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HIGHER-ORDER DUALITY IN NONDIFFERENTIABLE MULTIOBJECTIVE FRACTIONAL PROGRAMMING INVOLVING CONES

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Dedicated to Prof. Wataru Takahashi on his 70^{th}birthday

Abstract. We consider nondifferentiable multiobjective fractional programming problem involving cone constraints, where every denominator and numerator of the objective function contains a term involving the support function of a compact convex set. Necessary and sufficient optimality conditions are established for weakly efficient solutions under higher-order type I assumptions. We formulate duality theorems for multiobjective fractional programming problem under higher-order type I functions. Some special cases of our duality results are presented.

1. Introduction

Multiobjective fractional programming refers to a multiobjective problem where the objective functions are quotients, $f_i(x)/g_i(x)$. The fractional optimization problem with multiple objective functions have been the subject of intense investigations in the past few years, which have produced a number of optimality and duality for these problems.

Recently, optimality conditions and the duality for multiobjective programming have been studied under kinds of generalized convexity and some results for that had been obtained. Especially, necessary optimality conditions for vector minimization problem involving cones were formulated by Suneja et al. [14]. Furthermore, Fritz John and Karush-Kuhn-Tucker necessary optimality conditions for multiobjective fractional programming problems with support functions were established by Kim et al. [5]. Invexity is a generalization of the convexity property that extends the sufficiency of Fritz John conditions and duality theorem of convex programs to a more general class of optimization problems.

Second and higher-order duality provides tighter bounds for the value of the objective function of the primal problem when approximations are used because there are more parameters involved. Mangasarian [10] formulated a group of second and higher-order dual problems for a nonlinear programming problem involving twice differentiable functions. Higher-order generalized invexity and duality in nondifferentiable mathematical programming problem were studied by Mishra and Rueda [11]. Higher-order duality for a class of higher-order (F, ρ, σ)-type I functions in

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multiobjective fractional programming with support functions was introduced in the Paretian cone setting by Suneja et al. [15]. Higher-order duality in multiobjective programming involving cones under various higher-order type I functions was established by Kim and Lee [6].

Ratio invexity was formulated by Khan and Hanson [4] for optimality and duality in fractional programming. This concept seems to be new and it introduces a kind of modified characterization in sufficient optimality conditions. Recently, duality for multiobjective fractional programming problems with support functions was established under (V,ρ)-invexity by Kim et al. [5], in which, they used the generalized ratio invexity concept. Until then, many papers had shown the various ratio invexity concept such as [3, 4, 8] and [13]. Later, nondifferentiable multiobjective fractional programming problems with cone constraints over arbitrary closed convex cones were introduced and duality theorems for a weakly efficient solution were formulated by Kim et al. [7]. Subsequently, sufficient optimality conditions and duality for a class of multiobjective fractional programming problems under higher-order (F, α, ρ, d)-convexity assumptions were given by Chen [1].

In this paper, we consider nondifferentiable multiobjective fractional programming problem whose objective functions contain support functions of compact convex sets in $\mathbb{R}^n$ and whose constraints contain closed convex cones. We first show that the ratio of higher-order type I functions is still higher-order type I functions. In addition, necessary conditions for our primal problem with cone constraints and the sufficient optimality conditions under suitable higher-order type I assumptions are proposed. Higher-order Wolfe, Mond-Weir and Schaible type duals are formulated for the nondifferentiable multiobjective fractional program and various duality results for the weakly efficient solution are established. Moreover, some special cases of our duality results are given.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and let $\mathbb{R}^n_+$ be its non-negative orthant. The following convention for inequalities will be used in the paper:

$$x \leqq u \iff x_i \leqq u_i \text{ for all } i = 1, 2, \dots, n \ ;$$
$$x \leq u \iff x_i \leqq u_i \text{ for all } i = 1, 2, \dots, n, \text{ but } x \neq u \ ;$$
$$x \not< u \text{ is the negation of } x < u \ .$$

For x,$u \in \mathbb{R}$, $x \leqq u$ and $x < u$ have the usual meaning.

Consider the following nondifferentiable multiobjective fractional programming problem: (**MCFP**)

$$\begin{aligned} \text{Minimize} \quad & \frac{f(x) + s(x|D)}{g(x) - s(x|E)} \\ & = \left(\frac{f_1(x) + s(x|D_1)}{g_1(x) - s(x|E_1)}, \frac{f_2(x) + s(x|D_2)}{g_2(x) - s(x|E_2)}, \dots, \frac{f_l(x) + s(x|D_l)}{g_l(x) - s(x|E_l)} \right) \\ \text{subject to} \quad & h(x) \in C_2^*, x \in C_1, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}^l$, $g : \mathbb{R}^n \to \mathbb{R}^l$, and $h : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable over C_1. C_1 and C_2 are closed convex cones with nonempty interiors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. $X := \{x \in \mathbb{R}^n | h(x) \in C_2^*, x \in C_1\}$ is the feasible set. C_2^* is a polar cone of C_2. For each $i = 1, \ldots, l$, D_i and E_i are compact convex sets of $\mathbb{R}^n$. The support function of D_i and E_i at $x \in \mathbb{R}^n$ is defined by $s(x|D_i) := max\{x^T w_i : w_i \in D_i\}$ and $s(x|E_i) := max\{x^T z_i : z_i \in E_i\}$, respectively.

In addition, we use the usual definition of cone. A nonempty set C in $\mathbb{R}^n$ is said to be a cone with vertex zero, if $x \in C$ implies that $\lambda x \in C$ for all $\lambda \geqq 0$. Moreover, if C is convex, then C is called a convex cone. Also, the polar cone C^* of C is defined by

$$C^* := \{z \in \mathbb{R}^n \mid x^T z \leqq 0 \text{ for all } x \in C\}.$$

Throughout this paper, suppose that for all $x \in C_1$, $f(x) + x^T w \geqq 0$ and $g(x) - x^T z > 0$.

We recall some known facts about support functions [12].

Every sublinear function defined on $\mathbb{R}^n$ may be written as a support function and further any compact set D can be uniquely determined by its support function. The support function $s(x|D)$ of a compact convex set $D \subseteq \mathbb{R}^n$, being convex and everywhere finite, has a subgradient [2] at every x, that is, there exists $z \in D$ such that $s(y|D) \geq s(x|D) + z^T(y - x)$ for all $y \in D$, as the subdifferential of $s(x|D)$ is given by

$$\partial s(x|D) := \{z \in D : x^T z = s(x|D)\}.$$

For any set $S \subset \mathbb{R}^n$, the normal cone to S at any point $x \in S$ is defined by

$$N_S(x) := \{y \in \mathbb{R}^n : y^T(z - x) \leqq 0 \text{ for all } z \in S\}.$$

If D is a compact convex set, then $x \in N_D(z)$ if and only if $s(x|D) = x^T z$, or equivalently $z \in \partial s(x|D)$.

Definition 2.1. A feasible point $\bar{x}$ is said to be a weakly efficient solution of (**MCFP**), if there exists no other $x \in X$ such that $\frac{f(x)+s(x|D)}{g(x)-s(x|E)} < \frac{f(\bar{x})+s(\bar{x}|D)}{g(\bar{x})-s(\bar{x}|E)}$.

The following definition is about higher-order type I functions.

Definition 2.2. $\big(f(\cdot) + (\cdot)^T w,\ h(\cdot)\big)$ is said to be higher-order type I at u with respect to a function η, if for all x, the following inequalities hold:

$$f_i(x) + x^T w_i - f_i(u) - u^T w_i \geqq \eta_1(x,u)^T \nabla_p F_i(u,p) + F_i(u,p) - p^T \nabla_p F_i(u,p), \quad i = 1, 2, \ldots, l,$$

and

$$-h_j(u) \geqq \eta_2(x,u)^T \nabla_p H_j(u,p) + H_j(u,p) - p^T \nabla_p H_j(u,p), \quad j = 1, 2, \ldots, m,$$

where $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^l$ and $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions; $\nabla_p F_i(u,p)$, $i = 1, \ldots, l$ and $\nabla_p H_j(u,p)$, $j = 1, \ldots, m$ denote the $n \times 1$ gradient for F_i and H_j with respect to p, respectively.

In order to establish sufficient optimality conditions for our model, we develop the following ratio concept for generalized higher-order type I.

Lemma 2.3. *Assume that f and g are vector-valued differentiable functions on $\mathbb{R}^n$. Let $\big(f(\cdot) + (\cdot)^T w,\ h(\cdot)\big)$ and $\big(-g(\cdot) + (\cdot)^T z,\ h(\cdot)\big)$ be higher-order type I at $\bar{x}$ with respect to η. If $\nabla g_i(\bar{x}) - z_i = 0$ and $\nabla^2 \frac{f_i(\bar{x})+\bar{x}^T w_i}{g_i(\bar{x})-\bar{x}^T z_i}$ is negative semidefinite(or* *positive semidefinite), then* $\left(\frac{f(\cdot)+(\cdot)^T w}{g(\cdot)-(\cdot)^T z},\ h(\cdot)\right)$ *is higher-order type I at* $\bar{x}$ *with respect to* η, *where* $\eta_2(x,\bar{x}) = \frac{g_i(\bar{x})-\bar{x}^T z_i}{g_i(x)-x^T z_i}\eta_1(x,\bar{x})$. *Further,* $F, G, K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^l$ *and* $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ *are differentiable functions. Furthermore,* $\nabla_p F_i(u,p), \nabla_p G_i(u,p)$, *and* $\nabla_p K_i(u,p)$, $i = 1,\ldots,l$ *and* $\nabla_p y^T H(u,p)$ *denote the* $n\times 1$ *gradient of* F_i, G_i, K_i *and* $y^T H$ *with respect to p, respectively.*

Proof. By the higher-order type I assumptions for $\left(f(\cdot)+(\cdot)^T w,\ h(\cdot)\right)$ and $\left(-g(\cdot)+(\cdot)^T z,\ h(\cdot)\right)$, we have

$$\begin{aligned}
&\frac{f_i(x)+x^T w_i}{g_i(x)-x^T z_i} - \frac{f_i(\bar{x})+\bar{x}^T w_i}{g_i(\bar{x})-\bar{x}^T z_i}\\
=&\frac{f_i(x)+x^T w_i - f_i(\bar{x}) - \bar{x}^T w_i}{g_i(x)-x^T z_i} - (f_i(\bar{x})+\bar{x}^T w_i)\frac{g_i(x)-x^T z_i - g_i(\bar{x}) + \bar{x}^T z_i}{(g_i(x)-x^T z_i)\,(g_i(\bar{x})-\bar{x}^T z_i)}\\
\geqq&\frac{1}{g_i(x)-x^T z_i}\left[\eta_1(x,\bar{x})^T\nabla_p F_i(\bar{x},p) + F_i(\bar{x},p) - p^T\nabla_p F_i(\bar{x},p)\right]\\
&-\frac{f_i(\bar{x})+\bar{x}^T w_i}{(g_i(x)-x^T z_i)\,(g_i(\bar{x})-\bar{x}^T z_i)}\left[\eta_1(x,\bar{x})^T\nabla_p G_i(\bar{x},p) + G_i(\bar{x},p) - p^T\nabla_p G_i(\bar{x},p)\right]\\
=&\frac{g_i(\bar{x})-\bar{x}^T z_i}{g_i(x)-x^T z_i}\Bigg[\eta_1(x,\bar{x})^T\frac{\nabla_p F_i(\bar{x},p)\left(g_i(\bar{x})-\bar{x}^T z_i\right) - \left(f_i(\bar{x})+\bar{x}^T w_i\right)\nabla_p G_i(\bar{x},p)}{\{g_i(\bar{x})-\bar{x}^T z_i\}^2}\\
&+\frac{F_i(\bar{x},p)\left(g_i(\bar{x})-\bar{x}^T z_i\right) - \left(f_i(\bar{x})+\bar{x}^T w_i\right)G_i(\bar{x},p)}{\{g_i(\bar{x})-\bar{x}^T z_i\}^2}\\
&-p^T\frac{\nabla_p F_i(\bar{x},p)\left(g_i(\bar{x})-\bar{x}^T z_i\right) - \left(f_i(\bar{x})+\bar{x}^T w_i\right)\nabla_p G_i(\bar{x},p)}{\{g_i(\bar{x})-\bar{x}^T z_i\}^2}\Bigg].
\end{aligned}$$

Using the assumption $\nabla g_i(\bar{x}) - z_i = 0$ and

$$\begin{aligned}
\nabla^2\left\{\frac{f_i(\bar{x})+\bar{x}^T w_i}{g_i(\bar{x})-\bar{x}^T z_i}\right\} =&\frac{\nabla^2 f_i(\bar{x})(g_i(\bar{x})-\bar{x}^T z_i) - (f_i(\bar{x})+\bar{x}^T w_i)\nabla^2 g_i(\bar{x})}{\{g_i(\bar{x})-\bar{x}^T z_i\}^2}\\
&-2\nabla\left\{\frac{f_i(\bar{x})+\bar{x}^T w_i}{g_i(\bar{x})-\bar{x}^T z_i}\right\}\cdot\frac{\nabla g_i(\bar{x})-z_i}{g_i(\bar{x})-\bar{x}^T z_i}.
\end{aligned}$$

Denoting

$$\begin{aligned}
K_i(\bar{x},p) &= \frac{F_i(\bar{x},p)(g_i(\bar{x})-\bar{x}^T z_i) - (f_i(\bar{x})+\bar{x}^T w_i)G_i(\bar{x},p)}{\{g_i(\bar{x})-\bar{x}^T z_i\}^2}\\
\nabla_p K_i(\bar{x},p) &= \frac{\nabla_p F_i(\bar{x},p)(g_i(\bar{x})-\bar{x}^T z_i) - (f_i(\bar{x})+\bar{x}^T w_i)\nabla_p G_i(\bar{x},p)}{\{g_i(\bar{x})-\bar{x}^T z_i\}^2}.
\end{aligned}$$

Therefore, we get

$$\frac{f_i(x)+x^T w_i}{g_i(x)-x^T z_i} - \frac{f_i(\bar{x})+\bar{x}^T w_i}{g_i(\bar{x})-\bar{x}^T z_i}$$

$$\geqq \frac{g_i(\bar{x}) - \bar{x}^T z_i}{g_i(x) - x^T z_i}\eta_1(x,\bar{x})^T \nabla_p K_i(\bar{x},p) + \frac{g_i(\bar{x}) - \bar{x}^T z_i}{g_i(x) - x^T z_i}\left(K_i(\bar{x},p) - p^T \nabla_p K_i(\bar{x},p)\right)$$
$$= \eta_2(x,\bar{x})^T \nabla_p K_i(\bar{x},p) + K_i(\bar{x},p) - p^T \nabla_p K_i(\bar{x},p) \text{ (by the second assumption).}$$

This completes the proof. □

3. Optimality conditions

Now, we establish necessary and sufficient conditions for a weakly efficient solution of (**MCFP**).

Theorem 3.1 (Necessary Optimality Condition)**.** *Let $\bar{x}$ be a weakly efficient solution of* (**MCFP**)*, then there exist $\bar{\lambda} \geqq 0$, $\bar{y} \in C_2$, $((\bar{\lambda},\bar{y}) \neq 0)$, $\bar{w}_i \in D_i$ and $\bar{z}_i \in E_i (i = 1,\dots,l)$ such that*

$$\left[\bar{\lambda}^T \nabla \left\{\frac{f(\bar{x}) + \bar{x}^T \bar{w}}{g(\bar{x}) - \bar{x}^T \bar{z}}\right\} + \nabla \bar{y}^T h(\bar{x})\right]^T (x - \bar{x}) \geqq 0 \textit{ for all } x \in C_1,$$
$$\bar{y}^T h(\bar{x}) = 0,\ s(\bar{x}|D_i) = \bar{x}^T \bar{w}_i \textit{ and } s(\bar{x}|E_i) = \bar{x}^T \bar{z}_i,\ i = 1,\dots,l.$$

Proof. Let $k_i(x) = s(x|D_i)$ and $m_i(x) = s(x|E_i)$, $i = 1,\dots,l$. Since D_i and E_i are convex and compact, then k_i and $m_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions.

Hence, for all $d \in \mathbb{R}^n$, $k_i'(\bar{x};d) = \lim_{\lambda \to 0^+} \frac{k_i(\bar{x}+\lambda d) - k_i(\bar{x})}{\lambda}$ and $m_i'(\bar{x};d) = \lim_{\lambda \to 0^+} \frac{m_i(\bar{x}+\lambda d) - m_i(\bar{x})}{\lambda}$ are finite, then $\left(\frac{f_i + k_i}{g_i - m_i}\right)'(\bar{x};d) = \frac{1}{\{g_i(\bar{x}) - m_i(\bar{x})\}^2} \big[\{g_i(\bar{x}) - m_i(\bar{x})\}\langle \nabla f_i(\bar{x}), d\rangle + \{g_i(\bar{x}) - m_i(\bar{x})\} k_i'(\bar{x};d) - \{f_i(\bar{x}) + k_i(\bar{x})\}\langle \nabla g_i(\bar{x}), d\rangle - \{f_i(\bar{x}) + k_i(\bar{x})\} m_i'(\bar{x};d)\big]$.

Since $\bar{x}$ is a weakly efficient solution of (**MCFP**), then there exists no solution $x \in C_1$ such that the system

$$\text{(3.1)} \qquad \begin{cases} \left(\dfrac{f_i + k_i}{g_i - m_i}\right)'(\bar{x}; x - \bar{x}) < 0, \text{ for all } i = 1,\dots,l \\ \nabla h(\bar{x})^T (x - \bar{x}) + h(\bar{x}) \in \operatorname{int} C_2^* \end{cases}$$

holds.

Ab absurdo, suppose that there is a solution $x^* \in C_1$.

Now, using the first inequality of the system, we have

$$\begin{aligned} &\left(\frac{f_i + k_i}{g_i - m_i}\right)'(\bar{x}; x^* - \bar{x}) \\ =& \lim_{\alpha \to 0^+} \frac{1}{\alpha}\Big[\frac{f_i(\bar{x} + \alpha(x^* - \bar{x})) + k_i(\bar{x} + \alpha(x^* - \bar{x}))}{g_i(\bar{x} + \alpha(x^* - \bar{x})) - m_i(\bar{x} + \alpha(x^* - \bar{x}))} - \frac{f_i(\bar{x}) + k_i(\bar{x})}{g_i(\bar{x}) - m_i(\bar{x})}\Big] \\ <& 0, \qquad \text{for } 0 < \alpha < 1 \end{aligned}$$

and so there exists $\delta > 0$ such that for all $\alpha \in (0,\delta)$

$$\frac{1}{\alpha}\Big[\frac{f_i(\bar{x} + \alpha(x^* - \bar{x})) + k_i(\bar{x} + \alpha(x^* - \bar{x}))}{g_i(\bar{x} + \alpha(x^* - \bar{x})) - m_i(\bar{x} + \alpha(x^* - \bar{x}))} - \frac{f_i(\bar{x}) + k_i(\bar{x})}{g_i(\bar{x}) - m_i(\bar{x})}\Big] < 0.$$

Hence we get

$$\frac{f_i(\bar{x} + \alpha(x^* - \bar{x})) + k_i(\bar{x} + \alpha(x^* - \bar{x}))}{g_i(\bar{x} + \alpha(x^* - \bar{x})) - m_i(\bar{x} + \alpha(x^* - \bar{x}))} - \frac{f_i(\bar{x}) + k_i(\bar{x})}{g_i(\bar{x}) - m_i(\bar{x})} < 0.$$

Also, using the second statement of the system for $0 < \alpha < 1$, we have

$$\begin{aligned} h(\bar{x} + \alpha(x^* - \bar{x})) =& h(\bar{x}) + \nabla h(\bar{x})^T \alpha(x^* - \bar{x}) + o(\alpha) \\ =& \alpha(h(\bar{x}) + \nabla h(\bar{x})^T (x^* - \bar{x})) + (1-\alpha)h(\bar{x}) + o(\alpha) \in C_2^*, \end{aligned}$$

where $\lim_{x \to 0^+} o(\alpha) = 0$.

Since $h(\bar{x}) \in C_2^*$, we obtain $h(\bar{x} + \alpha(x^* - \bar{x})) \in C_2^*$, i.e. $\bar{x} + \alpha(x^* - \bar{x})$ is feasible of (**MCFP**), which contradicts that $\bar{x}$ is a weakly efficient solution.

Thus for the system (3.1), $\forall x \in C_1$, there exist no solution ($\bar{\lambda} \geqq 0, \bar{y} \in C_2$, $((\bar{\lambda}, \bar{y}) \neq 0)$) such that

$$\bar{\lambda}_i^T \Big(\frac{f_i + k_i}{g_i - m_i}\Big)'(\bar{x}; x - \bar{x}) + \bar{y}^T\{\nabla h(\bar{x})^T (x - \bar{x}) + h(\bar{x})\} < 0$$

holds, that is, $\forall x \in C_1$, there exist the solution $(\bar{\lambda} \geqq 0, \bar{y} \in C_2, ((\bar{\lambda}, \bar{y}) \neq 0))$ such that

$$\bar{\lambda}_i^T \Big(\frac{f_i + k_i}{g_i - m_i}\Big)'(\bar{x}; x - \bar{x}) + \bar{y}^T\{\nabla h(\bar{x})^T (x - \bar{x}) + h(\bar{x})\} \geqq 0$$

holds,

$$\begin{aligned} \text{i.e. } & \sum_{i=1}^{l} \frac{\bar{\lambda}_i}{\{g_i(\bar{x}) - m_i(\bar{x})\}^2} \Big\langle (g_i(\bar{x}) - m_i(\bar{x}))(\nabla f_i(\bar{x}) + w_i) \\ & - (f_i(\bar{x}) + k_i(\bar{x})(\nabla g_i(\bar{x}) + z_i), x - \bar{x} \Big\rangle + \bar{y}^T\{\nabla h(\bar{x})^T (x - \bar{x}) + h(\bar{x})\} \\ & \geqq 0 \end{aligned}$$

for all $\bar{w}_i \in \partial k_i(\bar{x})$, $\bar{z}_i \in \partial m_i(\bar{x}) (i = 1, \ldots, l)$ and $x \in C_1$.

Taking $x = \bar{x}$, the above relation gives $\bar{y}^T h(\bar{x}) \geqq 0$. Since $h(\bar{x}) \in C_2^*$ and $\bar{y} \in C_2$, we get $\bar{y}^T h(\bar{x}) \leqq 0$. Thus $\bar{y}^T h(\bar{x}) = 0$.

Hence, there exist $\bar{\lambda} \geqq 0$ and $\bar{y} \in C_2$, $((\bar{\lambda}, \bar{y}) \neq 0)$, $\bar{w}_i \in D_i$ and $\bar{z}_i \in E_i$, $i = 1, \ldots, l$ such that the desired results hold.

This completes the proof. □

Theorem 3.2 (Sufficient Optimality Conditions)**.** *Let $\bar{x}$ be a feasible solution of* (**MCFP**)*. Suppose that there exist $\bar{\lambda} \geq 0$, $\bar{y} \in C_2$, $\bar{w}_i \in D_i$, and $\bar{z}_i \in E_i (i = 1, \ldots, l)$ such that*

$$\Big[\bar{\lambda}^T \nabla \Big\{\frac{f(\bar{x}) + \bar{x}^T \bar{w}}{g(\bar{x}) - \bar{x}^T \bar{z}}\Big\} + \nabla \bar{y}^T h(\bar{x})\Big]^T (x - \bar{x}) \geqq 0 \text{ for all } x \in C_1, \tag{3.2}$$

$$\bar{y}^T h(\bar{x}) = 0, \ s(\bar{x}|D_i) = \bar{x}^T \bar{w}_i, \ \text{and } s(\bar{x}|E_i) = \bar{x}^T \bar{z}_i \text{ for } i = 1, \ldots, l.$$

Assume that

$$K(\bar{x}, 0) = 0, \ H(\bar{x}, 0) = 0, \ \nabla_p K(\bar{x}, 0) = \nabla \frac{f(\bar{x}) + \bar{x}^T \bar{w}}{g(\bar{x}) - \bar{x}^T \bar{z}}, \ \text{and}$$

$$\nabla_p H(\bar{x}, 0) = \nabla h(\bar{x}).$$

Let $(f(\cdot) + (\cdot)^T w,\ y^T h(\cdot)e)$ and $(-g(\cdot) + (\cdot)^T z,\ y^T h(\cdot)e)$ be higher-order type I at $\bar{x}$ with respect to a function η, where $\eta_2(x, \bar{x}) = \frac{g_i(\bar{x}) - \bar{x}^T \bar{z}_i}{g_i(x) - x^T z_i} \eta_1(x, \bar{x})$. Then $\bar{x}$ is a weakly efficient solution of (**MCFP**)*.*

Proof. Since $\bar{y} \in C_2$, for any feasible solution x of (**MCFP**), we obtain

$$\bar{y}^T h(x) \leqq 0. \tag{3.3}$$

We now suppose that $\bar{x}$ is not a weakly efficient solution of (**MCFP**). Then there exists a feasible solution $x \in C_1$ such that

$$\frac{f(x) + s(x|D)}{g(x) - s(x|E)} < \frac{f(\bar{x}) + s(\bar{x}|D)}{g(\bar{x}) - s(\bar{x}|E)}.$$

Since $\lambda \geq 0$, $s(x|D) \geqq x^T w$, $s(x|E) \geqq x^T z$, $s(\bar{x}|D_i) = \bar{x}^T \bar{w}_i$, and $s(\bar{x}|E_i) = \bar{x}^T \bar{z}_i$, we have

$$\lambda^T \left[\frac{f(x) + x^T w}{g(x) - x^T z} \right] < \lambda^T \left[\frac{f(\bar{x}) + \bar{x}^T \bar{w}}{g(\bar{x}) - x^T \bar{z}} \right].$$

From the higher-order type I hypotheses of $(f(\cdot) + (\cdot)^T w, y^T h(\cdot)e)$ and $(-g(\cdot) + (\cdot)^T z, y^T h(\cdot)e)$, by Lemma 2.3, we get

$$\eta_2(x, \bar{x})^T \nabla_p \lambda^T K(\bar{x}, p) + \lambda^T K(\bar{x}, p) - p^T \nabla_p \lambda^T K(\bar{x}, p) < 0.$$

Using the assumptions for $p = 0$, we can consider above inequality as following:

$$\eta_2(x, \bar{x})^T \left[\bar{\lambda}^T \nabla \left\{ \frac{f(\bar{x}) + \bar{x}^T \bar{w}}{g(\bar{x}) - \bar{x}^T \bar{z}} \right\} \right] < 0.$$

Since $x \in C_1$, $\bar{x} \in C_1$, and C_1 is a closed convex cone, we have $x + \bar{x} \in C_1$ and thus the inequality (3.2) implies

$$\left[\bar{\lambda}^T \nabla \left\{ \frac{f(\bar{x}) + \bar{x}^T \bar{w}}{g(\bar{x}) - \bar{x}^T \bar{z}} \right\} + \nabla \bar{y}^T h(\bar{x}) \right]^T x \geqq 0, \quad \text{for all} \quad x \in C_1, \text{ i.e.,}$$

$$\bar{\lambda}^T \nabla \left\{ \frac{f(\bar{x}) + \bar{x}^T \bar{w}}{g(\bar{x}) - \bar{x}^T \bar{z}} \right\} + \bar{y}^T \nabla h(\bar{x}) = 0.$$

Then we obtain $\eta_2(x, \bar{x})^T \nabla \bar{y}^T h(\bar{x}) > 0$.

Using the assumption, we can consider above inequality as

$$\eta_2(x, \bar{x})^T \nabla_p \bar{y}^T H(\bar{x}, 0) > 0.$$

By the type I hypotheses and $\bar{y}^T h(\bar{x}) = 0$, we get

$$0 < \bar{y}^T h(x) - \bar{y}^T h(\bar{x}) - \bar{y}^T H(\bar{x}, 0) = \bar{y}^T h(x),$$

which contradicts to (3.3).

Therefore, $\bar{x}$ is a weakly efficient solution of (**MCFP**). □

4. Duality

In this section, we would like to formulate Wolfe, Mond-Weir and Schaible type dual problems.

First, we propose Wolfe dual problem $(\mathbf{MCFD})_\mathbf{W}$ to (**MCFP**):

$(\mathbf{MCFD})_\mathbf{W}$

$$\begin{aligned} \text{Maximize} \quad & \frac{f(u) + u^T w}{g(u) - u^T z} + (\lambda^T K(u, p))e - p^T \nabla_p (\lambda^T K(u, p))e \\ & + y^T h(u)e + (y^T H(u, p))e - p^T \nabla_p (y^T H(u, p))e \end{aligned}$$

subject to

$$\nabla_p \lambda^T K(u,p) + \nabla_p y^T H(u,p) = 0, \tag{4.1}$$
$$w_i \in D_i,\ z_i \in E_i,\ i = 1, \ldots, l,$$
$$y \in C_2, \lambda \geq 0, \lambda^T e = 1, e = (1, \ldots, 1)^T \in \mathbb{R}^l.$$

Now we establish weak and strong duality theorems between (**MCFP**) and (**MCFD**)$_{\mathbf{W}}$.

Theorem 4.1 (Weak Duality)**.** *Let x and (u, y, λ, w, z, p) be feasible solutions of* **(MCFP)** *and* (**MCFD**)$_{\mathbf{W}}$*, respectively. Assume that $(f(\cdot) + (\cdot)^T w,\ y^T h(\cdot)e)$ and $(-g(\cdot) + (\cdot)^T z,\ y^T h(\cdot)e)$ are higher-order type I at $u \in \mathbb{R}^n$ with respect to a function η, where $\eta_2(x,u) = \frac{g_i(u) - u^T \bar{z}_i}{g_i(x) - x^T z_i} \eta_1(x,u)$, then*

$$\frac{f(x) + s(x|D)}{g(x) - s(x|E)} \not< \frac{f(u) + u^T w}{g(u) - u^T z} + (\lambda^T K(u,p))e - p^T \nabla_p (\lambda^T K(u,p))e$$
$$+ y^T h(u)e + (y^T H(u,p))e - p^T \nabla_p (y^T H(u,p))e.$$

Proof. Assume to the contrary that

$$\frac{f(x) + s(x|D)}{g(x) - s(x|E)} < \frac{f(u) + u^T w}{g(u) - u^T z} + (\lambda^T K(u,p))e - p^T \nabla_p (\lambda^T K(u,p))e$$
$$+ y^T h(u)e + (y^T H(u,p))e - p^T \nabla_p (y^T H(u,p))e.$$

Since $\lambda \geq 0$, $s(x|D) \geqq x^T w$, and $s(x|E) \geqq x^T z$,

$$\lambda^T \left[\frac{f(x) + x^T w}{g(x) - x^T z}\right] < \lambda^T \left[\frac{f(u) + u^T w}{g(u) - u^T z}\right] + \lambda^T K(u,p) - p^T \nabla_p \lambda^T K(u,p) \tag{4.2}$$
$$+ y^T h(u) + y^T H(u,p) - p^T \nabla_p y^T H(u,p).$$

Now, since $(f(\cdot) + (\cdot)^T w,\ y^T h(\cdot)e)$ and $(-g(\cdot) + (\cdot)^T z,\ y^T h(\cdot)e)$ are higher-order type I at u with respect to a function η and by Lemma 2.3, we have, for $\lambda \geq 0$,

$$\lambda^T \left[\frac{f(x) + x^T w}{g(x) - x^T z}\right] - \lambda^T \left[\frac{f(u) + u^T w}{g(u) - u^T z}\right]$$
$$\geqq \eta_2(x,u)^T \nabla_p \lambda^T K(u,p) + \lambda^T K(u,p) - p^T \nabla_p \lambda^T K(u,p)$$

and

$$-y^T h(u) \geqq \eta_2(x,u)^T \nabla_p y^T H(u,p) + y^T H(u,p) - p^T \nabla_p y^T H(u,p).$$

By (4.1), we obtain

$$\lambda^T \left[\frac{f(x) + x^T w}{g(x) - x^T z}\right] - \lambda^T \left[\frac{f(u) + u^T w}{g(u) - u^T z}\right]$$
$$\geqq \eta_2(x,u)^T (-\nabla_p y^T H(u,p)) + \lambda^T K(u,p) - p^T \nabla_p \lambda^T K(u,p)$$
$$\geqq y^T h(u) + y^T H(u,p) - p^T \nabla_p y^T H(u,p) + \lambda^T K(u,p) - p^T \nabla_p \lambda^T K(u,p),$$

which contradicts (4.2). $\square$

Theorem 4.2 (Strong Duality)**.** *If $\bar{x}$ is a weakly efficient solution of* (**MCFP**) *at which constraint qualification* [9] *is satisfied. Let*

$$K(\bar{x},0)=0,\ H(\bar{x},0)=0,\ \nabla_p K(\bar{x},0)=\nabla\frac{f(\bar{x})+\bar{x}^T\bar{w}}{g(\bar{x})-\bar{x}^T\bar{z}}\ \textit{and}$$

$$\nabla_p H(\bar{x},0)=\nabla h(\bar{x}). \tag{4.3}$$

Then there exist $\bar{\lambda}\geq 0$, $\bar{y}\in C_2$, $\bar{w}_i\in D_i$, and $\bar{z}_i\in E_i (i=1,\dots,l)$ such that $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{z},\bar{p}=0)$ is feasible for $(\mathbf{MCFD})_{\mathbf{W}}$ *and the objective values of* (**MCFP**) *and* $(\mathbf{MCFD})_{\mathbf{W}}$ *are equal. If in addition the type I assumptions of Theorem 4.1 are satisfied, then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{z},\bar{p}=0)$ is a weakly efficient solution of* $(\mathbf{MCFD})_{\mathbf{W}}$.

Proof. Since $\bar{x}$ is a weakly efficient solution of (**MCFP**), by Theorem 3.1, there exist $\bar{\lambda}\geq 0$, $\bar{y}\in C_2$, $\bar{w}_i\in D_i$, and $\bar{z}_i\in E_i(i=1,\dots,l)$ such that

$$\left[\bar{\lambda}^T\nabla\left\{\frac{f(\bar{x})+\bar{x}^T\bar{w}}{g(\bar{x})-\bar{x}^T\bar{z}}\right\}+\nabla\bar{y}^T h(\bar{x})\right]^T(x-\bar{x})\geqq 0,\ \text{for all } x\in C_1, \tag{4.4}$$

$$\bar{y}^T h(\bar{x})=0, \tag{4.5}$$

$$s(\bar{x}|D_i)=\bar{x}^T\bar{w}_i, s(\bar{x}|E_i)=\bar{x}^T\bar{z}_i,\ i=1,\dots,l. \tag{4.6}$$

Since $x\in C_1$, $\bar{x}\in C_1$, and C_1 is a closed convex cone, we have $x+\bar{x}\in C_1$ and thus the inequality (4.4) implies

$$\left[\bar{\lambda}^T\nabla\left\{\frac{f(\bar{x})+\bar{x}^T\bar{w}}{g(\bar{x})-\bar{x}^T\bar{z}}\right\}+\nabla\bar{y}^T h(\bar{x})\right]^T x\geqq 0,\quad\text{for all}\quad x\in C_1,$$

$$\text{i.e.,}\qquad \bar{\lambda}^T\nabla\left\{\frac{f(\bar{x})+\bar{x}^T\bar{w}}{g(\bar{x})-\bar{x}^T\bar{z}}\right\}+\nabla\bar{y}^T h(\bar{x})=0.$$

And (4.5) implies $\bar{y}^T h(\bar{x})\geqq 0$, then

$$-h(\bar{x})\in C_2^*.$$

Therefore, using the hypotheses (4.3) and (4.6), we obtain $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{z},\bar{p}=0)$ is feasible for $(\mathbf{MCFD})_{\mathbf{W}}$ and corresponding values of (**MCFP**) and $(\mathbf{MCFD})_{\mathbf{W}}$ are equal. If the assumptions of Theorem 4.1 are satisfied, then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{z},\bar{p}=0)$ is a weakly efficient solution of $(\mathbf{MCFD})_{\mathbf{W}}$. ☐

Now, we propose Mond-Weir dual problem $(\mathbf{MCFD})_{\mathbf{M}}$ to (**MCFP**):
$(\mathbf{MCFD})_{\mathbf{M}}$

$$\text{Maximize}\quad \frac{f(u)+u^T w}{g(u)-u^T z}+(\lambda^T K(u,p))e-p^T\nabla_p(\lambda^T K(u,p))e$$

subject to

$$\begin{aligned}&\nabla_p\lambda^T K(u,p)+\nabla_p y^T H(u,p)=0,\\ &-\Big[h(u)+H(u,p)-p^T\nabla_p H(u,p)\Big]\in C_2^*,\\ &w_i\in D_i,\ z_i\in E_i,\ i=1,\dots,l,\\ &y\in C_2,\lambda\geq 0,\lambda^T e=1,e=(1,\dots,1)^T\in\mathbb{R}^l.\end{aligned} \tag{4.7}$$

We establish weak and strong duality theorems between (**MCFP**) and $(\mathbf{MCFD})_{\mathbf{M}}$.

Theorem 4.3 (Weak Duality)**.** *Let x and (u, y, λ, w, z, p) be feasible solutions of* **(MCFP)** *and* $(\mathbf{MCFD})_\mathbf{M}$*, respectively. Assume that $(f(\cdot) + (\cdot)^T w,\ y^T h(\cdot)e)$ and $(-g(\cdot) + (\cdot)^T z,\ y^T h(\cdot)e)$ are higher-order type I at $u \in \mathbb{R}^n$ with respect to a function η, where $\eta_2(x,u) = \frac{g_i(u) - u^T \bar{z}_i}{g_i(x) - x^T z_i}\eta_1(x,u)$, then*

$$\frac{f(x) + s(x|D)}{g(x) - s(x|E)} \not< \frac{f(u) + u^T w}{g(u) - u^T z} + (\lambda^T K(u,p))e - p^T \nabla_p (\lambda^T K(u,p))e.$$

Proof. The proof is similar to the one in Theorem 4.1. □

Theorem 4.4 (Strong Duality)**.** *If $\bar{x}$ is a weakly efficient solution of* (**MCFP**) *at which constraint qualification* [9] *is satisfied. Let*

$$K(\bar{x}, 0) = 0,\ \ H(\bar{x}, 0) = 0,\ \ \nabla_p K(\bar{x}, 0) = \nabla \frac{f(\bar{x}) + \bar{x}^T \bar{w}}{g(\bar{x}) - \bar{x}^T \bar{z}},$$
$$\text{and}\ \ \nabla_p H(\bar{x}, 0) = \nabla h(\bar{x}).$$

Then there exist $\bar{\lambda} \geq 0$, $\bar{y} \in C_2$, $\bar{w}_i \in D_i$, and $\bar{z}_i \in E_i (i = 1, \ldots, l)$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{z}, \bar{p} = 0)$ is feasible for $(\mathbf{MCFD})_\mathbf{M}$ *and the objective values of* (**MCFP**) *and* $(\mathbf{MCFD})_\mathbf{M}$ *are equal. If the assumptions of Theorem 4.3 are satisfied, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{z}, \bar{p} = 0)$ is a weakly efficient solution of* $(\mathbf{MCFD})_\mathbf{M}$*.*

Proof. The proof is similar to the one in Theorem 4.3. □

We propose Schaible dual problem $(\mathbf{MCFD})_\mathbf{S}$ to (**MCFP**):
$(\mathbf{MCFD})_\mathbf{S}$

$$\begin{aligned}
&\text{Maximize} \quad \tau = (\tau_1, \ldots, \tau_l)\\
&\text{subject to}
\end{aligned}$$

$$\sum_{i=1}^{l} \lambda_i \nabla_p \Big((F_i(u,p) - \tau_i G_i(u,p) \Big) + \nabla_p y^T H(u,p) = 0, \tag{4.8}$$

$$\begin{aligned}
&f_i(u) + u^T w_i + F_i(u,p) - p^T \nabla_p F_i(u,p)\\
&\quad -\tau_i \Big(g_i(u) - u^T z_i + G_i(u,p) - p^T \nabla_p G_i(u,p) \Big) \geqq 0,\ \ i = 1, \ldots, l
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&-\Big[h(u) + H(u,p) - p^T \nabla_p H(u,p) \Big] \in C_2^*,\\
&w_i \in D_i,\ \ z_i \in E_i,\ \ i = 1, \ldots, l,\\
&\tau \geqq 0,\ \ y \in C_2, \lambda \geq 0, \lambda^T e = 1, e = (1, \ldots, 1)^T \in \mathbb{R}^l.
\end{aligned} \tag{4.10}$$

Now we establish weak and strong duality theorems between (**MCFP**) and $(\mathbf{MCFD})_\mathbf{S}$.

Theorem 4.5 (Weak Duality)**.** *Let x and $(u, y, \lambda, \tau, w, z, p)$ be feasible solutions of* **(MCFP)** *and* $(\mathbf{MCFD})_\mathbf{S}$*, respectively. Assume that $(f(\cdot) + (\cdot)^T w,\ y^T h(\cdot)e)$ and $(-g(\cdot) + (\cdot)^T z,\ y^T h(\cdot)e)$ are higher-order type I at $u \in \mathbb{R}^n$ with respect to a function η, where $\eta(x,u) = \eta_1(x,u) = \eta_2(x,u)$, then*

$$\frac{f(x) + s(x|D)}{g(x) - s(x|E)} \not< \tau.$$

Proof. Assume to the contrary that

$$\frac{f(x)+s(x|D)}{g(x)-s(x|E)}<\tau,$$

i.e.,

$$f_i(x)+s(x|D_i)<\tau_i\left(g_i(x)-s(x|E_i)\right), \quad \text{for all } i=1,\dots,l.$$

Since $s(x|D)\geqq x^Tw$ and $s(x|E)\geqq x^Tz$, for all $i=1,\dots,l$, we get

$$f_i(x)+x^Tw_i<\tau_i\left(g_i(x)-x^Tz_i\right). \tag{4.11}$$

Since $\left(f(\cdot)+(\cdot)x^Tw,\ y^Th(\cdot)e\right)$ and $\left(-g(\cdot)+(\cdot)^Tz,\ y^Th(\cdot)e\right)$ are higher-order type I at u with respect to a function η, for all $i=1,\dots,l$, we obtain

$$\begin{aligned} &f_i(x)+x^Tw_i-\tau_i\left(g_i(x)-x^Tz_i\right)\\ &\geqq f_i(u)+u^Tw_i+\eta(x,u)^T\nabla_pF_i(u,p)+F_i(u,p)-p^T\nabla_pF_i(u,p)\\ &\quad-\tau_i\left[g_i(u)-u^Tz_i+\eta(x,u)^T\nabla_pG_i(u,p)+G_i(u,p)-p^T\nabla_pG_i(u,p)\right] \end{aligned} \tag{4.12}$$

and

$$-y^Th(u)\geqq\eta(x,u)^T\nabla_py^TH(u,p)+y^TH(u,p)-p^T\nabla_py^TH(u,p). \tag{4.13}$$

By (4.9), (4.11) and (4.12), we have

$$\begin{aligned} 0 &> f_i(x)+x^Tw_i-\tau_i\left(g_i(x)-x^Tz_i\right)\\ &\geqq\eta(x,u)^T\nabla_p\left[F_i(u,p)-\tau_iG_i(u,p)\right]. \end{aligned}$$

As $\lambda\geq 0$ and by (4.8), we get

$$\begin{aligned} 0 &> \eta(x,u)^T\sum_{i=1}^{l}\lambda_i\nabla_p\left[F_i(u,p)-\tau_iG_i(u,p)\right]\\ &= -\eta(x,u)^T\nabla_py^TH(u,p). \end{aligned} \tag{4.14}$$

Now, by (4.10), $y\in C_2$, and (4.13), we obtain

$$\eta(x,u)^T\nabla_py^TH(u,p)\leqq 0,$$

which contradicts (4.14). □

Theorem 4.6 (Strong Duality)**.** *If $\bar{x}$ is a weakly efficient solution of* (**MCFP**) *at which constraint qualification* [9] *is satisfied. Let*

$$\begin{aligned} &K(\bar{x},0)=0,\ H(\bar{x},0)=0,\ \nabla_pF(\bar{x},0)=\nabla f(\bar{x})+\bar{w},\\ &\nabla_pG(\bar{x},0)=\nabla g(\bar{x})-\bar{z},\\ &\nabla_pH(\bar{x},0)=\nabla h(\bar{x}). \end{aligned} \tag{4.15}$$

Then there exist $\bar{\lambda}\geq 0$, $\bar{y}\in C_2$, $\bar{w}_i\in D_i$, and $\bar{z}_i\in E_i (i=1,\dots,l)$ such that $(\bar{x},\bar{y},\bar{\lambda},\bar{\tau},\bar{w},\bar{z},\bar{p}=0)$ is feasible for $(\mathbf{MCFD})_\mathbf{S}$ *and the objective values of* (**MCFP**) *and* $(\mathbf{MCFD})_\mathbf{S}$ *are equal. If the assumptions of Theorem 4.5 are satisfied, then $(\bar{x},\bar{y},\bar{\lambda},\bar{\tau},\bar{w},\bar{z},\bar{p}=0)$ is a weakly efficient solution of* $(\mathbf{MCFD})_\mathbf{S}$.

Proof. Since $\bar{x}$ is a weakly efficient solution of (**MCFP**), by Theorem 3.1, there exist $\bar{\lambda} \geq 0$, $\bar{y} \in C_2$, $\bar{w}_i \in D_i$, and $\bar{z}_i \in E_i (i = 1, \ldots, l)$ such that

$$\left[\lambda^{*T} \nabla \left\{ \frac{f(\bar{x}) + \bar{x}^T \bar{w}}{g(\bar{x}) - \bar{x}^T \bar{z}} \right\} + \nabla y^{*T} h(\bar{x}) \right]^T (x - \bar{x}) \geqq 0, \text{ for all } x \in C_1, \tag{4.16}$$

$$y^{*T} h(\bar{x}) = 0,$$

$$s(\bar{x}|D_i) = \bar{x}^T \bar{w}_i,\ s(\bar{x}|E_i) = \bar{x}^T \bar{z}_i,\ i = 1, \ldots, l. \tag{4.17}$$

Denote

$$\bar{\tau}_i = \frac{f_i(\bar{x}) + \bar{x}^T \bar{w}_i}{g_i(\bar{x}) - \bar{x}^T \bar{z}},\ i = 1, \ldots, l\ ; \tag{4.18}$$

$$\bar{\lambda}_i = \frac{\lambda_i^*}{g_i(\bar{x}) - \bar{x}^T \bar{z}_i} \Big/ \sum_{i=1}^{l} \frac{\lambda_i^*}{g_i(\bar{x}) - \bar{x}^T \bar{z}_i},\ i = 1, \ldots, l\ ; \text{ and} \tag{4.19}$$

$$\bar{y}_j = y_j^* \Big/ \sum_{i=1}^{l} \frac{\lambda_i^*}{g_i(\bar{x}) - \bar{x}^T \bar{z}_i},\ j = 1, \ldots, m. \tag{4.20}$$

Since $x \in C_1$, $\bar{x} \in C_1$, and C_1 is a closed convex cone, we have $x + \bar{x} \in C_1$ and thus the inequality (4.16) implies

$$\left[\lambda^{*T} \nabla \left\{ \frac{f(\bar{x}) + \bar{x}^T \bar{w}}{g(\bar{x}) - \bar{x}^T \bar{z}} \right\} + \nabla y^{*T} h(\bar{x}) \right]^T x \geqq 0, \quad \text{for all} \quad x \in C_1,$$

$$i.e., \qquad \sum_{i=1}^{l} \lambda_i^* \nabla \left\{ \frac{f_i(\bar{x}) + \bar{x}^T \bar{w}_i}{g_i(\bar{x}) - \bar{x}^T \bar{z}_i} \right\} + \sum_{j=1}^{m} y_j^* \nabla h_j(\bar{x}) = 0.$$

By (4.18), we get

$$\begin{aligned} \nabla \left\{ \frac{f_i(\bar{x}) + \bar{x}^T \bar{w}_i}{g_i(\bar{x}) - \bar{x}^T \bar{z}_i} \right\} &= \frac{(g_i(\bar{x}) - \bar{x}^T \bar{z}_i)(\nabla f_i(\bar{x}) + \bar{w}_i) - (f_i(\bar{x}) + \bar{x}^T \bar{w}_i)(\nabla g_i(\bar{x}) - \bar{z}_i)}{(g_i(\bar{x}) - \bar{x}^T \bar{z}_i)^2} \\ &= \frac{1}{g_i(\bar{x}) - \bar{x}^T \bar{z}_i} (\nabla f_i(\bar{x}) + \bar{w}_i - \bar{\tau}_i (\nabla g_i(\bar{x}) - \bar{z}_i)). \end{aligned}$$

Using above two equalities, we have

$$\sum_{i=1}^{l} \frac{\lambda_i^*}{g_i(\bar{x}) - \bar{x}^T \bar{z}_i} (\nabla f_i(\bar{x}) + \bar{w}_i - \bar{\tau}_i (\nabla g_i(\bar{x}) - \bar{z}_i)) + \sum_{j=1}^{m} y_j^* \nabla h_j(\bar{x}) = 0.$$

Taking $\bar{\tau}_i$, $\bar{\lambda}_i$, $\bar{y}_j$ for all $i = 1, \ldots, l$ and $j = 1, \ldots, m$ as (4.18), (4.19) and (4.20), we obtain

$$\begin{aligned} &\sum_{i=1}^{l} \bar{\lambda}_i (\nabla f_i(\bar{x}) + \bar{w}_i - \bar{\tau}_i (\nabla g_i(\bar{x}) - \bar{z}_i)) + \sum_{j=1}^{m} \bar{y}_j \nabla h_j(\bar{x}) = 0, \\ &\bar{y}^T h(\bar{x}) = 0,\ s(\bar{x}|D_i) = \bar{x}^T \bar{w}_i,\ s(\bar{x}|E_i) = \bar{x}^T \bar{z}_i,\ i = 1, \ldots, l, \\ &f_i(\bar{x}) + \bar{x}^T \bar{w}_i - \bar{\tau}_i \left(g_i(\bar{x}) - \bar{x}^T \bar{z}_i\right) = 0,\ \bar{\lambda} \geq 0 \text{ with } \bar{\lambda}^T e = 1, \\ &\bar{\tau} \geqq 0, \text{ and } \ \bar{y} \in C_2. \end{aligned}$$

By the hypothesis (4.15) and $-h(\bar{x}) \in C_2^*$, we can derive the following:

$$\sum_{i=1}^{l} \bar{\lambda}_i \nabla_p \left(F_i(\bar{x}, p) - \bar{\tau}_i G_i(\bar{x}, p) \right) + \sum_{j=1}^{m} \bar{y}_j \nabla_p H_j(\bar{x}, p) = 0,$$
$$f_i(\bar{x}) + \bar{x}^T \bar{w}_i - \bar{\tau}_i \left(g_i(\bar{x}) - \bar{x}^T \bar{z}_i \right) \geqq 0,$$
$$-\left[h(\bar{x}) + H(\bar{x}, p) \right] \in C_2^*.$$

Hence, we get $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\tau}, \bar{w}, \bar{z}, \bar{p} = 0)$ is feasible for $(\mathbf{MCFD})_\mathbf{S}$ and using the condition (4.17), corresponding values of (**MCFP**) and $(\mathbf{MCFD})_\mathbf{S}$ are equal. If the type I assumptions of Theorem 4.5 are satisfied, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\tau}, \bar{w}, \bar{z}, \bar{p} = 0)$ is a weakly efficient solution of $(\mathbf{MCFD})_\mathbf{S}$. □

5. Special cases

We give some special cases for our nondifferentiable multiobjective fractional programs.

(i) If $E_i = \{0\}, i = 1, \ldots, l$, then our primal and Wolfe dual model reduce to the corresponding ones in [7].

(ii) If $E_i = \{0\}, i = 1, \ldots, l$ and C_1 is an open set of $\mathbb{R}^n$, $C_2 = \mathbb{R}^m_+$, then our primal and Wolfe and Mond-Weir dual models reduce to the corresponding ones in [5].

(iii) If $D_i = \{0\}, E_i = \{0\}, i = 1, \ldots, l$ and C_1 is an open set of $\mathbb{R}^n$, $C_2 = \mathbb{R}^m_+$, then our primal and Mond-Weir dual model reduce to the corresponding ones in [1].

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