langle M|p(x)\rangle$, $\langle cL|p(x)\rangle=c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t)\in\mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle=a_n,\quad (n\geq 0). \tag{2.2}$$

In particular,

$$\left\langle t^k|x^n\right\rangle=n!\delta_{n,k}\quad (n,k\geq 0), \tag{2.3}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $f_L(t)=\sum_{k=0}^{\infty}\frac{\langle L|x^k\rangle}{k!}t^k$, we have $\langle f_L(t)|x^n\rangle=\langle L|x^n\rangle$. That is, $L=f_L(t)$. The map $L\mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O\big(f(t)\big)$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O\big(f(t)\big)=1$, then $f(t)$ is called a *delta series*; if $O\big(f(t)\big)=0$, then $f(t)$ is called an *invertible series*. For $f(t),g(t)\in\mathcal{F}$ with $O\big(f(t)\big)=1$ and $O\big(g(t)\big)=0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x)=n$) such that $\left\langle g(t)f(t)^k|s_n(x)\right\rangle=n!\delta_{n,k}$, for $n,k\geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $\big(g(t),f(t)\big)$ which is denoted by $s_n(x)\sim\big(g(t),f(t)\big)$.

For $f(t),g(t)\in\mathcal{F}$ and $p(x)\in\mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle=\langle f(t)|g(t)p(x)\rangle=\langle g(t)|f(t)p(x)\rangle \tag{2.4}$$

and

$$f(t)=\sum_{k=0}^{\infty}\left\langle f(t)|x^k\right\rangle\frac{t^k}{k!},\quad p(x)=\sum_{k=0}^{\infty}\left\langle t^k|p(x)\right\rangle\frac{x^k}{k!} \tag{2.5}$$

([8, Theorem 2.2.5]). Thus, by (2.5), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt}p(x) = p(x+y). \tag{2.6}$$

Sheffer sequences are characterized in the generating function ([8, Theorem 2.3.4]).

Lemma 2.1. *The sequence $s_n(x)$ is Sheffer for $\big(g(t), f(t)\big)$ if and only if*

$$\frac{1}{g\big(\bar{f}(t)\big)} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim \big(g(t), f(t)\big)$, we have the following equations ([8, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]):

$$f(t)s_n(x) = n s_{n-1}(x) \quad (n \geq 0), \tag{2.7}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\big(\bar{f}(t)\big)^{-1} \bar{f}(t)^j | x^n \right\rangle x^j, \tag{2.8}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \tag{2.9}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim \big(1, f(t)\big)$ and $q_n(x) \sim \big(1, g(t)\big)$. Then the transfer formula ([8, Corollary 3.8.2]) is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim \big(g(t), f(t)\big)$ and $r_n(x) \sim \big(h(t), l(t)\big)$, assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0).$$

Then we have ([8, p.132])

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\big(\bar{f}(t)\big)}{g\big(\bar{f}(t)\big)} l\big(\bar{f}(t)\big)^m \middle| x^n \right\rangle. \tag{2.10}$$

3. Main results

From (1.1), we see that $\widehat{D}_n^{(r,k)}(x)$ is the Sheffer sequence for the pair

$$\left(g(t) = \left(\frac{e^t - 1}{te^t}\right)^r \frac{1}{\mathrm{Lif}_k(-t)}, f(t) = e^t - 1 \right).$$

So,

$$\widehat{D}_n^{(r,k)}(x) \sim \left(\left(\frac{e^t - 1}{te^t}\right)^r \frac{1}{\mathrm{Lif}_k(-t)}, e^t - 1 \right). \tag{3.1}$$

Remark 3.1. *$\widehat{D}_n^{(r,k)}(x)$ will be called the higher-order Daehee of the second kind and poly-Cauchy of the second kind mixed-type polynomials. When $x=0$, $\widehat{D}_n^{(r,k)} = D_n^{(r,k)}(0)$ are called the higher-order Daehee of the second kind and poly-Cauchy of the second kind mixed-type numbers.*

3.1. **Explicit expressions.** First, we express $\widehat{D}_n^{(r,k)}(x)$ in terms of the Stirling numbers of the first kind and the higher-order Bernoulli polynomials, poly-Cauchy numbers of the second kind and poly-Cauchy polynomials of the second kind. Bernoulli polynomials $B_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty}\frac{B_n^{(r)}(x)}{n!}t^n \tag{3.2}$$

(see e.g. [8, Section 2.2]).

Theorem 3.2.

$$\widehat{D}_n^{(r,k)}(x) = \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}B_l^{(r)}(x+r)\,, \tag{3.3}$$

$$\widehat{D}_n^{(r,k)}(x) = \sum_{j=0}^{n}\left(\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{D}_l^{(r,k)}\right)x^j \tag{3.4}$$

$$= \sum_{j=0}^{n}\left(\sum_{l=0}^{n-j}\sum_{m=0}^{l}\binom{n}{l}\binom{l}{m}S_1(n-l,j)\widehat{c}_m^{(k)}\widehat{D}_{l-m}^{(r)}\right)x^j\,, \tag{3.5}$$

$$\widehat{D}_n^{(r,k)}(x) = \sum_{l=0}^{n}\binom{n}{l}\widehat{D}_{n-l}^{(r)}\widehat{c}_l^{(k)}(x)\,, \tag{3.6}$$

$$\widehat{D}_n^{(r,k)}(x) = \sum_{l=0}^{n}\binom{n}{l}\widehat{c}_{n-l}^{(k)}\widehat{D}_l^{(r)}(x)\,. \tag{3.7}$$

Proof. Denote $(x)_n$ the falling factorial defined by $(x)_n = x(x-1)\cdots(x-n+1)$ $(n\ge 1)$ with $(x)_0=1$. Notice that

$$\left(\frac{e^t-1}{te^t}\right)^r\frac{1}{\mathrm{Lif}_k(-t)}\widehat{D}_n^{(r,k)}(x) \sim (1, e^t-1)\,, \tag{3.8}$$

$$(x)_n = \sum_{m=0}^{n}S_1(n,m)x^m \sim (1,e^t-1)\,, \tag{3.9}$$

where $S_1(n,m)$ are the Stirling numbers of the first kind.

So,

$$\begin{aligned}\widehat{D}_n^{(r,k)}(x) &= \left(\frac{te^t}{e^t-1}\right)^r\mathrm{Lif}_k(-t)(x)_n\\ &= \sum_{m=0}^{n}S_1(n,m)\left(\frac{te^t}{e^t-1}\right)^r\mathrm{Lif}_k(-t)x^m\end{aligned}$$

$$= \sum_{m=0}^{n} S_1(n,m) \left(\frac{te^t}{e^t-1}\right)^r \sum_{l=0}^{m} \frac{(-1)^l}{l!(l+1)^k} t^l x^m$$

$$= \sum_{m=0}^{n} S_1(n,m) \left(\frac{te^t}{e^t-1}\right)^r \sum_{l=0}^{m} \frac{(-1)^l \binom{m}{l}}{(l+1)^k} x^{m-l}$$

$$= \sum_{m=0}^{n} S_1(n,m) \sum_{l=0}^{m} \frac{(-1)^l \binom{m}{l}}{(l+1)^k} e^{rt} \left(\frac{t}{e^t-1}\right)^r x^{m-l}$$

$$= \sum_{m=0}^{n} S_1(n,m) \sum_{l=0}^{m} \frac{(-1)^l \binom{m}{l}}{(l+1)^k} B_{m-l}^{(r)}(x+r)$$

$$= \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(n,m) \frac{(-1)^l \binom{m}{l}}{(l+1)^k} B_{m-l}^{(r)}(x+r)$$

$$= \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(n,m) \frac{(-1)^{m-l} \binom{m}{l}}{(m-l+1)^k} B_{l}^{(r)}(x+r)\,.$$

So, we get the identity (3.3).

Next, we shall use the conjugation formula (2.8) for (3.1). Since $\bar{f}(t) = \ln(1+t)$ and

$$g\big(\bar{f}(t)\big)^{-1} = \left(\frac{(1+t)\ln(1+t)}{t}\right)^r \operatorname{Lif}_k\big(-\ln(1+t)\big)\,,$$

we get

$$\left\langle g\big(\bar{f}(t)\big)^{-1} \bar{f}(t)^j \middle| x^n \right\rangle$$

$$= \left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^r \operatorname{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^j \middle| x^n \right\rangle$$

$$= \left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^r \operatorname{Lif}_k\big(-\ln(1+t)\big) \middle| \sum_{l=0}^{n-j} \frac{j!}{(l+j)!} S_1(l+j,j) t^{l+j} x^n \right\rangle$$

$$= \sum_{l=0}^{n-j} \frac{j!}{(l+j)!} S_1(l+j,j)(n)_{l+j}$$

$$\times \left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^r \operatorname{Lif}_k\big(-\ln(1+t)\big) \middle| x^{n-l-j} \right\rangle$$

$$= \sum_{l=0}^{n-j} \frac{j!}{(l+j)!} S_1(l+j,j)(n)_{l+j} \left\langle \sum_{i=0}^{\infty} \widehat{D}_i^{(r,k)} \frac{t^i}{i!} \middle| x^{n-l-j} \right\rangle$$

$$= \sum_{l=0}^{n-j} \frac{j!}{(l+j)!} S_1(l+j,j)(n)_{l+j} \widehat{D}_{n-l-j}^{(r,k)}\,.$$

Thus,

$$\widehat{D}_n^{(r,k)}(x) = \sum_{j=0}^{n} \left(\sum_{l=0}^{n-j} \binom{n}{l+j} S_1(l+j,j) \widehat{D}_{n-l-j}^{(r,k)} \right) x^j$$

$$= \sum_{j=0}^{n} \left(\sum_{l=0}^{n-j} \binom{n}{l} S_1(n-l,j) \widehat{D}_l^{(r,k)} \right) x^j ,$$

which is the identity (3.4).

In another direction, we have

$$\begin{aligned}
&\left\langle g\big(\bar{f}(t)\big)^{-1} \bar{f}(t)^j \Big| x^n \right\rangle \\
&= \sum_{l=0}^{n-j} \frac{j!}{(l+j)!} S_1(l+j,j)(n)_{l+j} \\
&\quad \times \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \Big| \mathrm{Lif}_k\big(-\ln(1+t)\big) x^{n-l-j} \right\rangle \\
&= \sum_{l=0}^{n-j} \frac{j!}{(l+j)!} S_1(l+j,j)(n)_{l+j} \\
&\quad \times \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \Big| \sum_{m=0}^{n-l-j} \widehat{c}_m^{(k)} \frac{t^m}{m!} x^{n-l-j} \right\rangle \\
&= \sum_{l=0}^{n-j} \frac{j!}{(l+j)!} S_1(l+j,j)(n)_{l+j} \sum_{m=0}^{n-l-j} \widehat{c}_m^{(k)} \frac{1}{m!} (n-l-j)_m \\
&\quad \times \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \Big| x^{n-l-j-m} \right\rangle \\
&= \sum_{l=0}^{n-j} \sum_{m=0}^{n-l-j} j! \binom{n}{l+j} \binom{n-l-j}{m} S_1(l+j,j) \widehat{c}_m^{(k)} \\
&\quad \times \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \Big| x^{n-l-j-m} \right\rangle \\
&= \sum_{l=0}^{n-j} \sum_{m=0}^{n-l-j} j! \binom{n}{l+j} \binom{n-l-j}{m} S_1(l+j,j) \widehat{c}_m^{(k)} \left\langle \sum_{i=0}^{\infty} \widehat{D}_i^{(r)} \frac{t^i}{i!} \Big| x^{n-l-j-m} \right\rangle \\
&= \sum_{l=0}^{n-j} \sum_{m=0}^{n-l-j} j! \binom{n}{l+j} \binom{n-l-j}{m} S_1(l+j,j) \widehat{c}_m^{(k)} \widehat{D}_{n-l-j-m}^{(r)} \\
&= \sum_{l=0}^{n-j} \sum_{m=0}^{l} j! \binom{n}{l} \binom{l}{m} S_1(n-l,j) \widehat{c}_m^{(k)} \widehat{D}_{l-m}^{(r)} .
\end{aligned}$$

Thus, we have the identity (3.5).

Next,

$$\begin{aligned}
\widehat{D}_n^{(r,k)}(y) &= \left\langle \sum_{i=0}^{\infty} \widehat{D}_i^{(r,k)}(y) \frac{t^i}{i!} \Big| x^n \right\rangle \\
&= \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| x^n \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \Big| \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y x^n \right\rangle \\
&= \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \Big| \sum_{l=0}^{n} \widehat{c}_l^{(k)}(y)\frac{t^l}{l!}x^n \right\rangle \\
&= \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \Big| \sum_{l=0}^{n} \widehat{c}_l^{(k)}(y)\binom{n}{l}x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \widehat{c}_l^{(k)}(y)\binom{n}{l} \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \widehat{c}_l^{(k)}(y)\binom{n}{l} \left\langle \sum_{i=0}^{\infty} \widehat{D}_i^{(r)}\frac{t^i}{i!} \Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} \widehat{D}_{n-l}^{(r)}\widehat{c}_l^{(k)}(y)\,.
\end{aligned}
$$

Therefore, we have the identity (3.6).

Finally,

$$
\begin{aligned}
\widehat{D}_n^{(r,k)}(y) &= \left\langle \sum_{i=0}^{\infty} \widehat{D}_i^{(r,k)}(y)\frac{t^i}{i!} \Big| x^n \right\rangle \\
&= \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| \left(\frac{(1+t)\ln(1+t)}{t} \right)^r (1+t)^y x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| \sum_{l=0}^{\infty} \widehat{D}_l^{(r)}(y)\frac{t^l}{l!}x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| \sum_{l=0}^{n} \widehat{D}_l^{(r)}(y)\binom{n}{l}x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \widehat{D}_l^{(r)}(y)\binom{n}{l} \left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \widehat{D}_l^{(r)}(y)\binom{n}{l} \left\langle \sum_{i=0}^{\infty} \widehat{c}_i^{(k)}\frac{t^i}{i!} \Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} \widehat{c}_{n-l}^{(k)}\widehat{D}_l^{(r)}(y)\,.
\end{aligned}
$$

Thus, we obtain the identity (3.7). □

3.2. **Sheffer identity.**

Theorem 3.3.

$$\widehat{D}_n^{(r,k)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} \widehat{D}_j^{(r,k)}(x)\,(y)_{n-j}\,. \tag{3.10}$$

Proof. We shall use (3.1) with

$$p_n(x) = \left(\frac{e^t-1}{te^t}\right)^r \frac{1}{\mathrm{Lif}_k(-t)} \widehat{D}_n^{(r,k)}(x) = (x)_n \sim (1, e^t-1)\,.$$

By (2.9), we get the identity (3.10). □

3.3. **Recurrence relation.**

Theorem 3.4.

$$\widehat{D}_n^{(r,k)}(x+1) - \widehat{D}_n^{(r,k)}(x) = n\widehat{D}_{n-1}^{(r,k)}(x)\,. \tag{3.11}$$

Proof. By (2.7) with (3.1), we have

$$(e^t-1)\widehat{D}_n^{(r,k)}(x) = n\widehat{D}_{n-1}^{(r,k)}(x)\,.$$

Namely, the identity (3.11) is obtained. □

3.4. **Recurrence.**

Theorem 3.5.

$$\begin{aligned}
\widehat{D}_{n+1}^{(r,k)}(x) &= (x+r)\widehat{D}_n^{(r,k)}(x-1) - r\sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(l+1)(m-l+1)^k} \\
&\quad\times \left(B_{l+1}^{(r+1)}(x+r) - B_{l+1}^{(r)}(x+r-1)\right) \\
&\quad - \sum_{m=0}^{n}\sum_{l=0}^{m} \frac{(-1)^l\binom{m}{l}}{(l+2)^k} S_1(n,m) B_{m-l}^{(r)}(x+r-1) \qquad (3.12) \\
&= (x+r)\widehat{D}_n^{(r,k)}(x-1) \\
&\quad - r\sum_{j=0}^{n}\frac{1}{j+1}\sum_{l=0}^{n-j}\binom{n}{l} S_1(n-l,j)\widehat{D}_l^{(r,k)}\left(B_{j+1}(x) - (x-1)^{j+1}\right) \\
&\quad - \sum_{m=0}^{n}\sum_{l=0}^{m} \frac{(-1)^l\binom{m}{l}}{(l+2)^k} S_1(n,m) B_{m-l}^{(r)}(x+r-1)\,. \qquad (3.13)
\end{aligned}$$

Proof. We shall use

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)} s_n(x)$$

([8, Corollary 3.7.2]) with (3.1). By (2.6),

$$\widehat{D}_{n+1}^{(r,k)}(x) = \left(x - \frac{g'(t)}{g(t)}\right) e^{-t}\widehat{D}_n^{(r,k)}(x)$$

$$= x\widehat{D}_n^{(r,k)}(x-1) - e^{-t}\frac{g'(t)}{g(t)}\widehat{D}_n^{(r,k)}(x)\,.$$

Now,

$$\begin{aligned}\frac{g'(t)}{g(t)} &= \big(\ln g(t)\big)' \\ &= \big(r\ln(e^t-1) - r\ln t - rt - \ln \mathrm{Lif}_k(-t)\big)' \\ &= r\frac{e^t}{e^t-1} - r\frac{1}{t} - r + \frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)} \\ &= r\frac{te^t-e^t+1}{t(e^t-1)} - r + \frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)}\,.\end{aligned}$$

So, by (2.6) again,

$$\begin{aligned}\widehat{D}_{n+1}^{(r,k)}(x) &= x\widehat{D}_n^{(r,k)}(x-1) + r\widehat{D}_n^{(r,k)}(x-1) \\ &\quad - re^{-t}\frac{te^t-e^t+1}{t(e^t-1)}\widehat{D}_n^{(r,k)}(x) - e^{-t}\frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)}\widehat{D}_n^{(r,k)}(x)\,. \end{aligned}\tag{3.14}$$

Since

$$\frac{te^t-e^t+1}{e^t-1} = \frac{1}{2}t + \frac{1}{12}t^2 - \cdots$$

is a delta series, by (3.3) we have

$$\begin{aligned}&\frac{te^t-e^t+1}{t(e^t-1)}\widehat{D}_n^{(r,k)}(x) \\ &= \frac{te^t-e^t+1}{t(e^t-1)}\left(\sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}B_l^{(r)}(x+r)\right) \\ &= \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}\frac{te^t-e^t+1}{t(e^t-1)}B_l^{(r)}(x+r) \\ &= \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}\frac{te^t-e^t+1}{e^t-1}\frac{B_{l+1}^{(r)}(x+r)}{l+1} \\ &= \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}\left(\frac{te^t}{e^t-1}-1\right)\frac{B_{l+1}^{(r)}(x+r)}{l+1} \\ &= \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}\frac{B_{l+1}^{(r+1)}(x+r+1)}{l+1} \\ &\quad - \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}\frac{B_{l+1}^{(r)}(x+r)}{l+1} \end{aligned}$$

$$= \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(l+1)(m-l+1)^k}\left(B_{l+1}^{(r+1)}(x+r+1) - B_{l+1}^{(r)}(x+r)\right)\tag{3.15}$$

or by (3.4)

$$\begin{aligned}
&\frac{te^t-e^t+1}{t(e^t-1)}\widehat{D}_n^{(r,k)}(x)\\
&=\frac{te^t-e^t+1}{t(e^t-1)}\sum_{j=0}^{n}\left(\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{D}_l^{(r,k)}\right)x^j\\
&=\sum_{j=0}^{n}\left(\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{D}_l^{(r,k)}\right)\frac{te^t-e^t+1}{t(e^t-1)}x^j\\
&=\sum_{j=0}^{n}\left(\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{D}_l^{(r,k)}\right)\left(\frac{te^t}{e^t-1}-1\right)\frac{x^{j+1}}{j+1}\\
&=\sum_{j=0}^{n}\frac{1}{j+1}\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{D}_l^{(r,k)}\left(B_{j+1}(x+1)-x^{j+1}\right). \qquad (3.16)
\end{aligned}$$

By substituting (3.15) and (3.16) into (3.14), respectively, using (2.6) we have

$$\begin{aligned}
\widehat{D}_{n+1}^{(r,k)}(x)&=x\widehat{D}_n^{(r,k)}(x-1)+r\widehat{D}_n^{(r,k)}(x-1)\\
&\quad-r\sum_{m=0}^{n}\sum_{l=0}^{m}S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(l+1)(m-l+1)^k}\left(B_{l+1}^{(r+1)}(x+r)-B_{l+1}^{(r)}(x+r-1)\right)\\
&\quad-e^{-t}\frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)}\widehat{D}_n^{(r,k)}(x)\\
&=x\widehat{D}_n^{(r,k)}(x-1)+r\widehat{D}_n^{(r,k)}(x-1)\\
&\quad-r\sum_{j=0}^{n}\frac{1}{j+1}\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{D}_l^{(r,k)}\left(B_{j+1}(x)-(x-1)^{j+1}\right)\\
&\quad-e^{-t}\frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)}\widehat{D}_n^{(r,k)}(x).
\end{aligned}$$

Since by (3.8) and (3.9)

$$\left(\frac{e^t-1}{te^t}\right)^r\frac{1}{\mathrm{Lif}_k(-t)}\widehat{D}_n^{(r,k)}(x)=\sum_{m=0}^{n}S_1(n,m)x^m,$$

we obtain

$$\begin{aligned}
\frac{1}{\mathrm{Lif}_k(-t)}\widehat{D}_n^{(r,k)}(x)&=\sum_{m=0}^{n}S_1(n,m)\left(\frac{te^t}{e^t-1}\right)^r x^m\\
&=\sum_{m=0}^{n}S_1(n,m)e^{rt}\left(\frac{t}{e^t-1}\right)^r x^m\\
&=\sum_{m=0}^{n}S_1(n,m)B_m^{(r)}(x+r).
\end{aligned}$$

So,

$$\begin{aligned}\frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)}\widehat{D}_n^{(r,k)}(x) &= \sum_{m=0}^{n} S_1(n,m)\mathrm{Lif}_k'(-t)B_m^{(r)}(x+r)\\ &= \sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l t^l}{l!(l+2)^k}B_m^{(r)}(x+r)\\ &= \sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l (m)_l}{l!(l+2)^k}B_{m-l}^{(r)}(x+r)\\ &= \sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+2)^k}S_1(n,m)B_{m-l}^{(r)}(x+r)\,.\end{aligned}$$

Altogether, we have

$$\begin{aligned}&\widehat{D}_{n+1}^{(r,k)}(x) = (x+r)\widehat{D}_n^{(r,k)}(x-1)\\ &\quad - r\sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(l+1)(m-l+1)^k}\left(B_{l+1}^{(r+1)}(x+r)-B_{l+1}^{(r)}(x+r-1)\right)\\ &\quad - \sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+2)^k}S_1(n,m)B_{m-l}^{(r)}(x+r-1)\\ &= (x+r)\widehat{D}_n^{(r,k)}(x-1)\\ &\quad - r\sum_{j=0}^{n}\frac{1}{j+1}\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{D}_l^{(r,k)}\left(B_{j+1}(x)-(x-1)^{j+1}\right)\\ &\quad - \sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+2)^k}S_1(n,m)B_{m-l}^{(r)}(x+r-1)\,.\end{aligned}$$

Therefore, we obtain the desired results. □

3.5. **Differentiation.**

Theorem 3.6.

$$\frac{d}{dx}\widehat{D}_n^{(r,k)}(x) = n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{D}_l^{(r,k)}(x)\,. \tag{3.17}$$

Proof. We shall use

$$\frac{d}{dx}s_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}\left\langle \bar{f}(t)|x^{n-l}\right\rangle s_l(x)$$

(*Cf.* [8, Theorem 2.3.12]) with (3.1). Hence,

$$\left\langle \bar{f}(t)|x^{n-l}\right\rangle = \left\langle \ln(1+t)|x^{n-l}\right\rangle$$

$$\begin{aligned}
&= \left\langle \sum_{m=1}^{\infty} \frac{(-1)^{m-1}t^m}{m} \Big| x^{n-l} \right\rangle \\
&= \sum_{m=1}^{n-l} \frac{(-1)^{m-1}}{m} \left\langle t^m | x^{n-l} \right\rangle \\
&= \sum_{m=1}^{n-l} \frac{(-1)^{m-1}}{m} (n-l)! \delta_{m,n-l} \\
&= (-1)^{n-l-1}(n-l-1)!\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{d}{dx}\widehat{D}_n^{(r,k)}(x) &= \sum_{l=0}^{n-1} \binom{n}{l} (-1)^{n-l-1}(n-l-1)! \widehat{D}_l^{(r,k)}(x) \\
&= n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} \widehat{D}_l^{(r,k)}(x)\,,
\end{aligned}$$

which is the identity (3.17). □

3.6. **A more recurrence relation.**

Theorem 3.7.

$$\begin{aligned}
\widehat{D}_n^{(r,k)}(x) &= \frac{n}{n+r} x \widehat{D}_{n-1}^{(r,k)}(x-1) + \frac{r-1}{n+r} \widehat{D}_n^{(r-1,k)}(x) + \frac{1}{n+r} \widehat{D}_n^{(r-1,k-1)}(x) \\
&\quad + \frac{r}{n+r} \sum_{l=1}^{n} (-1)^{l-1} (n)_l \widehat{D}_{n-l}^{(r,k)}(x)\,. \qquad (3.18)
\end{aligned}$$

Proof. For $n \geq 1$, we have

$$\begin{aligned}
\widehat{D}_n^{(r,k)}(y) &= \left\langle \sum_{l=0}^{\infty} \widehat{D}_l^{(r,k)}(y) \frac{t^l}{l!} \Big| x^n \right\rangle \\
&= \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| x^n \right\rangle \\
&= \left\langle \partial_t \left(\left(\frac{(1+t)\ln(1+t)}{t} \right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \right) \Big| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| x^{n-1} \right\rangle \\
&\quad + \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \big(\partial_t \mathrm{Lif}_k\big(-\ln(1+t)\big)\big) (1+t)^y \Big| x^{n-1} \right\rangle \\
&\quad + \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big) \big(\partial_t (1+t)^y\big) \Big| x^{n-1} \right\rangle \\
&= y \widehat{D}_{n-1}^{(r,k)}(y-1)
\end{aligned}$$

$$+\left\langle \left(\partial_t \left(\frac{(1+t)\ln(1+t)}{t}\right)^r\right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| x^{n-1}\right\rangle$$
$$+\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^r \left(\partial_t \mathrm{Lif}_k\big(-\ln(1+t)\big)\right)(1+t)^y \Big| x^{n-1}\right\rangle .$$

Since

$$\ln(1+t)+1-\frac{(1+t)\ln(1+t)}{t}=\frac{1}{2}t-\frac{1}{3}t^2+\cdots$$

is a delta series,

$$\left\langle \left(\partial_t \left(\frac{(1+t)\ln(1+t)}{t}\right)^r\right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| x^{n-1}\right\rangle$$
$$= r\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1}\right.$$
$$\left.\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| \frac{\ln(1+t)+1-\frac{(1+t)\ln(1+t)}{t}}{t} x^{n-1}\right\rangle$$
$$= r\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big|\right.$$
$$\left.\left(\ln(1+t)+1-\frac{(1+t)\ln(1+t)}{t}\right)\frac{x^n}{n}\right\rangle$$
$$= \frac{r}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big|\right.$$
$$\left.\left(\ln(1+t)+1-\frac{(1+t)\ln(1+t)}{t}\right) x^n\right\rangle$$
$$= \frac{r}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| \ln(1+t)x^n\right\rangle$$
$$+\frac{r}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| x^n\right\rangle$$
$$-\frac{r}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| x^n\right\rangle$$
$$= \frac{r}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| \frac{t}{1+t}x^n\right\rangle$$
$$+\frac{r}{n}\widehat{D}_n^{(r-1,k)}(y)-\frac{r}{n}\widehat{D}_n^{(r,k)}(y)$$
$$= \frac{r}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| \sum_{l=1}^{\infty}(-1)^{l-1}t^l x^n\right\rangle$$
$$+\frac{r}{n}\widehat{D}_n^{(r-1,k)}(y)-\frac{r}{n}\widehat{D}_n^{(r,k)}(y)$$

$$= \frac{r}{n}\sum_{l=1}^{n}(-1)^{l-1}(n)_l\widehat{D}_{n-l}^{(r,k)}(y) + \frac{r}{n}\widehat{D}_n^{(r-1,k)}(y) - \frac{r}{n}\widehat{D}_n^{(r,k)}(y)\,.$$

In addition,

$$\begin{aligned}
&\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^r \left(\partial_t \mathrm{Lif}_k\big(-\ln(1+t)\big)\right)(1+t)^y \middle| x^{n-1}\right\rangle \\
&= \left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^r \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{(1+t)\ln(1+t)}(1+t)^y \middle| x^{n-1}\right\rangle \\
&= \left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}(1+t)^y \middle| x^{n-1}\right\rangle \\
&= \left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} (1+t)^y \middle| \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t} x^{n-1}\right\rangle \\
&= \left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} (1+t)^y \middle| \left(\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)\right)\frac{x^n}{n}\right\rangle \\
&= \frac{1}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)(1+t)^y \middle| x^n\right\rangle \\
&\quad - \frac{1}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_{k}\big(-\ln(1+t)\big)(1+t)^y \middle| x^n\right\rangle \\
&= \frac{1}{n}\widehat{D}_n^{(r-1,k-1)}(y) - \frac{1}{n}\widehat{D}_n^{(r-1,k)}(y)\,.
\end{aligned}$$

Altogether, we get

$$\begin{aligned}
\left(1+\frac{r}{n}\right)\widehat{D}_n^{(r,k)}(y) &= y\widehat{D}_{n-1}^{(r,k)}(y-1) + \frac{r-1}{n}\widehat{D}_n^{(r-1,k)}(y) + \frac{1}{n}\widehat{D}_n^{(r-1,k-1)}(y) \\
&\quad + \frac{r}{n}\sum_{l=1}^{n}(-1)^{l-1}(n)_l\widehat{D}_{n-l}^{(r,k)}(y)\,.
\end{aligned}$$

Thus, we obtain the desired result. □

3.7. **Relations including the Stirling numbers of the first kind.**

Theorem 3.8. *For* $1 \le m \le n-1$,

$$\begin{aligned}
&\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{D}_l^{(r,k)} \\
&= \frac{r(m+1)}{n+r}\sum_{l=0}^{n-m-1}\binom{n}{l}S_1(n-l,m+1)\widehat{D}_l^{(r-1,k)} \\
&\quad + \frac{r-1}{n+r}\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{D}_l^{(r-1,k)}
\end{aligned}$$

$$+\frac{1}{n+r}\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{D}_l^{(r-1,k-1)}$$

$$+\frac{n}{n+r}\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l^{(r,k)}(-1)\,. \tag{3.19}$$

Proof. We shall compute the following in two different ways:

$$\left\langle\left(\frac{(1+t)\ln(1+t)}{t}\right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^n\right\rangle .$$

On the one hand,

$$\begin{aligned}
&\left\langle\left(\frac{(1+t)\ln(1+t)}{t}\right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^n\right\rangle\\
&=\left\langle\left(\frac{(1+t)\ln(1+t)}{t}\right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|\big(\ln(1+t)\big)^m x^n\right\rangle\\
&=\left\langle\left(\frac{(1+t)\ln(1+t)}{t}\right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}S_1(l+m,m)t^{l+m}x^n\right\rangle\\
&=\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}S_1(l+m,m)(n)_{l+m}\\
&\qquad\left\langle\left(\frac{(1+t)\ln(1+t)}{t}\right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|x^{n-l-m}\right\rangle\\
&=\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}S_1(l+m,m)(n)_{l+m}\left\langle\sum_{i=0}^{\infty}\widehat{D}_i^{(r,k)}\frac{t^i}{i!}\Big|x^{n-l-m}\right\rangle\\
&=\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}S_1(l+m,m)(n)_{l+m}\widehat{D}_{n-l-m}^{(r,k)}\\
&=\sum_{l=0}^{n-m}m!\binom{n}{l+m}S_1(l+m,m)\widehat{D}_{n-l-m}^{(r,k)}\\
&=\sum_{l=0}^{n-m}m!\binom{n}{l}S_1(n-l,m)\widehat{D}_l^{(r,k)}\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle\left(\frac{(1+t)\ln(1+t)}{t}\right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^n\right\rangle\\
&=\left\langle\partial_t\left(\left(\frac{(1+t)\ln(1+t)}{t}\right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\right)\Big|x^{n-1}\right\rangle\\
&=\left\langle\left(\partial_t\left(\frac{(1+t)\ln(1+t)}{t}\right)^r\right) \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^{n-1}\right\rangle
\end{aligned}$$

$$+\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^r \left(\partial_t \mathrm{Lif}_k\big(-\ln(1+t)\big)\right)\big(\ln(1+t)\big)^m \Big| x^{n-1}\right\rangle$$

$$+\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)\left(\partial_t \big(\ln(1+t)\big)^m\right) \Big| x^{n-1}\right\rangle . \tag{3.20}$$

The first term of (3.20) is

$$\begin{aligned}
&\left\langle \left(\partial_t \left(\frac{(1+t)\ln(1+t)}{t}\right)^r\right) \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^{n-1}\right\rangle \\
&= \frac{r}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| \right. \\
&\qquad \left. \left(\ln(1+t)+1-\frac{(1+t)\ln(1+t)}{t}\right) x^n\right\rangle \\
&= \frac{r}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^{m+1} \Big| x^n\right\rangle \\
&\quad + \frac{r}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^n\right\rangle \\
&\quad - \frac{r}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r} \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^n\right\rangle .
\end{aligned}$$

Observe that

$$\begin{aligned}
&\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^{m+1} \Big| x^n\right\rangle \\
&= \left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| \big(\ln(1+t)\big)^{m+1} x^n\right\rangle \\
&= \left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| \right. \\
&\qquad \left. \sum_{l=0}^{n-m-1} \frac{(m+1)!}{(l+m+1)!} S_1(l+m+1,m+1) t^{l+m+1} x^n\right\rangle \\
&= \sum_{l=0}^{n-m-1} \frac{(m+1)!}{(l+m+1)!} S_1(l+m+1,m+1)(n)_{l+m+1} \\
&\quad \times \left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| x^{n-l-m-1}\right\rangle \\
&= \sum_{l=0}^{n-m-1} \frac{(m+1)!}{(l+m+1)!} S_1(l+m+1,m+1)(n)_{l+m+1} \left\langle \sum_{i=0}^{\infty} \widehat{D}_i^{(r-1,k)} \frac{t^i}{i!} \Big| x^{n-l-m-1}\right\rangle \\
&= \sum_{l=0}^{n-m-1} (m+1)! \binom{n}{l+m+1} S_1(l+m+1,m+1) \widehat{D}_{n-l-m-1}^{(r-1,k)}
\end{aligned}$$

$$= \sum_{l=0}^{n-m-1} (m+1)! \binom{n}{l} S_1(n-l, m+1) \widehat{D}_l^{(r-1,k)}$$

and

$$\left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \middle| x^n \right\rangle$$
$$= \sum_{l=0}^{n-m} m! \binom{n}{l} S_1(n-l, m) \widehat{D}_l^{(r,k)} .$$

Thus,

$$\left\langle \left(\partial_t \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \middle| x^{n-1} \right\rangle$$
$$= \frac{r}{n} \sum_{l=0}^{n-m-1} (m+1)! \binom{n}{l} S_1(n-l, m+1) \widehat{D}_l^{(r-1,k)}$$
$$+ \frac{r}{n} \sum_{l=0}^{n-m} m! \binom{n}{l} S_1(n-l, m) \widehat{D}_l^{(r-1,k)}$$
$$- \frac{r}{n} \sum_{l=0}^{n-m} m! \binom{n}{l} S_1(n-l, m) \widehat{D}_l^{(r,k)} .$$

The second term of (3.20) is

$$\left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \big(\partial_t \mathrm{Lif}_k\big(-\ln(1+t)\big)\big) \big(\ln(1+t)\big)^m \middle| x^{n-1} \right\rangle$$
$$= \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^r \right.$$
$$\left. \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big) - \mathrm{Lif}_k\big(-\ln(1+t)\big)}{(1+t)\ln(1+t)} \big(\ln(1+t)\big)^m \middle| x^{n-1} \right\rangle$$
$$= \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^{r-1} \right.$$
$$\left. \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big) - \mathrm{Lif}_k\big(-\ln(1+t)\big)}{t} \big(\ln(1+t)\big)^m \middle| x^{n-1} \right\rangle$$
$$= \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^{r-1} \right.$$
$$\left. \big(\ln(1+t)\big)^m \middle| \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big) - \mathrm{Lif}_k\big(-\ln(1+t)\big)}{t} x^{n-1} \right\rangle$$
$$= \frac{1}{n} \left\langle \left(\frac{(1+t)\ln(1+t)}{t} \right)^{r-1} \big(\ln(1+t)\big)^m \right|$$
$$\left. \big(\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big) - \mathrm{Lif}_k\big(-\ln(1+t)\big)\big)\, x^n \right\rangle$$

$$= \frac{1}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^n \right\rangle$$
$$- \frac{1}{n}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1} \mathrm{Lif}_{k}\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^n \right\rangle$$
$$= \frac{1}{n}\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m)\widehat{D}_l^{(r-1,k-1)} - \frac{1}{n}\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m)\widehat{D}_l^{(r-1,k)}.$$

The third term of (3.20) is

$$\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r} \mathrm{Lif}_{k}\big(-\ln(1+t)\big)\left(\partial_t\big(\ln(1+t)\big)^m\right) \Big| x^{n-1} \right\rangle$$
$$= m\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r} \mathrm{Lif}_{k}\big(-\ln(1+t)\big)(1+t)^{-1} \Big| \big(\ln(1+t)\big)^{m-1} x^{n-1} \right\rangle$$
$$= m\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r} \mathrm{Lif}_{k}\big(-\ln(1+t)\big)(1+t)^{-1} \Big| \right.$$
$$\left. \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1,m-1) t^{l+m-1} x^{n-1} \right\rangle$$
$$= m\sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1,m-1)(n-1)_{l+m-1}$$
$$\times \left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r} \mathrm{Lif}_{k}\big(-\ln(1+t)\big)(1+t)^{-1} \Big| x^{n-l-m} \right\rangle$$
$$= m\sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1,m-1)(n-1)_{l+m-1}$$
$$\left\langle \sum_{i=0}^{\infty} \widehat{D}_i^{(r,k)}(-1)\frac{t^i}{i!} \Big| x^{n-l-m} \right\rangle$$
$$= m\sum_{l=0}^{n-m} (m-1)!\binom{n-1}{l+m-1} S_1(l+m-1,m-1)\widehat{D}_{n-l-m}^{(r,k)}(-1)$$
$$= m\sum_{l=0}^{n-m} (m-1)!\binom{n-1}{l} S_1(n-l-1,m-1)\widehat{D}_{l}^{(r,k)}(-1).$$

Thus, we have

$$\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m)\widehat{D}_l^{(r,k)}$$
$$= \frac{r}{n}\sum_{l=0}^{n-m-1} (m+1)!\binom{n}{l} S_1(n-l,m+1)\widehat{D}_l^{(r-1,k)}$$

$$+\frac{r}{n}\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m)\widehat{D}_l^{(r-1,k)}$$
$$-\frac{r}{n}\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m)\widehat{D}_l^{(r,k)}$$
$$+\frac{1}{n}\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m)\widehat{D}_l^{(r-1,k-1)}$$
$$-\frac{1}{n}\sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m)\widehat{D}_l^{(r-1,k)}$$
$$+m\sum_{l=0}^{n-m}(m-1)!\binom{n-1}{l} S_1(n-l-1,m-1)\widehat{D}_l^{(r,k)}(-1)\,.$$

So, we complete to prove the result. □

3.8. **Binomial relations.**

Theorem 3.9.

$$\widehat{D}_n^{(r,k)}(x)=\sum_{m=0}^{n}\binom{n}{m}\widehat{D}_{n-m}^{(r,k)}(x)_m\,. \tag{3.21}$$

Proof. For (3.1) and (3.9), assume that $\widehat{D}_n^{(r,k)}(x)=\sum_{m=0}^{n} C_{n,m}(x)_m$. By (2.10), we have

$$\begin{aligned}
C_{n,m}&=\frac{1}{m!}\left\langle\left(\frac{(1+t)\ln(1+t)}{t}\right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)t^m\Big|x^n\right\rangle\\
&=\frac{1}{m!}\left\langle\left(\frac{(1+t)\ln(1+t)}{t}\right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|t^m x^n\right\rangle\\
&=\binom{n}{m}\left\langle\left(\frac{(1+t)\ln(1+t)}{t}\right)^r \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|x^{n-m}\right\rangle\\
&=\binom{n}{m}\left\langle\sum_{i=0}^{\infty}\widehat{D}_i^{(r,k)}\frac{t^i}{i!}\Big|x^{n-m}\right\rangle\\
&=\binom{n}{m}\widehat{D}_{n-m}^{(r,k)}\,.
\end{aligned}$$

Thus, we obtain the identity (3.21). □

3.9. **Relations with higher-order Frobenius-Euler polynomials.** For $\lambda\in\mathbb{C}$ with $\lambda\neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\lambda)$ are defined by the generating function

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}(x|\lambda)\frac{t^n}{n!}$$

(see e.g. [2]).

Theorem 3.10.
(3.22)

$$\widehat{D}_n^{(r,k)}(x) = \sum_{m=0}^{n}\Big(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j(1-\lambda)^{-j}S_1(n-j-l,m)\widehat{D}_l^{(r,k)}\Big)H_m^{(s)}(x|\lambda)\,.$$

Proof. For (3.1) and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right),$$

assume that $\widehat{D}_n^{(r,k)}(x) = \sum_{m=0}^{n} C_{n,m}H_m^{(s)}(x|\lambda)$. By (2.10), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{\left(\frac{e^{\ln(1+t)}-\lambda}{1-\lambda}\right)^s}{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t)e^{\ln(1+t)}}\right)^r}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^n\right\rangle\\
&= \frac{1}{m!(1-\lambda)^s}\Bigg\langle\left(\frac{(1+t)\ln(1+t)}{t}\right)^r\\
&\qquad \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|(1-\lambda+t)^s x^n\Bigg\rangle\\
&= \frac{1}{m!(1-\lambda)^s}\sum_{j=0}^{n}\binom{s}{j}(1-\lambda)^{s-j}(n)_j\\
&\quad\times\left\langle\left(\frac{(1+t)\ln(1+t)}{t}\right)^r\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^{n-j}\right\rangle\\
&= \frac{1}{m!(1-\lambda)^s}\sum_{j=0}^{n-m}\binom{s}{j}(1-\lambda)^{s-j}(n)_j\sum_{l=0}^{n-j-m}m!\binom{n-j}{l}S_1(n-j-l,m)\widehat{D}_l^{(r,k)}\\
&= \sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j(1-\lambda)^{-j}S_1(n-j-l,m)\widehat{D}_l^{(r,k)}
\end{aligned}$$

Therefore, we obtain the identity (3.22). □

3.10. **Relations with higher-order Bernoulli polynomials.** We denote by $\widehat{A}_l^{(r,k)}(x)$ higher-order Cauchy of the second kind and poly-Cauchy of the second kind mixed-type polynomials, whose generating function is given by

$$\left(\frac{t}{(1+t)\ln(1+t)}\right)^r\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^x = \sum_{n=0}^{\infty}\widehat{A}_l^{(r,k)}(x)\frac{t^n}{n!}\quad (r,k\in\mathbb{Z}_{>0})\,.$$

Theorem 3.11. *If $s>r$, then*

$$\widehat{D}_n^{(r,k)}(x) = \sum_{m=0}^{n}\left(\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{A}_l^{(s-r,k)}(s)\right)B_m^{(s)}(x)\,. \tag{3.23}$$

If $s = r$, then

$$\widehat{D}_n^{(r,k)}(x) = \sum_{m=0}^{n} \left(\sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) \widehat{c}_l^{(k)}(s) \right) B_m^{(s)}(x) . \tag{3.24}$$

If $s < r$, then

$$\widehat{D}_n^{(r,k)}(x) = \sum_{m=0}^{n} \left(\sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) \widehat{D}_l^{(r-s,k)}(s) \right) B_m^{(s)}(x) . \tag{3.25}$$

Proof. For (3.1) and

$$B_n^{(s)}(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^s , t \right) ,$$

assume that $\widehat{D}_n^{(r,k)}(x) = \sum_{m=0}^{n} C_{n,m} B_m^{(s)}(x)$. By (2.10), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t)} \right)^s}{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t) e^{\ln(1+t)}} \right)^r} \mathrm{Lif}_k\big(-\ln(1+t)\big) \big(\ln(1+t)\big)^m \Big| x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \frac{\left(\frac{t}{\ln(1+t)} \right)^s}{\left(\frac{t}{(1+t)\ln(1+t)} \right)^r} \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| \big(\ln(1+t)\big)^m x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \frac{\left(\frac{t}{(1+t)\ln(1+t)} \right)^s}{\left(\frac{t}{(1+t)\ln(1+t)} \right)^r} \mathrm{Lif}_k\big(-\ln(1+t)\big) (1+t)^s \Big| \big(\ln(1+t)\big)^m x^n \right\rangle . \end{aligned}$$

If $s > r$, then

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \Bigg\langle \left(\frac{t}{(1+t)\ln(1+t)} \right)^{s-r} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^s \Big| \\ &\qquad \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m, m) t^{l+m} x^n \Bigg\rangle \\ &= \sum_{l=0}^{n-m} \frac{1}{(l+m)!} S_1(l+m, m) (n)_{l+m} \\ &\quad \times \left\langle \left(\frac{t}{(1+t)\ln(1+t)} \right)^{s-r} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^s \Big| x^{n-l-m} \right\rangle \\ &= \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m, m) \widehat{A}_{n-l-m}^{(s-r,k)}(s) \\ &= \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) \widehat{A}_l^{(s-r,k)}(s) . \end{aligned}$$

Thus, we get the identitiy (3.23).

If $s = r$, then

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^s \middle| \big(\ln(1+t)\big)^m x^n \right\rangle \\
&= \sum_{l=0}^{n-m} \frac{1}{(l+m)!} S_1(l+m,m)(n)_{l+m} \left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^s \middle| x^{n-l-m} \right\rangle \\
&= \sum_{l=0}^{n-m} \frac{1}{(l+m)!} S_1(l+m,m)(n)_{l+m} \left\langle \sum_{i=0}^{\infty} \widehat{c}_i^{(k)}(s)\frac{t^i}{i!} \middle| x^{n-l-m} \right\rangle \\
&= \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m,m)\widehat{c}_{n-l-m}^{(k)}(s) \\
&= \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m)\widehat{c}_l^{(k)}(s)\,.
\end{aligned}$$

Thus, we get the identitiy (3.24).

If $s < r$, then

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-s} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^s \middle| \big(\ln(1+t)\big)^m x^n \right\rangle \\
&= \sum_{l=0}^{n-m} \frac{1}{(l+m)!} S_1(l+m,m)(n)_{l+m} \\
&\quad \times \left\langle \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-s} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^s \middle| x^{n-l-m} \right\rangle \\
&= \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m,m)\widehat{D}_{n-l-m}^{(r-s,k)}(s) \\
&= \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m)\widehat{D}_l^{(r-s,k)}(s)\,.
\end{aligned}$$

Thus, we get the identitiy (3.25). □

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