Journal of Nonlinear and Convex Analysis
Volume 16, Number 10, 2015, 2059–2066

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

FIXED POINTS FOR GENERALIZED α-ψ-CONTRACTIONS IN b-METRIC SPACES

VILDAN OZTURK AND DURAN TURKOGLU

Dedicated to Prof. Wataru Takahashi on his 70th birthday

Abstract. In this paper, we give some fixed point theorems for generalized α-ψ-contractive mappings in b-metric spaces. Our results are generalizations of many existence results in the literature.

1. Introduction

In last years, many generalizations of the concept of metric spaces are defined and some fixed point theorems was proved in these spaces. In particular, in 1989, Bakhtin [3] introduced the consept of b-metric spaces. He proved a generalization of the Banach contraction principle in b-metric spaces. After many authors obtained several interesting results about the existence of a fixed point for single-valued and multi-valued operators in b-metric spaces ([1, 2, 7, 9–13, 15–21, 23]).

Definition 1.1 ([3]). Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, every metric is a b-metric with $s = 1$.

Example 1.2. Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

For example, if $X = R$ is the set of real numbers and $d(x, y) = |x - y|$ is usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a b-metric on R with $s = 2$. But is not a metric on R.

Definition 1.3 ([7]). Let $\{x_n\}$ be a sequence in a b-metric space (X, d).

(a) $\{x_n\}$ is called b-convergent if and only if there is $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.
(b) $\{x_n\}$ is a b-Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. b-metric space, generalized α-ψ-contraction, fixed point.

A b-metric space is said to be complete if and only if each b-Cauchy sequence in this space is b-convergent.

Proposition 1.4 ([7]). *In a b-metric space (X, d), the following assertions hold:*

(p1) *A b-convergent sequence has a unique limit.*
(p2) *Each b-convergent sequence is b-Cauchy.*
(p3) *In general, a b-metric is not continuous.*

In this paper we prove some fixed point theorems for generalized α-ψ-contractive mappings in b-metric spaces. The notion of α-ψ-contractive type mapping was introduced by Samet et al. [22]. They also established some fixed point results in complete metric space. Karapinar and Samet [14] introduced generalized α-ψ-contractive type mappings in complete metric spaces. Bota et al. in [8] gived α-ψ-contraction in b-metric spaces.

Now we give some definitions that will be used throughout this paper.

A mapping $\psi : [0, \infty) \to [0, \infty)$ is called a comparison function if it is increasing and $\lim_{n\to\infty} \psi^n (t) = 0$ for all $t > 0$.

Lemma 1.5 ([5]). *Let $\psi : [0, \infty) \to [0, \infty)$ is a comparison function then*

(a) *each iterate ψ^n of ψ, $n \geq 1$, is also a comparison function,*
(b) *ψ is continuous at $t = 0$,*
(c) *$\psi (t) < t$ for all $t > 0$.*

Definition 1.6 ([5]). A function $\psi : [0, \infty) \to [0, \infty)$ is said to be a (c)-comparison function if

(c1) ψ is increasing,
(c2) there exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum\limits_{k=1}^{\infty} v_k$, such that $\psi^{k+1} (t) \leq a\psi^k (t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

In [6] Berinde also defined (b)-comparison function.

Definition 1.7. Let $s \geq 1$ be a real number. A function $\psi : [0, \infty) \to [0, \infty)$ is said to be a (b)-comparison function if

(b1) ψ is monotonically increasing,
(b1) there exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum\limits_{k=1}^{\infty} v_k$, such that $s^{k+1}\psi^{k+1} (t) \leq as^k\psi^k (t)+v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

When $s = 1$, (b)-comparison function reduces to (c)-comparison function.

We denote by Ψ_b for the class of (b)-comparison function.

Lemma 1.8 ([4]). *If $\psi : [0, \infty) \to [0, \infty)$ is a (b)-comparison function then one has the following:*

(i) *$\sum\limits_{k=0}^{\infty} s^k\psi^k (t)$ converges to any $t \in R^+$,*
(ii) *the function $b_s : [0, \infty) \to [0, \infty)$ defined by $b_s (t) = \sum\limits_{k=0}^{\infty} s^k\psi^k (t)$, $t \in R^+$, increasing and continuous at 0.*

Any (b)-comparison function is a comparison function.

Definition 1.9 ([22])**.** For any nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. T is called α-admissible if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Bota et. al. in ([8]) gived the definition of α-ψ-contractive mapping of type (b) in b-metric space which is a generalization of Definition 1.9.

Definition 1.10. Let (X, d) be a b-metric space and $T : X \to X$ be a given mapping. T is called an α-ψ-contractive mapping of type (b), if there exists two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_b$ such that

$$\alpha(x, y)\, d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in X.$$

2. Main results

Definition 2.1. Let (X, d) be a b-metric space and $T : X \to X$ be a given mapping. T is called generalized α-ψ-contractive mapping of type (I), if there exists two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_b$ such that for all $x, y \in X$

$$\alpha(x, y)\, d(Tx, Ty)) \leq \psi(M_s(x, y)) \tag{2.1}$$

where,

$$M_s(x, y) = \max\left\{ d(x, y), d(Tx, x), d(Ty, y), \frac{d(Tx, y) + d(x, Ty)}{2s} \right\}.$$

Theorem 2.2. *Let (X, d) be a complete b-metric space. Suppose that $T : X \to X$ be a generalized α-ψ-contractive mapping of type (I) and satisfies:*

(i) *T is α-admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(iii) *T is continuous,*

then T has a fixed point.

Proof. By assumption (ii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T.

Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since T is α-admissible, then

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Longrightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By induction, we get for all $n \in \mathbb{N}$,

$$\alpha(x_n, x_n + 1) \geq 1. \tag{2.2}$$

Using (2.1) and (2.2)

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n)\, d(Tx_{n-1}, Tx_n) \leq \psi(M_s(x_{n-1}, x_n)). \tag{2.3}$$

where

$$M_s(x_{n-1}, x_n) \quad = \quad \max\left\{ \begin{array}{c} d(x_{n-1}, x_n), d(Tx_{n-1}, x_{n-1}), d(Tx_n, x_n), \\ \frac{d(Tx_{n-1}, x_n) + d(Tx_n, x_{n-1})}{2s} \end{array} \right\},$$

$$
\begin{aligned}
&= \max\left\{\begin{array}{c} d\left(x_{n-1},x_n\right), d\left(x_n,x_{n-1}\right), d\left(x_{n+1},x_n\right), \\ \frac{d(x_n,x_n)+d(x_{n+1},x_{n-1})}{2s} \end{array}\right\}, \\
&= \max\left\{\begin{array}{c} d\left(x_{n-1},x_n\right), d\left(x_{n+1},x_n\right), \\ \frac{s[d(x_{n+1},x_n)+d(x_n,x_{n-1})]}{2s} \end{array}\right\} \\
&\leq \max\left\{d\left(x_{n-1},x_n\right), d\left(x_{n+1},x_n\right)\right\}.
\end{aligned}
$$

If $M_s\left(x_{n-1},x_n\right) = d\left(x_n,x_{n+1}\right)$, then from (2.3) and definition of ψ,

$$d\left(x_n,x_{n+1}\right) \leq \psi\left(d\left(x_n,x_{n+1}\right)\right) < d\left(x_n,x_{n+1}\right)$$

a contradiction. Thus $M_s\left(x_{n-1},x_n\right) = d\left(x_{n-1},x_n\right)$. By (2.3) and definition of ψ,

$$d\left(x_n,x_{n+1}\right) \leq \psi\left(d\left(x_{n-1},x_n\right)\right) < d\left(x_{n-1},x_n\right)$$

for all $n \geq 1$. By induction we get

$$d\left(x_n,x_{n+1}\right) \leq \psi^n\left(d\left(x_0,x_1\right)\right). \tag{2.4}$$

Thus, for all $p \geq 1$,

$$
\begin{aligned}
d\left(x_n,x_{n+p}\right) &\leq sd\left(x_n,x_{n+1}\right) + s^2 d\left(x_{n+1},x_{n+2}\right) + \cdots \\
&\quad + s^{p-1} d\left(x_{n+p-2},x_{n+p-1}\right) + s^p d\left(x_{n+p-1},x_{n+p}\right) \\
&\leq s\psi^n\left(d\left(x_0,x_1\right)\right) + s^2\psi^{n+1}\left(d\left(x_0,x_1\right)\right) + \cdots \\
&\quad + s^{p-1}\psi^{n+p-2}\left(d\left(x_0,x_1\right)\right) + s^p\psi^{n+p-1}\left(d\left(x_0,x_1\right)\right) \\
&= \frac{1}{s^{n-1}}[s^n\psi^n\left(d\left(x_0,x_1\right)\right) + s^{n+1}\psi^{n+1}\left(d\left(x_0,x_1\right)\right) + \cdots \\
&\quad + s^{p-n-2}\psi^{p-n-2}\left(d\left(x_0,x_1\right)\right) + s^{p+n-1}\psi^{p+n-1}\left(d\left(x_0,x_1\right)\right)].
\end{aligned}
$$

Denoting $S_n = \sum\limits_{k=0}^{\infty} s^k\psi^k\left(d\left(x_0,x_1\right)\right)$, $n \geq 1$, we obtain

$$d\left(x_n,x_{n+p}\right) \leq \frac{1}{s^{n-1}}[S_{n+p-1} - S_{n-1}] \tag{2.5}$$

for $n \geq 1$, $p \geq 1$. From Lemma 1.8, we conclude that the series $\sum\limits_{k=0}^{\infty} s^k\psi^k\left(d\left(x_0,x_1\right)\right)$ is convergent. Thus there exists

$$S = \lim_{n\to\infty} S_n \in [0,\infty).$$

Regarding $s \geq 1$ and by (2.5) $\{x_n\}$ is a Cauchy sequence in b-metric space (X,d). Since (X,d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Using continuity of T,

$$x_{n+1} = Tx_n \to Tx^*$$

as $n \to \infty$. By the uniqueness of the limit, we get $x^* = Tx^*$. Hence x^* is a fixed point of T. □

Definition 2.3. Let (X,d) be a b-metric space and $T : X \to X$ be a given mapping. T is called generalized α-ψ-contractive mapping of type (II), if there exists two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi_b$ such that for all $x, y \in X$

$$\alpha\left(x,y\right) d\left(Tx,Ty\right)) \leq \psi(N_s\left(x,y\right)) \tag{2.6}$$

where,

$$N_s(x,y)=\max\left\{d(x,y),\frac{d(Tx,x)+d(Ty,y)}{2s},\frac{d(Tx,y)+d(Ty,x)}{2s}\right\}.$$

Theorem 2.4. *Let (X,d) be a complete b-metric space. Suppose that $T:X\to X$ be a generalized α-ψ-contractive mapping of type (II) and satisfies:*

(i) *T is α-admissible,*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$*
(iii) *T is continuous*

then T has a fixed point.

Proof. The proof is evident due to Theorem 2.2. Indeed ψ is nondecreasing and

$$\alpha(x,y)\,d(Tx,Ty))\leq\psi(N_s(x,y))\leq\psi(M_s(x,y)).$$

□

In the following two theorems we are able to remove the continuity condition for the α-ψ-contractive mappings of type (I) and type (II).

Theorem 2.5. *Let (X,d) be a complete b-metric space. Suppose that $T:X\to X$ be a generalized α-ψ-contractive mapping of type (I) and satisfies:*

(i) *T is α-admissible,*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in X$,as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha\left(x_{n(k)},x\right)\geq 1$, for all k.*

Then T has a fixed point.

Proof. Following the proof of Theorem 2.2 , we know that the sequence $\{x_n\}$ defined by $x_{n+1}=Tx_n$ for all $n\geq 0$, is Cauchy and converges to some $u\in X$.

We shall show that $Tu=u$. Suppose on the contrary that $d(Tu,u)>0$. From (2.2) and (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha\left(x_{n(k)},u\right)\geq 1$ for all k. By (2.1)

$$d\left(x_{n(k)+1},Tu\right)\leq\alpha\left(x_{n(k)},u\right)d\left(Tx_{n(k)},Tu\right)\leq\psi\left(M_s\left(x_{n(k)},u\right)\right), \tag{2.7}$$

where

$$M_s\left(x_{n(k)},u\right)=\max\left\{\begin{array}{c} d\left(x_{n(k)},u\right),d\left(Tx_{n(k)},x_{n(k)}\right),d(Tu,u),\\ \frac{d\left(Tx_{n(k)},u\right),d\left(Tu,x_{n(k)}\right)}{2s}\end{array}\right\}.$$

As $k\to\infty$, $\lim_{k\to\infty}M_s\left(x_{n(k)},u\right)=d(Tu,u)$.

In (2.7), as $k\to\infty$

$$d(u,Tu)\leq\psi(d(u,Tu))<d(u,Tu)$$

which is a contradiction. Hence, $u=Tu$ and u is a fixed point of T. □

Theorem 2.6. *Let (X,d) be a complete b-metric space. Suppose that $T:X\to X$ be a generalized α-ψ-contractive mapping of type (II) and satisfies:*

(i) *T is α-admissible,*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$,*

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$*,as* $n \to \infty$*, then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$*, for all* k*.*

then T *has a fixed point.*

Proof. Following the proof of Theorem 2.5, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, for all $n \geq 0$, is Cauchy and converges to some $u \in X$.

We shall show that $Tu = u$. Suppose on the contrary that $d(Tu, u) > 0$. From (2.2) and (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Applying (2.6),

$$d(x_{n(k)+1}, Tu) \leq \alpha(x_{n(k)}, u)\, d(Tx_{n(k)}, Tu) \leq \psi(N_s(x_{n(k)}, u)) \tag{2.8}$$

where

$$N_s(x_{n(k)}, u) = \max\left\{ \begin{array}{c} d(x_{n(k)}, u), \frac{d(Tx_{n(k)}, x_{n(k)}) + d(Tu,u)}{2s}, \\ \frac{d(Tx_{n(k)}, u), d(Tu, x_{n(k)})}{2s} \end{array} \right\}.$$

As $k \to \infty$, $\lim_{k\to\infty} N_s(x_{n(k)}, u) = \frac{d(Tu,u)}{2s}$, for $s \geq 1$.

In (2.8), as $k \to \infty$

$$d(u, Tu) \leq \psi\left(\frac{d(Tu,u)}{2s}\right) < \frac{d(Tu,u)}{2s}$$

which is a contradiction. Hence, $u = Tu$ and u is a fixed point of T. □

Example 2.7. Let $X = R^+$ endowed with b-metric

$$d : X \times X \to R^+, \quad d(x,y) = (x-y)^2$$

with constant $s = 2$. (X, d) is a complete b-metric space. Let the functions $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be defined by

$$\begin{aligned} T(x) &= 2x - \frac{3}{2}, \text{ if } x > 1 \text{ and } T(x) = \frac{x}{2} \text{ if } 0 \leq x \leq 1 \\ \alpha(x,y) &= 1 \text{ if } x, y \in [0,1] \text{ and } \alpha(x,y) = 0, \text{other} \end{aligned}$$

Clearly, T is α-admissible and continuous. Also α-ψ-contraction of type (I) is satisfied with $\psi(t) = \frac{t}{2}$, for all $t \geq 0$. In fact

if $x, y \in [0,1]$,

$$\begin{aligned} \alpha(x,y)\, d(Tx, Ty)) &= \left|\frac{x}{2} - \frac{y}{2}\right|^2 \leq \frac{1}{2}|x-y|^2 = \psi(d(x,y)) \\ &\leq \psi(M_s(x,y)). \end{aligned}$$

Then (2.1) is satisfied for all $x, y \in X$. 0 and $\frac{3}{2}$ are two fixed points of T.

Corollary 2.8. *Let* (X, d) *be a complete* b*-metric space and* $T : X \to X$ *be continuous mapping. Suppose that there exists a function* $\psi \in \Psi_b$ *such that*

$$d(Tx, Ty) \leq \psi(M_s(x,y))$$

for all $x, y \in X$*, then* T *has a fixed point.*

Similarly, be taken $\alpha(x,y) = 1$ in Theorem 2.4, the following result is obtained.

Corollary 2.9. *Let (X, d) be a complete b-metric space and $T : X \to X$ be continuous mapping. Suppose that there exists a function $\psi \in \Psi_b$ such that*

$$d(Tx, Ty) \leq \psi(N_s(x, y))$$

for all $x, y \in X$, then T has a fixed point.

References

[1] M. Akkouchi, *Common fixed point theorems for two selfmappings of a b-metric space under an implict relation*, Hacet. J. Math. Stat. **40** (2011), 805–810.

[2] H. Aydi, M. F. Bota, E. Karapinar and S. Moradi, *A common fixed point for weak φ-contractions on b-metric spaces*, Fixed Point Theory **13** (2012), 337–346.

[3] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Functional Analysis **30** (1989), 26–37.

[4] V. Berinde, *Generalized contractions in quasimetric spaces*, Seminar on Fixed Point Theory **3** (1993), 3–9.

[5] V. Berinde, *Contractii Generalizate şi Aplicatii*, vol. 2, Editura Cub Press, Baia Mare, Romania, 1997.

[6] V. Berinde, *Sequences of operators and fixed points in quasimetric spaces*, Mathematica **41** (1996), 23–27.

[7] M. Boriceanu, M. Bota and A. Petrusel, *Multivalued fractals in b-metric spaces*, Cent. Eur. J. Math. **8** (2010), 367–377.

[8] M. F. Bota, E. Karapinar and O. Mlesnite, *Ulam-Hyers stability results for fixed point problems via α-ψ contractive mapping in b-metric space*, Abstract and Applied Analysis **2013** (2013), Article ID 825293.

[9] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis **1** (1993), 5–11.

[10] M. Demma and P. Vetro, *Picard sequence and fixed point results on b-metric spaces*, J. Function Spaces **2015** (2015), Article ID 189861.

[11] H.-S. Ding, V. Ozturk and S. Radenović, *On some new fixed point results in b-rectangular metric spaces*, J. Nonlinear Sci. Appl. **8** (2015), 378–386.

[12] H. Huang and S. Xu, *Fixed point theorems of contractive mappings in cone b-metric spaces and applications*, Fixed Point Theory Appl. **2013** (2013), Article ID 113.

[13] R. George, C. Alaca and K. P. Reshma, *On best proximity Points in b-metric spaces*, J. Nonlinear Anal. Appl. **2015** (2015) Article ID jnaa-00278.

[14] E. Karapinar and B. Samet, *Generalized α-ψ contractive type mappings and related fixed point theorems with applications*, Abstract and Applied Anal. **2012** (2012), Article ID 793486.

[15] M. Kir and H. Kiziltunc, *On some well known fixed point theorems in b-metric spaces*, Turkish J. Anal. Number Theory **1** (2013), 13–16.

[16] M. Păcurar, *A fixed point result for contractions and fixed points on b-metric spaces without the boundness assumption*, Fasc. Math. **43** (2010), 127–136.

[17] H. K. Pathak, R. George, H. A. Nabwey, M. S. El-Paoumy and K. P. Reshma, *Some generalixed fixed point results in a b-metric space and application to matrix equation*, Fixed Point Theory Appl. **2015** (2015), Article ID 101.

[18] R. Plebaniak, *New generalized pseudodistance and coincidence point theorem in a b-metric space*, Fixed Point Theory Appl. **2013** (2013), Article ID 270.

[19] J. R. Roshan, V. Parvaneh and I. Altun, *Some coincidence point results in ordered b- metric spaces and applications in a system of integral equations*, Appl. Math. Comput. **226** (2014), 725–737.

[20] J. R. Roshan, V. Parvaneh, S. Radenović and M. Rajović, *Some coincidence point results for generalized (ψ, φ)-weakly contractions in ordered b-metric spaces,* Fixed Point Theory Appl. **2015** (2015), Article ID 68.

[21] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei and W. Shatanawi, *Common fixed points of almost generalized* $(\varphi, \psi)_s$*-contractive mappings in ordered b-metric spaces*, Fixed Point Theory Appl. **2013** (2013), Article ID 130.
[22] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for* α*-*ψ *contractive type mappings*, Nonlinear Analysis Theory, Methods Applications **75** (2012), 2154–2165.
[23] W. Sintunavarat, S. Plubtieng and P. Katchang, *Fixed point results and applications on a b-metric space endowed with an arbitrary binary relation*, Fixed Point Theory Appl. **2013** (2013), Article ID 296.

Manuscript received August 29, 2014
revised December 1, 2014

V. Ozturk
Department of Primary Education, Faculty of Education, University of Artvin Coruh, Artvin, Turkey
E-mail address: vildanozturk84@gmail.com

D. Turkoglu
Department of Mathematics, Faculty of Science, University of Gazi, Ankara, Turkey
E-mail address: dturkoglu@gazi.edu.tr