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LEVITIN-POLYAK WELL-POSEDNESS OF GENERALIZED VARIATIONAL INEQUALITY WITH GENERALIZED MIXED VARIATIONAL INEQUALITY CONSTRAINT

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Dedicated to Prof. Wataru Takahashi on the occasion of his 70th birth day

Abstract. In this paper, we first introduce the concept of Levitin-Polyak well-posedness of generalized variational inequality with generalized mixed variational inequality constraint(for short,GGVI) in Banach space. We establish some metric characterizations of Levitin-Polyak well-posedness for GGVI. Finally, we derive some conditions under which the GGVI is Levitin-Polyak well-posed.

1. Introduction

In 1966, Tykhonov [21] first gave the well-posedness of a minimization problem, which has been known as Tykhonov well-posedness. Roughly speaking, the Tykhonov well-posedness of a minimization problem means the existence and uniqueness of minimizers, and the convergence of every sequence toward the unique minimizer. In many practical situations, there are more than one minimizers for a minimization problem. In this case, the concept of Tykhonov well-posedness in the generalized sense was introduced, which means the existence of minimizers and the convergence of some subsequence of every minimizing sequence toward a minimizer. Clearly, the concept of well-posedness is motivated by the numerical methods producing optimizing sequences. Because of its importance in optimization problems, various concepts of well-posedness have been introduced in past decades. For details, we refer the readers to [24–26] and the references therein.

The Tykhonov well-posedness of a constrained minimization problem requires that every minimizing sequence should lie in the constraint set. In many practical situations, the minimizing sequence produced by a numerical optimization method usually fails to be feasible but gets closer and closer to the constraint set. Such a sequence is called a generalized minimizing sequence for constrained minimization problems. To take care of such a case, Levitin and Polyak [13] strengthened the concept of Tykhnov well-posedness by requiring the existence and uniqueness of minimizer, and the convergence of every generalized minimizing sequence toward the

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unique minimizer, which has been known as Levitin and Polyak (for short, LP) well-posedness. There are a very large number of results concerned with Tykhonov well-posedness, LP well-posedness and their generalizations for minimization problems. For details, we refer the readers to [2, 13, 15, 21, 24, 25].

In recent years, the concepts of well-posedness have been extended to other contexts: equilibrium problems, variational inequalities, inclusion problems and fixed point problems [1, 3–9, 12, 14, 16, 19, 20, 22, 23]. For Tykhonov well-posedness, Lemaire [12] discussed the relations among the well-posedness of minimization problems, inclusion problems and fixed point problems. In the setting of Hilbert space, Fang, Huang and Yao [5] proved that under suitable conditions the well-posedness of a general mixed variational inequality is equivalent to the existence and the unique of its solution. They also considered the relations of the well-posedness of mixed variational inequality, the corresponding inclusion problem and a corresponding fixed point problem. Recently, Ceng and Yao [3] derived some results for the well-posedness of the generalized mixed variational inequality, the corresponding inclusion problem and the corresponding fixed point problem.

Furthermore, Hu and Fang [8] considered the Levitin-Polyak well-posedness of general variational inequalities in $\mathbb{R}^n$. They derived a characterization of the LP well posedness by considering the size of LP approximating solution sets of general variational inequalities. They also proved that the LP well-posedness of general variational inequalities is closely related to the LP well-posedness of minimization problems and fixed point problems. Finally, they showed that under suitable conditions, the LP well-posedness of general variational inequality is equivalent to the uniqueness and existence of its solution. Li and Xia [14] derived some results for the Levitin-Polyak well-posedness of the generalized mixed variational inequality, the corresponding inclusion problem and the corresponding fixed point problem.

Our attention here will be focused on the following generalized variational inequality with generalized mixed variational inequality constraint problem associated with (F, G, ϕ, K) (denoted by $GGVI(F, G, \phi, K)$):

$$\text{find } x \in \Omega \text{ and } u \in F(x) \text{ such that}$$

$$\langle u, x - y \rangle \leq 0, \quad \forall y \in \Omega,$$

where $\Omega \triangleq \{x \in K : \exists v \in G(x) \text{ such that } \langle v, x - y \rangle + \phi(x) - \phi(y) \leq 0, \quad \forall y \in K\}$. X is a real reflexive Banach space with its dual X^* and K be a nonempty, closed and convex subset of X. $F, G : X \to 2^{X^*}$ are two set-valued mappings, and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function.

The $GGVI(F, G, \phi, K)$ is commonly referred to as of *bilevel* (or *hierarchical*) type, because the set of constraints Ω itself is given as a subproblem. A large variety of problems can be as special instances of problem $GGVI(F, G, \phi, K)$. For example, if $G = 0$ and $\phi = 0$, the $GGVI(F, G, \phi, K)$ reduces to the following generalized variational inequality problem: find $x \in K$ and $u \in F(x)$ such that

$$\langle u, x - y \rangle \leq 0, \quad \forall y \in K.$$

If $f : X \to \mathbb{R}$ be a Gâteaux differentiable function (over an open set containing K). Special instances of $GGVI(F, G, \phi, K)$ are given below:

When $F = \nabla f$ and $G = 0$, the $GGVI(F, G, \phi, K)$ reduces to the bi-level minimization problem:

$$\min_{x \in argmin\ \phi} f(x).$$

When $F = \nabla f$, $\phi = 0$, $K = \mathbb{R}^n_+$ and G is a single valued mapping, the $GGVI(F, G, \phi, K)$ reduces to the following $MPCC$:

$$\min_{x \geq 0,\ G(x) \geq 0,\ \langle x, G(x) \rangle = 0} f(x),$$

which enters the class of problems considered in [10].

Recently, Maingé [17] has constructed an efficient algorithm for solving $GGVI(F, G, \phi, K)$. However, to the best of our knowledge, there is no a result concerning the LP well-posedness for $GGVI(F, G, \phi, K)$. Therefore, it is worth studying implementable results for the Levitin-Polyak well-posedness of the $GGVI(F, G, \phi, K)$.

Motivated and inspired by the research work going on this field, we extend the notion of Levitin-Polyak well-posedness to the $GGVI(F, G, \phi, K)$ in Banach spaces, and give some characterizations of its Levitin-Polyak well-posedness. We also derive some conditions under which the $GGVI(F, G, \phi, K)$ is Levitin-Polyak well-posed.

2. Preliminaries

Let X be a real reflexive Banach space with its dual X^* and K be a nonempty, closed and convex subset of X. Let $F, G : X \to 2^{X^*}$ be two set-valued mappings, and let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. Denote by $dom\phi$ the domain of ϕ, i.e.,

$$dom\phi = \{x \in X : \phi(x) < +\infty\}.$$

In this paper we always assume that $dom\phi \bigcap K \neq \emptyset$. Consider the following $GGVI(F, G, \phi, K)$ problem:

$$\text{find } x \in \Omega \text{ and } u \in F(x) \text{ such that}$$

$$\langle u, x - y \rangle \leq 0, \quad \forall y \in \Omega,$$

where

$$\Omega \triangleq \{x \in K : \exists v \in G(x) \text{ such that} \langle v, x - y \rangle + \phi(x) - \phi(y) \leq 0, \forall y \in K\}.$$

First, we give some definitions and lemmas.

Definition 2.1. A nonempty set-valued mapping $F : X \to 2^{X^*}$ is said to be monotone, if for all $x, y \in X, u \in F(x), v \in F(y)$,

$$\langle u - v, x - y \rangle \geq 0.$$

Definition 2.2. Let A, B be nonempty subsets of Banach space X. The Hausdorff metric $H(\cdot, \cdot)$ between A and B is defined by

$$H(A, B) = \max\{e(A, B), e(B, A)\},$$

Where $e(A, B) = \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} \|a - b\|$.

Lemma 2.3 (Nadler's Theorem [18]). *Let $(X, \|\cdot\|)$ be a normed vector space and $H(\cdot,\cdot)$ be the Hausdorff metric on the collection $CB(X)$ of all nonempty, closed and bounded subsets of X. If U and V lie in $CB(X)$, then for any $\epsilon > 0$ and any $u \in U$, there exists $v \in V$ such that $\|u - v\| \leq (1+\epsilon)H(U, V)$. In particular, whenever U and V are compact subsets in X, one has $\|u - v\| \leq H(U, V)$.*

Definition 2.4 ([3]). A nonempty compact-valued mapping $F : X \to 2^{X^*}$ is said to be

(i) H-semicontinuous, if for any $x, y \in X$, the function $t \mapsto H(F(x + t(y - x)), F(x))$ from $[0, 1]$ into $\mathbb{R}^+ = [0, +\infty)$ is continuous at 0^+, where $H(\cdot,\cdot)$ is the Hausdorff metric defined on $CB(X)$.

(ii) H-uniformly continuous, if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$, with $\|x - y\| < \delta$, one has $H(F(x), F(y)) < \epsilon$, where $H(\cdot,\cdot)$ is the Hausdorff metric defined on $CB(X)$.

Definition 2.5. Let X and Y be two topological spaces and $x \in X$. A set-valued mapping $F : X \to 2^Y$ is said to be upper semicontinuous (for short, u.s.c) at x, if for any neighbourhood V of $F(x)$, there exists a neighbourhood U of x such that $F(y) \subset V$, $\forall y \in U$. If F is u.s.c at each point of X, we say that F is u.s.c on X.

Definition 2.6 ([11]). Let A be a nonempty subset of Banach space X. The measure of noncompactness u of the set A is defined by

$$u(A) = \inf\{\epsilon > 0 : A \subset \bigcup_{i=1}^{n} A_i, diamA_i < \epsilon, i = 1, 2, \ldots, n\},$$

where *diam* means the diameter of a set.

The following Proposition is a special case of Lemma 2.2 in Ceng and Yao [3].

Proposition 2.7. *Let K be a nonempty, closed and convex subset of Banach space X and $F : X \to 2^{X^*}$ be a nonempty compact-valued mapping which is H-hemicontinuous and monotone. Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper and convex function. Then for a given $x \in K$, the following statements are equivalent:*

(i) *there exists $u \in F(x)$ such that $\langle u, x - y\rangle + \phi(x) - \phi(y) \leq 0, \quad \forall y \in K;$*

(ii) $\langle v, x - y\rangle + \phi(x) - \phi(y) \leq 0, \quad \forall y \in K, \quad v \in F(y)$.

We first prove the following Proposition which is the key Proposition of this paper.

Proposition 2.8. Let K be a nonempty, closed and convex subset of Banach space X. Let F, $G : X \to 2^{X^*}$ be nonempty compact-valued mappings which are H-hemicontinuous and monotone, respectively. Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper and convex function. Let S be the soluton set of problem $GGVI(F, G, \phi, K)$. Then for a given $x^* \in K$, $x^* \in S$ if and only if there exists $u \in F(x^*)$ and $v \in G(x^*)$ such that

$$\langle u, x^* - y\rangle \leq \frac{\alpha}{2}\|x^* - y\|^2, \quad \forall y \in \Omega, \tag{2.1}$$

and

$$\langle v, x^* - y\rangle + \phi(x^*) - \phi(y) \leq \frac{\alpha}{2}\|x^* - y\|^2, \quad \forall y \in K, \tag{2.2}$$

where $\alpha > 0$ is constant.

Proof. If $x^* \in S$, it is easy to know that (2.1) and (2.2) hold. Conversely, for a given $x^* \in K$, suppose (2.1) and (2.2) hold. Let $y_t = ty + (1-t)x^* \in K, \forall y \in K, t \in (0,1)$. It follows from (2.2) that

$$\langle v, x^* - y_t \rangle + \phi(x^*) - \phi(y_t) \leq \frac{\alpha}{2}\|x^* - y_t\|^2, \quad \forall y \in K.$$

We deduce that

$$\langle v, x^* - y \rangle + \phi(x^*) - \phi(y) \leq \frac{t\alpha}{2}\|x^* - y\|^2, \quad \forall y \in K.$$

Let $t \to 0^+$, we have

$$\langle v, x^* - y \rangle + \phi(x^*) - \phi(y) \leq 0, \quad \forall y \in K.$$

Then $x^* \in \Omega$.

Now, we will prove that Ω is a convex set. Indeed, for any $x_1,\ x_2 \in \Omega$, there exists $u \in G(x_1)$ and $v \in G(x_2)$ such that

$$\langle u . x_1 - y \rangle + \phi(x_1) - \phi(y) \leq 0, \quad \forall y \in K, \tag{2.3}$$

and

$$\langle v . x_2 - y \rangle + \phi(x_2) - \phi(y) \leq 0, \quad \forall y \in K. \tag{2.4}$$

Let $x = \lambda x_1 + (1-\lambda)x_2$, $\forall \lambda \in (0,1)$. It follows from (2.3) and (2.4) that

$$\lambda\langle u, x_1 - y \rangle + (1-\lambda)\langle v, x_2 - y \rangle + \phi(x) - \phi(y) \leq 0, \quad \forall y \in K. \tag{2.5}$$

Since G is a monotone mapping, we obtain that

$$\langle u, x_1 - y \rangle \geq \langle w, x_1 - y \rangle, \quad \forall w \in G(y), \tag{2.6}$$

and

$$\langle v, x_2 - y \rangle \geq \langle w, x_2 - y \rangle, \quad \forall w \in G(y). \tag{2.7}$$

(2.5)-(2.7) implies that

$$\lambda\langle w, x_1 - y \rangle + (1-\lambda)\langle w, x_2 - y \rangle + \phi(x) - \phi(y) \leq 0, \quad \forall w \in G(y),\ y \in K.$$

That is,

$$\langle w, x - y \rangle + \phi(x) - \phi(y) \leq 0, \quad \forall w \in G(y),\ y \in K.$$

By Proposition 2.7, we know that there exists $r \in G(x)$ such that

$$\langle r, x - y \rangle + \phi(x) - \phi(y) \leq 0, \quad \forall y \in K.$$

This implies that $x \in \Omega$. So, Ω is a convex set.

Now, for any $y \in \Omega$, $t \in (0,1)$, let $y_t = ty + (1-t)x^*$. Since $x^* \in \Omega$ and Ω is a convex set, we know that $y_t \in \Omega$. It follows from (2.1) that

$$\langle u, x^* - y \rangle \leq \frac{t\alpha}{2}\|x^* - y\|^2, \quad \forall y \in \Omega.$$

Let $t \to 0^+$, we obtain that there exists $u \in F(x^*)$ such that

$$\langle u, x^* - y \rangle \leq 0, \quad \forall y \in \Omega. \tag{2.8}$$

By $x^* \in \Omega$ and (2.8), we deduce that $x^* \in S$. The proof is complete. □

3. Levitin-Polyak well-posedness of $GGVI(F, G, \phi, K)$

In this section, we extend the concepts of Levitin-Poylak well-posedness to the problem $GGVI(F, G, \phi, K)$ and establish its metric characterizations. In the sequel, we always denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. Let $\alpha \geq 0$ be a fixed number.

Definition 3.1. A sequence $\{x_n\} \subset X$ is said to be an Levitin-Polyak(for short, LP)α-approximating sequence for $GGVI(F, G, \phi, K)$, if there exist $w_n \in X$ with $w_n \to 0$ and $0 < \epsilon_n \to 0$ such that $x_n + w_n \in K$ for all $n \in N$, and there exists $u_n \in F(x_n)$, $v_n \in G(x_n)$ such that

$$\langle u_n, x_n - y\rangle \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n, \quad \forall y \in \Omega, n \in N, \tag{3.1}$$

and

$$x_n \in \mathrm{dom}\phi, \quad \langle v_n, x_n - z\rangle + \phi(x_n) - \phi(z) \leq \frac{\alpha}{2}\|x_n - z\|^2 + \epsilon_n, \quad \forall z \in K, n \in N. \tag{3.2}$$

Definition 3.2. We say that $GGVI(F, G, \phi, K)$ is strongly (resp., weakly) LP α-well-posed if $GGVI\ (F, G, \phi, K)$ has a unique solution and every LP α-approximating sequence converges strongly (resp., weakly) to the unique solution.

In the sequel, strongly (resp., weakly) LP 0-well-posed is always called as strongly (resp., weakly) LP well-posed. If $\alpha_1 > \alpha_2 \geq 0$, then strongly (resp., weakly) LP α_1-well-posed implies strongly (resp., weakly) LP α_2-well-posed.

Definition 3.3. We say that $GGVI(F, G, \phi, K)$ is strongly (resp., weakly) LP α-well-posed in the generalized sense if $GGVI$ has nonempty solution set S and every LP α-approximating sequence has subsequence which converges strongly (resp., weakly) to some point of S.

In the sequel, strongly (resp., weakly) LP 0-well-posed in the generalized sense is always called as strongly (resp., weakly) LP well-posed in the generalized sense.

To derive the metric characterizations of LP α-well-posedness, we consider the following approximating solution set of $GGVI(F, G, \phi, K)$: for any $\epsilon \geq 0$,

$$\Delta_\alpha(\epsilon) = \{x \in dom\phi : d(x, K) \leq \epsilon, \exists u \in F(x), v \in G(x) \text{ s.t. for all } y \in \Omega$$
$$\langle u, x - y\rangle \leq \frac{\alpha}{2}\|x - y\|^2 + \epsilon,\ \langle v, x - z\rangle + \phi(x) - \phi(z) \leq \frac{\alpha}{2}\|x - z\|^2 + \epsilon, \forall z \in K\}.$$

Theorem 3.4. Let K be a nonempty, closed and convex subset of X. Let F, $G : X \to 2^{X^*}$ be nonempty H-hemicontinuous, compact-valued and monotone mappings, respectively. Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Then $GGVI(F, G, \phi, K)$ is strongly LP α-well-posed if and only if

$$\Delta_\alpha(\epsilon) \neq \emptyset, \forall \epsilon > 0 \text{ and } diam(\Delta_\alpha(\epsilon)) \to 0 \text{ as } \epsilon \to 0. \tag{3.3}$$

Proof. Suppose that $GGVI(F, G, \phi, K)$ is strongly LP α-well-posed and $x^* \in \Omega$ is the unique solution of $GGVI(F, G, \phi, K)$. It is obvious that $x^* \in \Delta_\alpha(\epsilon) \neq \emptyset$, for all $\epsilon \geq 0$. If $diam(\Delta_\alpha(\epsilon))$ does not converge to 0 as $\epsilon \to 0$, then there exists $l > 0$ and $0 < \epsilon_n \to 0$ and $\{x_n^{(1)}\}, \{x_n^{(2)}\}$ with $x_n^{(1)}, x_n^{(2)} \in \Delta_\alpha(\epsilon_n)$ such that

$$\|x_n^{(1)} - x_n^{(2)}\| > l, \quad \forall n \in N. \tag{3.4}$$

Since $x_n^{(1)} \in \Delta_\alpha(\epsilon_n)$, by the definition of $\Delta_\alpha(\epsilon_n)$, we have

$$d(x_n^{(1)}, K) \le \epsilon_n < \epsilon_n + \frac{1}{n},$$

and there exists $u_n \in F(x_n^{(1)})$, $v_n \in G(x_n^{(1)})$ such that

$$\langle u_n, x_n^{(1)} - y\rangle \le \frac{\alpha}{2}\|x_n^{(1)} - y\|^2 + \epsilon_n, \quad \forall y \in \Omega,$$

$$\langle v_n, x_n^{(1)} - z\rangle + \phi(x_n^{(1)}) - \phi(z) \le \frac{\alpha}{2}\|x_n^{(1)} - z\|^2 + \epsilon_n, \quad \forall z \in K.$$

Since K is closed and convex, then there exists $\overline{x}_n^{(1)} \in K$ such that $\|x_n^{(1)} - \overline{x}_n^{(1)}\| < \epsilon_n + \frac{1}{n}$. Let $w_n = \overline{x}_n^{(1)} - x_n^{(1)}$, then we have $w_n + x_n^{(1)} = \overline{x}_n^{(1)} \in K$ and $\|w_n\| = \|x_n^{(1)} - \overline{x}_n^{(1)}\| \to 0$ as $n \to \infty$. This implies that $w_n \to 0$. Thus, $\{x_n^{(1)}\}$ is an LP approximating sequence for $GGVI(F, G, \phi, K)$. By the similar argument, we get $\{x_n^{(2)}\}$ is an LP approximating sequence for $GGVI(F, G, \phi, K)$. So they have to converge strongly to the unique solution of $GGVI(F, G, \phi, K)$ which is a contradiction to (3.4).

Conversely, suppose that condition (3.3) holds. Let $\{x_n\} \subset X$ be a LP α-approximating sequence for $GGVI(F, G, \phi, K)$. Then there exists $w_n \in X$ with $w_n \to 0$ such that $x_n + w_n \in K$, and there exists $0 < \epsilon_n' \to 0$, $u_n \in F(x_n)$, $v_n \in G(x_n)$ such that

$$\langle u_n, x_n - y\rangle \le \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n', \quad \forall y \in \Omega, n \in N. \tag{3.5}$$

$$x_n \in dom\phi, \langle v_n, x_n - z\rangle + \phi(x_n) - \phi(z) \le \frac{\alpha}{2}\|x_n - z\|^2 + \epsilon_n', \quad \forall z \in K, n \in N. \tag{3.6}$$

Since $x_n + w_n \in K$, then there exists $\overline{x}_n \in K$ such that $x_n + w_n = \overline{x}_n$. It is easy to see $d(x_n, K) \le \|x_n - \overline{x}_n\| = \|w_n\| \to 0$ as $n \to \infty$. Set $\epsilon_n = \max\{\epsilon_n', \|w_n\|\}$. We deduce that $x_n \in \Delta_\alpha(\epsilon_n)$. By (3.3), we get $\{x_n\}$ is a Cauchy sequence and so it converges to a point $\overline{x} \in K$.

Now, we will prove that $\overline{x} \in K$ is a solution of $GGVI(F, G, \phi, K)$. Since G is monotone and ϕ is lower semicontinuous, it follows from (3.6) that for any $z \in K$, $v \in G(z)$,

$$\begin{aligned}
\langle v, \overline{x} - z\rangle + \phi(\overline{x}) - \phi(z) &\le \liminf_{n\to\infty}\{\langle v, x_n - z\rangle + \phi(x_n) - \phi(z)\}\\
&\le \liminf_{n\to\infty}\{\langle v_n, x_n - z\rangle + \phi(x_n) - \phi(z)\}\\
&\le \liminf_{n\to\infty}\{\frac{\alpha}{2}\|x_n - z\|^2 + \epsilon_n'\}\\
&= \frac{\alpha}{2}\|\overline{x} - z\|^2. \qquad (3.7)
\end{aligned}$$

For any $z \in K$, let $z_t = \overline{x} + t(z - \overline{x})$ for all $t \in [0, 1]$. Since K is a nonempty, closed and convex subset, this implies that $z_t \in K$. Then, (3.7) implies that

$$\langle v_t, \overline{x} - z_t\rangle + \phi(\overline{x}) - \phi(z_t) \le \frac{\alpha}{2}\|\overline{x} - z_t\|^2, \quad \forall v_t \in G(z_t).$$

Since ϕ is convex,

$$\langle v_t, \overline{x} - z\rangle + \phi(\overline{x}) - \phi(z) \leq \frac{\alpha t}{2}\|\overline{x} - z\|^2, \quad \forall v_t \in G(z_t), \quad z \in K. \tag{3.8}$$

Since G is a nonempty compact-valued mapping which is H-hemicontinuous. According to Lemma 2.3, for each fixed $v_t \in G(z_t)$ and each $t \in (0,1)$, there exist $r_t \in G(\overline{x})$ such that $\|v_t - r_t\| \leq H(G(z_t), G(\overline{x}))$. Since G is H-hemicontinuous, one deduces that $\|v_t - r_t\| \leq H(G(z_t), G(\overline{x})) \to 0$ as $t \to 0^+$. Since G is compact-valued mapping, without loss of generality, we may assume that $r_t \to r \in G(\overline{x})$ as $t \to 0^+$. Thus, we deduces that

$$\|v_t - r\| \leq \|v_t - r_t\| + \|r_t - r\| \leq H(G(z_t), G(\overline{x})) + \|r_t - r\| \to 0 \text{ as } t \to 0^+.$$

This implies that $v_t \to r$ as $t \to 0^+$. It follows from (3.8) that

$$\langle r, \overline{x} - z\rangle + \phi(\overline{x}) - \phi(z) \leq 0, \quad \forall z \in K.$$

So $\overline{x} \in \Omega$. Since F is monotone, it follows from (3.5) that for any $y \in \Omega$, $u \in F(y)$,

$$\begin{aligned}\langle u, \overline{x} - y\rangle &\leq \liminf_{n\to\infty}\{\langle u, x_n - y\rangle\} \\ &\leq \liminf_{n\to\infty}\{\langle u_n, x_n - y\rangle\} \\ &\leq \liminf_{n\to\infty}\{\frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n'\} \\ &= \frac{\alpha}{2}\|\overline{x} - y\|^2.\end{aligned}$$

For any $y \in \Omega$, let $y_t = \overline{x} + t(y - \overline{x})$ for all $t \in [0,1]$. By the proof of Proposition 2.8, we know Ω is convex set. So, we deduce that $y_t \in \Omega$. By the similar arguments, we have that there exists $s \in F(\overline{x})$ such that

$$\langle s, \overline{x} - y\rangle \leq 0, \quad \forall y \in \Omega.$$

This implies that $\overline{x}$ is a solution of $GGVI(F, G, \phi, K)$.

To complete the proof, we need only to prove that $GGVI(F, G, \phi, K)$ has a unique solution. Assume by contradiction that $GGVI(F, G, \phi, K)$ has two distinct solution x_1 and x_2. Then it is easy to see that $x_1, x_2 \in \Delta_\alpha(\epsilon)$ for all $\epsilon > 0$ and

$$0 < \|x_1 - x_2\| \leq diam(\Delta_\alpha(\epsilon)) \to 0,$$

which is a contradiction to (3.3). The proof is complete. □

Theorem 3.5. Let K be a nonempty, closed and convex subset of X. Let F, $G : X \to 2^{X^*}$ be nonempty upper semicontinuous, compact-valued and monotone mappings, respectively. Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. Then $GGVI(F, G, \phi, K)$ is strongly LP α-well-posed in the generalized sense if and only if

$$\Delta_\alpha(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0 \text{ and } u(\Delta_\alpha(\epsilon)) \to 0 \text{ as } \epsilon \to 0. \tag{3.9}$$

Proof. Suppose that $GGVI(F, G, \phi, K)$ is strongly LP α-well-posed in the generalized sense. Let S be the solution set of $GGVI(F, \phi, K)$. Then S is nonempty and compact. Indeed, let $\{x_n\}$ be any sequence in S, then $\{x_n\}$ is a strongly LP α-approximating sequence for $GGVI(F, G, \phi, K)$. Since $GGVI(F, G, \phi, K)$ is strongly LP α-well-posed in the generalized sense, $\{x_n\}$ has a subsequence which converges strongly to some point of S. Thus S is compact. It is obvious that $\Delta_\alpha(\epsilon) \supset S \neq \emptyset$ for all $\epsilon > 0$. Now we show that

$$u(\Delta_\alpha(\epsilon)) \to 0 \text{ as } \epsilon \to 0.$$

Observe that for every $\epsilon > 0$,

$$H(\Delta_\alpha(\epsilon), S) = \max\{e(\Delta_\alpha(\epsilon), S), e(S, \Delta_\alpha(\epsilon))\} = e(\Delta_\alpha(\epsilon), S).$$

Taking into account the compactness of S, we get

$$u(\Delta_\alpha(\epsilon)) \leq 2H(\Delta_\alpha(\epsilon), S) + u(S) = 2e(\Delta_\alpha(\epsilon), S).$$

To prove (3.9), it is sufficient to show that

$$e(\Delta_\alpha(\epsilon), S) \to 0 \text{ as } \epsilon \to 0.$$

Indeed, if $e(\Delta_\alpha(\epsilon), S)$ does not converge to 0 as $\epsilon \to 0$, then there exist $l > 0$ and $\{\epsilon_n\} \subset \mathbb{R}^+$ with $\epsilon_n \to 0$, and $x_n \in \Delta_\alpha(\epsilon_n)$ such that

$$x_n \notin S + B(0, l), \quad \forall n \in N, \tag{3.10}$$

where $B(0, l)$ is the closed ball centered at 0 with radius l. By the definition of $\Delta_\alpha(\epsilon_n)$, we know $d(x_n, K) \leq \epsilon_n < \epsilon_n + \frac{1}{n}$, and there exist $u_n \in F(x_n)$, $v_n \in G(x_n)$ such that

$$\langle u_n, x_n - y\rangle \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n, \quad \forall y \in \Omega,$$

and

$$\langle v_n, x_n - z\rangle + \phi(x_n) - \phi(z) \leq \frac{\alpha}{2}\|x_n - z\|^2 + \epsilon_n, \quad \forall z \in K.$$

Thus, there exists $\overline{x}_n \in K$ such that $\|\overline{x}_n - x_n\| < \epsilon_n + \frac{1}{n}$. Let $w_n = \overline{x}_n - x_n$. Then we have $w_n + x_n \in K$ with $w_n \to 0$. Thus $\{x_n\}$ is an LP α-approximating sequence for $GGVI(F, G, \phi, K)$. Since $GGVI(F, G, \phi, K)$ is strongly LP α-well-posed in the generalized sense, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to some point of S. This contradicts (3.10) and so

$$e(\Delta_\alpha(\epsilon), S) \to 0 \text{ as } \epsilon \to 0.$$

Conversely, assume that (3.9) holds. We first show that $\Delta_\alpha(\epsilon)$ is closed . Let $\{x_n\} \subset \Delta_\alpha(\epsilon)$ and $x_n \to x$ as $n \to \infty$. Then there exists $u_n \in F(x_n)$, $v_n \in G(x_n)$ such that $d(x_n, K) \leq \epsilon$ and

$$\langle u_n, x_n - y\rangle \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon, \quad \forall y \in \Omega, n \in N, \tag{3.11}$$

$$\langle v_n, x_n - z\rangle + \phi(x_n) - \phi(z) \leq \frac{\alpha}{2}\|x_n - z\|^2 + \epsilon, \quad \forall z \in K, n \in N. \tag{3.12}$$

Since F, G are nonempty upper semicontinuous and compact-valued mappings, there exist sequences $\{u_{n_k}\}$ of $\{u_n\}$, $\{v_{n_k}\}$ of $\{v_n\}$ and some $u \in F(x)$, $v \in G(x)$ such that $u_{n_k} \to u$ and $v_{n_k} \to v$, respectively. Therefore, it follows from (3.11), (3.12) and the lower semicontinuity of ϕ that

$$\langle u, x - y\rangle \leq \frac{\alpha}{2}\|x - y\|^2 + \epsilon, \quad \forall y \in \Omega,$$

and

$$\langle v, x - z\rangle + \phi(x) - \phi(z) \leq \frac{\alpha}{2}\|x - z\|^2 + \epsilon, \quad \forall z \in K.$$

It is easy to see $d(x, K) \leq \epsilon$. This implies that $x \in \Delta_\alpha(\epsilon)$ and so $\Delta_\alpha(\epsilon)$ is nonempty closed for all $\epsilon > 0$.

Now, we will prove

$$S = \bigcap_{\epsilon>0} \Delta_\alpha(\epsilon).$$

Indeed, it is obvious that $S \subset \Delta_\alpha(\epsilon)$, for any $\epsilon > 0$. On the other hand, if $0 < \epsilon_1 < \epsilon_2$, it is easy to know that $\Delta_\alpha(\epsilon_1) \subset \Delta_\alpha(\epsilon_2)$. We deduce that

$$\bigcap_{\epsilon>0} \Delta_\alpha(\epsilon) = \lim_{\epsilon\to 0} \Delta_\alpha(\epsilon).$$

That is,

$$\bigcap_{\epsilon>0} \Delta_\alpha(\epsilon) = \{x \in K : \exists u \in F(x), \exists v \in G(x), \langle u, x - y\rangle \leq \frac{\alpha}{2}\|x - y\|^2, \quad \forall y \in \Omega;$$

$$\langle v, x - z\rangle + \phi(x) - \phi(z) \leq \frac{\alpha}{2}\|x - z\|^2, \quad \forall z \in K\}.$$

By Proposition 2.8, we know that $\bigcap_{\epsilon>0} \Delta_\alpha(\epsilon) \subset S$. So, we obtain that $S = \bigcap_{\epsilon>0} \Delta_\alpha(\epsilon)$.

Since $u(\Delta_\alpha(\epsilon)) \to 0$, the Theorem in page 412 of [11] can be applied and one concludes that S is nonempty and compact with

$$H(\Delta_\alpha(\epsilon), S) = e(\Delta_\alpha(\epsilon), S) \to 0 \text{ as } \epsilon \to 0.$$

Let $\{\widehat{x}_n\} \subset X$ be an LP α-approximating sequence for $GGVI(F, G, \phi, K)$. Then there exists $w_n \in X$ with $w_n \to 0$ such that $\widehat{x}_n + w_n \in K$, and there exist $\widehat{u}_n \in F(\widehat{x}_n)$, $\widehat{v}_n \in G(\widehat{x}_n)$ and $0 < \epsilon_n^{'} \to 0$ such that

$$\langle \widehat{u}_n, \widehat{x}_n - y\rangle \leq \frac{\alpha}{2}\|\widehat{x}_n - y\|^2 + \epsilon_n^{'}, \quad \forall y \in \Omega, n \in N,$$

$$\langle \widehat{v}_n, \widehat{x}_n - z\rangle + \phi(\widehat{x}_n) - \phi(z) \leq \frac{\alpha}{2}\|\widehat{x}_n - z\|^2 + \epsilon_n^{'}, \quad \forall z \in K, n \in N.$$

Since $\widehat{x}_n + w_n \in K$, there exists $\overline{x}_n \in K$ such that $\widehat{x}_n + w_n = \overline{x}_n$. It follows that

$$d(\widehat{x}_n, K) \leq \|\widehat{x}_n - \overline{x}_n\| = \|w_n\| \to 0 \text{ as } n \to \infty.$$

Set $\epsilon_n = \max\{\|w_n\|, \epsilon_n^{'}\}$. We get $\widehat{x}_n \in \Delta_\alpha(\epsilon_n)$. It follows from (3.9) and the definition of $\Delta_\alpha(\epsilon_n)$ that

$$d(\widehat{x}_n, S) \leq e(\Delta_\alpha(\epsilon_n), S) \to 0 \text{ as } n \to \infty.$$

Since S is compact, there exists $p_n \in S$ such that

$$\|p_n - \widehat{x}_n\| = d(\widehat{x}_n, S) \to 0.$$

Again from the compactness of S, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ which converges strongly to $\overline{p} \in S$. Hence the corresponding subsequence $\{x_{n_k}\}$ of $\{\widehat{x}_n\}$ converges strongly to $\overline{p} \in S$. Thus, $GGVI(F, G, \phi, K)$ is strongly LP α-well-posed in the generalized sense. The proof is complete. □

4. Conditions for Levitin-Poylak well-posedness

In this section, we derive some conditions under which the $GGVI(F, G, \phi, K)$ is Levitin-Poylak well-posed in Banach spaces.

Theorem 4.1. Let K be a nonempty, closed and convex subset of X. Let F, $G : X \to 2^{X^*}$ be nonempty compact-valued mappings which are H-semicontinuous and monotone, respectively. Let $\phi : X \to \mathbb{R}\bigcup\{+\infty\}$ be a proper, convex and uniformly continuous functional. Then $GGVI(F, G, \phi, K)$ is weakly LP well-posed if and only if it has a unique solution.

Proof. The necessity is obvious. For the sufficiency, suppose that $GGVI\ (F, G, \phi, K)$ has a unique solution x^*. If $GGVI(F, G, \phi, K)$ is not weakly LP well-posed, then there exists an LP approximating sequence $\{x_n\}$ such that $\{x_n\}$ is not weakly converging to x^*. Thus, there exist $w_n \in X$ with $w_n \to 0$ and $0 < \epsilon_n \to 0$ such that $x_n + w_n \in K$, and there exist $u_n \in F(x_n)$, $v_n \in G(x_n)$ such that

$$\langle u_n, x_n - y\rangle \le \epsilon_n, \quad \forall y \in \Omega, n \in N, \tag{4.1}$$

and

$$\langle v_n, x_n - z\rangle + \phi(x_n) - \phi(z) \le \epsilon_n, \quad \forall z \in K, n \in N. \tag{4.2}$$

Since $x_n + w_n \in K$, there exists $\overline{x}_n \in K$ such that $x_n + w_n = \overline{x}_n$. Thus,

$$d(x_n, K) \le \|x_n - \overline{x}_n\| = \|w_n\| \to 0 \text{ as } n \to \infty.$$

If $\{x_n\}$ is unbounded, then $\{\overline{x}_n\}$ is an unbounded sequence of K, without loss of generality, we can suppose that $\|\overline{x}_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|\overline{x}_n - x^*\|}, \quad z_n = x^* + t_n(\overline{x}_n - x^*).$$

Without loss of generality, we can suppose $t_n \in (0, 1]$ and $z_n \rightharpoonup z(\ne x^*)$. Then we have, for each $y \in K, v \in G(y)$,

$$\begin{aligned} \langle v, z - y\rangle &= \langle v, z - z_n\rangle + \langle v, z_n - x^*\rangle + \langle v, x^* - y\rangle \\ &= \langle v, z - z_n\rangle + t_n\langle v, \overline{x}_n - x^*\rangle + \langle v, x^* - y\rangle \\ &= \langle v, z - z_n\rangle + t_n\langle v, x_n + w_n - x^*\rangle + \langle v, x^* - y\rangle \\ &= \langle v, z - z_n\rangle + t_n\langle v, x_n - y\rangle + (1 - t_n)\langle v, x^* - y\rangle + t_n\langle v, w_n\rangle. \end{aligned} \tag{4.3}$$

Since x^* is the unique solution of $GGVI(F, G, \phi, K)$, there exist $u^* \in F(x^*)$, $v^* \in G(x^*)$ such that

$$\langle u^*, x^* - y\rangle \le 0, \quad \forall y \in \Omega, \tag{4.4}$$

and

$$\langle v^*, x^* - z\rangle + \phi(x^*) - \phi(z) \le 0, \quad \forall z \in K. \tag{4.5}$$

Since G is monotone,

$$\langle v, x^* - y\rangle \le \langle v^*, x^* - y\rangle, \quad \langle v, x_n - y\rangle \le \langle v_n, x_n - y\rangle. \tag{4.6}$$

It follows from (4.2),(4.3), (4.5), (4.6) and the convexity of ϕ that, for all $\forall y \in K$ and $v \in G(y)$,

$$\langle v, z - y\rangle \;\le\; \langle v, z - z_n\rangle + t_n\phi(y) - t_n\phi(x_n) + t_n\epsilon_n$$

$$\begin{aligned}
&\quad +(1-t_n)(\phi(y)-\phi(x^*))+t_n\langle v,w_n\rangle \\
&= \langle v,z-z_n\rangle+\phi(y)-[t_n\phi(x_n)+(1-t_n)\phi(x^*)] \\
&\quad +t_n\epsilon_n+t_n\langle v,w_n\rangle \\
&= \langle v,z-z_n\rangle+\phi(y)-[t_n\phi(\overline{x}_n)+(1-t_n)\phi(x^*)+t_n\phi(x_n)-t_n\phi(\overline{x}_n)] \\
&\quad +t_n\epsilon_n+t_n\langle v,w_n\rangle \\
&\leq \langle v,z-z_n\rangle+\phi(y)-\phi(z_n)-[t_n\phi(x_n)-t_n\phi(\overline{x}_n)]+t_n\epsilon_n+t_n\langle v,w_n\rangle. \qquad (4.7)
\end{aligned}$$

Since ϕ is uniformly continuous, we have

$$\begin{aligned}
&\langle v,z-y\rangle \\
&\leq \liminf_{n\to\infty}\{\langle v,z-z_n\rangle+\phi(y)-\phi(z_n)-[t_n\phi(x_n)-t_n\phi(\overline{x}_n)]+t_n\epsilon_n+t_n\langle v,w_n\rangle\} \\
&\leq \phi(y)-\phi(z)
\end{aligned}$$

This together with Proposition 2.7 implies that $z\in\Omega$. For any $y\in\Omega$, $u\in F(y)$,

$$\begin{aligned}
\langle u,z-y\rangle &= \langle u,z-z_n\rangle+\langle u,z_n-x^*\rangle+\langle u,x^*-y\rangle \\
&= \langle u,z-z_n\rangle+t_n\langle u,\overline{x}_n-x^*\rangle+\langle u,x^*-y\rangle \\
&= \langle u,z-z_n\rangle+t_n\langle u,x_n+w_n-x^*\rangle+\langle u,x^*-y\rangle \\
&= \langle u,z-z_n\rangle+t_n\langle u,x_n-y\rangle+(1-t_n)\langle u,x^*-y\rangle+t_n\langle u,w_n\rangle. \qquad (4.8)
\end{aligned}$$

Since F is monotone,

$$\langle u,x^*-y\rangle\leq\langle u^*,x^*-y\rangle,\quad \langle u,x_n-y\rangle\leq\langle u_n,x_n-y\rangle. \qquad (4.9)$$

It follows from (4.1), (4.4), (4.8), (4.9) that, for all $u\in F(y)$,

$$\langle u,z-y\rangle\leq\langle u,z-z_n\rangle+t_n\epsilon_n+t_n\langle u,w_n\rangle.$$

We deduce that

$$\langle u,z-y\rangle\leq 0,\quad \forall y\in\Omega.$$

This together with $z\in\Omega$ implies that z is a solution of $GGVI(F,G,\phi,K)$, a contradiction. Thus, $\{x_n\}$ is bounded.

Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\overline{x}$ as $k\to\infty$. Clearly $\overline{x}\in K$. It follows from (4.1), (4.2) that

$$\langle u_{n_k},y-x_{n_k}\rangle\leq\epsilon_{n_k},\quad \forall y\in\Omega,$$

and

$$\langle v_{n_k},z-x_{n_k}\rangle+\phi(x_{n_k})-\phi(z)\leq\epsilon_{n_k},\quad \forall z\in K.$$

Since F,G is monotone, ϕ is convex lower semicontinuous, we have

$$\begin{aligned}
\langle u,\overline{x}-y\rangle &\leq \liminf_{k\to\infty}\{\langle u,x_{n_k}-y\rangle\}\leq\liminf_{k\to\infty}\{\langle u_{n_k},x_{n_k}-y\rangle\} \\
&\leq \liminf_{k\to\infty}\epsilon_{n_k}=0,\quad y\in\Omega,u\in F(y), \qquad (4.10)
\end{aligned}$$

and

$$\begin{aligned}
\langle v,\overline{x}-z\rangle+\phi(\overline{x})-\phi(z)\} &\leq \liminf_{k\to\infty}\{\langle v,x_{n_k}-z\rangle+\phi(x_{n_k})-\phi(z)\} \\
&\leq \liminf_{k\to\infty}\{\langle v_{n_k},x_{n_k}-z\rangle+\phi(x_{n_k})-\phi(z)\} \\
&\leq \liminf_{k\to\infty}\epsilon_{n_k}=0,\quad \forall z\in K,v\in G(z). \qquad (4.11)
\end{aligned}$$

This together with Proposition 2.8 yields that $\overline{x}$ solves $GGVI(F, G, \phi, K)$. Since $GGVI(F, G, \phi, K)$ has a unique solution x^*, we have $\overline{x} = x^*$. Thus $x_n \rightharpoonup x^*$, a contradiction. So $GGVI(F, G, \phi, K)$ is weakly LP well-posed. The proof is complete. $\square$

Now, for any $\delta_0 \geq 0$, we denote $M(\delta_0) = \{x \in X : d(x, K) \leq \delta_0\}$. We have the following result.

Theorem 4.2. Let K be a nonempty, closed and convex subset of X. Let F, $G : X \to 2^{X^*}$ be nonempty upper semicontinuous and compact-valued mappings, respectively. Let $\phi : X \to \mathbb{R}\bigcup\{+\infty\}$ be a proper, convex and lower semicontinuous function. If there exists some $\delta_0 > 0$ such that $M(\delta_0)$ is compact, then $GGVI(F, G, \phi, K)$ is strongly LP α-well-posed in the generalized sense.

Proof. Let $\{x_n\}$ be an LP approximating sequence for $GGVI(F, G\phi, K)$. Then there exist $0 < \epsilon_n^{'} \to 0$ and $w_n \in X$ with $w_n \to 0$ such that

$$x_n + w_n \in K,$$

and there exist $u_n \in F(x_n)$, $v_n \in G(x_n)$ satisfying

$$\langle u_n, x_n - y\rangle \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n^{'}, \quad \forall y \in \Omega,\ n \in N, \tag{4.12}$$

and

$$\langle v_n, x_n - z\rangle + \phi(x_n) - \phi(z) \leq \frac{\alpha}{2}\|x_n - z\|^2 + \epsilon_n^{'}, \quad \forall z \in K,\ n \in N. \tag{4.13}$$

Since $x_n + w_n \in K$, there exists $\overline{x}_n \in K$ such that $x_n + w_n = \overline{x}_n$. Thus,

$$d(x_n, K) \leq \|x_n - \overline{x}_n\| = \|w_n\| \to 0 \text{ as } n \to \infty.$$

Set $\epsilon_n = \max\{\epsilon_n^{'}, \|w_n\|\}$, we can get $d(x_n, K) \leq \epsilon_n$. Since $\epsilon_n \to 0$ as $n \to \infty$, without loss of generality, we can assume that $\{x_n\} \subset M(\delta_0)$ for n sufficiently large. By the compactness of $M(\delta_0)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\overline{x} \in M(\delta_0)$ such that $x_{n_k} \to \overline{x}$. It is easy to see $\overline{x} \in K$. First, by the u.s.c. of G at $\overline{x}$ and compactness of $G(\overline{x})$, there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and some $\overline{v} \in G(\overline{x})$ such that $v_{n_k} \to \overline{v}$. Since ϕ is lower semicontinuous, it follows from (4.13) that

$$\langle \overline{v}, \overline{x} - z\rangle + \phi(\overline{x}) - \phi(z) \leq \frac{\alpha}{2}\|\overline{x} - z\|^2, \quad \forall z \in K. \tag{4.14}$$

Similarly, by (4.12), we can deduce that there exists some $\bar{u} \in F(\bar{x})$ such that

$$\langle \bar{u}, \bar{x} - y\rangle \leq \frac{\alpha}{2}\|\bar{x} - y\|^2, \quad \forall y \in \Omega. \tag{4.15}$$

It follows from (4.14), (4.15) and Proposition 2.8 that $\overline{x}$ solves $GGVI(F, G, \phi, K)$. Thus $GGVI(F, G, \phi, K)$ is strongly LP α-well-posed in the generalized sense. The proof is complete. $\square$

References

[1] L. Q. Anh, P. Q. Khanh, D. T. M. Van and J. C. Yao, *Well-posedness for vector quasi-equilibria,* Taiwanese J. Math.**13** (2009), 713–737.

[2] E. M. Bednarczuk, *Well-posedness of Optimization Problem,* in Recent Advances and Historical Development of Vector Optimization Problems, Jahn, J., Krabs, W.(ends.), Lecture Notes in Economics and Mathematical Systems, vol. 294, Springer, Berlin, 1987, pp. 51–61.

[3] L. C. Ceng and J. C. Yao, *Well-posed of generalized mixed variational inequalities, inclusion problems and fixed-point problems,* Nonlinear Anal. TMA, **69** (2008), 4585–4603.

[4] L. C. Ceng, N. Hadjisavvas, S. Schaible and J. C. Yao, *Well-posedness for mixed quasivariational-like inequalities,* J. Optim. Theory Appl. **139** (2008), 109–125.

[5] Y. P. Fang, N. J. Huang and J. C. Yao, *Well-posed of mixed variational inequalities, inclusion problems and fixed-point problems,* J. Glob Optim. **41** (2008), 117–133.

[6] Y. P. Fang, N. J. Huang and J. C. Yao, *Well-posed by perturbations of mixed variational inequalities in Banach spaces,* European J. Oper. Res. **201** (2010), 682–692.

[7] Y. Han and X. H. Gong, *Levitin-Polyak well-posedness of symmetric vector quasi-equilibrium problems,* Optim., DOI:10.1080/02331934.2014.886037.

[8] R. Hu and Y. P. Fang, *Levitin-Polyak well- posedness of variational inequalities,* Nonlinear Anal. TMA **72** (2010), 373–381.

[9] R. Hu and Y. P. Fang, *Levitin-Polyak well-posedness by perturbations of inverse variational inequalities,* Optim. Lett. **7** (2013), 343–359.

[10] A. F. Izmaelov and M. V. Solodov, *An active set Newton method for mathematical program with complementary constraints,* SIAM J. Optim. **19** (2008)1003–1027.

[11] K. Kuratowski, *Topology,* vols, 1 and 2, Academic, New York, NY, 1966.

[12] B. Lemaire, *Well-posedness, conditions, and regularization of minimization, inclusion,and fixed-point problems,* Pliska Studia Mathematic Bulgaria **12** (1998), 71–84.

[13] E. S. Levitin and B. T. Polyak, *Convergence of minimizing sequences in conditional extremum problem,* Soveit Math. Dokl. **7** (1996), 764–767.

[14] X. B. Li and F. Q. Xia, *Levitin-Polyak well-posedness of a generalized mixed variational inequality in Banach spaces,* Nonlinear Anal. TMA **75** (2012), 2139–2153.

[15] R. Lucchetti and F. Patrone, *A characterization of Tykhonov well-posedness for minimum problems with applications to variational inequalities,* Numer, Funct. Anal. Optim. **3** (1981), 461–476.

[16] R. Lucchetti and F. Patrone, *Hadamard and Tykhonov well-posedness of certain class of convex functions,* J. Math. Anal. Appl. **88** (1982), 204–215.

[17] P. E. Maingé, *Projected subgradient techniques and viscosity methods for optimization with variational inequality constraints,* European J. Oper. Res. **205** (2010), 501–506.

[18] S. B. Nadler, *Multi-valued contraction mappings,* Pacific J. Math. **30** (1969), 475–488.

[19] A. Petrusel, I. Rus and J. C. Yao, *Well-posedness in the generalized sense of the fixed point problems for multivalued operators,* Taiwanese J. Math. **11** (2007), 903–914.

[20] L. H. Peng, C. Li and J. C. Yao, *Well-posedness of a class of perturbed optimization problems in Banach spaces,* J. Math. Anal. Appl. **346** (2008), 384–394.

[21] A. N. Tykhonov, *On the stability of the functional optimization problem,* USSRJ. Comput. Math. Phys. **6** (1966), 631–634.

[22] S. H. Wang, N. J. Huang and D. O'Regan, *Well-posedness for generalized quasi-variational inclusion problems and for optimization problems with constraints,* J. Glob. Optim. **55** (2013), 189–208.

[23] Y. B. Xiao, X. M. Yang and N. J. Huang, *Some equivalence results for well-posedness of hemivariational inequalities,* J. Glob. Optim., DOI: 10.1007/s10898-014-0198-7.

[24] T. Zolezzi, *Extend well-posedness of optimization problems,* J. optim. Theory Appl. **91** (1996), 257–266.

[25] T. Zolezzi, *Well-posedness criteria in optimization with application to the calculus of variations,* Nonlinear Anal. TMA **25** (1995), 437–453.

[26] T. Zolezzi, *Well-posedness of optimal control problems,* Control and Cyber-netics **23** (1994), 289–301.

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