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PRICING ASIAN OPTIONS VIA TAYLOR APPROXIMATIONS

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This paper is dedicated to Professor Wataru Takahashi on the occasion of his 70th Birthday.

Abstract. Asian options are path dependent derivatives that have payoffs depending on some form of averaged prices of the underlying asset. The valuation of Asian options is complicated and no closed form solution exists, in general, due to the fact that the distribution of the arithmetic average is no longer lognormal. Several analytic approaches have been proposed in the literature, including, among others, partial differential equations, Laplace and Fourier transforms, and analytic approximations. In this paper we derive new analytic approximate formulas for the pricing of Asian options with arithmetic averages via higher order Taylor approximations. The resulted formulas are in closed form. Comparisons with the first- and quadratic-order approximations are included.

1. Introduction

Asian options are path dependent derivatives whose payoff depends on some form of averaging prices of the underlying asset. Since Black and Scholes (1973) [3] and Merton [24] introduced the pricing and hedging theory for the option market, their model has been the most popular model for option pricing. Pricing European options have a closed form called Black-Scholes formula. European options are path-independent. This means that their payoffs are solely determined by the price of the underlying asset at the expiration date of the options. However far insufficient from the need of the financial trading. Many exotic options have therefore been invented and traded in the financial markets. Among them are the Asian (or average rate) options. One important feature of an Asian option is that its payoff depends on the average of the underlying asset price over a certain time interval before and/or including the expiration time.

The valuation of Asian options is always complicated and no closed form solution exists, in general. The difficulties come from the fact that the distribution of the arithmetic average is no longer log normal and it is quite complex to analytically characterize it. Therefore the distribution cannot be given explicitly and the Black-Scholes method cannot be applied, so far no one has produced an exact formula for their values. Nevertheless, Asian options are popular in the financial community, because they are often cheaper than the equivalent classical European option.

Several analytic approaches have been proposed to price Asian options, we recall:

- Laplace transform approach, see [11, 17, 18], Dewynne and Shaw [12] found asymptotic solutions for the case of low volatility.

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- PDE approach, see Rogers and Shi [1,15,26] and Vecer [30] reduce the Asian pricing problem to one-dimensional PDE that can model both continuous and discrete average Asian option.
- Taylor approximations, see [4, 28] and Ju [19] utilizes the Taylor expansion of the ratio of the characteristic function of the average to that of the approximating lognormal variable around zero volatility.
- Fourier transform approach, see [8, 20].
- Approximation for the density, see [25, 29].
- Path integral approach, see [7] and Devreese [13] derive a closed-form solution for the price of an average strike as well as an average price geometric Asian option.
- Approximate analytic methods, see [31–33]. Most recently, Jianqiang [27] approximation of the arithmetic average with the geometric average of log-normal variables, Chang and Tsao [9] use quadratic approximated by the Chi-square distribution.

While most of the literature focuses on the log-normal dynamics and provides ad hoc methods for pricing Asian options in the special case of the Black-Scholes model, there are some notable exceptions given by the very recent papers in [21, 23] (for geometric Asian options), [2, 5] where models with jumps are considered, [6] consider on stochastic average and [16] consider in local volatility models.

In this paper, we develop approximations formulae expressed Taylor series for the pricing of arithmetic Asian options. The remainder of this paper is built up as follows. In Section 2, we giving some preliminary notions of stochastic analysis and pricing theory. We describe the arithmetic and geometric Asian options, and recall that symmetry results. In Section 3, we derive new analytic approximate formulas for the pricing of Asian option with arithmetic averages via higher order Taylor approximations. The resulting formulas are in closed form. Comparisons with first and quadratic orders are included in Section 4. Eventually we conclude prices of arithmetic Asian options in Section 5.

2. Prices of Asian options

We assume that the underlying asset (stock) S_t follows the geometric Brownian motion :

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t,$$

where $\widetilde{W}_t, 0 \leq t \leq T$, is a Brownian motion under the risk-neutral probability measure $\widetilde{\mathbb{P}}$, and the interest rate r and the volatility σ are both assumed to be constant.

Introducing the processes A_t and I_t defined by

$$A_t = \int_0^t S_u du, \quad I_t = \int_0^t \log S_u du,$$

we can classify Asian options as follows

(1) Arithmetic average floating strike call option:

$$V_T = \left(S_T - \frac{1}{T} A_T\right)^+.$$

(2) Arithmetic average fixed strike call option:

$$V_T = \Big(\frac{1}{T}A_T - K\Big)^+,$$

where K is the strike price.

(3) Geometric average floating strike call option:

$$V_T = \Big(S_T - e^{I_T/T}\Big)^+.$$

(4) Geometric average fixed strike call option:

$$V_T = \Big(e^{I_T/T} - K\Big)^+.$$

Consider the payoff of an Asian call option with arithmetic average floating strike is

$$V_T = \Big(S_T - \frac{1}{T}\int_0^T S_u du\Big)^+.$$

By usual no-arbitrage arguments, the value at time t of this option is

$$V_t = e^{-r(T-t)}\widetilde{\mathbb{E}}\Big[\Big(S_T - \frac{1}{T}\int_0^T S_u du\Big)^+\Big|\mathcal{F}_t\Big]. \tag{2.1}$$

The Black-Scholes equation for the arithmetic Asian option is

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial A} - rV = 0,\\ &V(T, S_T, A_T) = V_T, \end{aligned} \tag{2.2}$$

where S is a stock price, r is interest rate, σ is the volatility, T is the expiration date,

$$A = \int_0^t S_u du$$

is the running sum of the asset price, and V_T is the payoff function which depends on the type of the arithmetic Asian option.

By the Feynman-Kac representation, the price function V in (2.2) is the solution to the Cauchy problem. Under suitable regularity and growth conditions, the existence and the uniqueness of the solution to the Cauchy problem were proved in [1].

3. Arithmetic averages via Taylor series

Consider the payoff of an Asian call option with arithmetic average floating strike is

$$V_T = \Big(S_T - \frac{1}{T}\int_0^T S_u du\Big)^+.$$

The value at time t of Asian call option is

$$V_t = e^{-r(T-t)}\widetilde{\mathbb{E}}\Big[\Big(S_T - \frac{1}{T}\int_0^T S_u du\Big)^+\Big|\mathcal{F}_t\Big]. \tag{3.1}$$

Now we fix the $0 \le t < T$. We split the integral

$$\frac{1}{T}\int_0^T S_u du = M_t + \frac{1}{T}\int_t^T S_u du, \quad M_t = \frac{1}{T}\int_0^t S_u du.$$

The time-t value of the option is then given by

$$V_t = e^{-r(T-t)}\widetilde{\mathbb{E}}\Big[\Big(S_T - M_t - \frac{1}{T}\int_t^T S_u du\Big)^+\Big|\mathcal{F}_t\Big]. \tag{3.2}$$

Set

$$\begin{aligned} X_t &= e^{\sigma(\widetilde{W}_T-\widetilde{W}_t)+(r-\frac{1}{2}\sigma^2)(T-t)} \\ &\quad -\frac{1}{T}\int_t^T e^{\sigma(\widetilde{W}_u-\widetilde{W}_t)+(r-\frac{1}{2}\sigma^2)(u-t)}du. \end{aligned}$$

Since

$$S_u = S_t \exp\Big(\sigma(\widetilde{W}_u - \widetilde{W}_t) + \Big(r - \frac{1}{2}\sigma^2\Big)(u-t)\Big), \quad u \ge t$$

We have

$$\begin{aligned} S_T - M_t - \frac{1}{T}\int_t^T S_u du &= S_t e^{\sigma(\widetilde{W}_T-\widetilde{W}_t)+(r-\frac{1}{2}\sigma^2)(T-t)} - M_t \\ &\quad -\frac{1}{T}\int_t^T S_t e^{\sigma(\widetilde{W}_u-\widetilde{W}_t)+(r-\frac{1}{2}\sigma^2)(u-t)}du \\ &= S_t X_t - M_t \end{aligned}$$

and the pricing formula (3.2) can be rewritten as

$$V_t = e^{-r(T-t)}\widetilde{\mathbb{E}}\Big[\Big(S_t X_t - M_t\Big)^+\Big|\mathcal{F}_t\Big]. \tag{3.3}$$

Bouaziz et al.(1994) use a simple linearize procedure and propose an approximate closed-form solution to the pricing of floating-strike Asian options. Unfortunately their formula is not correct, there was a missing term in the pricing formula. Tsao et al. (2003) derive the correct Bouaziz et al. (1994) pricing formulae and then added the second-order term to the Taylor series using the normal distribution.

Bouaziz et al. (1994) linearize both exponential terms in X_t,

$$\begin{aligned} e^{\sigma(\widetilde{W}_T-\widetilde{W}_t)+(r-\frac{1}{2}\sigma^2)(T-t)} &\approx 1 + \Big(r - \frac{1}{2}\sigma^2\Big)(T-t) + \sigma(\widetilde{W}_T - \widetilde{W}_t). \\ e^{\sigma(\widetilde{W}_u-\widetilde{W}_t)+(r-\frac{1}{2}\sigma^2)(u-t)} &\approx 1 + \Big(r - \frac{1}{2}\sigma^2\Big)(u-t) + \sigma(\widetilde{W}_u - \widetilde{W}_t). \end{aligned}$$

In order to have efficient approximations, given the interest rate r and volatility σ, and fixed t and T, the term of $e^{(r-\frac{1}{2}\sigma^2)(T-t)}$ is a constant, we derive new liner approximates:

$$\begin{aligned} e^{\sigma(\widetilde{W}_T-\widetilde{W}_t)+(r-\frac{1}{2}\sigma^2)(T-t)} &\approx e^{(r-\frac{1}{2}\sigma^2)(T-t)}[1 + \sigma(\widetilde{W}_T - \widetilde{W}_t)]. \\ e^{\sigma(\widetilde{W}_u-\widetilde{W}_t)+(r-\frac{1}{2}\sigma^2)(u-t)} &\approx e^{(r-\frac{1}{2}\sigma^2)(u-t)}[1 + \sigma(\widetilde{W}_u - \widetilde{W}_t)]. \end{aligned}$$

Applying the new liner approximate to X_t as the following.

3.1. **First-Order Approximations.** To approximate V_t, we using liner approximate to the exponential function :
$e^x \approx 1 + x$, we get $\tau = T - t$, $\rho = r - \frac{1}{2}\sigma^2$,

$$\begin{aligned} X_t \approx X_{1,t} &= e^{\rho\tau}[1 + \sigma(\widetilde{W}_T - \widetilde{W}_t)] - \frac{1}{T}\int_t^T e^{\rho(u-t)}[1 + \sigma(\widetilde{W}_u - \widetilde{W}_t)]du \\ &= e^{\rho\tau}[1 + \sigma\widetilde{W}_\tau] - \frac{1}{T}\int_0^\tau e^{\rho v}[1 + \sigma\widetilde{W}_v]dv \quad ,let \quad v = u - t. \\ &= e^{\rho\tau}[1 + \sigma\widetilde{W}_\tau] - \frac{1}{\rho T}m_0 - \frac{\sigma}{T}\int_0^\tau e^{\rho v}\widetilde{W}_v dv, \end{aligned}$$

where $m_0 = \int_o^\tau e^{\rho v}dv$. We have

$$\mathbb{E}[X_{1,t}] = e^{\rho\tau} - \frac{1}{T}m_0 \tag{3.4}$$

To find the variance of $X_{1,t}$, we have

$$\mathrm{Var}(X_{1,t}) = \mathrm{Var}(X_{1,t} - \mathbb{E}[X_{1,t}]) = \mathbb{E}[(X_{1,t} - \mathbb{E}[X_{1,t}])^2],$$

where

$$X_{1,t} - \mathbb{E}[X_{1,t}] = \sigma e^{\rho\tau}\widetilde{W}_\tau - \frac{\sigma}{T}\int_0^\tau e^{\rho v}\widetilde{W}_v dv.$$

Next we compute

$$\mathrm{Var}(X_{1,t}) = \mathbb{E}\Big[\sigma^2 e^{2\rho\tau}\widetilde{W}_\tau^2 - \frac{2\sigma}{T}e^{\rho\tau}\widetilde{W}\int_0^\tau e^{\rho v}\widetilde{W}_v dv + \frac{\sigma^2}{T^2}\Big(\int_0^\tau e^{\rho v}\widetilde{W}_v dv\Big)^2\Big].$$

Since $\mathbb{E}[\widetilde{W}_u\widetilde{W}_v] = \min\{u, v\}$, we get

$$\begin{aligned} \mathbb{E}\Big[\int_0^\tau e^{\rho v}\widetilde{W}_v dv\Big]^2 &= \mathbb{E}\Big[\int_0^\tau\int_0^\tau e^{\rho(u+v)}\widetilde{W}_u\widetilde{W}_v dudv\Big] \\ &= \int_0^\tau\int_0^\tau e^{\rho(u+v)}\mathbb{E}\Big[\widetilde{W}_u\widetilde{W}_v\Big]dudv \\ &= \int_0^\tau\int_0^\tau e^{\rho(u+v)}\min\{u, v\}dudv \\ &= \int_0^\tau e^{\rho v}\Big\{\int_0^v ue^{\rho u}du + \int_v^\tau ve^{\rho u}du\Big\}dv \\ &= \int_0^\tau e^{\rho v}\Big\{\frac{ve^{\rho v}}{\rho} - \frac{e^{\rho v} - 1}{\rho^2} + \frac{v}{\rho}(e^{\rho\tau} - e^{\rho v})\Big\}dv \\ &= \frac{1}{\rho^2}\int_0^\tau e^{\rho v}dv - \frac{1}{\rho^2}\int_0^\tau e^{2\rho v}dv + \frac{e^{\rho\tau}}{\rho}\int_0^\tau ve^{\rho v}dv \\ &= \frac{1}{\rho^2}(m_0 - m_3) + \frac{e^{\rho\tau}}{\rho}m_1 \qquad (3.5) \end{aligned}$$

Let

$$m_1 = \int_o^\tau ve^{\rho v}dv, \quad m_2 = \int_o^\tau v^2 e^{\rho v}dv, \quad m_3 = \int_o^\tau e^{2\rho v}dv.$$

Then we have

$$\operatorname{Var}(X_{1,t}) = e^{2\rho\tau}\sigma^2\tau + \frac{\sigma^2}{T^2}\Big(\frac{1}{\rho^2}(m_0 - m_3) + \frac{e^{\rho\tau}}{\rho}m_1\Big)$$

$$= e^{2\rho\tau}\sigma^2\tau + \frac{\sigma^2}{T^2}\Big(\frac{-1 + 4e^{\rho\tau} + e^{2\rho\tau}(2\rho\tau - 3)}{2\rho^3}\Big) \tag{3.6}$$

Let

$$\alpha_{1,t} = \mathbb{E}[X_{1,t}], \quad \nu_{1,t} = \sqrt{\operatorname{Var}(X_{1,t})}$$

Then we can write

$$X_{1,t} = \alpha_{1,t} + \nu_{1,t}Z, \quad Z \sim N(0,1) \quad \text{independent of} \quad \mathcal{F}_t.$$

The value V_t in (3.3) is approximated by $V_{1,t}$ as follows:

$$V_t \approx V_{1,t} = e^{-r(T-t)}\widetilde{\mathbb{E}}[(S_t X_{1,t} - M_t)^+|\mathcal{F}_t]$$

$$= e^{-r(T-t)}\widetilde{\mathbb{E}}[(S_t(\alpha_{1,t} + \nu_{1,t}Z) - M_t)^+|\mathcal{F}_t]. \tag{3.7}$$

By the independence lemma, we get

$$V_{1,t} = v^*(t, S_t, M_t),$$

where

$$v^*(t,x,y) = e^{-r(T-t)}\widetilde{\mathbb{E}}[(x(\alpha_{1,t} + \nu_{1,t}Z) - y)^+].$$

We therefor have

$$v^*(t,x,y) = e^{-r\tau}\int_{-\infty}^{\infty}(x(\alpha_{1,t} + \nu_{1,t}z) - y)^+\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz$$

Solving the inequality for z:

$$x(\alpha_{1,t} + \nu_{1,t}z) - y > 0$$

yields

$$z > \frac{1}{\nu_{1,t}}\Big(\frac{y}{x} - \alpha_{1,t}\Big) =: \rho(t,x,y).$$

So we obtain, with $\tau = T - t$,

$$\begin{aligned} v^*(t,x,y) &= e^{-r\tau}\int_{\rho(t,x,y)}^{\infty}\frac{1}{\sqrt{2\pi}}(x(\alpha_{1,t} + \nu_{1,t}z) - y)e^{-\frac{1}{2}z^2}dz \\ &= e^{-r\tau}(\alpha_{1,t}x - y)\int_{\rho(t,x,y)}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz \\ &\quad + e^{-r\tau}\nu_{1,t}x\int_{\rho(t,x,y)}^{\infty}\frac{1}{\sqrt{2\pi}}ze^{-\frac{1}{2}z^2}dz \\ &= e^{-r\tau}(\alpha_{1,t}x - y)N(-\rho(t,x,y)) \\ &\quad + e^{-r\tau}\nu_{1,t}x\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\rho(t,x,y)^2}. \end{aligned}$$

Therefore, eventually, we get

$$V_t \approx V_{1,t} = e^{-r(T-t)}\Big\{(\alpha_{1,t}S_t - M_t)N\Big(\frac{1}{\nu_{1,t}}\Big(\alpha_{1,t} - \frac{M_t}{S_t}\Big)\Big)$$

$$+\frac{\nu_{1,t}S_t}{\sqrt{2\pi}}\exp\Big(\frac{-1}{2(\nu_{1,t})^2}\Big(\frac{M_t}{S_t}-\alpha_{1,t}\Big)^2\Big)\Big\}.$$

3.2. **Quadratic Approximations.** In this subsection we will consider quadratic expansions:

$$e^x \approx 1+x+\frac{1}{2}x^2.$$

Tsao et al. (2003) added the second terms to the Taylor series, as follows:

$$\begin{aligned} e^{\sigma(\widetilde{W}_T-\widetilde{W}_t)+(r-\frac{1}{2}\sigma^2)(T-t)} \quad &\approx \quad 1+\Big(r-\frac{1}{2}\sigma^2\Big)(T-t)+\sigma(\widetilde{W}_T-\widetilde{W}_t) \\ &+\frac{1}{2}\Big[\Big(r-\frac{1}{2}\sigma^2\Big)^2(T-t)^2+\sigma^2(\widetilde{W}_T-\widetilde{W}_t^2)\Big] \\ &+\sigma\Big(r-\frac{1}{2}\sigma^2\Big)(T-t)(\widetilde{W}_T-\widetilde{W}_t). \end{aligned}$$

Instead of expansion X_t we use the new quadratic approximation to derive more accurate approximate pricing formulae for Asian options.

$$e^{\rho\tau+\sigma\widetilde{W}_\tau} \quad \approx \quad e^{\rho\tau}\Big[1+\sigma\widetilde{W}_\tau+\frac{1}{2}\sigma^2\widetilde{W}_\tau^2\Big].$$

We obtain

$$\begin{aligned} X_t \approx X_{2,t} \quad &= \quad e^{\rho\tau}\Big[1+\sigma\widetilde{W}_\tau+\frac{1}{2}\sigma^2\widetilde{W}_\tau^2\Big]-\frac{1}{T}\int_0^\tau e^{\rho v}\Big[1+\sigma\widetilde{W}_v+\frac{1}{2}\sigma^2\widetilde{W}_v^2\Big]dv \\ &= \quad e^{\rho\tau}\Big[1+\sigma\widetilde{W}_\tau+\frac{1}{2}\sigma^2\widetilde{W}_\tau^2+\Big]-\frac{1}{T}m_0 \\ &\quad -\frac{\sigma}{T}\int_0^\tau e^{\rho v}\widetilde{W}_v dv-\frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}\widetilde{W}_v^2 dv \end{aligned}$$

We have

$$\mathbb{E}[X_{2,t}] \equiv \alpha_{2,t} = e^{\rho\tau}\Big(1+\frac{1}{2}\sigma^2\tau\Big)-\frac{1}{T}\Big(m_0+\frac{1}{2}\sigma^2 m_1\Big) \tag{3.8}$$

To find the variance of $X_{2,t}$, we have

$$\begin{aligned} X_{2,t}-\mathbb{E}[X_{2,t}] &= \frac{\sigma^2}{2}\Big(\frac{m_1}{T}-\tau e^{\rho\tau}\Big)+e^{\rho\tau}\Big[\sigma\widetilde{W}_\tau+\frac{1}{2}\sigma^2\widetilde{W}_\tau^2\Big] \\ &\quad -\frac{\sigma}{T}\int_0^\tau e^{\rho v}\widetilde{W}_v dv-\frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}\widetilde{W}_v^2 dv \\ &\equiv I_1-I_2 \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{\sigma^2}{2}\Big(\frac{m_1}{T}-\tau e^{\rho\tau}\Big)+e^{\rho\tau}\Big[\sigma\widetilde{W}_\tau+\frac{1}{2}\sigma^2\widetilde{W}_\tau^2\Big], \\ I_2 &= \frac{1}{T}\Big(\int_0^\tau e^{\rho v}\sigma\widetilde{W}_v dv+\int_0^\tau e^{\rho v}\frac{\sigma^2}{2}\widetilde{W}_v^2 dv\Big). \end{aligned}$$

Next we have $(X_{2,t}-\alpha_{t,2})^2 = I_1^2-2I_1I_2+I_2^2$. Moreover,

$$I_1^2 = \frac{\sigma^4}{4}\Big(\frac{m_1}{T}-\tau e^{\rho\tau}\Big)^2+\sigma^2\Big(\frac{m_1}{T}-\tau e^{\rho\tau}\Big)e^{\rho\tau}\Big[\sigma\widetilde{W}_\tau+\frac{1}{2}\sigma^2\widetilde{W}_\tau^2\Big]$$

$$+ e^{2\rho\tau}\Big[\sigma^2\widetilde{W}_\tau^2 + \sigma^3\widetilde{W}_\tau^3 + \frac{1}{4}\sigma^4\widetilde{W}_\tau^4\Big].$$

Since

$$\mathbb{E}[\widetilde{W}_t^{2k}] = \frac{(2k)!}{2^k k!}t^k \quad \text{for integer} \quad k \geq 1,$$
$$\mathbb{E}[\widetilde{W}_t^k] = 0 \quad \text{for odd integer} \quad k \geq 1.$$

We have

$$\mathbb{E}[I_1^2] = \frac{\sigma^4}{4}\Big(\frac{m_1}{T} - \tau e^{\rho\tau}\Big)^2 + \frac{\sigma^4\tau}{2}e^{\rho\tau}\Big(\frac{m_1}{T} - \tau e^{\rho\tau}\Big) + e^{2\rho\tau}\Big(\sigma^2\tau + \frac{3}{4}\sigma^4\tau^2\Big). \tag{3.9}$$

$$I_2^2 = \frac{1}{T^2}\Big(\Big(\int_0^\tau e^{\rho v}\sigma\widetilde{W}_v dv\Big)^2 + 2\int_0^\tau e^{\rho v}\sigma\widetilde{W}_v dv \cdot \int_0^\tau e^{\rho v}\frac{\sigma^2}{2}\widetilde{W}_v^2 dv$$
$$+ \Big(\int_0^\tau e^{\rho v}\frac{\sigma^2}{2}\widetilde{W}_v^2 dv\Big)^2\Big).$$

Since

$$\mathbb{E}[\widetilde{W}_u^2\widetilde{W}_v^2] = 2(\min\{u,v\})^2 + uv,$$

we get

$$\mathbb{E}\Big[\int_0^\tau e^{\rho v}\widetilde{W}_v^2 dv\Big]^2 = \int_0^\tau\int_0^\tau e^{\rho(u+v)}\mathbb{E}\Big[\widetilde{W}_u^2\widetilde{W}_v^2\Big]dudv$$
$$= \int_0^\tau\int_0^\tau e^{\rho(u+v)}[2(\min\{u,v\})^2 + uv]dudv$$
$$= 2\int_0^\tau e^{\rho v}\Big\{\int_0^v u^2 e^{\rho u}du + v^2\int_v^\tau e^{\rho u}du\Big\}dv + m_1^2$$
$$= 2\int_0^\tau e^{\rho v}\Big\{\frac{1}{\rho}\Big[\Big(v^2 - \frac{2v}{\rho} + \frac{2}{\rho^2}\Big)e^{\rho v} - \frac{2}{\rho^2}\Big] + \frac{v^2}{\rho}(e^{\rho\tau} - e^{\rho v})\Big\}dv + m_1^2$$
$$= \frac{2}{\rho}\Big(-\frac{2}{\rho}m_4 + \frac{2}{\rho^2}m_3 - \frac{2}{\rho}m_0 + e^{\rho\tau}m_2\Big) + m_1^2, \tag{3.10}$$

where

$$m_4 = \int_o^\tau v e^{2\rho v}dv.$$

Therefore, by (3.5) and (3.10), we obtain

$$\mathbb{E}[I_2^2] = \frac{1}{T^2}\Big(\frac{\sigma^2}{\rho^2}\Big(m_0 - m_3 + \frac{e^{\rho\tau}}{\rho}m_1\Big)$$
$$+ \frac{\sigma^4}{2\rho}\Big(-\frac{2}{\rho}m_4 + \frac{2}{\rho^2}m_3 - \frac{2}{\rho}m_0 + e^{\rho\tau}m_2\Big) + \frac{\sigma^4}{4}m_1^2\Big). \tag{3.11}$$

To evaluate $\mathbb{E}[I_1 I_2]$, we have

$$I_1 I_2 = [A + B\widetilde{W}_\tau + C\widetilde{W}_\tau^2]I_2. \tag{3.12}$$

where

$$A = \frac{\sigma^2}{2}\Big(\frac{m_1}{T} - \tau e^{\rho\tau}\Big), \quad B = e^{\rho\tau}\sigma, \quad C = \frac{1}{2}\sigma^2.$$

$$\mathbb{E}[I_2] = \frac{\sigma^2}{2T}\int_0^\tau v e^{\rho v} dv = \frac{\sigma^2}{2T} m_1. \tag{3.13}$$

$$\widetilde{W}_\tau I_2 = \frac{\sigma}{T}\int_0^\tau e^{\rho v}(\widetilde{W}_{T-u} + \widetilde{W}_v)\widetilde{W}_v dv + \frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}(\widetilde{W}_{T-u} + \widetilde{W}_v)\widetilde{W}_v^2 dv.$$

Since $\widetilde{W}_{T-u}$ and $\widetilde{W}_{u-t}\,(=\widetilde{W}_v)$ are independent, and since $\mathbb{E}[\widetilde{W}_t^k] = 0$ for odd integer $k \geq 1$, we arrive that

$$\mathbb{E}[\widetilde{W}_\tau I_2] = \frac{\sigma}{T}\int_0^\tau v e^{\rho v} dv = \frac{\sigma}{T} m_1. \tag{3.14}$$

$$\widetilde{W}_\tau^2 I_2 = \frac{\sigma}{T}\int_0^\tau e^{\rho v}(\widetilde{W}_{T-u} + \widetilde{W}_v)^2\widetilde{W}_v dv + \frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}(\widetilde{W}_{T-u} + \widetilde{W}_v)^2\widetilde{W}_v^2 dv.$$

where

$$(\widetilde{W}_{T-u} + \widetilde{W}_v)^2 = \widetilde{W}_{T-u}^2 + 2\widetilde{W}_{T-u}\widetilde{W}_v + \widetilde{W}_v^2.$$

It turns out that

$$\begin{aligned}
\mathbb{E}[\widetilde{W}_\tau^2 I_2] &= \frac{\sigma^2}{2T}\mathbb{E}\Big[\int_0^\tau e^{\rho v}\Big(\widetilde{W}_{T-u}^2 + 2\widetilde{W}_{T-u}\widetilde{W}_v + \widetilde{W}_v^2\Big)\widetilde{W}_v^2 dv\Big] \\
&= \frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}((\tau-u)v + 3v^2) dv \\
&= \frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}(\tau v + 2v^2) dv \\
&= \frac{\sigma^2}{2T}(\tau m_1 + 2m_2). \qquad (3.15)
\end{aligned}$$

From (3.13)-(3.15), we arrive at

$$\begin{aligned}
\mathbb{E}[I_1 I_2] &= A\mathbb{E}[I_2] + B\mathbb{E}[\widetilde{W}_\tau I_2] + C\mathbb{E}[\widetilde{W}_\tau^2 I_2] \\
&= A\frac{\sigma^2}{2T}m_1 + B\frac{\sigma}{T}m_1 + C\frac{\sigma^2}{2T}\Big((T-u)m_1 + 3m_2\Big) \\
&= \Big(\frac{\sigma^4}{4T}\Big(\frac{m_1}{T} + \tau(1-e^{\rho\tau})\Big) + \frac{\sigma^2}{T}e^{\rho\tau}\Big)m_1 + \frac{\sigma^4}{2T}m_2 \qquad (3.16)
\end{aligned}$$

Now, we get by combining (3.9), (3.11) and (3.16),

$$\begin{aligned}
\mathrm{Var}[X_{2,t}] &= \mathbb{E}[I_1^2] - 2\mathbb{E}[I_1 I_2] + \mathbb{E}[I_2^2] \\
&= \frac{\sigma^4}{4}\Big(\frac{m_1}{T} - \tau e^{\rho\tau}\Big)^2 + \frac{\sigma^4\tau}{2}e^{\rho\tau}\Big(\frac{m_1}{T} - \tau e^{\rho\tau}\Big) + e^{2\rho\tau}\Big(\sigma^2\tau + \frac{3}{4}\sigma^4\tau^2\Big) \\
&\quad - \Big(\frac{\sigma^4}{2T}\Big(\frac{m_1}{T} + \tau(1-e^{\rho\tau})\Big) + \frac{2\sigma^2}{T}e^{\rho\tau}\Big)m_1 - \frac{\sigma^4}{T}m_2 \\
&\quad + \frac{1}{T^2}\Big(\frac{\sigma^2}{\rho^2}\Big(m_0 - m_3 + \frac{e^{\rho\tau}}{\rho}m_1\Big) \\
&\quad + \frac{\sigma^4}{2\rho}\Big(-\frac{2}{\rho}m_4 + \frac{2}{\rho^2}m_3 - \frac{2}{\rho}m_0 + e^{\rho\tau}m_2\Big) + \frac{\sigma^4}{4}m_1^2\Big)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sigma^2}{4\rho^5 T^2}\{\rho - 3\sigma^2 + 3\rho\sigma^2 + \rho^2(-2+8\sigma^2 T) - 2\rho^3\sigma^2 T\tau \\
&\quad - 2e^{\rho\tau}[2(-1+\sigma^2) + \rho(1+\sigma^2) + 3\rho^4\sigma^2 T\tau^2 + \rho^3 T(4+\tau-7\sigma^2\tau) \\
&\quad + \rho^2(-2-\tau+\sigma^2(4T+\tau))] \\
&\quad + e^{2\rho\tau}[-4+7\sigma^2 + 2\rho^2(1+\tau)(-1+\sigma^2\tau) + \rho^5 T^2\tau(4+\tau+\sigma^2\tau) \\
&\quad + 2\rho^4 T\tau(-4+(-1+2\sigma^2)\tau) \\
&\quad - \rho(-1-4\tau+\sigma^2(1+6\tau)) + \rho^3(-(-1+\sigma^2)\tau^2 \\
&\quad + T(8+2\tau-4\sigma^2\tau))]\} \qquad (3.17)
\end{aligned}$$

Let

$$\mathbb{E}[X_{2,t}] = \alpha_{2,t}, \quad \mathrm{Var}(X_{2,t}) = \nu_{2,t}^2.$$

Then we can write

$$X_{2,t} = \alpha_{2,t} + \nu_{2,t} Z, \quad Z \sim N(0,1) \quad \text{independent of } \mathcal{F}_t.$$

The value V_t in (3.3) is approximated by $V_{2,t}$ as follows:

$$\begin{aligned}
V_t \approx V_{2,t} &= e^{-r(T-t)}\widetilde{\mathbb{E}}[(S_t X_{2,t} - M_t)^+ | \mathcal{F}_t] \\
&= e^{-r(T-t)}\widetilde{\mathbb{E}}[(S_t(\alpha_{2,t} + \nu_{2,t} Z) - M_t)^+ | \mathcal{F}_t].
\end{aligned}$$

Similarity, eventually , we get

$$\begin{aligned}
V_t \approx V_{2,t} = e^{-r(T-t)}\Big\{&(\alpha_{2,t} S_t - M_t) N\Big(\frac{1}{\nu_{2,t}}\Big(\alpha_{2,t} - \frac{M_t}{S_t}\Big)\Big) \\
&+ \frac{\nu_{2,t} S_t}{\sqrt{2\pi}} \exp\Big(\frac{-1}{2(\nu_{2,t})^2}\Big(\frac{M_t}{S_t} - \alpha_{2,t}\Big)^2\Big)\Big\}.
\end{aligned}$$

3.3. **Third-Order Approximations.** Intuitively, in order to approach more accurate pricing formulae for Asian options, incorporating higher order terms should once again improve the approximation. We use third-order approximate to the exponential function :

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3.$$

To approximate V_t, we adding third-order term to the Taylor series for both exponential terms in X_t,

$$\begin{aligned}
e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(\widetilde{W}_T - \widetilde{W}_t)} \;\approx\; & e^{(r-\frac{1}{2}\sigma^2)(T-t)}\Big[1 + \sigma(\widetilde{W}_T - \widetilde{W}_t) \\
&+ \frac{1}{2}\sigma^2(\widetilde{W}_T - \widetilde{W}_t)^2 + \frac{1}{6}\sigma^3(\widetilde{W}_T - \widetilde{W}_t)^3\Big]. \\
e^{(r-\frac{1}{2}\sigma^2)(u-t)+\sigma(\widetilde{W}_u - \widetilde{W}_t)} \;\approx\; & e^{(r-\frac{1}{2}\sigma^2)(u-t)}\Big[1 + \sigma(\widetilde{W}_u - \widetilde{W}_t) \\
&+ \frac{1}{2}\sigma^2(\widetilde{W}_u - \widetilde{W}_t)^2 + \frac{1}{6}\sigma^3(\widetilde{W}_u - \widetilde{W}_t)^3\Big].
\end{aligned}$$

Let $\tau = T - t$ and $\rho = r - \frac{1}{2}\sigma^2$. We obtain

$$X_t \approx X_{3,t} \;=\; e^{\rho\tau}\Big[1 + \sigma(\widetilde{W}_T - \widetilde{W}_t) + \frac{1}{2}\sigma^2(\widetilde{W}_T - \widetilde{W}_t)^2 + \frac{1}{6}\sigma^3(\widetilde{W}_T - \widetilde{W}_t)^3\Big]$$

$$
\begin{aligned}
&-\frac{1}{T}\int_t^T e^{\rho(u-t)}\Big[1+\sigma(\widetilde{W}_u-\widetilde{W}_t)+\frac{1}{2}\sigma^2(\widetilde{W}_u-\widetilde{W}_t)^2\\
&+\frac{1}{6}\sigma^3(\widetilde{W}_u-\widetilde{W}_t)^3\Big]du\\
= \;& e^{\rho\tau}\Big[1+\sigma\widetilde{W}_\tau+\frac{1}{2}\sigma^2\widetilde{W}_\tau^2+\frac{1}{6}\sigma^3\widetilde{W}_\tau^3\Big]\\
&-\frac{1}{T}\int_0^\tau e^{\rho v}\Big[1+\sigma\widetilde{W}_v+\frac{1}{2}\sigma^2\widetilde{W}_v^2+\frac{1}{6}\sigma^3\widetilde{W}_v^3\Big]dv \quad (v:=u-t)\\
= \;& e^{\rho\tau}\Big[1+\sigma\widetilde{W}_\tau+\frac{1}{2}\sigma^2\widetilde{W}_\tau^2+\frac{1}{6}\sigma^3\widetilde{W}_\tau^3\Big]+\frac{1}{\rho T}(1-e^{\rho\tau})\\
&-\frac{\sigma}{T}\int_0^\tau e^{\rho v}\widetilde{W}_v dv-\frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}\widetilde{W}_v^2 dv-\frac{\sigma^3}{6T}\int_0^\tau e^{\rho v}\widetilde{W}_v^3 dv\\
\equiv \;& J_1-J_2
\end{aligned}
$$

where

$$J_1=e^{\rho\tau}\Big[1+\sigma\widetilde{W}_\tau+\frac{1}{2}\sigma^2\widetilde{W}_\tau^2+\frac{1}{6}\sigma^3\widetilde{W}_\tau^3\Big]+\frac{1}{\rho T}(1-e^{\rho\tau}), \tag{3.18}$$

$$J_2=\frac{\sigma}{T}\int_0^\tau e^{\rho v}\widetilde{W}_v dv+\frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}\widetilde{W}_v^2 dv+\frac{\sigma^3}{6T}\int_0^\tau e^{\rho v}\widetilde{W}_v^3 dv. \tag{3.19}$$

Let

$$m_5=\int_o^\tau v^2e^{2\rho v}dv, \quad m_6=\int_o^\tau v^3e^{\rho v}dv, \quad m_7=\int_o^\tau v^3e^{2\rho v}dv.$$

We have

$$
\begin{aligned}
\mathbb{E}[X_{3,t}] &= e^{\rho\tau}\Big(1+\frac{1}{2}\sigma^2\tau\Big)+\frac{1}{\rho T}(1-e^{\rho\tau})\\
&\quad-\frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}\mathbb{E}[\widetilde{W}_v^2]dv\\
&= e^{\rho\tau}\Big(1+\frac{1}{2}\sigma^2\tau\Big)+\frac{1}{\rho T}(1-e^{\rho\tau})-\frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}v dv\\
&= e^{\rho\tau}\Big(1+\frac{1}{2}\sigma^2\tau\Big)+\frac{1}{\rho T}(1-e^{\rho\tau})-\frac{\sigma^2 m_1}{2T}. \qquad (3.20)
\end{aligned}
$$

Next we compute $\mathbb{E}[(X_{3,t})^2]$. We have $(X_{3,t})^2=J_1^2-2J_1J_2+J_2^2$. Moreover,

$$
\begin{aligned}
J_1^2 &= \Big[e^{\rho\tau}+\frac{1}{\rho T}(1-e^{\rho\tau})\Big]^2\\
&\quad+\sigma^2e^{2\rho\tau}\Big[\widetilde{W}_\tau+\frac{\sigma}{2}\widetilde{W}_\tau^2+\frac{\sigma^2}{6}\widetilde{W}_\tau^3\Big]^2\\
&\quad+2\sigma e^{\rho\tau}\Big[e^{\rho\tau}+\frac{1}{\rho T}(1-e^{\rho\tau})\Big]\cdot\Big[\widetilde{W}_\tau+\frac{\sigma}{2}\widetilde{W}_\tau^2+\frac{\sigma^2}{6}\widetilde{W}_\tau^3\Big].
\end{aligned}
$$

We have

$$\mathbb{E}[J_1^2]=\Big[e^{\rho\tau}+\frac{1}{\rho T}(1-e^{\rho\tau})\Big]^2+\sigma^2e^{2\rho\tau}\Big[\tau+\frac{3}{4}\sigma^2\tau^2+\frac{5}{12}\sigma^4\tau^3+\sigma^2\tau^2\Big]$$

$$+\sigma^2\tau e^{\rho\tau}\Big[e^{\rho\tau}+\frac{1}{\rho T}(1-e^{\rho\tau})\Big]. \tag{3.21}$$

$$\begin{aligned}J_2^2=&\frac{\sigma^2}{T^2}\Big[\int_0^\tau e^{\rho v}\widetilde{W}_v dv\Big]^2+\frac{\sigma^4}{4T^2}\Big[\int_0^\tau e^{\rho v}\widetilde{W}_v^2 dv\Big]^2+\frac{\sigma^6}{36T^2}\Big[\int_0^\tau e^{\rho v}\widetilde{W}_v^3 dv\Big]^2\\&+2\Big[\frac{\sigma^3}{2T^2}\int_0^\tau e^{\rho v}\widetilde{W}_v dv\cdot\int_0^\tau e^{\rho v}\widetilde{W}_v^2 dv+\frac{\sigma^5}{12T^2}\int_0^\tau e^{\rho v}\widetilde{W}_v^2 dv\cdot\int_0^\tau e^{\rho v}\widetilde{W}_v^3 dv\\&+\frac{\sigma^4}{6T^2}\int_0^\tau e^{\rho v}\widetilde{W}_v dv\cdot\int_0^\tau e^{\rho v}\widetilde{W}_v^3 dv\Big].\end{aligned} \tag{3.22}$$

Indeed, we have , for $\rho\neq 0$,

$$m_1=\int_0^\tau ue^{\rho u}=\frac{1}{\rho}\Big[\Big(\tau-\frac{1}{\rho}\Big)e^{\rho\tau}+\frac{1}{\rho}\Big],$$

$$m_2=\int_0^\tau u^2e^{\rho u}=\frac{1}{\rho}\Big[\Big(\tau^2-\frac{2\tau}{\rho}+\frac{2}{\rho^2}\Big)e^{\rho\tau}-\frac{2}{\rho^2}\Big],$$

$$m_6=\int_0^\tau u^3e^{\rho u}=\frac{1}{\rho}\Big[\frac{6}{\rho^3}+\Big(\frac{-6}{\rho^3}+\frac{6\tau}{\rho^2}-\frac{3\tau^2}{\rho}+\tau^3\Big)e^{\rho\tau}\Big].$$

Since it is not hard to find that

$$\mathbb{E}[\widetilde{W}_u\widetilde{W}_v^2]=0\quad,\quad\mathbb{E}[\widetilde{W}_u^2\widetilde{W}_v^3]=0\quad\text{and}\quad\mathbb{E}[\widetilde{W}_u\widetilde{W}_v^3]=3v\min\{u,v\}.$$

We get

$$\begin{aligned}&\mathbb{E}\Big[\int_0^\tau\int_0^\tau e^{\rho(u+v)}\widetilde{W}_u\widetilde{W}_v^3dv\Big]=\int_0^\tau\int_0^\tau e^{\rho(u+v)}\mathbb{E}\Big[\widetilde{W}_u\widetilde{W}_v^3\Big]dvdu\\&=\int_0^\tau\int_0^\tau e^{\rho(u+v)}\cdot 3v\min\{u,v\}dudv\\&=\int_0^\tau e^{\rho v}\Big(\int_0^v 3vue^{\rho u}du+\int_v^\tau 3v^2e^{\rho u}du\Big)dv\\&=\int_0^\tau 3ve^{\rho v}\frac{1}{\rho}\Big(\Big(v-\frac{1}{\rho}\Big)e^{\rho v}+\frac{1}{\rho}\Big)+3v^2e^{\rho v}\Big(\frac{1}{\rho}(e^{\rho\tau}-e^{\rho v})\Big)dv\\&=\int_0^\tau\Big(\frac{3v^2e^{2\rho v}}{\rho}-\frac{3ve^{2\rho v}}{\rho^2}+\frac{3ve^{\rho v}}{\rho^2}+\frac{3v^2e^{\rho(v+\tau)}}{\rho}-\frac{3v^2e^{2\rho v}}{\rho}\Big)dv\\&=\frac{-3}{\rho^2}m_4+\frac{3}{\rho^2}m_1+\frac{3e^{\rho\tau}}{\rho^2}m_5.\end{aligned} \tag{3.23}$$

Since

$$\mathbb{E}[\widetilde{W}_u^3\widetilde{W}_v^3]=3\Big(3uv\min\{u,v\}+2(\min\{u,v\})^3\Big),$$

we get

$$\begin{aligned}&\mathbb{E}\Big[\int_0^\tau e^{\rho v}\widetilde{W}_v^3dv\Big]^2=\int_0^\tau\int_0^\tau e^{\rho(u+v)}\mathbb{E}\Big[\widetilde{W}_u^3\widetilde{W}_v^3\Big]dvdu\\&=3\int_0^\tau\int_0^\tau e^{\rho(u+v)}\Big(3uv\min\{u,v\}+2(\min\{u,v\})^3\Big)dvdu\\&=9\int_0^\tau e^{\rho u}\Big(\int_0^u e^{\rho v}uv^2dv+\int_u^\tau e^{\rho v}u^2vdv\Big)du\end{aligned}$$

$$
\begin{aligned}
&+ 6\int_0^\tau e^{\rho u}\Big(\int_0^u e^{\rho v} v^3 dv + \int_u^\tau e^{\rho v} u^3 dv\Big) du \\
&= 9\int_0^\tau u e^{\rho u}\Big(\frac{e^{\rho u}}{\rho}\Big(u^2 - \frac{2u}{\rho} + \frac{2}{\rho^2}\Big) - \frac{2}{\rho^2}\Big) du \\
&+ 9\int_0^\tau u^2 e^{\rho u}\Big(\frac{\tau e^{\rho\tau}}{\rho} - \frac{u e^{\rho u}}{\rho} - \frac{e^{\rho\tau}}{\rho^2} + \frac{e^{\rho u}}{\rho^2}\Big) du \\
&+ 6\int_0^\tau e^{\rho u}\Big(\int_0^u v^3 e^{\rho v} dv\Big) du \\
&+ 6\int_0^\tau u^3 e^{\rho u}\Big(\frac{1}{\rho}(e^{\rho\tau} - e^{\rho u})\Big) du \\
&= 9\int_0^\tau \Big(\frac{u^3 e^{2\rho u}}{\rho} - \frac{2u^2 e^{2\rho u}}{\rho^2} + \frac{2u e^{2\rho u}}{\rho^3} - \frac{2u e^{\rho u}}{\rho^2}\Big) du \\
&+ 9\int_0^\tau \Big(\frac{u^2 \tau e^{\rho(u+\tau)}}{\rho} - \frac{u^3 e^{2\rho u}}{\rho} - \frac{u^2 e^{\rho(u+\tau)}}{\rho^2} + \frac{u^2 e^{2\rho u}}{\rho^2}\Big) du \\
&+ 6\int_0^\tau \Big(\frac{6e^{\rho u}}{\rho^4} - \frac{6e^{\rho(u+\tau)}}{\rho^4} + \frac{6\tau e^{\rho(u+\tau}}{\rho^3} - \frac{3\tau^2 e^{\rho(u+\tau)}}{\rho^2} + \frac{\tau^3 e^{\rho(u+\tau)}}{\rho}\Big) du \\
&+ 6\int_0^\tau \Big(\frac{u^3 e^{\rho(u+\tau)}}{\rho} - \frac{u^3 e^{2\rho u}}{\rho}\Big) du \\
&= 9\Big(-\frac{2m_2}{\rho^2} + \frac{2m_4}{\rho^3} - \frac{2m_1}{\rho^2} + \frac{\tau e^{\rho\tau} m_2}{\rho} - \frac{e^{\rho\tau} m_2}{\rho^2} + \frac{m_5}{\rho^2}\Big) \\
&+ 6\Big(\frac{6(1 - e^{\rho\tau})}{\rho^4} + \frac{6\tau e^{\rho\tau}}{\rho^3} - \frac{3\tau^2 e^{\rho\tau}}{\rho^2} + \frac{\tau^3 e^{\rho\tau}}{\rho}\Big) m_0 \\
&+ 6\Big(\frac{e^{\rho\tau}}{\rho} m_6 - \frac{1}{\rho} m_7\Big). \qquad (3.24)
\end{aligned}
$$

Therefore, by (3.5), (3.10), (3.22), (3.23) and (3.24), we obtain

$$
\begin{aligned}
\mathbb{E}[J_2^2 &= \frac{\sigma^2}{T^2}\Big(\frac{1}{\rho^2}(m_0 - m_3) + \frac{e^{\rho\tau}}{\rho} m_1\Big) \\
&+ \frac{\sigma^4}{4T^2}\Big[\frac{2}{\rho}\Big(-\frac{2}{\rho} m_4 + \frac{2}{\rho^2} m_3 - \frac{2}{\rho} m_0 + e^{\rho\tau} m_2\Big) + m_1^2\Big] \\
&+ \frac{\sigma^6}{4T^2}\Big(-\frac{2m_2}{\rho^2} + \frac{2m_4}{\rho^3} - \frac{2m_1}{\rho^2} + \frac{\tau e^{\rho\tau} m_2}{\rho} - \frac{e^{\rho\tau} m_2}{\rho^2} + \frac{m_5}{\rho^2}\Big) \\
&+ \frac{\sigma^6}{6T^2}\Big[\Big(\frac{6(1 - e^{\rho\tau})}{\rho^4} + \frac{6\tau e^{\rho\tau}}{\rho^3} - \frac{3\tau^2 e^{\rho\tau}}{\rho^2} + \frac{\tau^3 e^{\rho\tau}}{\rho}\Big) m_0 \\
&+ \Big(\frac{e^{\rho\tau}}{\rho} m_6 - \frac{1}{\rho} m_7\Big)\Big] \\
&+ \frac{\sigma^4}{6T^2}\Big(\frac{-3}{\rho^2} m_4 + \frac{3}{\rho^2} m_1 + \frac{3e^{\rho\tau}}{\rho^2} m_5\Big). \qquad (3.25)
\end{aligned}
$$

To evaluate $\mathbb{E}[J_1 J_2]$, we have

$$J_1 J_2 = [a + b\widetilde{W}_\tau + c\widetilde{W}_\tau^2 + d\widetilde{W}_\tau^3] J_2. \qquad (3.26)$$

where

$$a = e^{\rho\tau} + \frac{1}{\rho T}(1 - e^{\rho\tau}), \quad b = \sigma e^{\rho\tau}, \quad c = \frac{1}{2}\sigma^2 e^{\rho\tau}, \quad d = \frac{1}{6}\sigma^3 e^{\rho\tau}.$$

$$\mathbb{E}[J_2] = \frac{\sigma^2}{2T}\int_0^\tau v e^{\rho v} dv = \frac{\sigma^2}{2T} m_1. \tag{3.27}$$

$$\begin{aligned}\widetilde{W}_\tau J_2 = & \frac{\sigma}{T}\int_0^\tau e^{\rho v}(\widetilde{W}_{T-u} + \widetilde{W}_v)\widetilde{W}_v dv + \frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}(\widetilde{W}_{T-u} + \widetilde{W}_v)\widetilde{W}_v^2 dv \\ & + \frac{\sigma^3}{6T}\int_0^\tau e^{\rho v}(\widetilde{W}_{T-u} + \widetilde{W}_v)\widetilde{W}_v^3 dv.\end{aligned}$$

Since $\widetilde{W}_{T-u}$ and $\widetilde{W}_{u-t}\Big(= \widetilde{W}_v\Big)$ are independent, and since

$$\mathbb{E}[(\widetilde{W}_t^k] = 0 \quad \text{for odd integer} k \geq 1,$$

we arrive that

$$\begin{aligned}\mathbb{E}[\widetilde{W}_\tau J_2] &= \frac{\sigma}{T}\int_0^\tau v e^{\rho v} dv + \frac{\sigma^3}{6T}\int_0^\tau 3v^2 e^{\rho v} dv \\ &= \frac{\sigma}{T} m_1 + \frac{\sigma^3}{2T} m_2. \end{aligned} \tag{3.28}$$

$$\begin{aligned}\widetilde{W}_\tau^2 J_2 = & \frac{\sigma}{T}\int_0^\tau e^{\rho v}(\widetilde{W}_{T-u} + \widetilde{W}_v)^2\widetilde{W}_v dv + \frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}(\widetilde{W}_{T-u} + \widetilde{W}_v)^2\widetilde{W}_v^2 dv \\ & + \frac{\sigma^3}{6T}\int_0^\tau e^{\rho v}(\widetilde{W}_{T-u} + \widetilde{W}_v)^2\widetilde{W}_v^3 dv.\end{aligned}$$

where

$$(\widetilde{W}_{T-u} + \widetilde{W}_v)^2 = \widetilde{W}_{T-u}^2 + 2\widetilde{W}_{T-u}\widetilde{W}_v + \widetilde{W}_v^2.$$

It turns out that

$$\begin{aligned}\mathbb{E}[\widetilde{W}_\tau^2 J_2] &= \frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}\Big(\widetilde{W}_{T-u}^2 + 2\widetilde{W}_{T-u}\widetilde{W}_v + \widetilde{W}_v^2\Big)\widetilde{W}_v^2 dv \\ &= \frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}((\tau - v)v + 3v^2) dv \\ &= \frac{\sigma^2}{2T}(\tau m_1 + 2m_2). \end{aligned} \tag{3.29}$$

Last, we calculate

$$\begin{aligned}\widetilde{W}_\tau^3 J_2 = & \frac{\sigma}{T}\int_0^\tau e^{\rho v}\Big(\widetilde{W}_{T-u}^3 + 3\widetilde{W}_{T-u}^2\widetilde{W}_v + 3\widetilde{W}_{T-u}\widetilde{W}_v^2 + \widetilde{W}_v^3\Big)\widetilde{W}_v dv \\ & + \frac{\sigma^2}{2T}\int_0^\tau e^{\rho v}\Big(\widetilde{W}_{T-u}^3 + 3\widetilde{W}_{T-u}^2\widetilde{W}_v + 3\widetilde{W}_{T-u}\widetilde{W}_v^2 + \widetilde{W}_v^3\Big)\widetilde{W}_v^2 dv \\ & + \frac{\sigma^3}{6T}\int_0^\tau e^{\rho v}\Big(\widetilde{W}_{T-u}^3 + 3\widetilde{W}_{T-u}^2\widetilde{W}_v + 3\widetilde{W}_{T-u}\widetilde{W}_v^2 + \widetilde{W}_v^3\Big)\widetilde{W}_v^3 dv.\end{aligned}$$

Since

$$\mathbb{E}[\widetilde{W}_t^{2k}] = \frac{(2k)!}{2^k k!} t^k \quad \text{for integer} \quad k \geq 1,$$

it turns out that

$$\begin{aligned}\mathbb{E}[\widetilde{W}_\tau^3 J_2] &= \frac{\sigma}{T}\int_0^\tau e^{\rho v}(3(\tau - v)v + 3v^2)dv + \frac{\sigma^3}{6T}\int_0^\tau e^{\rho v}(3(\tau - v)\cdot 3v^2 + 15v^3)dv \\ &= \frac{\sigma}{T}(3\tau m_1 + 2m_2) + \frac{\sigma^3}{6T}(9\tau m_2 + 6m_6). \qquad (3.30)\end{aligned}$$

From (3.26)-(3.30), we arrive at

$$\begin{aligned}\mathbb{E}[J_1 J_2] &= a\mathbb{E}[J_2] + b\mathbb{E}[\widetilde{W}_\tau J_2] + c\mathbb{E}[\widetilde{W}_\tau^2 J_2] + d\mathbb{E}[\widetilde{W}_\tau^3 J_2] \\ &= a\frac{\sigma^2}{2T}m_1 + b\Big(\frac{\sigma}{T}m_1 + \frac{\sigma^3}{2T}m_2\Big) + c\frac{\sigma^2}{2T}(\tau m_1 + 2m_2) \\ &\quad + d\Big[\frac{\sigma}{T}(3\tau m_1 + 2m_2) + \frac{\sigma^3}{2T}(3\tau m_2 + 2m_6)\Big] \\ &= \frac{\sigma}{2T}[(a\sigma + 2b + (6d + c\sigma)\tau)m_1 \\ &\quad + (4d + 2c\sigma + (b + 3d\tau)\sigma^2)m_2 + 2d\sigma^2 m_6]. \qquad (3.31)\end{aligned}$$

Now, we get by combining (3.21), (3.25) and (3.31),

$$\begin{aligned}\mathbb{E}[(X_t^3)^2] &= \mathbb{E}[J_1^2] - 2\mathbb{E}[J_1 J_2] + \mathbb{E}[J_2^2] \\ &= \Big[e^{\rho\tau} + \frac{1}{\rho T}(1 - e^{\rho\tau})\Big]^2 + \sigma^2 e^{2\rho\tau}\Big[\tau + \frac{3}{4}\sigma^2\tau^2 + \frac{5}{12}\sigma^4\tau^3 + \sigma^2\tau^2\Big] \\ &\quad + \sigma^2\tau e^{\rho\tau}\Big[e^{\rho\tau} + \frac{1}{\rho T}(1 - e^{\rho\tau})\Big] \\ &\quad - \frac{\sigma}{T}\Big[(a\sigma + 2b + (6d + c\sigma)\tau)m_1 \\ &\quad + (4d + 2c\sigma + (b + 3d\tau)\sigma^2)m_2 + 2d\sigma^2 m_6\Big] \\ &\quad + \frac{\sigma^2}{T^2}\Big(\frac{1}{\rho^2}(m_0 - m_3) + \frac{e^{\rho\tau}}{\rho}m_1\Big) \\ &\quad + \frac{\sigma^4}{4T^2}\Big[\frac{2}{\rho}\Big(-\frac{2}{\rho}m_4 + \frac{2}{\rho^2}m_3 - \frac{2}{\rho^2}m_0 + e^{\rho\tau}m_2\Big) + m_1^2\Big] \\ &\quad + \frac{\sigma^6}{4T^2}\Big(-\frac{2m_2}{\rho^2} + \frac{2m_4}{\rho^3} - \frac{2m_1}{\rho^2} + \frac{\tau e^{\rho\tau}m_2}{\rho} - \frac{e^{\rho\tau}m_2}{\rho^2} + \frac{m_5}{\rho^2}\Big) \\ &\quad + \frac{\sigma^6}{6T^2}\Big[\Big(\frac{6(1 - e^{\rho\tau})}{\rho^4} + \frac{6\tau e^{\rho\tau}}{\rho^3} - \frac{3\tau^2 e^{\rho\tau}}{\rho^2} + \frac{\tau^3 e^{\rho\tau}}{\rho}\Big)m_0 \\ &\quad + \Big(\frac{e^{\rho\tau}}{\rho}m_6 - \frac{1}{\rho}m_7\Big)\Big] \\ &\quad + \frac{\sigma^4}{6T^2}\Big(\frac{-3}{\rho^2}m_4 + \frac{3}{\rho^2}m_1 + \frac{3e^{\rho\tau}}{\rho^2}m_5\Big). \qquad (3.32)\end{aligned}$$

Finally, by (3.20) and (3.32), we obtain

$$\begin{aligned}
\mathrm{Var}(X_{3,t}) &= \mathbb{E}[(X_{3,t})^2] - \mathbb{E}[X_{3,t}]^2 \\
&= \Big[e^{\rho\tau} + \frac{1}{\rho T}(1-e^{\rho\tau})\Big]^2 + \sigma^2 e^{2\rho\tau}\Big[\tau + \frac{3}{4}\sigma^2\tau^2 + \frac{5}{12}\sigma^4\tau^3 + \sigma^2\tau^2\Big] \\
&\quad + \sigma^2\tau e^{\rho\tau}\Big[e^{\rho\tau} + \frac{1}{\rho T}(1-e^{\rho\tau})\Big] \\
&\quad - \frac{\sigma}{T}\Big[(a\sigma + 2b + (6d + c\sigma)\tau)m_1 + (4d + 2c\sigma + (b + 3d\tau)\sigma^2)m_2 \\
&\quad + 2d\sigma^2 m_6\Big] \\
&\quad + \frac{\sigma^2}{T^2}\Big(\frac{1}{\rho^2}(m_0 - m_3) + \frac{e^{\rho\tau}}{\rho}m_1\Big) \\
&\quad + \frac{\sigma^4}{4T^2}\Big[\frac{2}{\rho}\Big(-\frac{2}{\rho}m_4 + \frac{2}{\rho^2}m_3 - \frac{2}{\rho^2}m_0 + e^{\rho\tau}m_2\Big) + m_1^2\Big] \\
&\quad + \frac{\sigma^6}{4T^2}\Big(-\frac{2m_2}{\rho^2} + \frac{2m_4}{\rho^3} - \frac{2m_1}{\rho^2} + \frac{\tau e^{\rho\tau}m_2}{\rho} - \frac{e^{\rho\tau}m_2}{\rho^2} + \frac{m_5}{\rho^2}\Big) \\
&\quad + \frac{\sigma^6}{6T^2}\Big[\Big(\frac{6(1-e^{\rho\tau})}{\rho^4} + \frac{6\tau e^{\rho\tau}}{\rho^3} - \frac{3\tau^2 e^{\rho\tau}}{\rho^2} + \frac{\tau^3 e^{\rho\tau}}{\rho}\Big)m_0 \\
&\quad + \Big(\frac{e^{\rho\tau}}{\rho}m_6 - \frac{1}{\rho}m_7\Big)\Big] \\
&\quad + \frac{\sigma^4}{6T^2}\Big(\frac{-3}{\rho^2}m_4 + \frac{3}{\rho^2}m_1 + \frac{3e^{\rho\tau}}{\rho^2}m_5\Big) \\
&\quad - \Big[e^{\rho\tau}\Big(1 + \frac{1}{2}\sigma^2\tau\Big) + \frac{1}{\rho T}(1-e^{\rho\tau}) - \frac{\sigma^2 m_1}{2T}\Big]^2 \qquad (3.33)
\end{aligned}$$

Let

$$\alpha_{3,t} = \mathbb{E}[X_{3,t}], \quad \nu_{3,t} = \sqrt{\mathrm{Var}(X_{3,t})}$$

Then we can write

$$X_{3,t} = \alpha_{3,t} + \nu_{3,t}Z, \quad Z \sim N(0,1) \quad \text{independent of} \quad \mathcal{F}_t.$$

The value V_t in (3.3) is approximated by $V_{3,t}$ as follows:

$$\begin{aligned}
V_t \approx V_{3,t} &= e^{-r(T-t)}\widetilde{\mathbb{E}}[(S_t X_{3,t} - M_t)^+|\mathcal{F}_t] \\
&= e^{-r(T-t)}\widetilde{\mathbb{E}}[(S_t(\alpha_{3,t} + \nu_{3,t}Z) - M_t)^+|\mathcal{F}_t]. \\
&= e^{-r(T-t)}\Big\{(\alpha_{3,t}S_t - M_t)N\Big(\frac{1}{\nu_{3,t}}\Big(\alpha_{3,t} - \frac{M_t}{S_t}\Big)\Big) \\
&\quad + \frac{\nu_{3,t}S_t}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2(\nu_{3,t})^2}\Big(\frac{M_t}{S_t} - \alpha_{3,t}\Big)^2\Big)\Big\}. \qquad (3.34)
\end{aligned}$$

4. Numeric results

In this section we show some numeric results for the price of an Asian option obtained with the method described above. The numeric results have been obtained using the computational software program matlab to carry on the computations showed in the precedent section.

The approximation of the prices function

$$V_t = e^{-r(T-t)}\Big\{(\alpha_t S_t - M_t)N\Big(\frac{1}{\nu_t}\Big(\alpha_t - \frac{M_t}{S_t}\Big)\Big) + \frac{\nu_t S_t}{\sqrt{2\pi}}\exp\Big(\frac{-1}{2\nu_t^2}\Big(\frac{M_t}{S_t} - \alpha_t\Big)^2\Big)\Big\}.$$

The case $t = 0$ can be considered without losing generality. In the following table our approximation formulae are compared with the value of Black-Scholes model(BSM) European call option, arithmetic average floating strike call option with first order ($\alpha_t = \alpha_{1,t}$, $\nu_t = \nu_{1,t}$) and quadratic orders ($\alpha_t = \alpha_{2,t}$, $\nu_t = \nu_{2,t}$). We first assume that the time to maturity $T = 1$, the initial asset price $S_0 = 100$, the strike K, the interest rate r and the volatility σ for some case.

TABLE 1. Related to the $K = 95$, $K = 100$ and $K = 105$.

σ	r	$K = 95$	$K = 100$	$K = 105$	1 order	2 order
	0.05	10.4053	6.8049	4.0461	6.2693	2.5220
0.1	0.09	13.5083	9.5663	6.2527	7.3806	4.3869
	0.15	18.3068	14.2008	10.3990	9.1896	7.1415
	0.05	13.3465	10.4506	8.0214	10.7964	3.2657
0.2	0.09	15.8027	12.6820	9.9879	11.7589	4.9102
	0.015	19.7163	16.3560	13.3415	13.2591	7.4475
	0.05	16.7953	14.2313	11.9703	14.8796	4.7862
0.3	0.09	18.8769	16.2193	13.7529	15.7570	6.2085
	0.15	21.8471	19.4027	16.2965	17.1000	8.4513

In the case, show the approximations of the prices function with quadratic orders is lower than first order and BSM, so the approximations of the quadratic orders is better than first order.

Next we tested our method with a low-volatility parameter $\sigma = 0.01$ and $\sigma = 0.005$ with the strike $K = 100$, the values of the quadratic orders and first order are not far-off.

TABLE 2. Tests with low volatility $\sigma = 0.01$, 0.005 and $K = 100$.

σ	r	BSM	1 order	2 order
	0.05	4.8771	2.4699	2.4588
0.01	0.09	8.6069	4.3655	4.3680
	0.15	13.9292	7.1360	7.1387
	0.05	4.8771	2.4582	2.4588
0.005	0.09	8.6069	4.3673	4.3680
	0.015	13.9292	7.1380	7.1387

With a low-volatility parameter $\sigma = 0.01$, we tried a variety of the time to maturity $T = 2$ and $T = 5$. As T increases, the approximations of the prices function are increases, and the quadratic orders and first order are low than BSM.

TABLE 3. Tests with different T with $\sigma = 0.01$.

T	r	BSM	1 order	2 order
	0.05	4.8771	2.4699	2.4588
1	0.09	8.6069	4.3655	4.3680
	0.15	13.9292	7.1360	7.1387
	0.05	9.5163	4.8343	4.8374
2	0.09	16.4730	8.4782	8.4835
	0.015	25.9182	13.6006	13.6061
	0.05	22.1199	11.5068	11.5203
5	0.09	36.2372	19.4587	19.4730
	0.15	52.7633	29.6338	29.6489

Comparisons with first and quadratic orders, as can be seen, the approximations of the prices function with quadratic orders is lower than first order in large volatility. However, as low-volatility case, the value of the first order and quadratic orders are closed.

5. Conclusions

We derived new analytic approximate formulas for the pricing of Asian option with arithmetic averages via higher order Taylor approximations. Comparisons the numeric results, our technique illustrates that adding order Taylor approximations can increase the accuracy of an Asian option pricing in general.

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