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FIXED POINT THEOREMS FOR CONTRACTIVE MAPPINGS OF PEROV TYPE

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Dedicated to Prof. Wataru Takahashi on his 70th birthday

Abstract. [Perov, A. I., *On Cauchy problem for a system of ordinary diferential equations*, (in Russian), Priblizhen. Metody Reshen. Difer. Uravn., **2** (1964), 115-134] used the concept of a vector valued metric space and obtained a Banach type fixed point theorem on such a complete generalized metric space. In this article we study fixed point results for new extensions of Park's contraction condition [S. Park,*A unified approach to fixed points of contractive maps*, J. Korean Math. Soc., **16** (1980), 95-106] to a cone metric space, and give some generalized versions of the fixed point theorem of Perov. As corollaries we generalized some results of [Zima, M., *A certain fixed point theorem and its applications to integral-functional equations*, Bull. Austral. Math. Soc. **46** (1992), 179-186] and [Borkowski, M., Bugajewski, D. and Zima, M.,*On some fixed-point theorems for generalized contractions and their perturbations*, J. Math. Anal. Appl. **367** (2010), 464-475] for a Banach space with a non-normal cone.

1. Introduction

There exist many generalizations of the concept of metric spaces in the literature. Perov [24] used the concept of a vector valued metric space, and obtained a Banach type fixed point theorem on such a complete generalized metric space. After that, fixed point results of Perov type in vector valued metric spaces were studied by many other authors (see e.g., [13, 14, 26, 27] for some works in this line of research). We remark that the Perov theorem and related results have many applications in coincidence problems, coupled fixed point problems and systems of semilinear differential inclusions.

L.G. Huang and X. Zhang [15] (see also [32]) used the concept of cone metric spaces as a generalization of metric spaces. They replaced the real numbers (as the co-domain of a "metric") by an ordered Banach space. The authors described convergence in cone metric spaces and introduced their completeness. Then they proved some fixed point theorems for contractive mappings on cone metric spaces. In [1,16] and [30] some common fixed point theorems were proved for maps on cone metric spaces. However, in [1,15,16] and [31] the authors usually used the normality property of cones in their results.

In [4] some fixed point theorems where proved which involve fairly general conditions in the setting of a cone metric space. In this article the authors give cone

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metric versions of some results previously published in [10,16,22,23,30]. We shall prove Perov type generalizations of these theorems in solid cone metric spaces, and, also, in the case when the cone is normal with appropriate assumptions. We study fixed point results for new extensions of Banach's contraction principle to cone metric spaces, and give some generalize versions of the fixed point theorem of Perov. As corollaries we generalize some results of Zima [33] and Borkowski, Bugajewski and Zima [8] for a Banach space with a non-normal cone. The theory is illustrated with some examples.It is worth mentioning that the main result of this paper cannot be derived from Park's result by the scalarization method, and hence, indeed, improves many recent results in cone metric spaces.

Consistent with [15] (see, e.g., [1–3,5,11,17,19,28,30] for more details and recent results), the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if:

(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subseteq E$, we define the partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$ (interior of P).

There exist two kinds of cones: normal and non-normal ones.

A cone P in a real Banach space E is called normal if

$$\inf\{\|x + y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0, \tag{1.1}$$

or, equivalently, if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K \|y\|. \tag{1.2}$$

The least positive number satisfying (1.2) is called the normal constant of P. It is clear that $K \geq 1$. A cone P is called solid if $\operatorname{int} P \neq \emptyset$.

Definition 1.1 ([15])**.** Let X be a nonempty set, and let P be a cone on a real ordered Banach space E. Suppose that the mapping $d : X \times X \mapsto E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is known that the class of cone metric spaces is larger than the class of metric spaces.

Example 1.2. Let $X = \mathbb{R}, E = \mathbb{R}^n$ and $P = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0\}$. it is easy to see that $d : X \times X \mapsto E$ defined by $d(x, y) = (|x - y|, k_1|x - y|, \dots, k_{n-1}|x - y|)$ generates a cone metric on X, where $k_i \geq 0$ for all $i \in \{1, \dots, n - 1\}$.

Example 1.3 ([11])**.** Let $E = C^1[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E : x(t) \geq 0 \text{ on } [0, 1]\}$. This is not a normal cone. For example

$$x_n(t) = \frac{1 - \sin nt}{n + 2} \quad \text{and} \quad y_n(t) = \frac{1 + \sin nt}{n + 2}.$$

Since, $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| = \frac{2}{n+2} \to 0$, it follows by (1.1) that P is non-normal.

Let X be a nonempty set and $n \in \mathbb{N}$.

Definition 1.4. A mapping $d : X \times X \mapsto \mathbb{R}^n$ is called a *vector-valued metric* on X if the following statements are satisfied for all $x, y, z \in X$.

(d1) $d(x, y) \geq 0_n$ and $d(x, y) = 0_n$ if and only if $x = y$ where $0_n = (0, \ldots, 0) \in \mathbb{R}^n$;
(d2) $d(x, y) = d(y, x)$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$.

If $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, then $X \leq Y$ means that $X_i \leq Y_i$, $i = 1, \ldots, n$. This partial order determines a normal cone $P = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, 2, \ldots, n\}$ on $\mathbb{R}^n$, with normal constant $K = 1$. A nonempty set X with a vector-valued metric d is called a *generalized metric space.*

Throughout this paper we denote by $\mathcal{M}_{n,n}$ the set of all $n \times n$ matrices, and by $\mathcal{M}_{n,n}(\mathbb{R}^+)$ the set of all $n \times n$ matrices with nonnegative elements. It is well known that if $A \in \mathcal{M}_{n,n}$, then $A(P) \subset P$ if and only if $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$. We write Θ for the zero $n \times n$ matrix and I_n for the identity $n \times n$ matrix. For the sake of simplicity we will identify row and column vectors in $\mathbb{R}^n$.

A matrix $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ is said to be convergent to zero if $A^n \to \Theta$ as $n \to \infty$.

Theorem 1.5 (Perov [24, 25])**.** *Let (X, d) be a complete generalized metric space, $f : X \mapsto X$ and $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ a matrix convergent to zero, such that*

$$d(f(x), f(y)) \leq A(d(x, y)), \quad x, y \in X.$$

Then:

(i) *f has a unique fixed point $x^* \in X$;*
(ii) *the sequence of successive approximations $x_n = f(x_{n-1}), n \in \mathbb{N}$ converges to x^* for all $x_0 \in X$;*
(iii) *$d(x_n, x^*) \leq A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}$;*
(iv) *if $g : X \mapsto X$ satisfies the condition $d(f(x), g(x)) \leq c$ for all $x \in X$ and some $c \in \mathbb{R}^n$, then, by considering the sequence $y_n = g^n(x_0), n \in \mathbb{N}$, one has*

$$d(y_n, x^*) \leq (I_n - A)^{-1}(c) + A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

For completeness of the paper and convenience of the reader, in the Preliminaries section we collect some basic definitions and facts needed in subsequent sections.

2. Preliminaries

In the following we shall assume that E is a Banach space, P is a cone in E with int$P \neq \emptyset$ whenever P is a non-normal cone and $\leq$ is the partial order on E with respect to P.

Let (x_n) be a sequence in X, and $x \in X$. If for every c in E with $0 \ll c$, there is an n_0 such that, for all $n > n_0$, $d(x_n, x) \ll c$, then it is said that $\{\mathrm{X}_n\}$ converges to x, and we denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$, $n \to \infty$. If, for every c in E with $0 \ll c$, there is an n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{\mathrm{X}_n\}$ is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

We us recall [15] that, if P is a normal cone, even in the case that $\text{int}P = \emptyset$, then $\{x_n\} \subseteq X$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$, $n \to \infty$. Further, $\{\mathrm{X}_n\} \subseteq X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$, $n, m \to \infty$.

Let (X, d) be a cone metric space. Then the following properties are often used (particulary when dealing with cone metric spaces in which the cone need not to be normal):

- (p_1) If $u \leq v$ and $v \ll w$ then $u \ll w$.
- (p_2) If $0 \leq u \ll c$ for each $c \in \operatorname{int} P$ then $u = 0$.
- (p_3) If $a \leq b + c$ for each $c \in \operatorname{int} P$ then $a \leq b$.
- (p_4) If $0 \leq x \leq y$, and $a \geq 0$, then $0 \leq ax \leq ay$.
- (p_5) If $0 \leq x_n \leq y_n$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $0 \leq x \leq y$.
- (p_6) If $0 \leq d(x_n, x) \leq b_n$ and $b_n \to 0$, then $x_n \to x$.
- (p_7) If E is a real Banach space with a cone P, and if $a \leq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = 0$.
- (p_8) If $c \in \operatorname{int} P$, $0 \leq a_n$ and $a_n \to 0$, then there exists an n_0 such that, for all $n > n_0$ we have $a_n \ll c$.

From (p_8) it follows that the sequence $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$, and $\{x_n\}$ is a Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. For a non-normal cone we have only one part of Lemmas 1 and 4 from [15]. Also, in this case, the fact that $d(x_n, y_n) \to d(x, y)$ if $x_n \to x$ and $y_n \to y$ is not applicable.

We write $\mathscr{B}(E)$ for the set of all bounded linear operators on E and $\mathscr{L}(E)$ for the set of all linear operators on E. $\mathscr{B}(E)$ is a Banach algebra and, if $A \in \mathscr{B}(E)$, define

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_n \|A^n\|^{1/n}$$

be the spectral radius of A. We remark that, if $r(A) < 1$, then the series $\sum\limits_{i=0}^{\infty} A^n$ is absolutely convergent, $I - A$ is invertible in $\mathscr{B}(E)$ and

$$\sum_{i=0}^{\infty} A^n = (I - A)^{-1}.$$

Also, $r((I - A)^{-1}) \leq \frac{1}{1 - r(A)}$.

If $A, B \in \mathscr{B}(E)$ and $AB = BA$, then $r(AB) \leq r(A)r(B)$. Furthermore, if $\|A\| < 1$, then $I - A$ is invertible and

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

If $T : X \mapsto X$, $z \in X$ and X is a cone metric space, the orbit of z is defined by $O(z) = \{z, Tz, T^2z, \ldots\}$. The closure of the orbit will be denoted by $\overline{O(z)}$.

3. Main results

In this section we prove our main results. We start with some auxiliary results.

Lemma 3.1. *Let (X, d) be a cone metric space. Suppose that $\{x_n\}$ is a sequence in X and that $\{b_n\}$ is a sequence in E with $b_n \to 0$ as $n \to \infty$. If there exists an $n_0 \in \mathbb{N}$ such that $0 \leq d(x_n, x_m) \leq b_n$ for each $n \geq n_0$ and each $m \geq n_0$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. For each $c \gg 0$ there exists an $n_1 \in \mathbb{N}$ such that $b_n \ll c, n > n_1$. It follows that $0 \leq d(x_n, x_m) \ll c$ for $m > n > \max\{n_0, n_1\}$; i.e., $\{x_n\}$ is a Cauchy sequence. □

Lemma 3.2. *Let E be Banach space, $P \subseteq E$ a cone in E and $A : E \mapsto E$ a linear operator. The following conditions are equivalent:*

i) *A is increasing; i.e., $x \leq y$ implies that $A(x) \leq A(y)$.*
ii) *A is positive; i.e., $A(P) \subseteq P$.*

Proof. If A is monotonically increasing and $p \in P$, then it follows that $p \geq 0$ and $A(p) \geq A(0) = 0$. Thus $A(p) \in P$, and $A(P) \subseteq P$.
To prove the other implication, let us assume that $A(P) \subseteq P$ and $x, y \in E$ are such that $x \leq y$. Now $y - x \in P$, and so $A(y - x) \in P$. Thus $A(x) \leq A(y)$. □

The following two theorems generalize Theorem 1 of [4] and, consequently, Theorem 2 of [23].

Theorem 3.3. *Let (X, d) be a cone metric space, $P \subseteq E$ a cone and $T : X \mapsto X$. If there exists a point $z \in X$ such that $\overline{O(z)}$ is complete, $A \in \mathscr{B}(E)$ a positive operator with $r(A) < 1$, and*

$$d(Tx, Ty) \leq A(d(x, y)), \quad \text{holds for any} \quad x, y = Tx \in O(z), \tag{3.1}$$

then $\{T^n z\}$ converges to some $u \in \overline{O(z)}$ and

$$d(T^n z, u) \leq A^n (I - A)^{-1}(d(z, Tz)), \ n \in \mathbb{N}. \tag{3.2}$$

If (3.1) holds for any $x, y \in \overline{O(z)}$, then u is a fixed point of T.

Proof. First, we will show that $\{T^n z\}$ is a Cauchy sequence.
Since $d(T^n z, T^{n+1} z) \leq A(d(T^{n-1} z, T^n z))$, and A is a positive operator, by Lemma 3.2, it follows that

$$d(T^n z, T^{n+1} z) \leq A^n(d(z, Tz)).$$

Hence, for $n, m \in \mathbb{N}$, $m > n$,

$$d(T^n z, T^m z) \leq \sum_{i=n}^{m-1} d(T^i z, T^{i+1} z) \leq \sum_{i=n}^{m-1} A^i(d(z, Tz)),$$

and, since $r(A) < 1$ and Lemma 3.1 holds, $\{T^n z\}$ is a Cauchy sequence. Because $\overline{O(z)}$ is complete, there exists a $u \in \overline{O(z)}$ such that $\lim_{n\to\infty} T^n z = u$.
Let $n \in \mathbb{N}$ be arbitrary and $m > n$. Then,

$$d(T^n z, u) \quad \leq \quad d(T^n z, T^m z) + d(T^m z, u)$$

$$\begin{aligned}
&\leq \sum_{i=n}^{m-1} A^i(d(z,Tz)) + d(T^m z, u)\\
&\leq A^n \sum_{i=0}^{\infty} A^i(d(z,Tz)) + d(T^m z, u)\\
&= A^n(I-A)^{-1}(d(z,Tz)) + d(T^m z, u).
\end{aligned}$$

Taking the limit as $n \to \infty$ of the above inequality yields to (3.2).

If (3.1) is true for $x, y \in \overline{O(z)}$, then

$$d(T^{n+1}z, Tu) \leq A(d(T^n z, u)),$$

and $A(d(T^n z, u)) \to 0, n \to \infty$, thus $(\mathbf{p}_6)$ implies $\lim_{n\to\infty} T^n z = Tu$. But $\lim_{n\to\infty} T^n z = u$ gives us that u is a fixed point of T. □

Remark 1. Du [12] has investigated the equivalence of vectorial versions of fixed point theorems in generalized cone metric spaces and scalar versions of fixed point theorems in (general) metric spaces (in the usual sense). He has shown that the Banach contra-ction principles in general metric spaces and in TVS-cone metric spaces are equivalent. His theorems also extend some results of Huang and Zhang [15], Rezapour and Hamlbarani [30] and others.

Du [12] has used the nonlinear scalarization function ξ_e and the function d_ξ as follows: Let $d_\xi = \xi_e \circ d$, where (X, d) is a cone metric space, and ξ_e is defined by

$$\xi_e(u) = \inf\{r \in \mathbb{R} : u \in re - P\},$$

for each $u \in E$, and some $e \in intP$. Then d_ξ is a metric on X by Theorem 2.1 of [12]. Let $T : X \mapsto X$ be such that there exists a point $z \in X$ for which $\overline{O(z)}$ is a complete, and a $\lambda \in (0,1)$ such that

$$d(Tx, Ty) \leq \lambda \cdot d(x,y) \quad \textit{holds for any} \quad x, y = Tx \in O(z). \tag{3.3}$$

Then, applying Lemma 1.1 of [12], we have

$$d_\xi(Tx, Ty) \leq \lambda \cdot d_\xi(x,y), \quad \textit{holds for any} \quad x, y = Tx \in O(z). \tag{3.4}$$

Therefore, Theorem 1 of [4] directly follows from Park's result by Theorem 2 of [23]. However, if T satisfies (3.1), restricted with a linear bounded mapping, we cannot conclude that there exists some $\lambda \in (0,1)$ such that (3.4) is satisfied, and so Theorem 3.3 cannot be derived from Park's result. Therefore Theorem 3.3 indeed improves the corresponding result of [23]. Similar observations are valid for the Ćirić's quasi-contraction ([10, 29]) of Perov type and for Banach's contraction of Perov type [9]. For some more recent results see [18, 20, 21].

Now we list several corollaries of our main result.

Corollary 3.4 (Theorem 2.2 of [9])**.** *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, $f : X \mapsto X$, $A \in \mathcal{B}(E)$, with $r(A) < 1$ and $A(P) \subseteq P$, such that*

$$d(fx, fy) \leq Ad(x,y), \quad x, y \in X. \tag{3.5}$$

Then:

(i) *f has a unique fixed point $z \in X$;*

(ii) *For any* $x_0 \in X$ *the sequence* $x_n = fx_{n-1}$, $n \in \mathbb{N}$, *converges to* z *and*

$$d(x_n, z) \leq A^n(I - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

Let us remark that the initial assumption $A \in \mathcal{M}_{n,n}(\mathbb{R}_+)$, in Perov theorem, is unnecessary (Example 2.5 of [9]) . If P is a normal cone, we can modify the conditions of Theorem 3.3.

Theorem 3.5. *Let* (X, d) *be a cone metric space,* $P \subseteq E$ *a normal cone with a normal constant* K *and* $T : X \mapsto X$. *If there exists a point* $z \in X$ *such that* $\overline{O(z)}$ *is a complete and* $A \in \mathscr{B}(E)$ *bounded linear operator,* $K\|A\| < 1$, *such that (3.1) holds, then* $\{T^n z\}$ *converges to some* $u \in \overline{O(z)}$ *and*

$$\|d(T^n z, u)\| \leq \frac{(K\|A\|)^n}{1 - K\|A\|}\|d(z, Tz)\|,\ n \in \mathbb{N}. \tag{3.6}$$

If (3.1) *holds for every* $x, y \in \overline{O(z)}$, *then* u *is a fixed point of* T.

Proof. Observe that (3.1) and the fact that P is a normal cone imply that

$$\|d(T^n z, T^{n+1}z)\| \leq K\|A\|\|d(T^{n-1}z, T^n z)\|,$$

and, inductively, $\|d(T^n z, T^{n+1}z)\| \leq (K\|A\|)^n\|d(z, Tz)\|$ for every $n \in \mathbb{N}$. If $n, m \in \mathbb{N}$ and $m > n$, we have

$$\begin{aligned}
\|d(T^n z, T^m z)\| &\leq K\|\sum_{i=n-1}^{m-2} A(d(T^i z, T^{i+1}z))\| \\
&\leq K\|A\| \sum_{i=n-1}^{m-2} \|d(T^i z, T^{i+1}z)\| \\
&\leq K\|A\| \sum_{i=n-1}^{m-2} (K\|A\|)^i\|d(z, Tz)\| \\
&\leq \sum_{i=n}^{\infty}(K\|A\|)^i\|d(z, Tz)\| \\
&\leq \frac{(K\|A\|)^n}{1 - K\|A\|}\|d(z, Tz)\|. \qquad (3.7)
\end{aligned}$$

Because $K\|A\| < 1$, $\{T^n z\}$ is a Cauchy sequence and $\lim_{n\to\infty} T^n z = u$ for some $u \in \overline{O(z)}$.

Notice that, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\|d(T^n z, u)\| &\leq K\|A(d(T^{n-1}z, T^{m-1}z))\| + K\|d(T^m z, u)\| \\
&\leq \frac{(K\|A\|)^n}{1 - K\|A\|}\|d(z, Tz)\| + K\|d(T^m z, u)\|.
\end{aligned}$$

Last inequality is obtained from (3.7) for any $m \in \mathbb{N}$ and, because $K\|d(T^m z, u)\| \to 0$, $m \to \infty$, (3.6) holds.

If we include $x, y \in \overline{O(z)}$ in the condition (3.1), then

$$d(T^n z, Tu) \leq A(d(T^{n-1}z, u)),\ n \in \mathbb{N}$$

and $d(T^{n-1}z, u) \to 0$, $n \to \infty$, so $T^n z \to Tu$, $n \to \infty$. However, limit of convergent sequence is unique, thus $Tu = u$ □

Corollary 3.6 ([Theorem 2.4 of [9]). *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, P a normal cone with normal constant K, $A \in \mathcal{B}(E)$ and $K\|A\| < 1$. If condition (3.5) holds for a mapping $f : X \mapsto X$, then f has a unique fixed point $z \in X$ and the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$, converges to z for any $x_0 \in X$.*

Following the work of Berinde ([6,7]), in the next result we investigate the weak contraction of Perov type.

Corollary 3.7 (Theorem 2.5 of [9]). *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, $f : X \mapsto X$, $A \in \mathcal{B}(E)$, with $r(A) < 1$ and $A(P) \subseteq P$, $B \in \mathcal{L}(E)$ with $B(P) \subseteq P$, such that*

$$d(f(x), f(y)) \leq A(d(x, y)) + B(d(x, f(y))), \quad x, y \in X. \tag{3.8}$$

Then $f : X \mapsto X$ has a fixed point in X and, for any $x_0 \in X$, the sequence $x_n = fx_{n-1}$, $n \in \mathbb{N}$ converges to a fixed point of f.

Corollary 3.8. *Let (X, d) be a complete cone metric space and $T : X \mapsto X$ a mapping satisfying*

$$d(Tx, Ty) \leq A(d(Tx, x) + d(Ty, y)), \ x, y \in X \tag{3.9}$$

for some positive operator $A \in \mathscr{B}(E)$ with $r(A) < \frac{1}{2}$. Then T has a unique fixed point $u \in X$ and $\{T^n x\}$ converges to u for any $x \in X$.

Proof. Since,

$$d(Tx, T^2x) \leq A(I - A)^{-1}(d(Tx, x))$$

and $A(I - A)^{-1}$ is a positive operator,

$$r(A(I - A)^{-1}) \leq r(A)r((I - A)^{-1}) \leq \frac{r(A)}{1 - r(A)} < 1,$$

condition (3.1) of the Theorem 3.3 holds. Hence, T has a fixed point $u \in X$. Uniqueness of the fixed point follows from (3.9). If $v \in X$ and $T(v) = v$, then $d(u, v) = d(Tu, Tv) \leq A(d(Tu, u) + d(Tv, v)) = A(0) = 0$. □

Corollary 3.9. *Let (X, d) be a complete cone metric space, P a normal cone with a normal constant K and $T : X \mapsto X$ a mapping satisfying (3.9) for some operator $A \in \mathscr{B}(E)$ with $K\|A\| < \frac{1}{2}$. Then T has a unique fixed point $u \in X$ and $\{T^n x\}$ converges to u for any $x \in X$.*

Proof. Obviously

$$\|d(Tx, T^2x)\| \leq \frac{K\|A\|}{1 - K\|A\|}\|d(x, Tx)\|,$$

and $K\|A\|/(1 - K\|A\|) < 1$. Therefore, analogously to the proof of Theorem 3.5, it is easy to show that T has a fixed point and (3.9) implies uniqueness. □

Corollary 3.10. *Let (X, d) be a complete cone metric space and $T : X \mapsto X$ a mapping satisfying*

$$d(Tx, Ty) \leq A(d(x, T^m z) + d(y, T^m z)) \tag{3.10}$$

for some $m \in \mathbb{N}$, $A \in \mathscr{B}(E)$ positive operator, $r(A) < 1$ and for all $x, y, z \in X$. Then the iterative sequence $\{T^n x\}$ converges to a unique fixed point of T for any $x \in X$.

Proof. If, for any $z \in X$ and $m \in \mathbb{N}$, set $x = T^{m-1}z$ and $y = T^m z$ in (3.10)

$$d(T^m z, T^{m+1} z) \leq A(d(T^{m-1}z, T^m z)),$$

then

$$d(T^n z, T^{n+1} z) \leq A(d(T^{n-1}z, T^n z)), \ n \geq m,$$

so, as in the proof of Corollary 3.8, T has a fixed point. Condition (3.10) gives uniqueness. □

Corollary 3.11. *Let (X, d) be a complete cone metric space, $P \subseteq E$ a normal cone with a normal constant K and $T : X \mapsto X$ a mapping satisfying (3.10) for some $m \in \mathbb{N}$, $A \in \mathscr{B}(E)$ such that $K\|A\| < 1$ and for all $x, y, z \in X$. Then the iterative sequence $\{T^n x\}$ converges to a unique fixed point of T for any $x \in X$.*

Proof. For any $z \in X$ and $m \in \mathbb{N}$, setting $x = T^{m-1}z$ and $y = T^m z$ in (3.10) gives

$$\|d(T^m z, T^{m+1} z)\| \leq K\|A\|\|d(T^{m-1}z, T^m z)\|,$$

and, by similar observations as in the proofs of Corollary 3.10 and Corollary 3.8, T has a unique fixed point in X. □

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