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THE STRUCTURAL CHARACTERISTICS OF CHOQUET FUNCTIONALS

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

Abstract. In this paper we show the hereditary nature of the structural characteristics between the Choquet functionals and their representing measures. In fact we prove that, if a Choquet functional is weakly asymptotic null-additive (asymptotic null-additive, autocontinuous from above, uniformly autocontinuous from above, pseudometric generating, submodular, supermodular), then there is a representing measure having the same property, and vice versa. We also give a partial result for the hereditary nature of the (weak) null-additivity.

1. Introduction

In 1982, Greco [7] gave a most general type of the Choquet integral representation theorem as a successful nonadditive analogue of the famous Daniell-Stone integral representation theorem [15]. It gives a correspondence between a Choquet functional I on an appropriate family $\mathcal{F}$ of functions on a non-empty set X and a nonadditive measure μ on the power set 2^X through the Choquet integral

$$I\colon f \in \mathcal{F} \mapsto I(f) = (\mathrm{C})\int_X f\, d\mu.$$

The purpose of the paper is to show the hereditary nature of the structural characteristics between the Choquet functionals and their representing measures. More precisely, we prove that, if a Choquet functional I is weakly asymptotic null-additive (asymptotic null-additive, autocontinuous from above, uniformly autocontinuous from above, pseudometric generating, submodular, supermodular), then there is a representing measure μ having the same property, and vice versa.

As a matter of fact, some of the forward direction to the hereditary nature of those characteristics were already studied by Denneberg [4] for the submodularity and the supermodularity and by Pap [14] for the uniform autocontinuity from above; see also Bassanezi and Greco [1]. They proved those characteristics by a method based on subgraphs of functions, a method applied by Kindler [11] for his simple proof of the Daniell-Stone representation theorem. All other cases will be shown by the same method. So, our main contribution in this paper is the reverse direction to the hereditary nature.

The paper is organized as follows. In Section 2 we recall some structural characteristics of nonadditive measures and define the corresponding characteristics of

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functionals. In Section 3 we show the hereditary nature of some structural characteristics between the Choquet functionals and their representing measures. In Section 4, after indicating that the method based on subgraphs of functions does not work for the (weak) null-additivity case, we shall give a partial result for the hereditary nature of the (weak) null-additivity using an alternative approach.

2. Notation and preliminaries

Let X be a non-empty set and 2^X denote the family of all subsets of X. For each $A \subset X$, let 1_A denote the characteristic function of A. Let $[0,\infty]$ be the set of all non-negative extended real numbers with usual total order. For any $a, b \in [0,\infty]$, let $a \vee b := \max(a,b)$ and $a \wedge b := \min(a,b)$. As usual, we assume the standard convention $\infty \cdot 0 = 0 \cdot \infty = 0$ and $\inf \emptyset = \infty$. For any functions $f, g\colon X \to [0,\infty]$, $f \leq g$ denotes the partial order, called the *pointwise order*, defined by $f(x) \leq g(x)$ for every $x \in X$. Let $f \vee g$ and $f \wedge g$ be lattice operations defined by $(f \vee g)(x) := \max(f(x), g(x))$ and $(f \wedge g)(x) := \min(f(x), g(x))$ for every $x \in X$.

Let $\mathcal{L}$ be a lattice of subsets of X, that is, if $A, B \in \mathcal{L}$, then $A \cup B, A \cap B \in \mathcal{L}$. Assume that $\mathcal{L}$ contains the empty set $\emptyset$. A set function $\mu\colon \mathcal{L} \to [0,\infty]$ is called a *nonadditive measure* on $\mathcal{L}$ if $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathcal{L}$ and $A \subset B$. A function $f\colon X \to [0,\infty]$ is called *$\mathcal{L}$-measurable* if $\{f > t\}, \{f \geq t\} \in \mathcal{L}$ for every $t \in (0,\infty]$. Then the distribution functions $t \in (0,\infty) \to \mu(\{f > t\})$ and $t \in (0,\infty) \to \mu(\{f \geq t\})$ are decreasing, so that they are Lebesgue measurable. Thus the following formalization is well-defined; see Choquet [2].

Definition 2.1. Let $\mu\colon \mathcal{L} \to [0,\infty]$ be a nonadditive measure. Let $f\colon X \to [0,\infty]$ be an $\mathcal{L}$-measurable function. The *Choquet integral* of f with respect to μ is defined by

$$(\mathrm{C})\int_X f d\mu := \int_0^\infty \mu(\{f > t\}) dt,$$

where the integral on the right-hand side is the usual Lebesgue integral.

Remark 2.2. The function $\mu(\{f > t\})$ in the above definition may be replaced with the function $\mu(\{f \geq t\})$, since $\mu(\{f \geq t\}) \geq \mu(\{f > t\}) \geq \mu(\{f \geq t + \varepsilon\})$ for every $\varepsilon > 0$ and $t \in (0,\infty)$. This fact will be used in this paper without mentioning it explicitly.

In this paper we treat the following structural characteristics of nonadditive measures and monotone functionals.

Definition 2.3. Let $\mu\colon \mathcal{L} \to [0,\infty]$ be a nonadditive measure.

(1) μ is called *weakly asymptotic null-additive* if $\mu(A_n \cup B_n) \downarrow 0$ whenever $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$ are decreasing sequences with $\mu(A_n) \downarrow 0$ and $\mu(B_n) \downarrow 0$ [8].
(2) μ is called *asymptotic null-additive* if $\mu(A \cup B_n) \downarrow \mu(A)$ whenever $A \in \mathcal{L}$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$ is a decreasing sequence with $\mu(B_n) \downarrow 0$ [8].
(3) μ is called *autocontinuous from above* if, for every $\varepsilon > 0$ and $A \in \mathcal{L}$, there is $\delta > 0$ such that $\mu(A \cup B) \leq \mu(A) + \varepsilon$ whenever $B \in \mathcal{L}$ and $\mu(B) < \delta$ [16].

(4) μ is called *uniformly autocontinuous from above* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $\mu(A \cup B) \le \mu(A) + \varepsilon$ whenever $A, B \in \mathcal{L}$ and $\mu(B) < \delta$ [16].
(5) μ is called *pseudometric generating* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $\mu(A \cup B) < \varepsilon$ whenever $A, B \in \mathcal{L}$ and $\mu(A) \vee \mu(B) < \delta$ [5].
(6) μ is called *weakly null-additive* if $\mu(A \cup B) = 0$ whenever $A, B \in \mathcal{L}$ and $\mu(A) = \mu(B) = 0$ [17].
(7) μ is called *null-additive* if $\mu(A \cup B) = \mu(A)$ whenever $A, B \in \mathcal{L}$ and $\mu(B) = 0$ [16].
(8) μ is called *submodular* if $\mu(A \cup B) + \mu(A \cap B) \le \mu(A) + \mu(B)$ for every $A, B \in \mathcal{L}$.
(9) μ is called *supermodular* if $\mu(A \cup B) + \mu(A \cap B) \ge \mu(A) + \mu(B)$ for every $A, B \in \mathcal{L}$.

Let $\mathcal{F}$ be a lattice of functions $f\colon X \to [0, \infty]$ with pointwise order, that is, if $f, g \in \mathcal{F}$, then $f \vee g, f \wedge g \in \mathcal{F}$. Assume that $0 \in \mathcal{F}$. Let $\mathcal{F}_1 := \{f \in \mathcal{F}\colon 0 \le f \le 1\}$. A functional $I\colon \mathcal{F} \to [0, \infty]$ is called *monotone* if $I(f) \le I(g)$ whenever $f, g \in \mathcal{F}$ and $f \le g$.

Definition 2.4. Let $I\colon \mathcal{F} \to [0, \infty]$ be a monotone functional.

(1) I is called *weakly asymptotic null-additive* if $I(f_n \vee g_n) \downarrow 0$ whenever $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{F}_1$ and $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_1$ are decreasing sequences with $I(f_n) \downarrow 0$ and $I(g_n) \downarrow 0$.
(2) I is called *asymptotic null-additive* if $I(f \vee g_n) \downarrow I(f)$ whenever $f \in \mathcal{F}_1$ and $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_1$ is a decreasing sequence with $I(g_n) \downarrow 0$.
(3) I is called *autocontinuous from above* if, for every $\varepsilon > 0$ and every $f \in \mathcal{F}_1$, there is $\delta > 0$ such that $I(f \vee g) \le I(f) + \varepsilon$ whenever $g \in \mathcal{F}_1$ and $I(g) < \delta$.
(4) I is called *uniformly autocontinuous from above* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $I(f \vee g) \le I(f) + \varepsilon$ whenever $f, g \in \mathcal{F}_1$ and $I(g) < \delta$ [14].
(5) I is called *pseudometric generating* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $I(f \vee g) < \varepsilon$ whenever $f, g \in \mathcal{F}_1$ and $I(f) \vee I(g) < \delta$.
(6) I is called *weakly null-additive* if $I(f \vee g) = 0$ whenever $f, g \in \mathcal{F}_1$ and $I(f) = I(g) = 0$.
(7) I is called *null-additive* if $I(f \vee g) = I(f)$ whenever $f, g \in \mathcal{F}_1$ and $I(g) = 0$ [14].
(8) I is called *submodular* if $I(f \vee g) + I(f \wedge g) \le I(f) + I(g)$ for every $f, g \in \mathcal{F}_1$ [4].
(9) I is called *supermodular* if $I(f \vee g) + I(f \wedge g) \ge I(f) + I(g)$ for every $f, g \in \mathcal{F}_1$ [4].

The following assertions hold for both a nonadditive measure $\mu\colon \mathcal{L} \to [0, \infty]$ and a monotone functional $I\colon \mathcal{F} \to [0, \infty]$:

- submodular $\Rightarrow$ uniformly autocontinuous from above $\Rightarrow$ autocontinuous from above $\Rightarrow$ asymptotic null-additive $\Rightarrow$ null-additive.
- uniformly autocontinuous from above $\Rightarrow$ pseudometric generating $\Rightarrow$ weakly asymptotic null-additive $\Rightarrow$ weakly null-additive.
- asymptotic null-additive $\Rightarrow$ weakly asymptotic null-additive.

Furthermore the following properties hold:

- If $\mathcal{L}$ is closed for countable intersections and μ is continuous from above, that is, $\mu(A_n) \downarrow \mu(A)$ whenever $A \in \mathcal{L}$ and $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$ is a decreasing sequence with $A_n \downarrow A$, then μ is (weakly) asymptotic null-additive $\Leftrightarrow$ μ is (weakly) null-additive.

- If $\mathcal{F}$ is closed for countable infimum and if I is continuous from above, that is, $I(f_n) \downarrow I(f)$ whenever $f \in \mathcal{F}$ and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ is a decreasing sequence with $f_n \downarrow f$, then I is (weakly) asymptotic null-additive $\Leftrightarrow$ I is (weakly) null-additive.
- μ is autocontinuous from above if and only if $\mu(A \cup B_n) \to \mu(A)$ whenever $A \in \mathcal{L}$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$ is a sequence with $\mu(B_n) \to 0$.
- I is autocontinuous from above if and only if $I(f \vee g_n) \to I(f)$ whenever $f \in \mathcal{F}_1$ and $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_1$ is a sequence with $I(g_n) \to 0$.
- μ is pseudometric generating if and only if $\mu(A_n \cup B_n) \to 0$ whenever $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$ are sequences with $\mu(A_n) \to 0$ and $\mu(B_n) \to 0$.
- I is pseudometric generating if and only if $I(f_n \vee g_n) \to 0$ whenever $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_1$ and $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_1$ are sequences with $I(f_n) \to 0$ and $I(g_n) \to 0$.

Every nonadditive measure $\mu\colon \mathcal{L} \to [0,\infty]$ has its nonadditive extensions defined on 2^X: The largest and the smallest ones are given by

$$\mu^*(A) := \inf\{\mu(L)\colon A \subset L, L \in \mathcal{L}\}$$
$$\mu_*(A) := \sup\{\mu(L)\colon L \subset A, L \in \mathcal{L}\}.$$

for every $A \in 2^X$. The measures μ^* and μ_* are called the *outer extension* and the *inner extension* of μ, respectively.

3. Structural characteristics of Choquet functionals

Let X be a non-empty set. Recall that two functions $f, g\colon X \to [0,\infty]$ are *comonotonic* and they are written by $f \sim g$ if, for every $x, x' \in X$, $f(x) < f(x')$ implies $g(x) \leq g(x')$; see Dellacherie [3].

Throughout this paper, we assume that $\mathcal{F}$ is a non-empty family of functions $f\colon X \to [0,\infty]$ with pointwise order and satisfies

(F1) if $f \in \mathcal{F}$ and $c \in [0,\infty)$, then $cf, f \wedge c, f - f \wedge c \in \mathcal{F}$ (Stone condition) and thus $0 \in \mathcal{F}$, and
(F2) if $f, g \in \mathcal{F}$, then $f \vee g, f \wedge g \in \mathcal{F}$ (lattice condition).

For instance, the positive cones of the space $B(X)$ of all bounded real functions on X and the space $C(X)$ of all continuous real functions on a Hausdorff space X satisfy these conditions.

We also assume that $I\colon \mathcal{F} \to [0,\infty]$ is a *Choquet functional*, that is, it satisfies

(I1) $I(0) = 0$,
(I2) if $f, g \in \mathcal{F}$ and $f \leq g$, then $I(f) \leq I(g)$ (monotonicity),
(I3) if $f, g \in \mathcal{F}, f + g \in \mathcal{F}$ and $f \sim g$, then $I(f + g) = I(f) + I(g)$ (comonotonic additivity),
(I4) $\sup_{a>0} I(f - f \wedge a) = I(f)$ for every $f \in \mathcal{F}$ (lower marginal continuity), and
(I5) $\sup_{b>0} I(f \wedge b) = I(f)$ for every $f \in \mathcal{F}$ (upper marginal continuity).

Every functional $I\colon \mathcal{F} \to [0,\infty]$ given by the Choquet integral

$$I(f) := \text{(C)}\int_X f d\mu, \quad f \in \mathcal{F},$$

with respect to a nonadditive measure μ on 2^X (or a lattice $\mathcal{L}$ such that every $f \in \mathcal{F}$ is $\mathcal{L}$-measurable), is a Choquet functional.

For each $A \subset X$, define the set functions $\alpha, \beta\colon 2^X \to [0,\infty]$ by

$$\alpha(A) := \sup\{I(f)\colon f \in \mathcal{F}, f \le \chi_A\},$$
$$\beta(A) := \inf\{I(f)\colon f \in \mathcal{F}, \chi_A \le f\}.$$

Then α and β are nonadditive measures on 2^X with $\alpha \le \beta$. By the Greco representation theorem [7], for any nonadditive measure $\mu\colon 2^X \to [0,\infty]$, the following two conditions are equivalent:

(a) $\alpha \le \mu \le \beta$.

(b) $I(f) = \text{(C)}\int_X f d\mu$ for every $f \in \mathcal{F}$.

In this case μ is called a *representing measure* of the functional I.

Remark 3.1. (1) Every functional $I\colon \mathcal{F} \to [0,\infty]$ satisfying (I1)–(I3) is positively homogeneous, that is, $I(cf) = cI(f)$ for every $f \in \mathcal{F}$ and $c \in [0,\infty)$; see, for instance, [4, p. 159] and [13, Proposition 4.2].

(2) (I4) is satisfied if, for every $f \in \mathcal{F}$, there is $g \in \mathcal{F}$ such that $1_{\{f>0\}} \le g$ and $I(g) < \infty$ (in particular, $1 \in \mathcal{F}$ and $I(1) < \infty$). (I5) is also satisfied if every $f \in \mathcal{F}$ is bounded; see [9, Lemma 1] and [10].

The method based on subgraphs of functions: The above measures α and β may be constructed by the following steps:

Firstly, for any function $f\colon X \to [0,\infty]$, let G_f denote the subgraph of f, that is, $G_f := \{(x,t) \in X \times [0,\infty)\colon f(x) > t\}$. Then the family $\mathcal{G} := \{G_f\colon f \in \mathcal{F}_1\}$ is a lattice of subsets of $X \times [0,\infty)$ containing $\emptyset$.

Secondly, let $\omega(G_f) := I(f)$ for every $f \in \mathcal{F}_1$. This $\omega\colon \mathcal{G} \to [0,\infty]$ is a well-defined nonadditive measure and has the outer and the inner extensions ω^* and ω_* on 2^X.

Finally, let $\varphi(A) := \omega^*(G_{1_A})$ and $\psi(A) := \omega_*(G_{1_A})$ for every $A \subset X$. Then $\varphi = \beta$ and $\psi = \alpha$.

This construction is owing to Kindler [11] and was applied for his efficient approach to the Daniell-Stone representation theorem.

When the domain $\mathcal{F}$ of a functional I is rather large and thus it contains all characteristic functions of subsets of X, the nonadditive measure $\mu(A) := I(1_A)$, $A \subset X$, is a representing measure of I and satisfies each of the structural characteristics in Definition 2.3 if and only if I has the same property in Definition 2.4. In practical applications we often have only restricted information, that is, the domain $\mathcal{F}$ is as small as it does not necessarily contain all characteristic functions. The following theorem covers those cases.

Theorem 3.2. *If I is weakly asymptotic null-additive (asymptotic null-additive, autocontinuous from above, uniformly autocontinuous from above, pseudometric generating, submodular, supermodular), then there is a representing measure μ on 2^X of I having the same property.*

Conversely, if μ is any representing measure on 2^X of I such that

$$Q := \sup_{h\in\mathcal{F}_1} \mu(\{h > 0\}) < \infty,$$

in particular, $\mu(X) < \infty$, and it is weakly asymptotic null-additive (asymptotic null-additive, autocontinuous from above, uniformly autocontinuous from above, pseudometric generating), then I has the same property.

Furthermore, if μ is any submodular (supermodular) representing measure on 2^X of I, then I has the same property.

The forward direction of the above theorem will be proved by the method based on subgraphs of functions. As a matter of fact, it was already proved by Denneberg [4, Corollary 13.4] for the submodularity and the supermodularity cases and by Pap [14, Theorem 10.8] for the uniform autocontinuity from above case; see also Bassanezi and Greco [1]. To prove all other cases, we prepare the following proposition.

Proposition 3.3. *Let X be a non-empty set. Let $\mathcal{L}$ be a lattice of subsets of X containing $\emptyset$. Let $\mu\colon \mathcal{L} \to [0,\infty]$ be a nonadditive measure. If μ is weakly asymptotic null-additive (asymptotic null-additive, autocontinuous from above, uniformly autocontinuous from above, pseudometric generating, submodular), then so is its outer extension μ^*. By contrast, if μ is supermodular, then so is its inner extension μ_*.*

Proof. To begin with, we show that μ^* is weakly asymptotic null-additive. Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ be decreasing sequences of subsets of X such that $\mu^*(A_n)\downarrow 0$ and $\mu^*(B_n)\downarrow 0$. Then, there are subsequences $\{A_{n_k}\}_{k\in\mathbb{N}}$, $\{B_{n_k}\}_{k\in\mathbb{N}}$, and decreasing sequences $\{L_k\}_{k\in\mathbb{N}}\subset\mathcal{L}$, $\{M_k\}_{k\in\mathbb{N}}\subset\mathcal{L}$ such that $A_{n_k}\subset L_k$, $B_{n_k}\subset M_k$ for every $k\in\mathbb{N}$ and $\mu(L_k)\downarrow 0$, $\mu(M_k)\downarrow 0$. Since μ is weakly asymptotic null-additive, $\mu(L_k\cup M_k)\downarrow 0$ and thus

$$\inf_{n\in\mathbb{N}} \mu^*(A_n\cup B_n) \le \inf_{k\in\mathbb{N}} \mu^*(A_{n_k}\cup B_{n_k}) \le \inf_{k\in\mathbb{N}} \mu(L_k\cup M_k) = 0,$$

which implies the weak asymptotic null-additivity of μ^*.

Next we show that μ^* is asymptotic null-additive. Let $A\subset X$. Let $\{B_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of subsets of X such that $\mu^*(B_n)\downarrow 0$. If $\mu^*(A)=\infty$, then the statement is true. Assume that $\mu^*(A)<\infty$. Let $\varepsilon>0$ and find $L_0\in\mathcal{L}$ such that $A\subset L_0$ and $\mu(L_0)<\mu^*(A)+\varepsilon$. Then there are a subsequence $\{B_{n_k}\}_{k\in\mathbb{N}}$ and a decreasing sequence $\{M_k\}_{k\in\mathbb{N}}\subset\mathcal{L}$ such that $B_{n_k}\subset M_k$ for every $k\in\mathbb{N}$ and $\mu(M_k)\downarrow 0$. Since μ is asymptotic null-additive, $\mu(L_0\cup M_k)\downarrow\mu(L_0)$, so that

$$\inf_{n\in\mathbb{N}} \mu^*(A\cup B_n) \le \inf_{k\in\mathbb{N}} \mu(L_0\cup M_k) = \mu(L_0) < \mu^*(A)+\varepsilon$$

and the asymptotic null-additivity of μ^* follows.

The autocontinuity of μ^* from above was already proved in [14, Proposition 10.4] and its uniform version can be proved in a similar way.

The proof of the pseudometric generating property is as follows. Let $\varepsilon > 0$ and find $\delta > 0$ satisfying the following property (†): $\mu(L \cup M) < \varepsilon$ whenever $L, M \in \mathcal{L}$ and $\mu(L) \vee \mu(M) < \delta$. Let $A, B \subset X$ and assume that $\mu^*(A) \vee \mu^*(B) < \delta$. Then there are $L_0, M_0 \in \mathcal{L}$ such that $A \subset L_0$, $B \subset M_0$ and $\mu(L_0) \vee \mu(M_0) < \delta$, and hence, by (†), we have $\mu^*(A \cup B) \leq \mu(L_0 \cup M_0) < \varepsilon$. Thus μ^* is pseudometric generating.

The submodularity of μ^* and the supermodularity of μ_* are easy to prove; see [4, Proposition 2.4]. □

On the other hand, to prove the reverse direction, we need the following elementary result that is presented only for the completeness of the paper.

Lemma 3.4. *Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a sequence of decreasing functions $\varphi_n \colon [0, \infty) \to [0, \infty]$. If $\int_0^\infty \varphi_n(t)dt \to 0$, then $\varphi_n(t) \to 0$ for every $t \in (0, \infty)$.*

Proof. Assume that there is $t_0 \in (0, \infty)$ such that $\varphi_n(t_0) \not\to 0$. Then there are $\varepsilon_0 > 0$ and a subsequence $\{\varphi_{n_i}\}_{i \in \mathbb{N}}$ such that $\varphi_{n_i}(t_0) > \varepsilon_0$ for every $i \in \mathbb{N}$. Since each φ_{n_i} is decreasing, $\varphi_{n_i}(t) > \varepsilon_0$ for every $t \in [0, t_0]$ and every $i \in \mathbb{N}$ and thus

$$\int_0^\infty \varphi_{n_i}(t)dt \geq \int_0^{t_0} \varphi_{n_i}(t)dt \geq \int_0^{t_0} \varepsilon_0 dt = t_0 \cdot \varepsilon_0 > 0,$$

which is a contradiction. □

The proof of Theorem 3.2: The forward direction can be proved by the same method used in Denneberg [4, Corollary 13.4] and Pap [14, Theorem 10.8] together with Proposition 3.3. So we only show the case of the autocontinuity from above for the reader's convenience.

Let G_f, $\mathcal{G}$, ω, and φ be given in the method based on subgraphs of functions. Assume that I is autocontinuous from above. To begin with, we show that ω has the same property. Let $E \in \mathcal{G}$ and let $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{G}$ be a sequence with $\omega(F_n) \to 0$. Since $E = G_f$ and $F_n = G_{g_n}$ for some $f, g_n \in \mathcal{F}_1$, we have $I(f) = \omega(G_f) = \omega(E)$ and $I(f \vee g_n) = \omega(G_{f \vee g_n}) = \omega(G_f \cup G_{g_n}) = \omega(E \cup F_n)$ for every $n \in \mathbb{N}$, so that $\omega(E \cup F_n) = I(f \vee g_n) \to I(f) = \omega(E)$. Thus ω is autocontinuous from above and so is its outer extension ω^* by Proposition 3.3. In the same way as the case of ω, we can show the autocontinuity of φ from above. Since $\varphi = \beta$, this φ is a representing measure what we seek.

Our main contribution to Theorem 3.2 is the reverse direction. Let μ be a representing measure on 2^X of I such that $Q := \sup_{h \in \mathcal{F}_1} \mu(\{h > 0\}) < \infty$.

In the first place we show the weak asymptotic null-additivity of I. Let $\{f_n\}_{n \in \mathbb{N}}$, $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_1$ be decreasing sequences with $I(f_n) \downarrow 0$ and $I(g_n) \downarrow 0$. By Lemma 3.4, for every $t > 0$, we have $\mu(\{f_n > t\}) \downarrow 0$ and $\mu(\{g_n > t\}) \downarrow 0$, so that $\mu(\{f_n \vee g_n > t\}) = \mu(\{f_n > t\} \cup \{g_n > t\}) \downarrow 0$ since μ is weakly asymptotic null-additive. Noting that $0 \leq \mu(\{f_n \vee g_n > t\}) \leq Q$ for every $t \geq 0$ and every $n \in \mathbb{N}$, by the bounded convergence theorem,

$$I(f_n \vee g_n) = (\mathrm{C})\int_X (f_n \vee g_n)d\mu = \int_0^1 \mu(\{f_n \vee g_n > t\})dt \to 0$$

and the weak asymptotic null-additivity of I follows.

The asymptotic null-additivity and the autocontinuity from above can be proved in the same way.

Next we show the uniform autocontinuity of I from above. Let $\varepsilon > 0$ and find $n_0 \in \mathbb{N}$ such that $Q/n_0 < \varepsilon/2$. Since μ is uniformly autocontinuous from above, there is $\delta > 0$ satisfying the following property (†): $\mu(A \cup B) \le \mu(A) + \varepsilon/2$ whenever $A, B \subset X$ and $\mu(B) < n_0\delta$. Let $f, g \in \mathcal{F}_1$ and assume that $I(g) < \delta$. Since $1_{\{g>t\}} \le g/t$ for every $t > 0$, we have

$$\begin{aligned}\mu(\{g > t\}) &= \int_0^1 \mu(\{g > t\})ds \le \int_0^1 \mu(\{g/t > s\})ds \\ &\le \text{(C)}\int_X \frac{g}{t} d\mu = \frac{1}{t} \cdot \text{(C)}\int_X g d\mu = \frac{I(g)}{t} < \frac{\delta}{t}.\end{aligned}$$

Therefore, for every $t \ge 1/n_0$, we have $\mu(\{g > t\}) < n_0\delta$, so that (†) yields

$$\mu(\{f \vee g > t\}) = \mu(\{f > t\} \cup \{g > t\}) \le \mu(\{f > t\}) + \frac{\varepsilon}{2}.$$

Consequently,

$$\begin{aligned}I(f \vee g) &= \int_0^1 \mu(\{f \vee g > t\})dt \\ &= \int_0^{1/n_0} \mu(\{f \vee g > t\})dt + \int_{1/n_0}^1 \mu(\{f \vee g > t\})dt \\ &\le \frac{Q}{n_0} + \int_{1/n_0}^1 \left\{\mu(\{f > t\}) + \frac{\varepsilon}{2}\right\} dt \\ &\le \frac{\varepsilon}{2} + \int_0^1 \mu(\{f > t\})dt + \frac{\varepsilon}{2} \\ &= I(f) + \varepsilon,\end{aligned}$$

which implies the uniform autocontinuity of I from above.

The pseudometric generating property can be proved along the same lines as the uniform autocontinuity from above. The submodularity and the supermodularity are easy to prove.

4. Null-additivity of Choquet functionals

In this section we consider the (weakly) null-additive case. To obtain a similar hereditary nature to Theorem 3.2 for the (weak) null-additivity by the method based on subgraphs of functions, for a given (weakly) null-additive measure μ, at least one of the outer extension μ^* and the inner extension μ_* needs to have the same property. But this is not the case as the following example indicates:

Example 4.1 (Murofushi [12]). Let $\mathcal{J} := \{(a, b]\colon -\infty < a < b < \infty\}$ of all bounded left half-open intervals. Let $\mathcal{R}$ be the ring generated by $\mathcal{J}$. Let λ be the Lebesgue measure on the real line $\mathbb{R}$. Define the nonadditive measure $\mu\colon \mathcal{R} \to [0, \infty]$ by

$$\mu(A) := \begin{cases} \infty & \text{if } \{0, 1\} \subset A, \\ \lambda(A) & \text{otherwise} \end{cases}$$

for every $A \in \mathcal{R}$.

(1) μ is null-additive and thus weakly null-additive.
(2) $\mu^*(\{0\}) = \mu^*(\{1\}) = 0$, but $\mu^*(\{0,1\}) = \infty$, so that μ^* is neither weakly null-additive nor null-additive.
(3) $\mu_*(\mathbb{Q}) = \mu_*(\mathbb{R} \setminus \mathbb{Q}) = 0$, but $\mu_*(\mathbb{R}) = \infty$, so that μ_* is neither weakly null-additive nor null-additive, where $\mathbb{Q}$ is the set of all rational numbers.

Nevertheless, we can obtain the (weak) null-additivity on the restricted family of subsets, that is, the family of all open subsets of a Hausdorff space.

In what follows, X is a Hausdorff space, $\mathcal{U}$ is the family of all open subsets of X, and $\mathcal{K}$ is the family of all compact subsets of X.

Lemma 4.2. *Let $K \in \mathcal{K}$ and $U, V \in \mathcal{U}$. Assume that $K \subset U \cup V$. Then there are $L, M \in \mathcal{K}$ such that $K = L \cup M$, $L \subset U$ and $M \subset V$.*

Proof. Since $K \setminus U$ and $K \setminus V$ are disjoint compact sets, by [6, Theorem 3.1.6], there are disjoint open sets G and H satisfying $K \setminus V \subset G$ and $K \setminus U \subset H$. Let $L := K \setminus H$ and $M := K \setminus G$. Then L, M are compact and $K = L \cup M$. Since $K \setminus U \subset H$, we have $L \cap U^c = (K \cap H^c) \cap U^c = (K \cap U^c) \cap H^c = (K \setminus U) \cap H^c = \emptyset$ and thus $L \subset U$. Similarly we have $M \subset V$. □

Proposition 4.3. *Let $\mu\colon \mathcal{K} \to [0, \infty]$ be a nonadditive measure. If μ is (weakly) null-additive, then its inner extension μ_* is (weakly) null-additive on $\mathcal{U}$.*

Proof. Let $U, V \in \mathcal{U}$ and assume $\mu_*(V) = 0$. Let $K \in \mathcal{K}$ with $K \subset U \cup V$. By Lemma 4.2, there are $L, M \in \mathcal{K}$ satisfying $K = L \cup M$, $L \subset U$, and $M \subset V$. Since $\mu_*(V) = 0$, we have $\mu(M) = 0$. By the null-additivity of μ, we have $\mu(K) = \mu(L \cup M) = \mu(L) \le \mu_*(U)$ and thus $\mu_*(U \cup V) \le \mu_*(U)$. The reverse inequality is obvious.

The weak null-additivity of μ_* can be proved in a similar way. □

From this point forwards, we assume that $\mathcal{F}$ is a non-empty family of functions $f\colon X \to [0, \infty]$ with pointwise order, which satisfies, in addition to (F1) and (F2) in Section 3,

(F3) $\{f > t\} \in \mathcal{U}$ for every $t \ge 0$,
(F4) $\{f \ge t\} \in \mathcal{K}$ for every $t > 0$, and
(F5) if $K \in \mathcal{K}$, $U \in \mathcal{U}$ and $K \subset U$, then there is $f \in \mathcal{F}$ such that $1_K \le f \le 1_U$.

When X is locally compact, typical examples of such an $\mathcal{F}$ is the positive cones of the space $C_c(X)$ of all continuous real functions on X with compact support and the space $C_0(X)$ of all continuous real functions on X vanishing at infinity.

We also assume that $I\colon \mathcal{F} \to [0, \infty]$ is a Choquet functional, that is, it satisfies (I1)–(I5) in Section 3.

Now we have a similar result to Theorem 3.2 except that the representing measure is not necessarily (weakly) null-additive on the power set 2^X.

Theorem 4.4. *If I is (weakly) null-additive, then there is a representing measure μ on 2^X of I that is (weakly) null-additive on $\mathcal{U}$.*

Conversely, if μ is a representing measure on 2^X of I that is (weakly) null-additive on $\mathcal{U}$, then I is (weakly) null-additive.

Proof. Let $\alpha\colon 2^X \to [0,\infty]$ be the smallest representing measure of I, that is, $\alpha(A) := \sup\{I(f)\colon f \in \mathcal{F}, f \leq 1_A\}$ for every $A \subset X$. Let U, V be open and assume that $\alpha(V) = 0$. Let $h \in \mathcal{F}$ and assume that $h \leq 1_{U\cup V}$. Fix $n \in \mathbb{N}$. Then the set $\{h \geq 1/n\}$ is compact by (F4) and is contained in $U \cup V$. Therefore by Lemma 4.2, there are compact sets L_n, M_n such that $L_n \subset U, M_n \subset V$ and $\{h \geq 1/n\} = L_n \cup M_n$. By (F5), there are functions $f_n, g_n \in \mathcal{F}$ such that $1_{L_n} \leq f_n \leq 1_U$ and $1_{M_n} \leq g_n \leq 1_V$.

To begin with, we show that $\alpha(\{h \geq 1/n\}) \leq \alpha(U)$. Since $\{h \geq 1/n\} \subset \{f_n \vee g_n \geq t\}$ for every $t \in [0,1]$,

$$\begin{aligned}\alpha(\{h \geq 1/n\}) &= \int_0^1 \alpha(\{h \geq 1/n\}dt \\ &\leq \int_0^1 \alpha(\{f_n \vee g_n \geq t\})dt \\ &= (\mathrm{C})\int_X (f_n \vee g_n)d\alpha = I(f_n \vee g_n).\end{aligned}$$

Since $\alpha(V) = 0$ and $g_n \leq 1_V$, we have $I(g_n) = 0$ and thus $I(f_n \vee g_n) = I(f_n)$ by the null-additivity of I. Since $f_n \leq 1_U$, we have $I(f_n) \leq \alpha(U)$. Thus, $\alpha(\{h \geq 1/n\}) \leq I(f_n \vee g_n) = I(f_n) \leq \alpha(U)$. Consequently,

$$I(h) = \int_0^1 \alpha(\{h \geq t\})dt \leq \sup_{n\in\mathbb{N}} \int_{1/n}^1 \alpha(\{h \geq 1/n\})dt \leq \alpha(U),$$

so that $I(h) \leq \alpha(U)$ for every $h \in \mathcal{F}$ with $h \leq \chi_{U\cup V}$. Thus $\alpha(U \cup V) \leq \alpha(U)$. The reverse inequality is obvious and hence α is null-additive.

We show the reverse direction. Let $f, g \in \mathcal{F}_1$ and assume that $I(g) = 0$. Since μ is a representing measure of I,

$$\int_0^1 \mu(\{g > t\})dt = (\mathrm{C})\int_X g d\mu = I(g) = 0,$$

so that $\mu(\{g > t\}) = 0$ for almost all $t \in [0,1]$. Since μ is null-additive on open sets, noting (F3), we have $\mu(\{f \vee g > t\}) = \mu(\{f > t\} \cup \{g > t\}) = \mu(\{f > t\})$ for almost all $t \in [0,1]$, so that

$$I(f \vee g) = \int_0^1 \mu(\{f \vee g > t\})dt = \int_0^1 \mu(\{f > t\})dt = I(f).$$

This implies the null-additivity of I.

The proof of the weak null-additivity can be done in a similar way. □

Remark 4.5. We may replace $\mathcal{K}$ with the family $\mathcal{K}_\sigma$ of all compact G_δ-subsets of X in Lemma 4.2 and Proposition 4.3. Therefore, Theorem 4.4 remains valid in the case that $\mathcal{K}$ is replaced with $\mathcal{K}_\sigma$ in (F4) and (F5).

We end the paper with suggestive examples illustrating the correspondence between the Choquet functionals and their representing measures. Let $C[0,1]$ denote the space of all continuous real-valued functions on $[0,1]$ and $C^+[0,1] := \{f \in C[0,1]\colon f \geq 0\}$. Let $\mathcal{L}$ be the σ-field of all Borel subsets of $[0,1]$ and λ the Lebesgue measure on $\mathbb{R}$.

Example 4.6. Let $0 < p < \infty$. Define the nonadditive measure $\mu_p\colon \mathcal{L} \to [0,1]$ by

$$\mu_p(A) := \lambda(A)^p$$

for every $A \in \mathcal{L}$ and the functional $I_p\colon C^+[0,1] \to [0,\infty)$ by

$$I_p(f) := (\mathrm{C})\int_{[0,1]} f d\mu_p$$

for every $f \in C^+[0,1]$.

(1) μ_p is submodular if $p \leq 1$ and supermodular if $p \geq 1$.
(2) The outer extension $(\mu_p)^*$ is submodular if $p \leq 1$ and the inner extension $(\mu_p)_*$ is supermodular if $p \geq 1$. Both of them are representing measures of I_p.
(3) I_p is submodular if $p \leq 1$ and supermodular if $p \geq 1$.
(4) I_p satisfies the moment condition

$$I_p(1) = 1, \quad I_p(x^m) = \frac{\Gamma(m+1)\,\Gamma(p+1)}{\Gamma(m+p+1)}, \quad m = 1, 2, \ldots,$$

where Γ is the Gamma function.

Example 4.7. Define the nonadditive measure $\mu\colon \mathcal{L} \to [0,3]$ by

$$\mu(A) := \begin{cases} 0 & \text{if } A = \emptyset, \\ \lambda(A)^2 + \sqrt{\lambda(A)} + 1 & \text{if } A \neq \emptyset \end{cases}$$

for every $A \in \mathcal{L}$ and the functional $I\colon C^+[0,1] \to [0,\infty)$ by

$$I(f) := \int_{m_f}^{M_f} \left\{ \lambda(\{f > t\})^2 + \sqrt{\lambda(\{f > t\})} \right\} dt + M_f + 2m_f$$

for every $f \in C^+[0,1]$, where $M_f := \max_{x\in[0,1]} f(x)$ and $m_f := \min_{x\in[0,1]} f(x)$.

(1) The outer extension μ^* is a representing measure of I.
(2) μ^* and I are uniformly autocontinuous.

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