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STABLE LAGRANGE DUALITIES FOR ROBUST CONICAL PROGRAMMING

DONGHUI FANG, CHONG LI, AND JEN-CHIH YAO*

Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

Abstract. In this paper, we present the Lagrange duality for conical programming problems with data uncertainty within the framework of robust optimization. By using the infimal convolution of the conjugate functions, we give a new constraint qualification which completely characterizes the stable Lagrange duality for robust conical programming problems in real locally convex Hausdorff topological vector spaces.

1. Introduction

Let X and Y be real locally convex Hausdorff topological vector spaces, $C \subseteq X$ be a nonempty convex set. Let $S \subseteq Y$ be a closed convex cone and $S^{\oplus}$ the positive dual cone of S. Let $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a proper convex function and $g : X \to Y$ a proper S-convex mapping with respect to the cone S. The classic form of conical programming problem in the absence of data uncertainty is the problem (see for example [3, 10, 12–18] and the references therein)

$$(\mathcal{P}) \qquad \begin{array}{ll} \inf & f(x), \\ \text{s.t.} & x \in C,\ g(x) \in -S, \end{array}$$

and its Lagrange dual problem is

$$(\mathcal{D}) \qquad \sup_{\lambda \in S^{\oplus}} \inf_{x \in C} \{f(x) + (\lambda g)(x)\}.$$

It is well-known that the optimal values of these problems, $v(\mathcal{P})$ and $v(\mathcal{D})$, respectively, satisfy the so-called weak Lagrange duality, that is, $v(\mathcal{P}) \geq v(\mathcal{D})$, but a duality gap may occur (i.e., we may have $v(\mathcal{P}) > v(\mathcal{D})$). A challenge in convex analysis has been to give sufficient conditions which guarantee the Lagrange duality, i.e., $v(\mathcal{P}) = v(\mathcal{D})$. Various sufficient conditions, and complete characterizations of $(\mathcal{P})$ in terms of the value function for the Lagrange duality have been given in the literature (see [1–3, 17, 18, 22, 26, 27] and the references therein). Especially, in the case when f is lower semicontinuous (lsc in brief), g is star lsc and continuous at

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*Corresponding author.

some point in the closed subset C, Jeyakumar and Li in [17] presented some constraint qualifications which completely characterize the Lagrange duality in Banach spaces; and they established necessary and sufficient conditions for stable Lagrange duality in [18] under the assumptions that $C = X$ and g is continuous.

Recently, mathematical programming problems under uncertainty have received much attention (cf. [4–7,19,21,24] and the references therein). As mentioned in [19], the study of convex programming problems that are affected by data uncertainty is becoming increasingly important in optimization due to the reality of uncertainty in many real-world optimization problems and the importance of identifying and locating solutions that are immunized against data uncertainty. In particular, many authors considered the following uncertain convex programming problem (cf. [4,19, 21,24] and the references therein),

$$\begin{array}{ll} \inf & f(x), \\ \text{s.t.} & g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m, \\ & x \in C, \end{array} \tag{1.1}$$

and its dual problems

$$\sup_{\lambda_i \geq 0, v_i \in \mathcal{V}_i} \inf_{x \in C} \Big\{ f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i) \Big\}, \tag{1.2}$$

and

$$\sup_{\lambda_i \geq 0} \inf_{x \in C} \sup_{v_i \in \mathcal{V}_i} \Big\{ f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i) \Big\}, \tag{1.3}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex, $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is the uncertain parameter which belongs to the uncertainty set $\mathcal{V}_i \subseteq \mathbb{R}^q$, $i = 1, \ldots, m$. Under some additional assumptions, they established the Lagrange duality and the strong Lagrange duality between (1.1) and (1.2) and, between (1.1) and (1.3).

Inspired by the works mentioned above, we continue to study the conical optimization problem $(\mathcal{P})$ but with data uncertainty in the constraints, that is, the problem defined by

$$(UP) \quad \begin{array}{ll} \inf & f(x), \\ \text{s.t.} & x \in C,\ g_u(x) \in -S, \end{array}$$

where u is the uncertain parameter which belongs to the set $\mathcal{U}$ and, for each $u \in \mathcal{U}$, $g_u : X \to Y$ is a proper S-convex mapping with respect to the cone S. The Lagrange dual problem of (UP) is given by

$$(UD) \quad \sup_{\lambda \in S^{\oplus}} \inf_{x \in C} \{ f(x) + \lambda g_u(x) \}.$$

Following [24], we study the Lagrange duality for the uncertain conical programming problem (UP) by examining its robust counterpart, where the constraints are enforced for every parameter u in the prescribed set $\mathcal{U}$,

$$(RP) \quad \begin{array}{ll} \inf & f(x), \\ \text{s.t.} & x \in C,\ g_u(x) \in -S,\ \forall u \in \mathcal{U}, \end{array} \tag{1.4}$$

the optimistic counterpart of the uncertain Lagrange dual problem (UD)

$$(1.5)\qquad (OLD)\qquad \sup_{\lambda\in S^{\oplus}}\sup_{u\in\mathcal{U}}\inf_{x\in C}\{f(x)+(\lambda g_u)(x)\},$$

and the (standard) Lagrangian dual of the robust counterpart

$$(1.6)\qquad (RLD)\qquad \sup_{\lambda\in S^{\oplus}}\inf_{x\in C}\sup_{u\in\mathcal{U}}\{f(x)+(\lambda g_u)(x)\}.$$

In particular, in the case when $\mathcal{U}$ is a singleton, problem (RP) is reduced to problem $(\mathcal{P})$ and problems (OLD) and (RLD) are coincided with problem $(\mathcal{D})$. Moreover, in the case when $S=\mathbb{R}^m_+ := \{(x_1,\ldots,x_m)\in\mathbb{R}^m : x_i\geq 0\}$, $\mathcal{U}=\prod_{i=1}^m \mathcal{V}_i$ and, for each $u=(u_1,\ldots,u_m)\in\mathcal{U}$, define the function $g_u:\mathbb{R}^n\to\mathbb{R}^m$ by

$$g_u(x) := (g_1(x,u_1),\ldots,g_m(x,u_m)) \quad \text{for each } x\in\mathbb{R}^n,$$

then problem (1.1) can be viewed as an example of (RP).

Clearly, the optimal values of these problems, $v(RP)$, $v(OLD)$ and $v(RLD)$, respectively, satisfy the so-called weak Lagrange duality, that is, $v(RP)\geq v(RLD)\geq v(OLD)$. The Lagrange duality between primal problem and dual problem, that is, $v(RP)=v(OLD)$ (or $v(RP)=v(RLD)$), is a key ingredient of duality theory, which often reveals deep information that is not explicit in the original problem. Recently, the strong Lagrange duality between (RP) and (OLD), that is, $v(RP)=v(OLD)$ and problem (OLD) has an optimal solution, was established in [24] in the case when $X=C$, f is lsc and g_u is continuous for each $u\in\mathcal{U}$. Obviously, the strong Lagrange duality ensures the Lagrange duality. However, the converse is often not true. In this paper, we focus our interest on the stable Lagrange duality between (RP) and (OLD) and, between (RP) and (RLD), that is, the situation when for each $p\in X^*$,

$$(1.7)\qquad \inf_{x\in A}\{f(x)-\langle p,x\rangle\} = \sup_{\lambda\in S^{\oplus},u\in\mathcal{U}}\inf_{x\in C}\{f(x)+\lambda g_u(x)-\langle p,x\rangle\},$$

and

$$(1.8)\qquad \inf_{x\in A}\{f(x)-\langle p,x\rangle\} = \sup_{\lambda\in S^{\oplus}}\inf_{x\in C}\sup_{u\in\mathcal{U}}\{f(x)+\lambda g_u(x)-\langle p,x\rangle\}.$$

Our main aim in the present paper is to give some new regularity conditions which completely characterize the stable Lagrange dualities. In general, we do not impose any topological assumption on C, $\mathcal{U}$ or on f and g_u, $u\in\mathcal{U}$, that is, C is not necessarily closed, $\mathcal{U}$ is not necessarily compact and f, $g_u(u\in\mathcal{U})$ are not necessarily lsc. Most results obtained in this paper seem new and are proper extensions of the known results in [17,18,21], in particular, even in the special case when $\mathcal{U}$ is a singleton, our Corollary 4.4 and Theorem 4.5 improve the corresponding results in [18, Theorem 3.1] and [17, Theorem 4.1], respectively; and our Theorem 4.5 extends and improves the result in [21, Theorem 3.1] for the uncertain convex programming problem (1.1).

This paper is organized as follows. The next section contains some necessary notations and preliminary results. In Section 3, a new regularity condition is provided and several properties of this condition are given. The stable Lagrange dualities between (RP) and (OLD) and, between (RP) and (RLD), are obtained in Section 4.

2. Notations and preliminary results

The notation used in the present paper is standard (cf. [28]). In particular, we assume throughout the whole paper that X and Y are real locally convex Hausdorff topological vector spaces, and let X^* denote the dual space, endowed with the weak*-topology $w^*(X^*, X)$. By $\langle x^*, x\rangle$ we denote the value of the functional $x^* \in X^*$ at $x \in X$, i.e. $\langle x^*, x\rangle = x^*(x)$. Let Z be a set in X. The interior (*resp.* closure, convex hull) of Z is denoted by $\operatorname{int} Z$ (*resp.* $\operatorname{cl} Z$, $\operatorname{co} Z$). If $W \subseteq X^*$, then $\operatorname{cl} W$ denotes the weak*-closure of W. For the whole paper, we endow $X^* \times \mathbb{R}$ with the product topology of $w^*(X^*, X)$ and the usual Euclidean topology.

The indicator function δ_Z of the nonempty set Z is defined by

$$\delta_Z(x) := \begin{cases} 0 & x \in Z, \\ +\infty & \text{otherwise.} \end{cases}$$

Let $f : X \to \overline{\mathbb{R}}$ be a proper function. The effective domain and the epigraph of f are respectively defined by $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}$. As usual, the conjugate function $f^* : X^* \to \overline{\mathbb{R}}$ of f is defined by

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in X\} \quad \text{for each } x^* \in X^*.$$

Clearly, f^* is a proper convex lsc function and $\operatorname{epi} f^*$ is weak*-closed. Moreover, $\operatorname{epi}(\alpha f)^* = \alpha \operatorname{epi} f^*$ for any $\alpha > 0$. The lsc hull, the lsc convex hull of f, denoted respectively by $\operatorname{cl} f$ and $\operatorname{cl}(\operatorname{co} f)$, are defined by

$$\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f),$$

and

$$\operatorname{epi}(\operatorname{cl}(\operatorname{co} f)) = \operatorname{cl}(\operatorname{co}(\operatorname{epi} f)).$$

Then (cf. [28, Theorems 2.3.1(iv)]),

$$f^* = (\operatorname{cl} f)^* = (\operatorname{cl}(\operatorname{co} f))^* \quad \text{and} \quad f^{**} \le \operatorname{cl}(\operatorname{co} f) \le \operatorname{cl} f \le f. \tag{2.1}$$

By definition, the Young-Fenchel inequality below holds:

$$f(x) + f^*(x^*) \ge \langle x, x^*\rangle \text{ for each pair } (x, x^*) \in X \times X^*. \tag{2.2}$$

If g, h are proper functions, then

$$\operatorname{epi} g^* + \operatorname{epi} h^* \subseteq \operatorname{epi}(g + h)^*, \tag{2.3}$$

and

$$g \le h \Rightarrow g^* \ge h^* \Leftrightarrow \operatorname{epi} g^* \subseteq \operatorname{epi} h^*. \tag{2.4}$$

Furthermore, the infimal convolution of g and h as the function $g \square h : X \to \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$(g \square h)(x) := \inf_{z \in X} \{g(z) + h(x - z)\}.$$

If g and h are proper and $\operatorname{dom} g \cap \operatorname{dom} h \ne \emptyset$, then by [28, Theorem 2.3.1(ix)], we have that

$$(g \square h)^* = g^* + h^*. \tag{2.5}$$

Moreover, by definition,

$$\operatorname{epi} g + \operatorname{epi} h \subseteq \operatorname{epi}(g \square h). \tag{2.6}$$

Note that an element $p \in X^*$ can be naturally regarded as a function on X in such a way that

$$p(x) := \langle p, x\rangle \quad \text{for each } x \in X. \tag{2.7}$$

Thus the following facts are clear for any $a \in \mathbb{R}$ and any function $h : X \to \overline{\mathbb{R}}$:

$$(h + p + a)^*(x^*) = h^*(x^* - p) - a \quad \text{for each} \quad x^* \in X^*; \tag{2.8}$$

$$\text{epi}(h + p + a)^* = \text{epi}h^* + (p, -a). \tag{2.9}$$

Let $\{f_t : t \in T\}$ be a family of proper lsc convex functions on X, where T is an arbitrary index set. The infimum and supremum function of the family $\{f_t : t \in T\}$ are denoted by $\inf_{t\in T} f_t$ and $\sup_{t\in T} f_t$ and are defined by

$$(\inf_{t\in T} f_t)(x) := \inf_{t\in T} f_t(x) \quad \text{and} \quad (\sup_{t\in T} f_t)(x) := \sup_{t\in T} f_t(x) \quad \text{for each } x \in X,$$

respectively. The following lemma will be useful in the sequel. In particular, statement (i) is well known in [16,23] and statements (ii) and (iii) were used in [28, Theorem 2.13(i)] and [23, (2.5)], respectively.

Lemma 2.1. *Let T be an index set and let $\{f_t : t \in T\}$ be a family of functions. Suppose that the supremum function $\sup_{t\in T} f_t$ is proper. Then the following statements hold.*

(i) $\text{epi}(\sup_{t\in T} f_t)^* = \text{cl}\Big(\text{co} \bigcup_{t\in T} \text{epi} f_t^*\Big)$.

(ii) $\text{epi}\,(\sup_{t\in T} f_t) = \bigcap_{t\in T} \text{epi}\, f_t$.

(iii) $(\inf_{t\in T} f_t)^* = \sup_{t\in T} f_t^*$; *consequently,* $\text{epi}\,(\inf_{t\in T} f_t)^* = \bigcap_{t\in T} \text{epi}\, f_t^*$.

The following lemma is known in [28].

Lemma 2.2. *Let $g, h : X \to \overline{\mathbb{R}}$ be proper convex functions satisfying $\text{dom}\, g \cap \text{dom}\, h \neq \emptyset$.*

(i) *If g, h are lsc, then*

$$\text{epi}\,(g + h)^* = \text{cl}\,(\text{epi}\, g^* + \text{epi}\ h^*). \tag{2.10}$$

(ii) *If either g or h is continuous at some point of $\text{dom}\, g \cap \text{dom}\, h$, then*

$$\text{epi}(g^* \Box h^*) = \text{epi}\,(g + h)^* = \text{epi}\, g^* + \text{epi}\ h^*. \tag{2.11}$$

3. Regular condition for robust conical programming

Throughout this paper, let X, Y, Z be real locally convex Hausdorff topological vector spaces, $C \subseteq X$ be a nonempty convex set and $\mathcal{U}$ be a convex subset of Z. Let $S \subseteq Y$ be a convex cone. Its dual cone $S^\oplus$ is defined by

$$S^\oplus := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \quad \text{for each} \quad y \in S\}.$$

Define an order on Y by saying that $y \leq_S x$ if $y - x \in -S$. We attach a greatest element ∞ with respect to $\leq_S$ and denote $Y^\bullet := Y \cup \{+\infty\}$. The following operations are defined on $Y^\bullet$: for any $y \in Y$, $y + \infty = \infty + y = \infty$ and $t\infty = \infty$ for any $t \geq 0$. Let $f : X \to \overline{\mathbb{R}}$ be a proper convex function, and for each $u \in \mathcal{U}$, $g_u : X \to Y^\bullet$

be S-convex in the sense that for every $x, y \in \text{dom}g_u := \{x \in X : g_u(x) \in Y\}$ and every $t \in [0,1]$,

$$g_u(tx + (1-t)y) \leq_S tg_u(x) + (1-t)g_u(y)$$

(see [16]). Also, we always assume that g_u is proper for each $u \in \mathcal{U}$. Following [12, 23], we define for each $\lambda \in S^{\oplus}$,

$$(\lambda g_u)(x) := \begin{cases} \langle \lambda, g_u(x) \rangle & \text{if } x \in \text{dom}\, g_u, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that g_u is S-convex if and only if $(\lambda g_u)(\cdot) : X \to \overline{\mathbb{R}}$ is a convex function for each $\lambda \in S^{\oplus}$. For each $u \in \mathcal{U}$, denote $g_u^{-1}(-S) := \{x \in \text{dom}g_u : g_u(x) \in -S\}$ and set

$$D := \bigcap_{u \in \mathcal{U}} g_u^{-1}(-S) = \{x \in X : g_u(x) \in -S \quad \text{for each} \quad u \in \mathcal{U}\}.$$

Let A denote the solution set of the system $\{x \in C : \ g_u(x) \in -S, u \in \mathcal{U}\}$, that is

$$A := C \cap D = \{x \in C : g_u(x) \in -S \quad \text{for each} \quad u \in \mathcal{U}\}.$$

To avoid trivially, we always assume that $A \cap \text{dom}f \neq \emptyset$. Let $\lambda \in S^{\oplus}$. Recall that the supremum function of the family $\{\lambda g_u : \ u \in \mathcal{U}\}$ is denoted by $\sup_{u \in \mathcal{U}} \lambda g_u$, that is

$$(\sup_{u \in \mathcal{U}} \lambda g_u)(x) := \sup_{u \in \mathcal{U}} (\lambda g_u(x)) \quad \text{for each } x \in X.$$

Motivated by [17, 21], we define the characteristic function $g^{\Diamond} : X^* \to \overline{\mathbb{R}}$ by

$$g^{\Diamond}(x^*) = \inf_{\lambda \in S^{\oplus}, u \in \mathcal{U}} (\lambda g_u)^*(x^*) \quad \text{for each} \quad x^* \in X^*.$$

Then, by definition, $(\lambda g_u)^* \geq g^{\Diamond}$ for each $\lambda \in S^{\oplus}$ and $u \in \mathcal{U}$. This together with (2.4) implies that

$$\bigcup_{\lambda \in S^{\oplus}, u \in \mathcal{U}} \text{epi}(\lambda g_u)^* \subseteq \text{epi}g^{\Diamond}. \tag{3.1}$$

Recall from [23, 25] that a function $h : X \to Y^{\bullet}$ is said to S-epi-closed if

$$\text{epi}_S(h) := \{(x, y) \in X \times Y : y \in g(x) + S\}$$

is closed. The following proposition, which will be useful in our study, gives some properties of the function $g^{\Diamond}$.

Proposition 3.1. (i) *$g^{\Diamond}$ is a proper function on X^*.*

(ii) *$\text{epi}g^{\Diamond}$ is a cone.*

(iii) *If C is closed and g_u is S-epi-closed for each $u \in \mathcal{U}$, then*

$$\text{epi}\delta_D^* = \text{cl}(\text{co}(\text{epi}g^{\Diamond})) = \text{cl}\Big(\text{co}\Big(\bigcup_{\lambda \in S^{\oplus}, u \in \mathcal{U}} \text{epi}\,(\lambda g_u)^*\Big)\Big) \tag{3.2}$$

and

$$\text{epi}\delta_A^* = \text{cl}(\text{co}(\text{epi}\delta_C^* + \text{epi}g^{\Diamond})) = \text{cl}\Big(\text{co}\Big(\text{epi}\delta_C^* + \bigcup_{\lambda \in S^{\oplus}, u \in \mathcal{U}} \text{epi}\,(\lambda g_u)^*\Big)\Big). \tag{3.3}$$

Proof. (i) Since for each $u \in \mathcal{U}$ and $\lambda \in S^{\oplus}$, $\lambda g_u \leq \delta_D$, it follows form (2.4) that

$$(\lambda g_u)^* \geq \delta_D^* \quad \text{for each } \lambda \in S^{\oplus} \text{ and } u \in \mathcal{U}, \tag{3.4}$$

this implies that $g^{\Diamond} \geq \delta_D^* > -\infty$. Moreover,

$$g^{\Diamond}(0) = -\sup_{\lambda \in S^{\oplus}, u \in \mathcal{U}} \inf_{x \in X} (\lambda g_u)(x) \leq -\inf_{x \in X} (0 \cdot g_{u_0})(x) = 0, \tag{3.5}$$

where $u_0 \in \mathcal{U}$. This implies that $0 \in \text{dom} g^{\Diamond}$. Hence, $g^{\Diamond}$ is proper.

(ii) Note by (3.5) that $(0,0) \in \text{epi} g^{\Diamond}$. Let $(x^*, r) \in \text{epi} g^{\Diamond}$ and let $\alpha > 0$. Then

$$g^{\Diamond}(\alpha x^*) = \inf_{\lambda \in S^{\oplus}, u \in \mathcal{U}} (\lambda g_u)^*(\alpha x^*) = \alpha \inf_{\lambda \in S^{\oplus}, u \in \mathcal{U}} \left(\frac{\lambda}{\alpha} g_u\right)^*(x^*) = \alpha g^{\Diamond}(x^*).$$

Thus, $\alpha(x^*, r) \in \text{epi} g^{\Diamond}$. This implies that $\text{epi} g^{\Diamond}$ is a cone.

(iii) Suppose that C is closed and g_u is S-epi-closed for each $u \in \mathcal{U}$. Let $u \in \mathcal{U}$ and let $D_u := \{x \in C : g_u(x) \in -S\}$. Then D_u is closed and convex and $\text{epi}\delta_{D_u}^* = \text{cl}(\cup_{\lambda \in S^{\oplus}} \text{epi}(\lambda g_u)^*)$ (cf. [12, Proposition 6.4]). Note that $\delta_D = \sup_{u \in \mathcal{U}} \delta_{D_u}$. It follows from Lemma 2.1(i) that

$$\text{epi}\delta_D^* = \text{cl}\Big(\text{co}\Big(\bigcup_{u \in \mathcal{U}} \text{epi}\delta_{D_u}^*\Big)\Big) = \text{cl}\Big(\text{co}\Big(\bigcup_{\lambda \in S^{\oplus}, u \in \mathcal{U}} \text{epi}\,(\lambda g_u)^*\Big)\Big). \tag{3.6}$$

This together with (3.1) implies $\text{epi}\delta_D^* \subseteq \text{cl}(\text{co}(\text{epi} g^{\Diamond}))$; while, by (2.4) and (3.4), $\text{epi} g^{\Diamond} \subseteq \text{epi}\delta_D^*$ and hence $\text{cl}(\text{co}(\text{epi} g^{\Diamond})) \subseteq \text{epi}\delta_D^*$ as $\text{epi}\delta_D^*$ is weak*-closed and convex. Thus,

$$\text{epi}\delta_D^* = \text{cl}(\text{co}(\text{epi} g^{\Diamond})). \tag{3.7}$$

Combing this with (3.6), we see that (3.2) holds. Moreover, note that D and C are closed and so δ_D^* and δ_C^* are lsc. Then by Lemma 2.2(i), we have

$$\begin{aligned} \text{epi}\delta_A^* &= \text{cl}(\text{epi}\delta_C^* + \text{epi}\delta_D^*) = \text{cl}(\text{epi}\delta_C^* + \text{cl}(\text{co}(\text{epi} g^{\Diamond}))) = \text{cl}(\text{co}(\text{epi}\delta_C^* + \text{epi} g^{\Diamond})) \\ &= \text{cl}\Big(\text{co}\Big(\text{epi}\delta_C^* + \bigcup_{\lambda \in S^{\oplus}, u \in \mathcal{U}} \text{epi}\,(\lambda g_u)^*\Big)\Big). \end{aligned}$$

Hence, (3.3) holds and the proof is complete. □

To study the Lagrange dualities, we introduce the following new constraint qualification.

Definition 3.2. It is said that the family $\{f, \delta_C; g_u : u \in \mathcal{U}\}$ satisfies the constraint qualification (CQ) if

$$\text{epi}(f + \delta_A)^* = \text{epi}((f + \delta_C)^* \Box g^{\Diamond}). \tag{3.8}$$

The following proposition presents some equivalent conditions to ensure the (CQ) to hold.

Proposition 3.3. (i) *The following inclusion holds:*

$$\text{epi}((f + \delta_C)^* \Box g^{\Diamond}) \subseteq \text{epi}(f + \delta_A)^*. \tag{3.9}$$

Consequently, the family $\{f, \delta_C; g_u : u \in \mathcal{U}\}$ *satisfies the* (CQ) *if and only if*

$$\text{epi}(f + \delta_A)^* \subseteq \text{epi}((f + \delta_C)^* \Box g^{\Diamond}). \tag{3.10}$$

(ii) *Suppose that*

$$f \text{ is lsc, } C \text{ is closed and } g_u \text{ is } S\text{-epi-closed for each } u \in \mathcal{U}. \tag{3.11}$$

Then the family $\{f, \delta_C; g_u : u \in \mathcal{U}\}$ *satisfies the* (CQ) *if and only if* $\operatorname{epi}((f+\delta_C)^*\Box g^\Diamond)$ *is weak***-closed and convex.*

Proof. (i) Let

$$E := \{x \in X : \operatorname{cl}(\lambda g_u)(x) \le 0 \quad \text{for each } u \in \mathcal{U} \text{ and } \lambda \in S^\oplus\}.$$

Then, $D \subseteq E$ and

$$(g^\Diamond)^* = (\inf_{\lambda\in S^\oplus, u\in\mathcal{U}} (\lambda g_u)^*)^* = \sup_{\lambda\in S^\oplus, u\in\mathcal{U}} (\operatorname{cl}(\lambda g_u)) = \delta_E,$$

where the second equality holds by Lemma 2.1(iii). Thus, by (2.5), we have

$$((f+\delta_C)^*\Box(g^\Diamond)^{**})^{**} = (\operatorname{cl}(f+\delta_C) + (g^\Diamond)^*)^* = (\operatorname{cl}(f+\delta_C)+\delta_E)^*. \tag{3.12}$$

While, by definitions,

$$\operatorname{cl}(f+\delta_C) + \delta_E \le f + \delta_C + \delta_D \le f + \delta_A, \tag{3.13}$$

and

$$((f+\delta_C)^*\Box(g^\Diamond)^{**})^{**} = \operatorname{cl}((f+\delta_C)^*\Box(g^\Diamond)^{**}) \le (f+\delta_C)^*\Box g^\Diamond. \tag{3.14}$$

Hence, by (2.4) and (3.12)-(3.14), we have that

$$\operatorname{epi}((f+\delta_C)^*\Box g^\Diamond) \subseteq \operatorname{epi}(\operatorname{cl}(f+\delta_C)+\delta_E)^* \subseteq \operatorname{epi}(f+\delta_A)^*.$$

(ii) To show the equivalence of (CQ) and the closedness and convexity of $\operatorname{epi}((f+\delta_C)^*\Box g^\Diamond)$, we only need to show that

$$\operatorname{epi}(f+\delta_A)^* = \operatorname{cl}(\operatorname{co}(\operatorname{epi}((f+\delta_C)^*\Box g^\Diamond))). \tag{3.15}$$

To do this, by (3.9) and the convexity and closedness of $\operatorname{epi}(f+\delta_A)^*$, one has that

$$\operatorname{cl}(\operatorname{co}(\operatorname{epi}((f+\delta_C)^*\Box g^\Diamond))) \subseteq \operatorname{epi}(f+\delta_A)^*. \tag{3.16}$$

Conversely, since f and δ_A are lsc by (3.11), it follows from Lemma 2.2(i) that

$$\operatorname{epi}(f+\delta_A)^* = \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi}\delta_A^*) = \operatorname{cl}(\operatorname{co}(\operatorname{epi} f^* + \operatorname{epi}\delta_C^* + \operatorname{epi} g^\Diamond)), \tag{3.17}$$

where the last equality holds by Proposition 3.1(iii). Moreover, by (2.3),

$$\operatorname{epi} f^* + \operatorname{epi}\delta_C^* \subseteq \operatorname{epi}(f+\delta_C)^*.$$

This together with (3.17) and (2.6) implies that

$$\operatorname{epi}(f+\delta_A)^* \subseteq \operatorname{cl}(\operatorname{co}(\operatorname{epi}((f+\delta_C)^*\Box g^\Diamond))).$$

Combining this with (3.16), we see that (3.15) holds, which completes the proof. □

To study the strong Lagrange duality of problems (RP) and (OLD), the authors in [24] introduced the following condition:

$$\operatorname{epi} f^* + \operatorname{epi}\delta_C^* + \bigcup_{\lambda\in S^\oplus, u\in\mathcal{U}} \operatorname{epi}(\lambda g_u)^* \quad \text{is weak}^*\text{-closed and convex.} \tag{3.18}$$

Under the assumptions that $C = X$, f is lsc and g_u is continuous for each $u \in \mathcal{U}$, they proved that the strong Lagrange duality holds between (RP) and (OLD). Note that under the above assumptions, (3.18) is equivalent to the following condition:

$$\bigcup_{\lambda \in S^{\oplus}, u \in \mathcal{U}} \mathrm{epi}(f + \delta_C + \lambda g_u)^* \quad \text{is weak}^*\text{-closed and convex.} \tag{3.19}$$

The following proposition shows that the condition (3.18) is stronger than the (CQ).

Proposition 3.4. *Suppose that* (3.11) *holds. Then*

$$(3.18) \Longrightarrow (CQ). \tag{3.20}$$

Proof. Suppose that (3.18) holds. By Lemma 2.2(i) and proposition 3.1(iii), we see that

$$\mathrm{epi}(f + \delta_A)^* = \mathrm{cl}(\mathrm{epi} f^* + \mathrm{epi}\delta_A^*) = \mathrm{cl}(\mathrm{co}(\mathrm{epi}\, f^* + \mathrm{epi}\delta_C^* + \bigcup_{\lambda \in S^{\oplus}, u \in \mathcal{U}} \mathrm{epi}\,(\lambda g_u)^*)).$$

This together with (3.18) implies that

$$\mathrm{epi}(f + \delta_A)^* = \mathrm{epi}\, f^* + \mathrm{epi}\delta_C^* + \bigcup_{\lambda \in S^{\oplus}, u \in \mathcal{U}} \mathrm{epi}\,(\lambda g_u)^*; \tag{3.21}$$

while, by (2.3), (3.1) and (2.6),

$$\mathrm{epi} f^* + \mathrm{epi}\delta_C^* + \bigcup_{\lambda \in S^{\oplus}, u \in \mathcal{U}} \mathrm{epi}(\lambda g_u)^* \subseteq \mathrm{epi}(f + \delta_C)^* + \mathrm{epi} g^{\Diamond} \subseteq \mathrm{epi}((f + \delta_C)^* \Box g^{\Diamond}).$$

Hence, (3.10) holds. Therefore, by Proposition 3.3(i), the (CQ) holds and the proof is complete. □

The following example shows that the converse of Proposition 3.4 does not necessarily hold in general.

Example 3.5. Let $X = Y = C := \mathbb{R}$, $S := [0, +\infty)$ and $\mathcal{U} := [-1, 0]$. Let $f, g_u : \mathbb{R} \to \overline{\mathbb{R}}$ be defined by $f := \delta_{[0,+\infty)}$ and $g_u(x) := x^2 + u$ for each $x \in \mathbb{R}$ and $u \in \mathcal{U}$. Then $\mathrm{epi} f^* = (-\infty, 0] \times [0, +\infty)$ and $A := \{x \in C : g_u(x) \in -S \text{ for each } u \in [-1, 0]\} = \{0\}$. Thus, $\mathrm{epi}(f + \delta_A)^* = \mathbb{R} \times [0, +\infty)$. Moreover, since for each $\lambda \geq 0$ and $u \in [-1, 0]$,

$$(\lambda g_u)^*(x^*) = \begin{cases} \frac{(x^*)^2}{4\lambda} - \lambda u, & \lambda > 0, \\ \delta_{\{0\}}(x^*), & \lambda = 0, \end{cases}$$

it follows that $g^{\Diamond} = 0$. Hence,

$$\mathrm{epi}((f + \delta_C)^* \Box g^{\Diamond}) = \mathbb{R} \times [0, +\infty) = \mathrm{epi}(f + \delta_A)^*,$$

that is, the (CQ) holds. However, note that

$$\bigcup_{\lambda \geq 0, u \in [-1,0]} \mathrm{epi}(\lambda g_u)^* = \mathbb{R} \times (0, +\infty) \cup \{(0, 0)\}.$$

Then

$$\mathrm{epi} f^* + \mathrm{epi}\delta_C^* + \bigcup_{\lambda \geq 0, u \in [-1,0]} \mathrm{epi}(\lambda g_u)^* = \mathbb{R} \times (0, +\infty) \cup \{(0, 0)\}$$

and so (3.18) does not hold.

4. Stable Lagrange duality for robust conical programming

Let $p \in X^*$. Consider the primal problem

$$(4.1)\qquad (RP)_p \qquad \begin{array}{ll} \inf & f(x) - \langle p, x\rangle, \\ \text{s.t.} & x \in C,\ g_u(x) \in -S \quad \text{for each} \quad u \in \mathcal{U}, \end{array}$$

and its Lagrange dual problems defined respectively by

$$(4.2)\qquad (OLD)_p \qquad \sup_{\lambda \in S^{\oplus}} \sup_{u \in \mathcal{U}} \inf_{x \in C} \{f(x) - \langle p, x\rangle + (\lambda g_u)(x)\}$$

and

$$(4.3)\qquad (RLD)_p \qquad \sup_{\lambda \in S^{\oplus}} \inf_{x \in C} \sup_{u \in \mathcal{U}} \{f(x) - \langle p, x\rangle + (\lambda g_u)(x)\}.$$

In the case when $p = 0$, problem $(RP)_p$ and its dual problem $(OLD)_p$ (resp. $(RLD)_p$) are reduced to problem (RP) and its dual problem (OLD) (resp. (RLD)) defined in (1.4) and (1.5) (resp. (1.6)), respectively. Let $v((RP)_p)$, $v((OLD)_p)$ and $v((RLD)_p)$ denote the optimal values of $(RP)_p$, $(OLD)_p$ and $(RLD)_p$, respectively. Then,

$$(4.4)\qquad v((OLD)_p) \le v((RLD)_p) \le v((RP)_p) \quad \text{for each} \quad p \in X^*.$$

This implies that the stable weak Lagrange duality holds between (RP) and (OLD) and, between (RP) and (RLD). This section is devoted to the study of the stable Lagrange duality between (RP) and (OLD) and, between (RP) and (RLD), which are defined as follows.

Definition 4.1. We say that

(a) the Lagrange duality holds between (RP) and (OLD) (resp. (RLD)) if $v(RP) = v(OLD)$ (resp. $v(RP) = v(RLD)$);

(b) the stable Lagrange duality holds between (RP) and (OLD) (resp. (RLD)) if for each $p \in X^*$, the Lagrange duality holds between $(RP)_p$ and $(OLD)_p$ (resp. $(RLD)_p$) for each $p \in X^*$.

Note by the definition of conjugate function that

$$\inf_{x \in A} \{f(x) - \langle p, x\rangle\} = -(f + \delta_A)^*(p) \quad \text{for each} \quad p \in X^*.$$

Then for each $r \in \mathbb{R}$ and $p \in X^*$, the following equivalence holds:

$$(4.5)\qquad (p, r) \in \operatorname{epi}(f + \delta_A)^* \Longleftrightarrow v((RP)_p) \ge -r.$$

The following theorem characterizes completely the stable Lagrange dualities in terms of the condition (CQ).

Theorem 4.2. *Consider the following statements.*

(i) *The family* $\{f, \delta_C; g_u : u \in \mathcal{U}\}$ *satisfies the* (CQ).

(ii) *The stable Lagrange duality holds between* (RP) *and* (OLD).

(iii) *The stable Lagrange duality holds between* (RP) *and* (RLD).

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii). *If for each* $\lambda \in S^{\oplus}$ *and* $u \in \mathcal{U}$,

$$(4.6)\qquad \operatorname{epi}(f + \delta_C + \lambda g_u)^* \subseteq \operatorname{epi}(f + \delta_C)^* + \operatorname{epi}(\lambda g_u)^*,$$

then (i)⇔(ii). *If for each* $\lambda \in S^{\oplus}$,

$$\text{cl}\Big(\text{co}\Big(\bigcup_{u\in\mathcal{U}} \text{epi}(\lambda g_u)^*\Big)\Big) \subseteq \text{epi} g^{\Diamond} \tag{4.7}$$

and

$$\text{epi}(f + \delta_C + \sup_{u\in\mathcal{U}} \lambda g_u)^* \subseteq \text{epi}(f + \delta_C)^* + \text{epi}(\sup_{u\in\mathcal{U}} \lambda g_u)^*, \tag{4.8}$$

then (i)⇔(iii).

Proof. (i)⇒(ii) Suppose that (i) holds. Let $p \in X^*$. If $v((RP)_p) = -\infty$, then $v((RP)_p) = v((OLD)_p)$ holds trivially by (4.4). Below we assume that $-r := v((RP)_p) \in \mathbb{R}$. Then, by (4.5),

$$(p, r) \in \text{epi}(f + \delta_A)^* = \text{epi}((f + \delta_C)^* \Box g^{\Diamond}), \tag{4.9}$$

where the last equality holds by the assumed (CQ). Thus,

$$((f + \delta_C)^* \Box g^{\Diamond})(p) \leq r. \tag{4.10}$$

While, by definition,

$$\begin{array}{rl} & ((f + \delta_C)^* \Box g^{\Diamond})(p) \\ = & \inf_{x^*\in X^*}\{(f + \delta_C)^*(p + x^*) + g^{\Diamond}(-x^*)\} \\ = & \inf_{x^*\in X^*} \inf_{\lambda\in S^{\oplus},u\in\mathcal{U}}\{(f + \delta_C)^*(p + x^*) + (\lambda g_u)^*(-x^*)\} \\ = & -\sup_{x^*\in X^*} \sup_{\lambda\in S^{\oplus},u\in\mathcal{U}}\{-(f + \delta_C)^*(p + x^*) - (\lambda g_u)^*(-x^*)\}. \end{array} \tag{4.11}$$

Moreover, by the Young-Fenchel inequality (2.2), we see that for each $\lambda \in S^{\oplus}, u \in \mathcal{U}$ and $x^* \in X^*$,

$$-(f + \delta_C)^*(p + x^*) - (\lambda g_u)^*(-x^*) \leq f(x) - \langle p, x\rangle + (\lambda g_u)(x) \quad \text{for each} \quad x \in C. \tag{4.12}$$

This implies that

$$v((OLD)_p) \geq \sup_{x^*\in X^*} \sup_{\lambda\in S^{\oplus},u\in\mathcal{U}} \{-(f + \delta_C)^*(p + x^*) - (\lambda g_u)^*(-x^*)\}.$$

Combing this with (4.10) and (4.11), we obtain that $v((OLD)_p) \geq -r = v((RP)_p)$. This together with (4.4) implies that $v((OLD)_p) = v((RP)_p)$. Hence, by the arbitraryness of $p^* \in X^*$, we see that (ii) holds.

(ii)⇒(iii) Suppose that (ii) holds. Then for each $p \in X^*$, $v((RP)_p) = v((OLD)_p)$ and hence $v((RP)_p) = v((RLD)_p)$, thanks to (4.4). Thus, (iii) holds.

Assume that for each $\lambda \in S^{\oplus}$ and $u \in \mathcal{U}$, (4.6) holds. Below we show that (ii)⇒(i). Suppose that (ii) holds. To show (i), by Proposition 3.3(i), it suffices to show that (3.10) holds. To do this, let $(p, r) \in \text{epi}(f + \delta_A)^*$. Then, by (4.5), $v((RP)_p) \geq -r$ and hence $v((OLD)_p) \geq -r$ by (ii). Let $\epsilon > 0$, then there exist $\lambda_\epsilon \in S^{\oplus}$ and $u_\epsilon \in \mathcal{U}$ such that for each $x \in X$,

$$f(x) - \langle p, x\rangle + \delta_C(x) + \lambda_\epsilon g_{u_\epsilon}(x) \geq -r - \epsilon.$$

This implies that $(f + \delta_C + \lambda_\epsilon g_{u_\epsilon})^*(p) \leq r + \epsilon$. Thus,

$$(p, r + \epsilon) \in \text{epi}(f + \delta_C + \lambda_\epsilon g_{u_\epsilon})^* \subseteq \bigcup_{\lambda\in S^{\oplus},u\in\mathcal{U}} (f + \delta_C + \lambda g_u)^*$$

and hence, by (4.6),

$$(p, r+\epsilon) \in \mathrm{epi}(f+\delta_C)^* + \bigcup_{\lambda \in S^{\oplus}, u \in \mathcal{U}} \mathrm{epi}(\lambda g_u)^*.$$

This together with (3.1) and (2.6) implies that

$$(p, r+\epsilon) \in \mathrm{epi}(f+\delta_C)^* + \mathrm{epi} g^{\Diamond} \subseteq \mathrm{epi}((f+\delta_C)^* \Box g^{\Diamond}). \tag{4.13}$$

Thus,

$$((f+\delta_C)^* \Box g^{\Diamond})(p) \le r+\epsilon. \tag{4.14}$$

Letting $\epsilon \to 0$ in (4.14), we get $((f+\delta_C)^* \Box g^{\Diamond})(p) \le r$ and so $(p, r) \in \mathrm{epi}((f+\delta_C)^* \Box g^{\Diamond})$. This implies that (3.10) holds and hence the implication (ii)⇒(i) is proved.

Finally, assume that for each $\lambda \in S^{\oplus}$, (4.7) and (4.8) hold. Suppose that (iii) holds. Let $(p, r) \in \mathrm{epi}(f+\delta_A)^*$. Then, by (4.5), $v((RP)_p) \ge -r$ and hence $v((RLD)_p) \ge -r$ by (iii). Let $\epsilon > 0$, then there exists $\lambda_\epsilon \in S^{\oplus}$ such that

$$f(x) - \langle p, x\rangle + \delta_C(x) + \sup_{u \in \mathcal{U}} \lambda_\epsilon g_u(x) \ge -r-\epsilon \quad \text{for each } x \in X.$$

This implies that $(f + \delta_C + \sup_{u\in\mathcal{U}} \lambda_\epsilon g_u)^*(p) \le r+\epsilon$, that is,

$$(p, r+\epsilon) \in \mathrm{epi}(f+\delta_C+\sup_{u \in \mathcal{U}} \lambda_\epsilon g_u)^*.$$

Thus,

$$(p, r+\epsilon) \in \bigcup_{\lambda \in S^{\oplus}} \mathrm{epi}(f+\delta_C+\sup_{u\in\mathcal{U}} \lambda g_u)^* \subseteq \mathrm{epi}(f+\delta_C)^* + \bigcup_{\lambda \in S^{\oplus}} \big(\mathrm{epi}(\sup_{u\in\mathcal{U}} \lambda g_u)^*\big),$$

where the last inclusion holds by (4.8); while, by Lemma 2.1(i) and (4.7),

$$\bigcup_{\lambda \in S^{\oplus}} \big(\mathrm{epi}(\sup_{u\in\mathcal{U}} \lambda g_u)^*\big) = \bigcup_{\lambda \in S^{\oplus}} \mathrm{cl}\Big(\mathrm{co}\Big(\bigcup_{u\in\mathcal{U}} \mathrm{epi}(\lambda g_u)^*\Big)\Big) \subseteq \mathrm{epi} g^{\Diamond}.$$

Hence, (4.13) holds and so does (4.14). Letting $\epsilon \to 0$ in (4.14), we see $((f+\delta_C)^* \Box g^{\Diamond})(p) \le r$ and so $(p, r) \in \mathrm{epi}((f+\delta_C)^* \Box g^{\Diamond})$. This implies that (3.10) holds and hence the (CQ) is proved by Proposition 3.3(i). Thus, the implication (iii)⇒(i) holds and the proof is complete. □

Remark 4.3. Let cont$h$ denote the set of all points at which h is continuous, that is,

$$\mathrm{cont} h = \{x \in X : \ h \text{ is continuous at } x\}.$$

If $\mathrm{cont}(f+\delta_C) \cap A \ne \emptyset$, then for each $\lambda \in S^{\oplus}$ and each $u \in \mathcal{U}$, $\mathrm{cont}(f+\delta_C) \cap \mathrm{dom}(\lambda g_u) \ne \emptyset$. Thus, by Lemma 2.2(ii), we see that (4.6) and (4.8) hold.

By Theorem 4.2 and Proposition 3.3(ii), we get the following corollary straightforwardly, which was established in [18, Theorem 3.1] for the case when $C = X$, $\mathcal{U}$ is a singleton, f is lsc and g is continuous. Thus, our Theorem 4.2 extends and improves the corresponding result in [18, Theorem 3.1].

Corollary 4.4. *Suppose that* (3.11) *and* (4.6) *hold. Then the stable Lagrange duality holds between* (RP) *and* (OLD) *if and only if* $\mathrm{epi}((f+\delta_C)^* \Box g^{\Diamond})$ *is weak***-closed and convex.*

Theorem 4.5. *Suppose that for each $\lambda \in S^{\oplus}$ and $u \in \mathcal{U}$,*

$$\operatorname{epi}(\delta_C + \lambda g_u)^* \subseteq \operatorname{epi}(\delta_C^* \Box g^{\Diamond}). \tag{4.15}$$

Then the following assertions are equivalent.

(i) *The following condition holds:*

$$\operatorname{epi}\delta_A^* = \operatorname{epi}(\delta_C^* \Box g^{\Diamond}). \tag{4.16}$$

(ii) *If the proper lsc convex function φ is such that*

$$\operatorname{epi}(\varphi + \delta_A)^* = \operatorname{epi}\varphi^* + \operatorname{epi}\delta_A^*, \tag{4.17}$$

then

$$\inf_{x\in A} \varphi(x) = \sup_{\lambda\in S^{\oplus}} \sup_{u\in\mathcal{U}} \inf_{x\in C} \{\varphi(x) + (\lambda g_u)(x)\}. \tag{4.18}$$

(iii) *If the proper convex function φ is continuous at some point in A, then* (4.18) *holds.*

(iv) *If $p \in X^*$, then*

$$\inf_{x\in A} p(x) = \sup_{\lambda\in S^{\oplus}} \sup_{u\in\mathcal{U}} \inf_{x\in C} \{p(x) + (\lambda g_u)(x)\}. \tag{4.19}$$

Proof. (i)⇒(ii). Suppose that (i) holds and let φ be such that (4.17) is satisfied. Then, by (4.16),

$$\operatorname{epi}(\varphi + \delta_A)^* = \operatorname{epi}\varphi^* + \operatorname{epi}(\delta_C^* \Box g^{\Diamond}) \subseteq \operatorname{epi}(\varphi^* \Box \delta_C^* \Box g^{\Diamond}), \tag{4.20}$$

where the last inclusion holds by (2.6). Note by (2.1) and (2.5) that

$$\varphi^* \Box \delta_C^* \geq (\varphi^* \Box \delta_C^*)^{**} = (\varphi^{**} + \delta_C^{**})^* \geq (\varphi + \delta_C)^*.$$

Then

$$\varphi^* \Box \delta_C^* \Box g^{\Diamond} \geq (\varphi + \delta_C)^* \Box g^{\Diamond}$$

and so

$$\operatorname{epi}(\varphi^* \Box \delta_C^* \Box g^{\Diamond}) \subseteq \operatorname{epi}((\varphi + \delta_C)^* \Box g^{\Diamond})$$

by (2.4). This together with (4.20) implies that

$$\operatorname{epi}(\varphi + \delta_A)^* \subseteq \operatorname{epi}((\varphi + \delta_C)^* \Box g^{\Diamond})$$

and

$$\operatorname{epi}(\varphi + \delta_A)^* = \operatorname{epi}((\varphi + \delta_C)^* \Box g^{\Diamond})$$

by Proposition 3.3(i). Hence, applying the implication (i)⇒(ii) of Theorem 4.2 to φ in place of f, we see that (4.18) holds.

(ii)⇒(iii). Note that (4.17) is satisfied if φ is continuous at some point in A (see Lemma 2.2(ii)). Thus, it is immediate that (ii)⇒(iii).

(iii)⇒(iv). It is trivial.

(iv)⇒(i). To do this, suppose that (iv) holds. Then applying the implication (ii)⇒(i) of Theorem 4.2 to $f = 0$, we have that (4.16) holds. The proof is complete. □

The implication (i)⇒(ii) of the following theorem follows from Theorem 4.5 and (4.4) (φ in place of f) directly and the proofs of the other implications are similar to that of Theorem 4.5. So we omit the proof of Theorem 4.6 here.

Theorem 4.6. *Suppose that for each* $\lambda \in S^{\oplus}$, (4.7) *holds and*

$$\text{epi}(\delta_C + \sup_{u\in\mathcal{U}} \lambda g_u)^* \subseteq \text{epi}\delta_C^* + \text{epi}(\sup_{u\in\mathcal{U}} \lambda g_u)^*. \tag{4.21}$$

Then the following assertions are equivalent.

(i) *The following condition* (4.16) *holds.*

(ii) *If the proper lsc convex function* φ *is such that* (4.17) *holds, then*

$$\inf_{x\in A} \varphi(x) = \sup_{\lambda\in S^{\oplus}} \inf_{x\in C} \sup_{u\in\mathcal{U}} \{\varphi(x) + (\lambda g_u)(x)\}. \tag{4.22}$$

(iii) *If the proper convex function* φ *is continuous at some point in* A*, then* (4.22) *holds.*

(iv) *If* $p \in X^*$*, then*

$$\inf_{x\in A} p(x) = \sup_{\lambda\in S^{\oplus}} \inf_{x\in C} \sup_{u\in\mathcal{U}} \{p(x) + (\lambda g_u)(x)\}. \tag{4.23}$$

Remark 4.7. (a) Applying Proposition 3.3 (to 0 in place of f), we have that (4.16) holds if and only if

$$\text{epi}\delta_A^* \subseteq \text{epi}(\delta_C^* \Box g^{\Diamond}). \tag{4.24}$$

(b) In the case where $\mathcal{U}$ is a singleton, the authors in [17] introduced the following condition

$$\text{cl}(\text{epi}\delta_C^* + \text{epi}g^{\Diamond}) = \text{epi}(\delta_C^* \Box g^{\Diamond}) \tag{4.25}$$

to study the Lagrange duality between $(\mathcal{P})$ and $(\mathcal{D})$. Under the assumptions that C is closed, f and λg, $\lambda \in S^{\oplus}$ are lsc and $\text{cont}g \cap C \neq \emptyset$, they proved in [17, Theorem 4.1] that (4.25), assertions (iii) and (iv) in Theorem 4.5 are equivalent to each other. In this case, by Proposition 3.1(iii) and noting the fact that g is S-epi-closed if λg is lsc for each $\lambda \in S^{\oplus}$, we see that (4.25) is equivalent to (4.16). Moreover, the assumption $\text{cont}g \cap C \neq \emptyset$ implies that (4.15) holds. Thus, our Theorem 4.5 extends and improves the corresponding result in [17, Theorem 4.1].

Remark 4.8. As mentioned in Section 1, let $S = \mathbb{R}^m_+ := \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0\}$ and $\mathcal{U} = \prod_{i=1}^m \mathcal{V}_i$. For each $u = (u_1, \ldots, u_m) \in \mathcal{U}$, define the function $g_u : \mathbb{R}^n \to \mathbb{R}^m$ by

$$g_u(x) := (g_1(x, u_1), \ldots, g_m(x, u_m)) \quad \text{for each } x \in \mathbb{R}^n.$$

Then problem (1.1) introduced in Section 1 can be viewed as an example of (RP). Thus, all corresponding results for the Lagrange dualities between problem (1.1) and its dual problems (1.2) and (1.3) can be established. In particular, in the case when

$$X = C = \mathbb{R}^n \text{ and } g_i, i = 1, \ldots, m, \text{ are continuous,} \tag{4.26}$$

the authors in [21] proved that $\text{epi}g^{\Diamond}$ is convex and weak*-closed if and only if for each continuous function f on $\mathbb{R}^n$,

$$\begin{aligned} &\inf\{f(x) : g_i(x, v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\} \\ = &\sup_{\lambda_i \geq 0, v_i \in \mathcal{V}_i} \inf_{x\in\mathbb{R}^n} \Big\{ f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i) \Big\}. \end{aligned}$$

Note by Proposition 3.3(ii) that under the assumption (4.26),

$$\mathrm{epi}\delta_A^* = \mathrm{epi}g^\Diamond \Longleftrightarrow \mathrm{epi}g^\Diamond \text{ is convex and weak}^*\text{-closed.}$$

Then, our Theorem 4.5 improves the corresponding result in [21, Theorem 3.1].

We end this paper with an example. Consider the uncertainty sets $\mathcal{U}_i \subseteq L^2[0,1] \times \mathbb{R}, i = 1, \ldots, m$ and the best approximation problem:

$$(4.27) \qquad \inf_{x \in L^2[0,1]} \Big\{ \frac{1}{2} \int_0^1 x^2(t)dt : \int_0^1 \alpha_i(t)x(t)dt \le \beta_i, i = 1, \ldots, m \Big\},$$

where the date $(\alpha_i, \beta_i) \in \mathcal{U}_i$ is uncertain for each $i = 1, \ldots, m$. Problem (4.27) has been studied in [20] and also studied in [8, 9, 11] for the special case when the data (α_i, β_i), $i = 1, \ldots, m$ are fixed. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the inner product and the norm on $L^2[0,1]$, respectively. Using the robust optimization approach in [5, 7], problem (4.27) can be recasted into the robust optimization problem as follows

$$(4.28) \qquad (RP) \qquad \inf_{x \in L^2[0,1]} \Big\{ \frac{1}{2} \|x^2\| : \langle \alpha_i, x \rangle \le \beta_i, \forall (\alpha_i, \beta_i) \in \mathcal{U}_i, i = 1, \ldots, m \Big\}.$$

Corresponding, the dual problems of (RP) can be defined respectively by

$$(4.29) \quad (OLD) \qquad \sup_{(\alpha_i,\beta_i) \in \mathcal{U}_i, \lambda_i \ge 0, i=1,\ldots,m} \ \inf_{x \in L^2[0,1]} \Big\{ \frac{1}{2}\|x^2\| + \sum_{i=1}^m \lambda_i(\langle \alpha_i, x \rangle - \beta_i) \Big\},$$

and
(4.30)

$$(RLD) \qquad \sup_{\lambda_i \ge 0, i=1,\ldots,m} \ \inf_{x \in L^2[0,1]} \ \sup_{(\alpha_i,\beta_i) \in \mathcal{U}_i, i=1,\ldots,m} \Big\{ \frac{1}{2}\|x^2\| + \sum_{i=1}^m \lambda_i(\langle \alpha_i, x \rangle - \beta_i) \Big\}.$$

As before, we use A to denote the feasible solution set of problem (RP), that is,

$$A := \{ x \in L^2[0,1] : \langle \alpha_i, x \rangle \le \beta_i, \forall (\alpha_i, \beta_i) \in \mathcal{U}_i, i = 1, \ldots, m \}.$$

Below we give a sufficient condition to ensure the Lagrange duality between (RP) and (OLD) and, between (RP) and (RLD).

Theorem 4.9. *Suppose that the sets $\mathcal{U}_i, i = 1, \ldots, m$, are convex and that*

$$(4.31) \qquad \mathrm{epi}\delta_A^* = \bigcup_{(\alpha_i,\beta_i) \in \mathcal{U}_i, \lambda_i \ge 0, i=1,\ldots,m} \ \sum_{i=1}^m \big(\{\lambda_i \alpha_i\} \times [\lambda_i \beta_i, +\infty) \big).$$

Then the Lagrange duality holds between (RP) and (OLD) and, between (RP) and (RLD).

Proof. Let $u_i := (\alpha_i, \beta_i)$ for each $i = 1, \ldots, m$. Define $f : L^2[0,1] \to \mathbb{R}$ and $g_{u_i} : L^2[0,1] \to \mathbb{R}$ respectively by

$$f(x) := \frac{1}{2}\|x\|^2 \quad \text{and} \quad g_{u_i}(x) := \langle \alpha_i, x \rangle - \beta_i \quad \text{for each} \quad x \in L^2[0,1].$$

Obviously, f and each g_{u_i} are proper convex and continuous functions on $L^2[0,1]$. By definition,

$$g^{\diamondsuit}(x^*) = \inf_{\lambda_i\geq 0, u_i\in\mathcal{U}_i, i=1,\ldots,m} \Big(\sum_{i=1}^m \lambda_i g_{u_i}\Big)^*(x^*) \quad \text{for each} \quad x^*\in L^2[0,1].$$

Note by (3.1) and Proposition 3.1(iii) that

$$\bigcup_{\lambda_i\geq 0, u_i\in\mathcal{U}_i, i=1,\ldots,m} \operatorname{epi}\Big(\sum_{i=1}^m \lambda_i g_{u_i}\Big)^* \subseteq \operatorname{epi} g^{\diamondsuit} \subseteq \operatorname{epi}\delta_A^*.$$

Then, by Theorem 4.5 and Theorem 4.6, to show the Lagrange duality between (RP) and (OLD) and, between (RP) and (RLD), it suffices to show that

$$\operatorname{epi}\delta_A^* = \bigcup_{\lambda_i\geq 0, u_i\in\mathcal{U}_i, i=1,\ldots,m} \operatorname{epi}\Big(\sum_{i=1}^m \lambda_i g_{u_i}\Big)^*. \tag{4.32}$$

To do this, note that each g_{u_i} is continuous, it follows from Lemma 2.2(ii) that

$$\bigcup_{\lambda_i\geq 0, u_i\in\mathcal{U}_i, i=1,\ldots,m} \operatorname{epi}\Big(\sum_{i=1}^m \lambda_i g_{u_i}\Big)^* = \bigcup_{\lambda_i\geq 0, u_i\in\mathcal{U}_i, i=1,\ldots,m} \sum_{i=1}^m \operatorname{epi}(\lambda_i g_{u_i})^*. \tag{4.33}$$

While, for each $i=1,\ldots,m$,

$$\operatorname{epi} g_{u_i}^* = \{\alpha_i\}\times[\beta_i,+\infty).$$

Thus,

$$\bigcup_{\lambda_i\geq 0, u_i\in\mathcal{U}_i, i=1,\ldots,m} \sum_{i=1}^m \operatorname{epi}(\lambda_i g_{u_i})^* = \bigcup_{\lambda_i\geq 0, u_i\in\mathcal{U}_i, i=1,\ldots,m} \sum_{i=1}^m \big(\{\lambda_i\alpha_i\}\times[\lambda_i\beta_i,+\infty)\big).$$

Combining this with (4.31) and (4.33), one sees that (4.32) holds. Thus, the proof is complete. □

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D. H. Fang
College of Mathematics and Statistics, Jishou University, Jishou 416000, China
E-mail address: dh_fang@jsu.edu.cn

C. Li
Department of Mathematics, Zhejiang University, Hangzhou 310027, China
E-mail address: cli@zju.edu.cn

J. C. Yao
Corresponding author. Center for General Education, China Medical University, Taichung 40402, Taiwan; Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: yaojc@mail.cmu.edu.tw