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MINIMIZATION OF VECTOR-VALUED CONVEX FUNCTIONS

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Dedicated to Prof. Wataru Takahashi on his 70th birthday

Abstract. In this paper, we introduce some generalized semi-continuity for vector-valued functions to obtain the existence of a solution to Pareto's optimization problem for vector-valued convex functions in reflexive Banach spaces.

1. Introduction

Let E be a topological vector space and $C \subset E$ be a closed cone (i.e., C is closed convex, $\lambda C \subseteq C$ for all $\lambda > 0$, $C \cap (-C) = \{0\}$ and define a partial order $\leq$ induced by C on E by $x \leq y$ if and only if $y - x \in C$). Vector optimization problems have been extensively studied by many authors in the past years (see Cesari and Suryanarayana [6, 7, 8], Corley [10, 11], Hartley [12], Jahn [13], Yu [25] and Wagner [24]. In 1980, Corley [10] introduced very interesting notions, C-semicontinuity and C-semicompactness, to obtain the existence of a solution to a generalized Pareto's maximization problem. We note that a C-semicompact subset is essentially weaker than a compact subset since a compact subset is necessarily closed, but a C-semicompact subset may not be closed. For example, if $F = R^2$, $C = \{(x, x), x \geq 0\} \subset R^2$ is a cone and $B = \{(z, 0) : z \in (0, 1)\}$ is not closed, then, for any open covering of $B \subset \cup_{i \in I, z_i \in (0,1)}[(z_i, 0) + C]^c$, one can easily see that

$$B \subset ((z_1, 0) + P)^c \cup ((z_2, 0) + C)^c,$$

where $z_1 \neq z_2$, and so B is $C-$semicompact. In [9], Chen et al. used a generalized semi-continuity which is called *lower semi-continuous from above* to study the minimization of a convex function in reflexive Banach spaces. In fact, the concept of a generalized semi-continuity has been generalized by Khanh and Quy [14, 15, 16] to study variational problems for vector functions.

Definition 1.1 (see [25]). Let X, Y be topological vector spaces, $C \subset F$ be a cone and $\leq$ be the partial order induced by C on F.

(1) A vector valued function $\phi : X \to F$ is said to be *C-convex* if

$$\phi(\alpha x + \beta y) \leq \alpha \phi(x) + \beta \phi(y)$$

for all $x, y \in D(\phi)$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$;

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(2) A vector valued function $\phi : X \to F$ is said to be *C-concave* if

$$\phi(\alpha x + \beta y) \geq \alpha\phi(x) + \beta\phi(y)$$

for all $x, y \in D(\phi)$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$.

The following definition is essentially introduced by Corley [10]:

Definition 1.2. Let X be a topological space, Y be a topological vector space, $C \subset F$ be a cone and $\leq$ be the partial order induced by C on F.

(1) A vector valued function $f : X \to Y$ is said to *C-lower semi-continuous* if $f^{-1}(y - P)$ is closed for any $y \in Y$;

(2) A vector valued function $f : X \to Y$ is said to *C-upper semi-continuous* if $f^{-1}(y + P)$ is closed for any $y \in Y$.

Definition 1.3 (see [9])**.** Let X be a topological space.

(1) A function $f : X \to R$ is said to be *sequentially lower semi-continuous from above* at x_0 if, for any sequence $\{x_n\}$ with $x_n \to x_0$, $f(x_{n+1}) \leq f(x_n)$ implies that $f(x_0) \leq \lim_{n\to\infty} f(x_n)$.

(2) f is said to be *sequentially upper semi-continuous from below* at x_0 if, for any sequence $\{x_n\}$ with $x_n \to x_0$, $f(x_{n+1}) \geq f(x_n)$ implies that $f(x_0) \leq \lim_{n\to\infty} f(x_0)$.

Lower semi-continuous from above has been used and generalized by many authors to study variational problem, optimization problem and fixed point problem (see [2, 3, 4, 5, 14, 15, 16, 17, 18, 19, 20, 21]) and it has been generalized by Khanh and Quy to study variational problem for vector valued functions.

Definition 1.4 (see [14])**.** Let X be a topological space, F be a topological vector space, $C \subset F$ be a cone and $\leq$ be the partial order induced by C on F.

(1) A function $f : X \to F$ is said to be *sequentially C-lower semi-continuous from above* at x_0 if, for any sequence $\{x_n\}$ with $x_n \to x_0$, $f(x_{n+1}) \leq f(x_n)$ implies that $f(x_0) \leq f(x_n)$ for each $n \geq 1$;

(2) f is said to be *sequentially C-upper semi-continuous from below* at x_0 if, for any sequence $\{x_n\}$ with $x_n \to x_0$, $f(x_{n+1}) \geq f(x_n)$ implies that $f(x_0) \geq f(x_n)$ for each $n \geq 1$.

Clearly, the C-lower semi-continuity introduce by [10] implies that the C-lower semi-continuity from above, but the reverse is not true. See [9] and [14].

Definition 1.5. Let X be a topological space, F be a topological vector space, $C \subset F$ be a cone and $\leq$ be the partial order induced by C on F.

(1) A function $f : X \to F$ is said to be *sequentially strongly C-lower semi-continuous* from above at x_0 if, for any sequences $\{x_n\}$ and $\{y_n\}$ with $f(x_{n+1}) \leq f(x_n)$ and $y_n \to x_0$, respectively, $f(y_n) \leq f(x_n)$ implies that $f(x_0) \leq f(x_n)$ for each $n \geq 1$;

(2) f is said to be *sequentially strongly C-upper semi-continuous* from below at x_0 if, for any sequences $\{x_n\}$ and $\{y_n\}$ with $f(x_{n+1}) \leq f(x_n)$ and $y_n \to x_0$, respectively, $f(y_n) \geq f(x_n)$ implies that $f(x_0) \geq f(x_n)$ for each $n \geq 1$.

2. The existence of a solution to Pareto's optimization problem

Now, we give the main results in this paper.

Theorem 2.1. *Let E be a reflexive Banach space, F be a separable norm space, $C \subset F$ be a cone and $\phi : D(\phi) \subset E \to F$ be C-convex and strongly C-lower semi-continuous from above. Then, for any $r > 0$ such that $D(\phi) \cap \overline{B(0,r)} \neq \emptyset$, there exists $y_0 \in D(\phi) \cap \overline{B(0,r)}$ such that*

$$\phi(y_0) = \min\{\phi(x) : x \in D(\phi) \cap \overline{B(0,r)}\}.$$

Proof. For each totally ordered subset $\{\phi(x_\tau)\}_{\tau \in T}$ of $\{\phi(x) : x \in D(\phi) \cap \overline{B(0,r)}\}$, we prove that it has a lower bound in $\{\phi(x) : x \in D(\phi) \cap \overline{B(0,r)}\}$.

Since F is a separable norm space, there exists $\{\phi(x_{\tau_n}) :\geq 1\}$ such that $\{\phi(x_{\tau_n}) : n \geq 1\}$ is dense in $\{\phi(x_\tau)\}_{\tau \in T}$, which is simply denoted by $\{\phi(x_n) : n \geq 1\}$. We may also assume that $\phi(x_1) \geq \phi(x_2) \geq \cdots$. Since E is reflexive, $\{x_n\}$ has a weak convergence subsequence. For simplicity, we still denote it by $x_n \rightharpoonup x_0$. Obviously, we have $x_0 \in \overline{B(0,r)}$ By Banach-Mazur Theorem (see [24]), we know that $x_0 \in \overline{co\{x_n : n \geq 1\}}$ and so there exists $y_1 = \Sigma_{i=1}^{k_1} \alpha_i x_{n_i} \in co\{x_n : n \geq 1\}$ such that $|y_1 - x_0\| \leq 1$, where $\alpha_i \geq 0$ for each $i = 1, 2, \ldots, k_1$, and $\Sigma_{i=1}^{k_1} \alpha_i = 1$. Again, by $x_0 \in \overline{co\{x_n : n \geq n_{k_1}\}}$, there exists $y_2 = \Sigma_{j=k_1}^{k_2} \lambda_j x_{n_j} \in co\{x_n : n \geq n_{k_1}\}$ such that $\|y_2 - x_0\| \leq \frac{1}{2}$, where $\lambda_j \geq 0$ and $\Sigma_{j=k_1}^{k_2} \lambda_j = 1$. By using the convexity of ϕ, we have

$$\phi(y_1) \leq \Sigma_{i=k_0}^{k_1} \alpha_i \phi(x_i) \leq \phi(x_{k_1})$$

and, similarly, $\phi(y_2) \leq \phi(x_{k_2})$.

Continuously, by using $x_0 \in \overline{co\{x_n : n \geq n_{k_{m-1}}\}}$, we get a sequence $\{y_m\}$ with $y_m \in co\{x_n : n \geq n_{k_{m-1}}\}$ such that $\|y_m - x_0\| < \frac{1}{m}$ and $\phi(y_m) \leq \phi(x_{k_m})$ for each $m \geq 3$. By assumption, we get $\phi(x_0) \leq \phi(x_{n_{k_m}})$ for each $m \geq 1$ and thus $x_0 \in D\phi)$. Consequently, we have $\phi(x_0) \leq \phi(x_n)$ for each $n \geq 1$.

Now, we prove $\phi(x_0)$ is a lower bound for $\{\phi(x_\tau) : \tau \in T\}$, i.e., $\phi(x_0) \leq \phi(x_\tau)$ for all $\tau \in T$. In fact, for each $\tau \in T$, there exists $\{\phi(x_{n_j})\}$ such that $\phi(x_{n_j}) \to \phi(x_\tau)$. By using $\phi(x_0) \leq \phi(x_{n_j})$, we have $\phi(x_{n_j}) = \phi(x_0) + z_j$, where $z_j \in P$ for each $j \geq 1$. By letting $j \to \infty$ and noting that P is closed, we get $\phi(x_\tau) = \phi(x_0) + z_0$ for some $z_0 \in P$ and so $\phi(x_0) \leq \phi(x_\tau)$. Therefore, by Zorn's lemma, there exists $y_0 \in D(\phi) \cap \overline{B(0,r)}$ such that

$$\phi(y_0) = \min\{\phi(x) : x \in D(\phi) \cap \overline{B(0,r)}\}.$$

This completes the proof. □

By the similar argument, we get the following:

Theorem 2.2. *Let E be a reflexive Banach space, F be a separable norm space, $C \subset F$ be a closed convex cone and $\phi : D(\phi) \subset E \to F$ be P-concave and strongly $C-$upper semi-continuous from below. Then, for any $r > 0$ such that $D(\phi) \cap \overline{B(0,r)} \neq \emptyset$, there exists $y_0 \in D(\phi) \cap \overline{B(0,r)}$ such that*

$$\phi(y_0) = \max\{\phi(x) : x \in D(\phi) \cap \overline{B(0,r)}\}.$$

Theorem 2.3. *Let E be a reflexive Banach space, F be a separable norm space, $C \subset F$ be a closed convex cone and $\phi : D(\phi) \subset E \to F$ be C-lower semi-continuous from above in the weak topology. Then, for any $r > 0$ such that $D(\phi) \cap \overline{B(0,r)} \neq \emptyset$, there exists $y_0 \in D(\phi) \cap \overline{B(0,r)}$ such that*

$$\phi(y_0) = \min\{\phi(x) : x \in D(\phi) \cap \overline{B(0,r)}\}.$$

Proof. For each totally ordered subset $\{\phi(x_\tau)\}_{\tau \in T}$ of $\{\phi(x) : x \in D(\phi) \cap \overline{B(0,r)}\}$, we prove that it has a lower bound in $\{\phi(x) : x \in D(\phi) \cap \overline{B(0,r)}\}$.

Since F is separable, there exists $\{\phi(x_{\tau_n}) : n \geq 1\}$ such that $\{\phi(x_{\tau_n}) : n \geq 1\}$ is dense in $\{\phi(x_\tau)\}_{\tau \in T}$, which is simply denoted by $\{\phi(x_n) : n \geq 1\}$. We may also assume that $\phi(x_1) \geq \phi(x_2) \geq \cdots$. Since E is reflexive, $\{x_n\}$ has a weak convergence subsequence $\{x_{n_k}\}$ with $x_{n_k} \rightharpoonup x_0$. Obviously, we have $x_0 \in \overline{B(0,r)}$. Using the assumption that ϕ is C-lower semi-continuous from above in the weak topology, we get $\phi(x_0) \leq \phi(x_{n_k})$. Consequently, we have $\phi(x_0) \leq \phi(x_n)$.

By the similar argument as in the proof of Theorem 2.1, we get $\phi(x_0)$ is a lower bound for $\{\phi(x_\tau)\}_{\tau \in T}$. Therefore, by Zorn's lemma, there exists $y_0 \in D(\phi) \cap \overline{B(0,r)}$ such that

$$\phi(y_0) = \min\{\phi(x) : x \in D(\phi) \cap \overline{B(0,r)}\}.$$

This completes the proof. □

Remark 2.4. The conclusions of Theorems 2.1-2.3 are also true if $B(0,r)$ is replaced by a bounded closed convex subset in E.

Recall that $u_0 \in B(0,r)$ is a *generalized Pareto solution* of the problem (E1) if there exists $u_1 \in B(0,r)$ such that $G(u_1) \leq G(u_0)$, then $u_1 = u_0$.

Example 2.5. Let $g_1(x), g_2(x) : R^n \to R$ be two continuous convex functions. Consider the following multi-objective programming:

Find $u_0 \in R^n$ with $\|u_0\| \leq r$ such that

$$\text{(E1)} \qquad \begin{cases} g_1(u_0) \leq g_1(u), \\ g_2(u_0) \leq g_2(u) \text{ for all } u \in R^n \text{ with } \|u\| \leq r. \end{cases}$$

Set $C = \{(x_1, x_2) \in R^2 : x_i \geq 0, u = 1, 2\}$. Then C is a closed convex cone of R^2 and set $G : R^n \to R^2$ by $G(u) = (g_1(u), g_2(u))$ for all $u \in R^n$. It is easy to see that the continuity and convexity of g_1 and g_2 imply that G is C-convex and strongly C-lower semi-continuous from above. Tus, by Theorem 2.1, we know that the problem (E1) has a generalized Pareto solution.

Recall that $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a *generalized Pareto solution* to the problem (E2) if $G(u_1, v_1) \leq G(u_0, v_0)$ implies that $(u_1, v_1) = (u_0, v_0)$.

Example 2.6. Let $f(x,y), g(x,y) : R^2 \to R$ be two continuous convex functions. Suppose that the following conditions are satisfied:

(1) $|f(x,y)| \leq L\|x\| + b$ for all $(x,y) \in R^2$, where $L > 0$ and $b > 0$ are constants;
(2) $|g(x,y)| \leq K\|y\| + c$ for all $(x,y) \in R^2$, where $K > 0$ and $c > 0$ are constants.

Let $\Omega \subset R^n$ be an open bounded subset with smooth boundary and $H_0^1(\Omega)$ be the Sobolev space. Let

$$I(u,v) = \int_\Omega [\Sigma_{i=1}^n u_{x_i}^2(x) + f(u(x), v(x))]dx$$

and

$$J(u,v) = \int_\Omega [\Sigma_{i=1}^n v_{x_i}^2(x) + g(u(x), v(x))]dx$$

for all $u, v \in H_0^1(\Omega)$. We consider the following problem:

Find $u_0, v_0 \in H_0^1(\Omega)$ such that $\|u_0\| \le r_0$, $\|v_0\| \le r_0$ and

$$\text{(E2)} \qquad \begin{cases} I(u_0, v_0) = \min_{u,v \in H_0^1(\Omega)} \int_\Omega [\Sigma_{i=1}^n u_{x_i}^2(x) + f(u(x), v(x))]dx, \\ J(u_0, v_0) = \min_{u,v \in H_0^1(\Omega)} \int_\Omega [\Sigma_{i=1}^n v_{x_i}^2(x) + g(u(x), v(x))]dx. \end{cases}$$

Set $C = \{(x_1, x_2) \in R^2 : x_i \ge 0,\ u = 1, 2\}$. Then C is a closed convex cone of R^2 and define $G : H_0^1(\Omega) \times H_0^1(\Omega) \to R^2$ by $G(u,v) = (I(u,v), J(u,v))$ for all $u, v \in H_0^1(\Omega)$.

Under the assumptions, we know that both I and J are continuous and convex and so G is continuous and C-convex. Thus G is strongly C-lower semi-continuous from above. Therefore, by Theorem 2.1, we know that the problem (E2) has a generalized Pareto solution.

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