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TOPOLOGICAL METHOD FOR A CLASS OF THE SEMILINEAR ELLIPTIC SYSTEMS

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Dedicated to Prof. Wataru Takahashi on his 70th birthday

Abstract. We get a theorem which shows the existence of at least two nontrivial weak solutions for a class of the systems of the elliptic equations with some nonlinearity and boundary condition. We obtain this result by approaching the variational method, the critical point theory and the topological method. Among the topological methods we use the relative category theory on the manifold.

1. Introduction

Let Ω be a bounded subset of R^n with smooth boundary $\partial\Omega$, $n \geq 3$. Let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \ldots$ be the eigenvalues of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, and ϕ_k be the eigenfunction belonging to the eigenvalue λ_k, $k \geq 1$. Let $H : R^n \times R^n \to R$ be a C^2 function such that $H(x,\theta) = 0$, $\theta = (0,\ldots,0)$. In this paper we consider the number of the weak solutions for a class of the systems of the elliptic equations with Dirichlet boundary condition

$$
\begin{aligned}
-\Delta u_1 &= H_{u_1}(x,u_1,\ldots,u_n) \qquad \text{in } \Omega, \\
-\Delta u_2 &= H_{u_2}(x,u_1,\ldots,u_n) \qquad \text{in } \Omega, \\
&\vdots \\
-\Delta u_n &= H_{u_n}(x,u_1,\ldots,u_n) \qquad \text{in } \Omega, \\
u_i(x) = 0, &\qquad i = 1,\ldots,n, \qquad \text{on } \partial\Omega,
\end{aligned} \tag{1.1}
$$

where $u_i(x) \in W_0^{1,2}(\Omega)$ and $H_{u_i}(x,u_1,\ldots,u_n) = \frac{\partial H(x,u_1,\ldots,u_n)}{\partial u_i}$, $i = 1,\ldots,n$. Let $U = (u_1,\ldots,u_n)$ and $\|\cdot\|_{R^n}$ denote the Euclidean norm in R^n. Let us denote $H_U(x,U) = \text{grad}_U H(x,U) = (H_{u_1}(x,u_1,\ldots,u_n),\ldots,H_{u_n}(x,u_1,\ldots,u_n))$. Let E be a cartesian product of the Sobolev spaces $W_0^{1,2}(\Omega,R)$, i. e., $E = W_0^{1,2}(\Omega,R) \times \cdots \times W_0^{1,2}(\Omega,R)$. We endow the Hilbert space E with the norm

$$\|U\|^2 = \sum_{i=1}^{n} \|u_i\|^2,$$

where $\|u_i\|^2 = \int_\Omega |\nabla u_i(x)|^2 dx$.

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We assume that H satisfies the following conditions:
$(H1)$ $H \in C^2(R^n \times R^n, R)$, $H(x,\theta) = 0$, $\theta = (0, \dots, 0)$, $H_U(x,\theta) = \theta$,
$(H2)$ There exist constants α and β (α, β are not eigenvalues of the elliptic eigenvalue problem) such that $\alpha < \beta$ and

$$\alpha I \le d_U^2 H(x,U) \le \beta I \qquad \forall (x,U) \in R^n \times R^n$$

and there exists $k \in N^*$ such that $\alpha I < \lambda_k I < d_U^2 F(x,U) < \lambda_{k+1} I < \beta I$ for every U, where $U = (u_1, \dots, u_n)$,
$(H3)$ There exist eigenvalues $\lambda_{h+1}, \dots, \lambda_{h+m}$ such that

$$\lambda_h < \alpha < \lambda_{h+1} < \cdots < \lambda_{h+m} < \beta < \lambda_{h+m+1},$$

where $h \ge 1$, $m \ge 1$.
$(H4)$ There exist γ and C such that $\lambda_{h+m} < \gamma < \beta$ and

$$H(x,U) \ge \frac{1}{2}\gamma \|U\|^2 - C, \qquad \forall (x,U) \in R^n \times R^n.$$

Some papers of Lee [7, 8, 9] concerning the semilinear elliptic system and some papers of the other several authors [4, 6] have treated the system of this like nonlinear elliptic equations. Some papers of Chang [1] and Choi and Jung [2] considered the existence and the multiplicity of the weak solutions for the nonlinear boundary value problems with asymptotically linear term. The authors obtained some results for those problems by approaching the variational method, the critical point theory and the topological method.

The system (1.1) can be rewritten by

$$-\Delta U = \mathrm{grad}_U H(x,U) \qquad \text{in} \ \ \Omega,$$

$$U = \theta \qquad \text{on} \ \ \partial\Omega,$$

where $U = (u_1, \dots, u_n)$ and $\theta = (0, \dots, 0)$.

In this paper we are looking for the weak solutions of system (1.1) in E, that is, $U = (u_1, \dots, u_n) \in E$ such that

$$\int_\Omega [-\Delta U \cdot V] dx - \int_\Omega H_U(x,U) \cdot V = 0, \ \text{for all} \ \ V \in E.$$

Our main result is the following:

Theorem 1.1. *Assume that H satisfies the conditions $(H1)$-$(H4)$. Then system (1.1) has at least two nontrivial weak solutions.*

The proof of Theorem 1.1 is organized as follows: We approach the variational method, the critical point theory and the topological method. In section 2, we recall the relative category theory on the manifold as the topological method which is a crucial role for the proof of the main theorem. In section 3, we prove that the corresponding functional of (1.1) satisfies the geometric conditions of the multiplicity theorem, and prove Theorem 1.1.

2. Variational and topological approach

Lemma 2.1. *Let* $grad_U H(x,U) \in L^2(\Omega)$. *Then all the solutions of*

$$-\Delta U = grad_U H(x,U)$$

belong to E.

Proof. Let $\mathrm{grad}_U H(x,U) \in L^2(\Omega)$. We note that $\{\lambda_n : \ |\lambda_n| < |c|\}$ is finite. Then $\mathrm{grad}_{u_i} H(x,u_1,\dots,u_n) \in L^2(\Omega)$, $i = 1,\dots,n$, can be expressed by

$$\mathrm{grad}_{u_i} H(x,u_1,\dots,u_n) = \sum_{k=1}^{\infty} h_k \phi_k, \qquad \sum_{k=1}^{\infty} h_k^2 < \infty, \text{ for each } \ i = 1,\dots,n.$$

Then

$$(-\Delta)^{-1}\mathrm{grad}_{u_i} H(x,u_1,\dots,u_n) = \sum \frac{1}{\lambda_k} h_k \phi_k.$$

Hence we have the inequality

$$\|(-\Delta)^{-1}\mathrm{grad}_{u_i} H(x,u_1,\dots,u_n)\|^2 = \sum \lambda_k^2 \frac{1}{\lambda_k^2} h_k^2 \le \sum h_k^2,$$

which means that

$$\|(-\Delta)^{-1}\mathrm{grad}_{u_i} H(x,u_1,\dots,u_n)\| \le \|\mathrm{grad}_{u_i} H(x,u_1,\dots,u_n)\|_{L^2(\Omega)}.$$

□

By the following Lemma 2.2, the weak solutions of system (1.1) coincide with the critical points of the associated functional I

$$I \in C^{1,1}(E,R),$$

$$I(U) = \int_\Omega \Big[\frac{1}{2}|\nabla U|^2 - H(x,U)\Big] dx, \tag{2.1}$$

where $U = (u_1,\dots,u_n)$ and $\int_\Omega \|\nabla U\|^2_{R^n} dx = \sum_{i=1}^n \int_\Omega |\nabla u_i|^2 dx$, $n \ge 1$.

Lemma 2.2. *Assume that* H *satisfies the conditions* $(H1)$-$(H4)$. *Then the functional* $I(U)$ *is continuous, Fréchet differentiable with Fréchet derivative*

$$DI(U) \cdot V = \int_\Omega [(-\Delta U) \cdot V - H_U(x,U) \cdot V] dx.$$

Moreover $DI \in C$. *That is* $I \in C^1$.

Proof. First we shall prove that $I(U)$ is continuous. For $U, V \in E$,

$$\begin{aligned} |I(U+V) - I(U)| &= \Big|\frac{1}{2}\int_\Omega (-\Delta U - \Delta V)\cdot(U+V)dx - \int_\Omega H(x,U+V)dx \\ &\quad -\frac{1}{2}\int_\Omega (-\Delta U)\cdot U dx + \int_\Omega H(x,U)dx\Big| \\ &= \Big|\frac{1}{2}\int_\Omega [(-\Delta U \cdot V - \Delta V \cdot U - \Delta V \cdot V) dx \\ &\quad - \int_\Omega (H(x,U+V) - H(x,U))dx\Big|. \end{aligned}$$

We have

$$\Big|\int_\Omega [H(x,U+V)-H(x,U)]dx\Big| \le \Big|\int_\Omega [H_U(x,U)\cdot V + O(\|V\|_{R^n})]dx|$$

$$= O(\|V\|_{R^n}). \tag{2.2}$$

Thus we have

$$|I(U+V)-I(U)| = O(\|V\|_{R^n}). \tag{2.3}$$

$$|I(U+V)-I(U)-DI(U)\cdot V| = O(\|V\|^2_{R^n}). \tag{2.4}$$

Next we shall prove that $I(U)$ is *Fréchet* differentiable. For $U, V \in E$,

$$\begin{aligned}
&|I(U+V)-I(U)-DI(U)\cdot V|\\
&=\Big|\frac{1}{2}\int_\Omega(-\Delta U-\Delta V)\cdot(U+V)dx-\int_\Omega H(x,U+V)dx\\
&-\frac{1}{2}\int_\Omega(-\Delta U)\cdot Udx+\int_\Omega H(x,U)dx-\int_\Omega(-\Delta U-H_U(x,U))\cdot Vdx\Big|\\
=&\Big|\frac{1}{2}\int_\Omega[-\Delta U\cdot V-\Delta V\cdot U-\Delta V\cdot V]dx\\
&-\int_\Omega[H(x,U+V)-H(x,U)]dx-\int_\Omega[(-\Delta U-H_U(x,U))\cdot V]dx\Big|.
\end{aligned}$$

By (2.2),

$$||I(U+V)-I(U)-DI(U)\cdot V|| = O(\|V\|^2_{R^n}).$$

Thus $I \in C^1$. □

Lemma 2.3. *Let $H : R^n \times R^n \to R$ be a C^2 function satisfying the condition that there exists $k \in N^*$ such that $\lambda_k I < d^2_U H(x,U) < \lambda_{k+1} I$ for every U. Then there exist $M > 0$, $\bar{M}$ such that*

$$\frac{1}{2}H_U(x,U)\cdot U - \bar{M} < H(x,U) < \frac{1}{2}H_U(x,U)\cdot U + M.$$

Proof. We define a Borel function

$$c(U) = \begin{cases} \frac{H_U(x,U)\cdot U}{\|U\|^2_{R^n}} & \text{if } U \neq (0,\dots,0),\\ \lambda_k + \frac{\lambda_{k+1}-\lambda_k}{2} & \text{if } U = (0,\dots,0), \end{cases}$$

Then $c(U)$ is a C^1 function from $E\backslash\{0,\dots,0\}$ to R and

$$H_U(x,U)\cdot U = c(U)U^2.$$

Thus we have

$$H(x,U) = \frac{1}{2}H_U(x,U)\cdot U - \frac{1}{2}\int_\Omega U^2 d(c(U)),$$

where $d(c(U)) = \frac{\partial c(U)}{\partial u_1}du_1 + \cdots + \frac{\partial c(U)}{\partial u_n}du_n$. Since $c(U) \in C^1$, there exist $M > 0$ and $\bar{M} > 0$ such that

$$\bar{M} < \int_\Omega U^2 d(c(U)) < M,$$

so we prove the lemma. □

Now, we recall the critical point theory on the manifold with boundary. Let E be a Hilbert space and M be the closure of an open subset of E such that M can be endowed with the structure of C^2 manifold with boundary. Let $f : W \to R$ be a $C^{1,1}$ functional, where W is an open set containing M. For applying the usual topological methods of critical points theory we need a suitable notion of critical point for f on M. We recall the following notions: lower gradient of f on M, $(P.S.)_c$ condition and the relative category (see [5]).

Definition 2.4. If $u \in M$, the lower gradient of f on M at u is defined by

$$grad_{\bar{M}} f(u) = \begin{cases} \nabla f(u) & \text{if } u \in int(M), \\ \nabla f(u) + [< \nabla f(u), \nu(u) >]^- \nu(u) & \text{if } u \in \partial M, \end{cases} \tag{2.5}$$

where we denote by $\nu(u)$ the unit normal vector to ∂M at the point u, pointing outwards.

We say that u is a lower critical for f on M, if $grad_{\bar{M}} f(u)$.

Definition 2.5. Let $c \in R$. We say that f satisfies the $(P.S.)_c$ condition on M if for any sequence $(u_n)_n$ in M such that $f(u_n) \to c$ and $grad_{\bar{M}} f(u_n) \to 0$ there exists a subsequence $(u_{n_k})_k$ which converges to a point u in M such that $grad_{\bar{M}} f(u) = 0$.

Let Y be a closed subspace of M.

Definition 2.6. Let B be a closed subset of M with $Y \subset B$. We define the relative category $cat_{M,Y}(B)$ of B in (M,Y), as the least integer h such that there exist $h+1$ closed subsets $U_0, U_1, \ldots, U_h$ with the following properties:

$B \subset U_0 \cup U_1 \cup \cdots \cup U_h$;

$U_1, \ldots, U_h$ are contractible in M;

$Y \subset U_0$ and there exists a continuous map $F : U_0 \times [0,1] \to M$ such that

$$\begin{aligned} F(x,0) &= x & \forall x \in U_0, \\ F(x,t) &\in Y & \forall x \in Y, \forall t \in [0,1], \\ F(x,1) &\in Y & \forall x \in U_0. \end{aligned}$$

If such an h does not exist, we say that $cat_{M,Y}(B) = +\infty$.

Now we recall a theorem which gives an estimate of the number of critical points of a functional, in terms of the relative category of its sublevels (see [10]).

Theorem 2.7. *Let Y be a closed subset of M. For any integer i we set*

$$c_i = \inf\{\sup f(B) |\ B \text{ is closed, } Y \subset B, cat_{M,Y}(B) \geq i\}.$$

Assume that $(P.S.)_c$ holds for $c = c_i$ and that $\sup f(Y) < c_i < +\infty$. Then c_i is a lower critical level for f, that is, there exists u in M such that $f(u) = c_i$ and $grad_{\bar{M}} f(u) = 0$. Moreover, if

$$c_i = c_{i+1} = \cdots = c_{i+k-1} = c,$$

then

$$cat_M(\{u \in M |\ f(u) = c,\ grad_{\bar{M}} f(u) = 0\}) \geq k.$$

We recall the "nonsmooth" version of the classical Deformation Lemma in [3].

Lemma 2.8 (Deformation Lemma)**.** *Let $h : H \to R \cup \{+\infty\}$ be a lower semi-continuous function and assume h to be φ-convex of order 2. Let $c \in R$, $\delta > 0$ and D be a closed set in H such that*

$$\inf\{\|grad_{\bar{M}} h(x)\|| \; c-\delta \le h(x) \le c+\delta, \qquad dist(x,D) < \delta\} > 0.$$

Then there exists $\epsilon > 0$ and a continuous deformation $\eta : h^{c+\epsilon} \cap D \times [0,1] \to h^{c+\epsilon} \cap D_\delta$ (D_δ is the δ-neighborhood of D and $h^c = \{x | h(x) \le c\}$) such that

(i) $\eta(x,0) = x \qquad \forall x \in h^{c+\epsilon} \cap D,$
(ii) $\eta(x,t) = x \qquad \forall x \in h^{c-\epsilon} \cap D, \forall t \in [0,1],$
(iii) $\eta(x,1) \in h^{c-\epsilon} \qquad \forall x \in h^{c+\epsilon} \cap D, \forall t \in [0,1].$

We recall the following multiplicity result which can be obtain from Theorem 2.5 in [10], which will be used in the proofs of our main theorems.

Theorem 2.9. *Let E be a Hilbert space and let $E = X_1 \oplus X_2 \oplus X_3$, where X_1, X_2, X_3 are three closed subspaces of E with X_2 of finite dimension. Moreover for a given subspace X of E, let P_X be the orthogonal projection from E onto X. Set*

$$C = \{x \in X| \; \|P_{X_2} x\| \ge 1\}$$

and let $f : W \to R$ be a $C^{1,1}$ function defined on a neighborhood W of C. Let $1 < \rho < R$, $\rho > 0$ be a small number and $R_1 > 0$ and we define

$$S_2(r) = \{U \in X_2| \; \|z\| = r\},$$

$$B_{12}(r) = \{U \in X_1 \oplus X_2| \; \|z\| \le r\},$$

$$S_{12}(r) = \{U \in X_1 \oplus X_2| \; \|z\| = r\},$$

$$\Delta_{23}(S_2(\rho), X_3) = \{U = U_2 + U_3 \in X_2 \oplus X_{32}| \; U_2 \in S_2(\rho), \; U_3 \in X_3, \|U_1 + U_2\| \le R\},$$

$$\Sigma_{23}(S_2(\rho), X_3) = \{U = U_2 + U_3 \in X_2 \oplus X_3| \; U_2 \in S_2(\rho), \; U_3 \in X_3, \|U_2 + U_3\| = R\|\}.$$

Let

$$a = \inf f(\Sigma_{23}(S_2(\rho), X_3)), \qquad b = \sup f(B_{12}(r)).$$

Assume that

$$\sup f(S_{12}(r)) < \inf f(\Sigma_{23}(S_2(\rho), X_3)).$$

Assume that the $(P.S.)_c$ condition holds for f on C, $\forall c \in [a,b]$. Assume that $f|_{X_1 \oplus X_3}$ has no critical points with $a \le f(u) \le b$. Moreover we assume $b < +\infty$. Then there exist two lower critical points u_1, u_2 for f on Int C such that $\inf f(\Sigma_{23}(S_2(\rho), X_3)) \le f(u_i) \le \sup f(B_{12}(r))$, $i = 1, 2$.

3. Proof of Theorem 1.1

Assume that H satisfies the conditions $(H1) - (H4)$ hold. Let us consider the eigenvalue problem on E

$$-\Delta U(x) = \lambda U(x) \qquad \text{in } \Omega, \tag{3.1}$$

$$U = \theta \qquad \text{on } \partial\Omega,$$

where $U = (u_1, \ldots, u_n)$, $\theta = (0, \ldots, 0)$, $\lambda \in R$ and $U \in E$.

Let us set

$X_1 = \text{span}\{\text{eigenvectors belonging to the eigenvalue } \lambda \text{ of } (3.1)\ ,\ \lambda \le \lambda_h\},$
$X_2 = \text{span}\{\text{eigenvectors belonging to the eigenvalue } \lambda \text{ of } (3.1)\ , \lambda_{h+1} \le \lambda \le \lambda_{h+m}\},$
$X_3 = \text{span}\{\text{eigenvectors belonging to the eigenvalue } \lambda \text{ of } (3.1)\ ,\ \lambda \ge \lambda_{h+m+1}\}.$

Then X_i, $i = 1, 2, 3$, are subspaces of E and $E = X_1 \oplus X_2 \oplus X_3$, where X_2 is $m-$dimensional subspace.

Lemma 3.1. *Let α and β be any number with $\lambda_h < \alpha < \lambda_{h+1} < \cdots < \lambda_{j+m} < \beta < \lambda_{j+m+1}$. If U is a critical point for $I|_{X_1 \oplus X_3}$, then $I(U) = 0$.*

Proof. We notice that from Lemma 3.1, for fixed $U_1 \in X_1$, the functional $U_3 \mapsto I(U_1 + U_3)$ is weakly convex in X_3, while, for fixed $U_3 \in X_3$, the functional $U_1 \mapsto I(U_1 + U_3)$ is strictly concave in X_1. Moreover θ is the critical point in $X_1 \oplus X_3$ with $I(\theta) = 0$. So if $U = U_1 + U_3$ is another critical point for $I|_{X_1 \oplus X_3)}$, then we have

$$0 = I(\theta) \le I(U_3) \le I(U_1 + U_3) \le I(U_1) \le I(\theta) = 0.$$

So we have $I(U) = I(\theta) = 0$. □

Let P_{X_2} be the orthogonal projection from E onto X_2 and

$$C = \{Z \in E|\ \|P_{X_2} Z\| \ge 1\}.$$

Then C is the smooth manifold with boundary. Let us define a functional $\Psi : E \setminus (\{X_1 \oplus X_3) \to E$ by

$$\Psi(Z) = Z - \frac{P_{X_2} Z}{\|P_{X_2} Z\|} = P_{X_1 \oplus X_3} Z + \Big(1 - \frac{1}{\|P_{X_2} Z\|}\Big) P_{X_2} Z. \tag{3.2}$$

We have

$$\nabla \Psi(Z) \cdot W = W - \frac{1}{\|P_{X_2} Z\|}\Big(P_{X_2} W - \Big\langle \frac{P_{X_2} Z}{\|P_{X_2} Z\|}, W\Big\rangle \frac{P_{X_2} Z}{\|P_{X_2} Z\|}\Big). \tag{3.3}$$

Let us define the functional $\tilde{I} : C \to R$ by

$$\tilde{I} = I \circ \Psi.$$

Then $\tilde{I} \in C^{1,1}_{loc}$. We note that if $\tilde{Z}$ is the critical point of $\tilde{I}$ and lies in the interior of C, then $Z = \Psi(\tilde{Z})$ is the critical point of I. We also note that

$$\|grad_{\bar{C}} \tilde{I}(\tilde{Z})\| \ge \|P_{X_1 \oplus X_3} DI(\Psi(\tilde{Z}))\| \qquad \forall \tilde{Z} \in \partial C. \tag{3.4}$$

Let us set

$$\begin{aligned} \tilde{S_2}(r) &= \Psi^{-1}(S_2(r)),\\ \tilde{S_2}(r) &= \Psi^{-1}(B_{12}(r)),\\ \tilde{S_{12}}(r) &= \Psi^{-1}(S_{12}(r)),\\ \Delta_{23}(\tilde{S_2}(\rho), X_3) &= \Psi^{-1}(\Delta_{23}(S_2(\rho), X_3)),\\ \Sigma_{23}(\tilde{S_2}(\rho), X_3) &= \Psi^{-1}(\Sigma_{23}(S_2(\rho), X_3)). \end{aligned}$$

We note that $\tilde{S_2}(r)$, $\tilde{B_{12}}(r)$, $\tilde{S_{12}}(r)$, $\Delta_{23}(\tilde{S_2}(\rho), X_3)$ and $\Sigma_{23}(\tilde{S_2}(\rho), X_3)$ have the same topological structure as $S_2(r)$, $B_{12}(r)$, $S_{12}(r)$, $\Delta_{23}(S_2(\rho), X_3)$ and $\Sigma_{23}(S_2(\rho), X_3)$ respectively.

Lemma 3.2. *Assume that H satisfies the conditions $(H1)-(H4)$ hold. Then the functional I satisfies $(P.S.)_c$ condition for every $c \in R$.*

Proof. Let $(U_n)_n$ be a sequence in E such that $I(U_n) \to c$ and $DI(U_n) \to 0$. We shall show that $(U_n)_n$ has a convergent subsequence. We claim that $(U_n)_n$ is bounded. By contradiction, we suppose that $\|U_n\| \to +\infty$ and set $W_n = \frac{U_n}{\|U_n\|}$. Up to a subsequence $W_n \rightharpoonup W_0$ weakly for some $W_0 \in E$. By the asymptotically linearity of $DI(U_n)$ we have

$$\begin{aligned}&\Big\langle DI(U_n), \frac{U_n}{\|U_n\|}\Big\rangle\\ &= \frac{2I(U_n)}{\|U_n\|} + \int_\Omega \Big[\frac{2H(x,U_n)}{\|U_n\|} - \frac{H_{U_{n_1}}(x,U_n) + \cdots + (H_{U_{n_1}}, \ldots, H_{U_{n_n}}) \cdot U_n}{\|U_n\|}\Big]dx,\end{aligned}$$

where $U_n = (U_{n_1}, \ldots, U_{n_n})$. Passing to the limit we get

$$\lim_{n\to\infty} \int_\Omega \Big[\frac{2H(x,H_n)}{\|U_n\|} - \frac{(H_{U_{n_1}}, \ldots, H_{U_{n_n}}) \cdot U_n}{\|U_n\|}\Big]dx = 0.$$

Since H, $H_{U_{n_i}}$ are bounded and $\|U_n\| \to \infty$ in Ω, $W_0 = 0$. Moreover we have

$$\begin{aligned}\Big\langle \frac{DI(U_n)}{\|U_n\|}, W_n\Big\rangle &= \int_\Omega \Big[\frac{-\Delta U_n}{\|U_n\|} W_n - \frac{(H_{U_{n_1}}, \ldots, H_{U_{n_n}}) \cdot W_n}{\|U_n\|}\Big]dx\\ &= \int_\Omega \Big[-\Delta W_n \cdot W_n - \frac{(H_{U_{n_1}}, \ldots, H_{U_{n_n}}) \cdot W_n}{\|U_n\|}\Big]dx.\end{aligned}$$

Since W_n converges to 0 weakly and $-\Delta$ and $H_{U_{n_i}}$ are compact, $\int_\Omega -\Delta W_n \cdot W_n dx = \|W_n\|^2 \to 0$. Thus W_n converges to 0 strongly, which is a contradiction. Thus (U_n) is bounded. Up to a subsequence, U_n converges to U for some $U \in E$. We claim that U_n converges to U strongly. We have

$$\langle DI(U_n), U_n\rangle = \int_\Omega [-\Delta U_n \cdot U_n - (H_{U_{n_1}}, \ldots, H_{U_{n_n}}) \cdot U_n]dx \longrightarrow 0.$$

By the compactness of $-\Delta$ and $H_{U_{n_i}}$,

$$\int_\Omega -\Delta U_n \cdot U_n dx = \|U_n\|^2 \longrightarrow \int_\Omega -\Delta U \cdot U dx = \|U\|^2.$$

Thus we have that U_n converges to U strongly. Thus we have

$$DI(U) = \lim_{n\to\infty} DI(U_n) = 0.$$

Thus we prove the lemma. □

Lemma 3.3. *Assume that H satisfies the conditions $(H1)-(H4)$. Then there exist $r > 0$, $\rho > 0$ a small number and $R > 0$ such that $r < R$, and for any α and β with $\lambda_h < \alpha < \lambda_{h+1} < \cdots < \lambda_{j+m} < \beta < \lambda_{j+m+1}$,*

$$\sup_{\tilde{U} \in S_{1\tilde{2}}(r)} \tilde{I}(\tilde{U}) < 0 < \inf_{\tilde{U} \in \Sigma_{23}(S_{\tilde{2}}(\rho), X_3)} \tilde{I}(\tilde{U}),$$

$$\inf_{\tilde{U} \in \Delta_{23}(S_{\tilde{2}}(\rho), X_3)} \tilde{I}(\tilde{U}) > -\infty, \qquad \sup_{\tilde{U} \in B_{1\tilde{2}}(r)} \tilde{I}(\tilde{U}) < \infty.$$

Proof. Since

$$\sup_{\tilde{U}\in \tilde{S_{12}}(r)} \tilde{I}(\tilde{U}) = \sup_{U\in S_{12}(r)} I(U), \qquad \inf_{U\in \Sigma_{23}(\tilde{S_2}(\rho),X_3)} \tilde{I}(U) = \inf_{U\in \Sigma_{23}(S_2(\rho),X_3)} I(U)$$

and

$$\inf_{\tilde{U}\in \Delta_{23}(\tilde{S_2}(\rho),X_3)} \tilde{I}(\tilde{U}) = \inf_{U\in \Delta_{23}(S_2(\rho),X_3)} I(U), \qquad \sup_{U\in \tilde{B_{12}}(r)} \tilde{I}(\tilde{U}) = \sup_{U\in B_{12}(r)} I(U),$$

it suffices to show that

$$\sup_{U\in S_{12}(r)} I(U) < 0 < \inf_{U\in \Sigma_{23}(S_2(\rho),X_3)} I(U),$$

$$\inf_{U\in \Delta_{23}(S_2(\rho),X_3)} I(U) > -\infty, \qquad \sup_{U\in B_{12}(r)} I(U) < \infty.$$

Let $U = U_1 + U_2 \in X_1 \oplus X_2$. By $(H4)$ and Lemma 2.3, there exists a constant $\bar{M}$ such that

$$\begin{aligned} I(U) &= \frac{1}{2}\int_\Omega [-\Delta U\cdot U]dx - \int_\Omega H(x,U(x))dx \\ &\le \frac{1}{2}\|U_1+U_2\|^2 - \frac{\gamma}{2}\|U_1+U_2\|^2_{L^2} + \bar{M}|\Omega| \\ &\le \frac{1}{2}(\lambda_{h+m}-\gamma)\|U_1+U_2\|^2_{L^2} + \bar{M}|\Omega|. \end{aligned}$$

Since $\lambda_{h+m}-\gamma<0$, there exists $r>0$ such that if $U_1+U_2\in S_{12}(r)$, then $I(z)<0$. Thus $\sup_{U\in S_{12}(r)} I(U) < 0$. Moreover, if $U\in B_{12}(r)$, then $I(U)\le \bar{M}|\Omega|<\infty$, so we have $\sup_{U\in B_{12}(r)} I(U)<\infty$. Next we will show that there exist $r>0$, $\rho>0$ a small number and $R>0$ such that if $\lambda_h<\alpha<\lambda_{h+1}<\cdots<\lambda_{h+m}<\beta<\lambda_{h+m+1}$, then $\inf_{U\in\Sigma_{23}(S_2(\rho),X_3)} I(U)>0$. Let $U=U_2+U_3\in X_2\oplus X_3$ with $U_2\in X_2$, $U_3\in X_3$. Let α and β be any number such that $\lambda_h<\alpha<\lambda_{h+1}\le\cdots<\lambda_{h+m}<\beta<\lambda_{h+m+1}$, $\alpha>0$. By $(H2)$ and Lemma 2.3, there exist $M>0$, $\bar{M}$ such that

$$\frac{1}{2}H_U(x,U)\cdot U - \bar{M} < H(x,U) < \frac{1}{2}H_U(x,U)\cdot U + M.$$

Thus we have

$$\begin{aligned} I(U) &= \frac{1}{2}\int_\Omega [-\Delta U\cdot U]dx - \int_\Omega H(x,U(x))dx \\ &\ge \frac{1}{2}\int_\Omega [-\Delta U\cdot U]dx - \frac{1}{2}\int_\Omega H_U(x,U(x))\cdot U dx - M|\Omega| \\ &= \frac{1}{2}\|U_2\|^2 + \frac{1}{2}\|U_3\|^2 - \frac{1}{2}\int_\Omega H_U(x,U(x))\cdot U_2 dx \\ &\quad -\frac{1}{2}\int_\Omega H_U(x,U(x))\cdot U_3 dx - M|\Omega|. \end{aligned}$$

By $(H2)$, we have

$$I(U) \ge \frac{1}{2}(\lambda_{h+1}-\beta)\|U_2\|^2_{L^2} + \frac{1}{2}(\lambda_{h+m+1}-\beta)\|U_3\|^2_{L^2} - d.$$

Since $\lambda_{h+1}-\beta<0$ and $\lambda_{h+1}-\beta<0$, there exists a small number $\rho>0$ and $R>0$ such that if $U\in\Sigma_{23}(S_2(\rho),X_3)$, $I(U)\ge\frac{1}{2}(\lambda_{h+1}-\beta)\rho^2+\frac{1}{2}(\lambda_{h+m+1}-\beta)\|U_3\|^2_{L^2}-d$.

Thus we have $\inf I(U) > 0$. Moreover if $U \in \Delta_{23}(S_2(\rho), X_3)$, then $I(U) \geq \frac{1}{2}(\lambda_{h+1} - \beta)\rho^2 - d > -\infty$. Thus we prove the lemma. □

Lemma 3.4. *Assume that H satisfies the conditions $(H1)-(H4)$ hold. Then the functional $\tilde{I}$ satisfies $(P.S.)_c$ condition with respect to C for any c such that*

$$0 < \inf_{\tilde{Z} \in \Sigma_{23}(\tilde{S_2}(\rho), X_3)} \tilde{I}(\tilde{Z}) \leq c \leq \sup_{\tilde{U} \in \tilde{B_{12}}(r)} \tilde{I}(\tilde{U}),$$

where $r > 0$, $\rho > 0$ and $R > 0$ are introduced in Lemma 3.3.

Proof. Let $(\tilde{Z_n})_n$ be a sequence in C such that $\tilde{I}(\tilde{Z_n}) \to c$ and $grad_C^- \tilde{I}(\tilde{Z_n}) \to 0$. Set $Z_n = \Psi(\tilde{Z_n})$ (and hence $Z_n \in E$) and $I(z_n) \to c$. We first consider the case in which $Z_n \notin X_1 \oplus X_3$. We have

$$D\tilde{I}(\tilde{Z_n}) = \Psi'(\tilde{Z_n})(DI(Z_n)) = \Psi'(\tilde{Z_n})(DI(z_n) \longrightarrow 0.$$

By (3.3) and (3.4),

$$DI(Z_n) \to 0 \qquad \text{or}$$

$$P_{X_1 \oplus X_3} DI(Z_n) \to 0 \qquad \text{and} \qquad P_{X_2} Z_n \to 0. \tag{3.5}$$

In the first case the claim follows from the Palais-Smale condition for I. In the second case $P_{X_1 \oplus X_3)} DI(Z_n) \to 0$. We claim that $(Z_n)_n$ is bounded. By contradiction, we suppose that $\|Z_n\| \to +\infty$ and set $W_n = \frac{Z_n}{\|Z_n\|}$. Up to a subsequence $W_n \rightharpoonup W_0$ weakly for some $W_0 \in X_1 \oplus X_3$. By the asymptotically linearity of $DI(Z_n)$ we have

$$\Big\langle \frac{DI(Z_n)}{\|Z_n\|}, W_n \Big\rangle = \Big\langle P_{X_1 \oplus X_3)} \frac{DI(Z_n)}{\|Z_n\|}, W_n \Big\rangle + \Big\langle \frac{DI(Z_n)}{\|Z_n\|^2}, P_{X_1} Z_n \Big\rangle \longrightarrow 0.$$

We have

$$\Big\langle \frac{DI(Z_n)}{\|Z_n\|}, W_n \Big\rangle = \frac{2I(Z_n)}{\|Z_n\|^2} + \int_\Omega \Big[- \frac{2H(t, Z_n)}{\|Z_n\|^2} + \frac{H_Z(x, Z_n) \cdot W_n}{\|Z_n\|} \Big] dx,$$

where $Z_n = ((Z_n)_1, \ldots, (Z_n)_{2n})$. Passing to the limit we get

$$\lim_{n \to \infty} \int_\Omega \Big[\frac{2H(x, Z_n)}{\|Z_n\|^2} - \frac{H_z(x, Z_n) \cdot W_n}{\|Z_n\|} \Big] dx = 0.$$

Since H and $H_Z(x, Z_n) \cdot Z_n$ are bounded and $\|Z_n\| \to \infty$ in Ω, $W_0 = 0$. On the other hand we have

$$\Big\langle P_{X_1 \oplus X_3} \frac{DI(Z_n)}{\|Z_n\|}, W_n \Big\rangle = \int_\Omega \Big[- \Delta W_n \cdot W_n - P_{X_1 \oplus X_3} \frac{H_Z(x, Z_n)}{\|Z_n\|}) \cdot W_n \Big] dx.$$

Moreover we have

$$\Big\langle P_{X_1 \oplus X_3} \frac{DI(Z_n)}{\|Z_n\|}, W_n \Big\rangle = \|P_{X_1 \oplus X_3} W_n\|^2 - \int_\Omega P_{X_1 \oplus X_3} \frac{H_Z(x, Z_n)}{\|Z_n\|} \cdot W_n dx.$$

Since W_n converges to 0 weakly and $H_Z(x, Z_n) \cdot W_n$ is bounded, $\|P_{X_1 \oplus X_3} W_n\|^2 \to 0$. Since $\|P_{X_1 \oplus X_3} W_n\|^2 \to 0$, W_n converges to 0 strongly, which is a contradiction.

Hence $(Z_n)_n$ is bounded. Up to a subsequence, we can suppose that Z_n converges to Z_0 for some $Z_0 \in X_1 \oplus X_3$. We claim that Z_n converges to Z_0 strongly. We have

$$\langle P_{X_1 \oplus X_3} DI(Z_n), Z_n \rangle$$
$$= \|P_{X_1 \oplus X_3} Z_n\|^2 - P_{X_1 \oplus X_3} \int_\Omega H_Z(x, Z_n) \cdot Z_n.$$

By $(H1)$ and the boundedness of $H_Z(x, Z_n)(Z_n)$,

$$\|P_{X_1 \oplus X_3} Z_n\|^2 \longrightarrow P_{X_1 \oplus X_3} \int_\Omega H_Z(x, Z) \cdot Z.$$

That is, $\|P_{X_1 \oplus X_3} Z_n\|^2$ converges. Since $\|P_{X_2} Z_n\|^2 \to 0$, $\|Z_n\|^2$ converges, so Z_n converges to z strongly. Therefore we have

$$\begin{aligned} \mathrm{grad}_C^- \tilde{I}(\tilde{Z}) = grad_C^- I(Z) &= \lim_{n \to \infty} \mathrm{grad}_C^- I(Z_n) \\ &= \lim_{n \to \infty} \mathrm{grad}_C^- \tilde{I}(\tilde{Z}_n) = 0. \end{aligned}$$

So we proved the first case. We consider the case $P_{X_2} Z_n = 0$, i.e., $Z_n \in X_1 \oplus X_3$. Then $\tilde{Z}_n \in \partial C$, $\forall n$. In this case $Z_n = \Psi(\tilde{Z}_n) \in X_1 \oplus X_3$ and $P_{X_1 \oplus X_3} DI(Z_n) \to 0$. Thus by the same argument as the first case we obtain the conclusion. So we prove the lemma. □

Proof of Theorem 1.1. By Lemma 2.1, the functional $\tilde{I}(\tilde{Z})$ is continuous, *Fréchet* differentiable and $\tilde{I} \in C^1$. By Lemma 3.3, there exist $r > 0$, $\rho > 0$ a small number and $R > 0$ such that $r < R$, and for any α and β with $\lambda_h < \alpha < \lambda_{h+1} < \cdots < \lambda_{j+m} < \beta < \lambda_{j+m+1}$,

$$\sup_{\tilde{U} \in S_{1\tilde{2}}(r)} \tilde{I}(\tilde{U}) < 0 < \inf_{\tilde{U} \in \Sigma_{23}(S_{\tilde{2}}(\rho), X_3)} \tilde{I}(\tilde{U}), \tag{3.1}$$

$$\inf_{\tilde{U} \in \Delta_{23}(S_{\tilde{2}}(\rho), X_3)} \tilde{I}(\tilde{U}) > -\infty, \qquad \sup_{\tilde{U} \in B_{1\tilde{2}}(r)} \tilde{I}(\tilde{U}) < \infty.$$

By Lemma 3.4, $\tilde{I}$ satisfies $(P.S.)_c$ condition with respect to C for any c such that

$$0 < \inf_{\tilde{Z} \in \Sigma_{23}(S_{\tilde{2}}(\rho), X_3)} \tilde{I}(\tilde{Z}) \le c \le \sup_{\tilde{U} \in B_{1\tilde{2}}(r)} \tilde{I}(\tilde{U}).$$

By Theorem 2.9, there exist two critical points $\tilde{Z}_1$, $\tilde{Z}_2$ for the functional $\tilde{I}$ such that

$$0 < \inf_{\tilde{Z} \in \Sigma_{23}(S_{\tilde{2}}(\rho), X_3)} \tilde{I}(\tilde{Z}) \le \tilde{I}(\tilde{Z}_i) \le \sup_{\tilde{Z} \in B_{1\tilde{2}}(r)} \tilde{I}(Z).$$

Setting $Z_i = \tilde{\Psi}(\tilde{Z}_i)$, $i = 1, 2$, we have

$$0 < \inf_{Z \in \Sigma_{23}(S_2(\rho), X_3)} I(Z) \le I(Z_i) \le \sup_{Z \in B_{12}(r)} I(Z).$$

We claim that $\tilde{Z}_i \notin \partial \tilde{C}$, that is $Z_i \notin X_1 \oplus X_3$, which implies that z_i are the critical points for I in X_2. For this we assume by contradiction that $Z_i \in X_1 \oplus X_3$. From (3.5), $P_{X_1 \oplus X_3} DI(Z_i) = 0$, namely, Z_i, $i = 1, 2$, are the critical points for $I|_{X_1 \oplus X_3}$. By Lemma 3.1, $I(Z_i) = 0$, which is a contradiction for the fact that

$$0 < \inf_{Z \in \Sigma_{23}(S_2(\rho), X_3)} I(Z) \le I(Z_i) \le \sup_{Z \in B_{12}(r)} I(Z).$$

Lemma 3.1 implies that there is no critical point $z \in X_1 \oplus X_3$ such that

$$0 < \inf_{Z \in \Sigma_{23}(S_2(\rho), X_3)} I(Z) \leq I(Z) \leq \sup_{Z \in B_{12}(r)} I(Z).$$

Hence $Z_i \notin X_1 \oplus X_3$, $i = 1, 2$. Thus we prove Theorem 1.1. □

References

[1] K. C. Chang, *Infinite dimensional Morse theory and multiple solution problems,* Birkhäuser, Boston, 1993.

[2] Q. H. Choi and T. Jung, *Multiple periodic solutions of a semilinear wave equation at double external resonances,* Communications in Applied Analysis **3** (1999), 73–84 .

[3] M. Degiovanni, *Homotopical properties of a class of nonsmooth functions,* Ann. Mat. Pura Appl. **156** (1990), 37–71.

[4] D.R. Dunninger and H. Wang, *Multiplicity of positive radial solutions for an elliptic system on an annulus domain,* Nonlinear Analysis TMA **42** (2000), 803–811.

[5] G. Fournier, D. Lupo, M. Ramos and M. Willem, *Limit relative category and critical point theory,* Dynam. Report **3** (1993), 1–23.

[6] K. Lan and R.L. Webb, *Positive solutions of semilinear equation with singularities,* J. Differential Equations **148** (1998), 407-421.

[7] Y. H. Lee, *Existence of multiple positive radial solutions for a semilinear elliptic system on an unbounded domain,* Nonlinear Analysis, TMA **47** (2001), 3649–3660.

[8] Y. H. Lee, *A multiplicity result of positive radial solutions for a multiparameter elliptic system on an exterior domain,* Nonlinear Analysis, TMA **45** (2001), 597–611.

[9] Y. H. Lee, *Multiplicity of positive radial solutions for multiparameter semilinear elliptic systems on an annulus,* J. Differential Equations **174** (2001), 420–441.

[10] A. M. Micheletti and C. Saccon, *Multiple nontrivial solutions for a floating beam equation via critical point theory,* J. Differential Equations **170** (2001), 157–179.

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