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EXISTENCE AND APPROXIMATION OF COMMON FIXED POINTS OF TWO HYBRID MAPPINGS IN HILBERT SPACES

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday.

Abstract. We study the problem of finding a common fixed point of two commutative hybrid mappings in Hilbert spaces. Among other things, we obtain a common fixed point theorem for such mappings and a uniform convergence theorem for the average of such mappings.

1. Introduction

The aim of the present paper is to study the existence and approximation of common fixed points of two commutative hybrid mappings in Hilbert spaces. We formally state the problem considered as follows: Find a point $u \in C$ such that

$$Su = Tu = u, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H, $S\colon C \to C$ is a λ-hybrid mapping for some $\lambda \in \mathbb{R}$, $T\colon C \to C$ is a μ-hybrid mapping for some $\mu \in \mathbb{R}$, and $ST = TS$.

The notion of λ-hybrid mapping first introduced by Aoyama, Iemoto, Kohsaka, and Takahashi [2] is a generalization of the notions of nonexpansive mappings, nonspreading mappings in the sense of Kohsaka and Takahashi [19], and hybrid mappings in the sense of Takahashi [24]. It is known that every firmly nonexpansive mapping is λ-hybrid for each $\lambda \in [0, 1]$. Several existence and convergence theorems for such a mapping were obtained in [1, 2, 3, 5, 18]. See also Kocourek, Takahashi, and Yao [17] and Djafari Rouhani [12] for related results on generalized hybrid mappings and hybrid sequences in Hilbert spaces, respectively.

On the other hand, applying nonlinear ergodic theory for nonexpansive mappings, Shimizu and Takahashi [22] obtained strong convergence theorems for commutative families of nonexpansive mappings in Hilbert spaces. Later, Atsushiba and Takahashi [6] also showed weak convergence theorems for two commutative nonexpansive mappings in Banach spaces.

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The convergence analysis given in [6, 22] motivates us to discuss the asymptotic behavior of a sequence $\{V_n\}$ of mappings of C into itself defined by

$$V_n = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l \quad (n = 0, 1, \dots). \tag{1.2}$$

If T is particularly the identity mapping on C, then (1.2) is reduced to

$$V_n = \frac{1}{n+1} \sum_{k=0}^{n} S^k \quad (n = 0, 1, \dots). \tag{1.3}$$

According to Baillon's nonlinear ergodic theorem [7, Théorème], the sequence

$$\left\{ \frac{1}{n+1} \sum_{k=0}^{n} S^k x \right\} \tag{1.4}$$

converges weakly to a fixed point of S for all $x \in C$ whenever S is nonexpansive and C is bounded; see also Takahashi [23, Theorem 3.2.3].

This paper is organized as follows. In Section 2, we recall some definitions and results needed in this paper. In Section 3, we obtain a common fixed point theorem for S and T and a uniform convergence theorem for the sequences $\{V_n - SV_n\}$ and $\{V_n - TV_n\}$. We also show that the sequence $\{V_n\}$ satisfies the condition (S) with respect to the common fixed point set of S and T in the sense of Aoyama [1]. In Section 4, combining the results obtained in this paper and convergence theorems in [1, 18], we show three convergence theorems for S and T. In Section 5, we show some examples of two commutative hybrid mappings.

2. Preliminaries

Throughout this paper, we denote by H a real Hilbert space, $\langle \cdot, \cdot \rangle$ an inner product on H, $\|\cdot\|$ the induced norm on H, δB_H the closed ball with radius $\delta > 0$ centered at 0, C a nonempty closed convex subset of H, I, S^0, and T^0 the identity mapping on C, $\mathbb{N}$ the set of all nonnegative integers, $\mathbb{R}$ the set of all real numbers, $x_n \to x$ the strong convergence of a sequence $\{x_n\}$ to $x \in H$, and $x_n \rightharpoonup x$ the weak convergence of a sequence $\{x_n\}$ to $x \in H$, respectively.

Let $T\colon C \to C$ be a mapping. The set of all fixed points of T is denoted by $\mathcal{F}(T)$. A point $p \in C$ is said to be an asymptotic fixed point [21] of T if there exists a sequence $\{z_n\}$ of C such that $z_n \rightharpoonup p$ and $z_n - Tz_n \to 0$. The set of all asymptotic fixed points of T is denoted by $\widehat{\mathcal{F}}(T)$. Let $\lambda \in \mathbb{R}$ be given. Following [2], we say that a mapping $T\colon C \to C$ is λ-hybrid if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2(1-\lambda) \langle x - Tx, y - Ty \rangle \tag{2.1}$$

for all $x, y \in C$. It is obvious that T is 1-hybrid if and only if T is nonexpansive; T is 0-hybrid if and only if T is nonspreading in the sense of [19]; T is 1/2-hybrid if and only if T is hybrid in the sense of [24]; if $\lambda > 1$, then T is λ-hybrid if and only if $T = I$. It is known [4, Proposition 2.2] that if $\lambda < 2$ and $\alpha = (1-\lambda)/(2-\lambda)$, then T is λ-hybrid if and only if it is α-nonexpansive in the sense of [4], that is,

$$\|Tx - Ty\|^2 \le \alpha\big(\|x - Ty\|^2 + \|Tx - y\|^2\big) + (1-2\alpha)\|x - y\|^2 \tag{2.2}$$

for all $x, y \in C$. A mapping $T: C \to C$ is said to be quasi-nonexpansive if $\mathcal{F}(T)$ is nonempty and $\|w - Tx\| \le \|w - x\|$ for all $w \in \mathcal{F}(T)$ and $x \in C$. By Dotson [13, Theorem 1] and Itoh and Takahashi [16, Corollary 1], we know that $\mathcal{F}(T)$ is closed and convex whenever T is quasi-nonexpansive. Every λ-hybrid mapping with a fixed point is clearly quasi-nonexpansive. Thus the fixed point set of each λ-hybrid mapping is closed and convex. The mapping T is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \le \|x - y\|^2 \tag{2.3}$$

for all $x, y \in C$; see [10, 11, 14, 15] for more details on firmly nonexpansive mappings. It is known [2, Lemma 3.1] that if T is firmly nonexpansive, then T is λ-hybrid for each $\lambda \in [0, 1]$. We also know the following lemma:

Lemma 2.1 ([2, Lemma 3.2]). *If $T: C \to C$ is a λ-hybrid mapping for some $\lambda \in \mathbb{R}$, then $\widehat{\mathcal{F}}(T) = \mathcal{F}(T)$.*

Using some ideas in [9, Theorem 1] and [22, Lemma 1], we can show the following lemma:

Lemma 2.2. *If $n \in \mathbb{N}$, $x_0, x_1, \dots, x_n \in H$, and $z = (n+1)^{-1} \sum_{k=0}^{n} x_k$, then*

$$\|z - u\|^2 = \frac{1}{n+1} \sum_{k=0}^{n} \left(\|x_k - u\|^2 - \|x_k - z\|^2 \right) \tag{2.4}$$

for all $u \in H$.

Proof. Let $u \in H$ be given. Then we have

$$\|x_k - u\|^2 = \|x_k - z\|^2 + \|z - u\|^2 + 2 \langle x_k - z, z - u \rangle \tag{2.5}$$

for all $k \in \{0, 1, \dots, n\}$. This gives us that

$$\begin{aligned} \sum_{k=0}^{n} \|x_k - u\|^2 = \sum_{k=0}^{n} \|x_k - z\|^2 + (n+1) \|z - u\|^2 \\ + 2 \left\langle \sum_{k=0}^{n} x_k - (n+1)z, z - u \right\rangle. \end{aligned} \tag{2.6}$$

Thus the result follows. □

Let F be a nonempty closed convex subset of H. Then, for each $x \in H$, there exists a unique $\hat{x} \in F$ such that $\|\hat{x} - x\| \le \|y - x\|$ for all $y \in F$. The metric projection P_F of H onto F is defined by $P_F x = \hat{x}$ for all $x \in H$. It is well known that P_F is firmly nonexpansive and

$$\langle y - P_F x, x - P_F x \rangle \le 0 \tag{2.7}$$

for $y \in F$ and $x \in H$.

Motivated by [25, Lemma 3.2], we show the following lemma:

Lemma 2.3. *Let F be a nonempty closed convex subset of H and $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ a net of H such that $\|p - x_{\alpha'}\| \le \|p - x_\alpha\|$ whenever $p \in F$, $\alpha, \alpha' \in \mathcal{A}$, and $\alpha \le \alpha'$. Then the following hold:*

(i) $\|P_F x_{\alpha'} - x_{\alpha'}\| \le \|P_F x_\alpha - x_\alpha\|$ *whenever $\alpha, \alpha' \in \mathcal{A}$ and $\alpha \le \alpha'$;*

(ii) $\{P_F x_\alpha\}_{\alpha\in\mathcal{A}}$ *converges strongly to an element of* F.

Proof. We first show the part (i). By the definition of P_F and assumption, we know that

$$\|P_F x_{\alpha'} - x_{\alpha'}\| \le \|P_F x_\alpha - x_{\alpha'}\| \le \|P_F x_\alpha - x_\alpha\| \tag{2.8}$$

for all $\alpha, \alpha' \in \mathcal{A}$ with $\alpha \le \alpha'$.

We next show the part (ii). By (i), the net $\{\|P_F x_\alpha - x_\alpha\|^2\}$ is convergent and hence it is a Cauchy net. Thus for each $\varepsilon > 0$, we have $\alpha_0 \in \mathcal{A}$ such that

$$\Big| \|P_F x_\alpha - x_\alpha\|^2 - \|P_F x_\beta - x_\beta\|^2 \Big| < \frac{\varepsilon^2}{4} \tag{2.9}$$

whenever $\alpha, \beta \in \mathcal{A}$, $\alpha_0 \le \alpha$, and $\alpha_0 \le \beta$. Thus, by the firm nonexpansiveness of P_F and (2.9), we know that

$$\begin{aligned} \|P_F x_\alpha - P_F x_\gamma\|^2 &\le \|P_F x_\alpha - x_\gamma\|^2 - \|P_F x_\gamma - x_\gamma\|^2 \\ &\le \|P_F x_\alpha - x_\alpha\|^2 - \|P_F x_\gamma - x_\gamma\|^2 < \frac{\varepsilon^2}{4} \end{aligned} \tag{2.10}$$

whenever $\alpha, \gamma \in \mathcal{A}$, $\alpha_0 \le \alpha$, and $\alpha \le \gamma$. If $\alpha, \beta \in \mathcal{A}$, $\alpha_0 \le \alpha$, and $\alpha_0 \le \beta$, then there exists $\gamma \in \mathcal{A}$ such that $\alpha \le \gamma$ and $\beta \le \gamma$. Thus it follows from (2.10) that

$$\|P_F x_\alpha - P_F x_\beta\| \le \|P_F x_\alpha - P_F x_\gamma\| + \|P_F x_\gamma - P_F x_\beta\| < \varepsilon. \tag{2.11}$$

This implies that $\{P_F x_\alpha\}$ is a Cauchy net. Since F is closed in H, it converges strongly to an element of F. □

Remark 2.4. In the proof of Theorem 4.1, we apply Lemma 2.3 to the case where $\mathcal{A}$ is the directed set $\mathbb{N}^2$ with a binary relation $\le$ on $\mathbb{N}^2$ given by $(k,l) \le (k',l')$ if $k \le k'$ and $l \le l'$. We also note that if S is a right reversible semitopological semigroup and $\{T_s\}_{s\in S}$ is a continuous representation of S as nonexpansive mappings on C such that $F = \bigcap_{s\in S} \mathcal{F}(T_s)$ is nonempty, then S is a directed set with a binary relation $\le$ on S given by $s \le s'$ if

$$\{s\} \cup \overline{Ss} \supset \{s'\} \cup \overline{Ss'} \tag{2.12}$$

and it is obvious that

$$\|p - T_{s'}x\| \le \|p - T_s x\| \tag{2.13}$$

whenever $x \in C$, $p \in F$, $s, s' \in S$, and $s \le s'$; see [20, 23] for more details. Thus Lemma 2.3 implies that $\{P_F T_s x\}_{s\in S}$ converges strongly to an element of F for each $x \in C$.

The following lemma ensures that every hybrid mapping is bounded on bounded sets:

Lemma 2.5. *Let* $T\colon C \to C$ *be a* λ*-hybrid mapping for some* $\lambda \in \mathbb{R}$. *Then* $T(U)$ *is bounded for each nonempty bounded subset* U *of* C.

Proof. Suppose that the conclusion does not hold. Then there exists a bounded sequence $\{z_n\}$ of C such that $\|Tz_n\| > 0$ for all $n \in \mathbb{N}$ and $\|Tz_n\| \to \infty$. Fix $y \in C$. Since T is λ-hybrid, we have

$$\begin{aligned} \|Tz_n - Ty\|^2 &\le \|z_n - y\|^2 + 2(1-\lambda)\langle z_n - Tz_n, y - Ty\rangle \\ &\le \|z_n - y\|^2 + 2\,|1-\lambda|\,\|z_n - Tz_n\|\,\|y - Ty\| \end{aligned} \tag{2.14}$$

and hence

$$\begin{aligned} &\|Tz_n\| - 2\|Ty\| + \frac{\|Ty\|^2}{\|Tz_n\|} \\ &\quad \le \frac{(\|z_n\| + \|y\|)^2}{\|Tz_n\|} + 2\,|1-\lambda| \left(\frac{\|z_n\|}{\|Tz_n\|} + 1 \right) \|y - Ty\| \end{aligned} \tag{2.15}$$

for all $n \in \mathbb{N}$. This is a contradiction. □

Let $\{S_n\}$ be a sequence of mappings of C into itself and F a nonempty closed convex subset of H. Following Aoyama [1], we say that $\{S_n\}$ satisfies the condition (S) with respect to F if each weak subsequential limit of $\{S_n z_n\}$ belongs to F whenever $\{z_n\}$ is a bounded sequence of C.

We know the following strong and weak convergence theorems for a sequence of quasinonexpansive type mappings in Hilbert spaces:

Theorem 2.6 ([1, Theorem 1]). *Let H be a real Hilbert space, C and F nonempty closed convex subsets of H such that $F \subset C$, and $\{S_n\}$ a sequence of mappings of C into itself. Suppose that*

- *$\|w - S_n x\| \le \|w - x\|$ for all $n \in \mathbb{N}$, $w \in F$, and $x \in C$;*
- *$\{S_n\}$ satisfies the condition (S) with respect to F.*

Let u be an element of C and $\{x_n\}$ a sequence defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n) S_n x_n \quad (n = 0, 1, \dots), \tag{2.16}$$

where $\{\alpha_n\}$ is a sequence of $[0,1]$ such that $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_F u$.

Theorem 2.7 ([18, Theorem 3.1]). *Let H, C, F, and $\{S_n\}$ be the same as in Theorem 2.6 and $\{x_n\}$ a sequence defined by $x_0 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S_n x_n \quad (n = 0, 1, \dots), \tag{2.17}$$

where $\{\alpha_n\}$ is a sequence of $[0,1]$ such that $\sup_n \alpha_n < 1$. Then $\{x_n\}$ converges weakly to the strong limit of $\{P_F x_n\}$.

3. Fundamental results for two commutative hybrid mappings

In this section, among other things, we show a common fixed point theorem and a uniform convergence theorem for two commutative hybrid mappings.

Throughout this section, we suppose the following:

- C is a nonempty closed convex subset of a real Hilbert space H;
- $S\colon C \to C$ is λ-hybrid and $T\colon C \to C$ is μ-hybrid for some $\lambda, \mu \in \mathbb{R}$;
- $\{V_n\}$ is a sequence of mappings of C into itself defined by (1.2).

Motivated by [22, Lemma 1], we first show the following lemma:

Lemma 3.1. *Let D be a nonempty subset of C such that*

$$\left\{S^k T^l y : y \in D,\, k, l \in \mathbb{N}\right\} \tag{3.1}$$

is bounded. Then the following hold:

(i) $\lim_n \sup_{y \in D} \|V_n y - SV_n y\| = 0$;
(ii) *if* $ST = TS$, *then* $\lim_n \sup_{y \in D} \|V_n y - TV_n y\| = 0$.

Proof. We first show the part (i). By assumption, the set $\{V_n y : y \in D,\, n \in \mathbb{N}\}$ is bounded. Thus Lemma 2.5 implies that so is $\{SV_n y : y \in D,\, n \in \mathbb{N}\}$. Hence there exists a positive real number M such that

$$\left\|S^k T^l y\right\| \le M \quad \text{and} \quad \|SV_n y\| \le M \tag{3.2}$$

for all $k, l, n \in \mathbb{N}$, and $y \in D$. Note that $\|V_n y\| \le M$ for all $n \in \mathbb{N}$ and $y \in D$.

Let $y \in D$ and $n \in \mathbb{N} \setminus \{0\}$ be given. By Lemma 2.2, we have

$$\begin{aligned} &\|V_n y - SV_n y\|^2 \\ &= \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} \left(\left\|S^k T^l y - SV_n y\right\|^2 - \left\|S^k T^l y - V_n y\right\|^2 \right) \\ &= \frac{1}{(n+1)^2} \Big(\sum_{l=0}^{n} \left\|T^l y - SV_n y\right\|^2 + \sum_{k=0}^{n-1} \sum_{l=0}^{n} \left\|S^{k+1} T^l y - SV_n y\right\|^2 \\ &\quad - \sum_{k=0}^{n} \sum_{l=0}^{n} \left\|S^k T^l y - V_n y\right\|^2 \Big). \end{aligned} \tag{3.3}$$

Since S is λ-hybrid, we have

$$\begin{aligned} &\sum_{k=0}^{n-1} \sum_{l=0}^{n} \left\|S^{k+1} T^l y - SV_n y\right\|^2 \\ &\le \sum_{k=0}^{n-1} \sum_{l=0}^{n} \Big(\left\|S^k T^l y - V_n y\right\|^2 + 2(1-\lambda) \left\langle S^k T^l y - S^{k+1} T^l y, V_n y - SV_n y \right\rangle \Big) \\ &= \sum_{k=0}^{n-1} \sum_{l=0}^{n} \left\|S^k T^l y - V_n y\right\|^2 + 2(1-\lambda) \sum_{l=0}^{n} \left\langle T^l y - S^n T^l y, V_n y - SV_n y \right\rangle . \end{aligned} \tag{3.4}$$

Using (3.3) and (3.4), we obtain

$$\begin{aligned} &\|V_n y - SV_n y\|^2 \\ &\le \frac{1}{(n+1)^2} \sum_{l=0}^{n} \Big(\left\|T^l y - SV_n y\right\|^2 + 2(1-\lambda) \left\langle T^l y - S^n T^l y, V_n y - SV_n y \right\rangle \Big). \end{aligned} \tag{3.5}$$

Using (3.2) and (3.5), we obtain

$$\begin{aligned} \|V_n y - SV_n y\|^2 &\le \frac{1}{(n+1)^2} \sum_{l=0}^{n} \left((2M)^2 + 2\,|1-\lambda|\,(2M)^2 \right) \\ &= \frac{4M^2}{(n+1)} \left(1 + 2\,|1-\lambda|\right). \end{aligned} \tag{3.6}$$

Thus we obtain

$$\sup_{y\in D}\|V_n y - SV_n y\| \leq 2M\sqrt{\frac{1+2\,|1-\lambda|}{n+1}} \to 0. \tag{3.7}$$

Therefore, we obtain the conclusion.

We next show the part (ii). Suppose that $ST = TS$. Then we have

$$V_n = \frac{1}{(n+1)^2}\sum_{l=0}^{n}\sum_{k=0}^{n} T^l S^k \tag{3.8}$$

for all $n \in \mathbb{N}$. Thus, the part (i) implies the conclusion. □

Using Lemmas 2.1 and 3.1, we next show the following common fixed point theorem:

Theorem 3.2. *Suppose that $ST = TS$. Then $\mathcal{F}(S)\cap\mathcal{F}(T)$ is nonempty if and only if $\{S^kT^lx : k,l \in \mathbb{N}\}$ is bounded for some $x \in C$.*

Proof. The only if part is obvious since

$$\left\{S^kT^lp : k,l \in \mathbb{N}\right\} = \{p\} \tag{3.9}$$

for all $p \in \mathcal{F}(S) \cap \mathcal{F}(T)$. We show the if part. Suppose that $\{S^kT^lx : k,l \in \mathbb{N}\}$ is bounded for some $x \in C$. Setting $D = \{x\}$, we know that

$$\left\{S^kT^ly : y \in D,\, k,l \in \mathbb{N}\right\} \tag{3.10}$$

is bounded and hence Lemma 3.1 implies that

$$\lim_{n\to\infty}\|V_nx - SV_nx\| = \lim_{n\to\infty}\|V_nx - TV_nx\| = 0. \tag{3.11}$$

Since C is weakly closed and $\{V_nx\}$ is a bounded sequence of C, there exist $u \in C$ and a subsequence $\{V_{n_i}x\}$ of $\{V_nx\}$ such that $V_{n_i}x \rightharpoonup u$. Thus we know that u is an element of $\widehat{\mathcal{F}}(S) \cap \widehat{\mathcal{F}}(T)$. Consequently, by Lemma 2.1, we know that u is an element of $\mathcal{F}(S) \cap \mathcal{F}(T)$. □

Letting $T = I$ in Theorem 3.2, we obtain the following corollary:

Corollary 3.3 ([2, Theorem 4.1])**.** *If $S\colon C \to C$ is a λ-hybrid mapping for some $\lambda \in \mathbb{R}$, then $\mathcal{F}(S)$ is nonempty if and only if $\{S^nx\}$ is bounded for some $x \in C$.*

Using Lemma 3.1, we next obtain the following uniform convergence theorem:

Theorem 3.4. *Suppose that $ST = TS$ and $\mathcal{F}(S) \cap \mathcal{F}(T)$ is nonempty. Then*

$$\lim_{n\to\infty}\sup_{y\in D}\|V_ny - SV_ny\| = \lim_{n\to\infty}\sup_{y\in D}\|V_ny - TV_ny\| = 0 \tag{3.12}$$

for each nonempty bounded subset D of C.

Proof. Let D be a nonempty bounded subset of C and fix $w \in F$. Since S and T are quasi-nonexpansive, we know that

$$\left\|w - S^kT^ly\right\| \leq \|w - y\| \tag{3.13}$$

for all $k,l \in \mathbb{N}$ and $y \in D$. This implies that the set

$$\left\{S^kT^ly : y \in D,\, k,l \in \mathbb{N}\right\} \tag{3.14}$$

is bounded. Hence the result follows from Lemma 3.1. □

Remark 3.5. In the case where S and T are nonexpansive in Theorem 3.4, we obtain a result which is similar to [22, Lemma 1].

Using Lemma 2.1 and Theorem 3.4, we also show the following corollary:

Corollary 3.6. *Suppose that $ST = TS$ and $F = \mathcal{F}(S) \cap \mathcal{F}(T)$ is nonempty. Then the following hold:*

(i) $\|w - V_n x\| \leq \|w - x\|$ *for all $n \in \mathbb{N}$, $w \in F$, and $x \in C$;*
(ii) $\{V_n\}$ *satisfies the condition (S) with respect to F.*

Proof. Since S and T are quasi-nonexpansive, we know that

$$\|w - V_n x\| \leq \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} \left\| w - S^k T^l x \right\| \leq \|w - x\| \tag{3.15}$$

for all $n \in \mathbb{N}$, $w \in F$, and $x \in C$. Thus the part (i) holds.

We next show the part (ii). Let $\{z_n\}$ be a bounded sequence of C and u a weak subsequential limit of $\{V_n z_n\}$. Then we have a subsequence $\{V_{n_i} z_{n_i}\}$ of $\{V_n z_n\}$ such that $V_{n_i} z_{n_i} \rightharpoonup u$. Since C is weakly closed, we have $u \in C$. Let ρ be a positive real number such that $\|z_n\| \leq \rho$ for all $n \in \mathbb{N}$. By Theorem 3.4, we obtain

$$\|V_n z_n - S V_n z_n\| \leq \sup_{y \in C \cap \rho B_H} \|V_n y - S V_n y\| \to 0 \tag{3.16}$$

and

$$\|V_n z_n - T V_n z_n\| \leq \sup_{y \in C \cap \rho B_H} \|V_n y - T V_n y\| \to 0. \tag{3.17}$$

Hence u is an element of $\widehat{\mathcal{F}}(S) \cap \widehat{\mathcal{F}}(T)$. Accordingly, Lemma 2.1 implies that $u \in F$. Therefore, $\{V_n\}$ satisfies the condition (S) with respect to F. □

4. Three convergence theorems

In this section, applying the results obtained in Section 3, we show three convergence theorems for two commutative hybrid mappings in Hilbert spaces.

We define a binary relation $\leq$ on $\mathbb{N}^2$ by $(k, l) \leq (k', l')$ if $k \leq k'$ and $l \leq l'$. Then $(\mathbb{N}^2, \leq)$ is obviously a directed set.

Using some ideas in [2, Lemma 5.1] and [23, Theorem 3.2.3], we first obtain the following mean convergence theorem:

Theorem 4.1. *Let H be a real Hilbert space, C a nonempty closed convex subset of H, $S\colon C \to C$ a λ-hybrid mapping for some $\lambda \in \mathbb{R}$, and $T\colon C \to C$ a μ-hybrid mapping for some $\mu \in \mathbb{R}$. Suppose that $ST = TS$ and $F = \mathcal{F}(S) \cap \mathcal{F}(T)$ is nonempty. Let x be an element of C and $\{x_n\}$ a sequence defined by*

$$x_n = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x \quad (n = 0, 1, \dots). \tag{4.1}$$

Then the following hold:

(i) $\{P_F S^k T^l x\}_{(k,l) \in \mathbb{N}^2}$ *converges strongly to an element of F;*
(ii) $\{x_n\}$ *converges weakly to the strong limit of* $\{P_F S^k T^l x\}_{(k,l) \in \mathbb{N}^2}$.

Proof. We denote P_F by P and let $\{V_n\}$ be a sequence of mappings of C into itself defined by (1.2). Note that $x_n = V_n x$ for all $n \in \mathbb{N}$.

We first show the part (i). Since S and T are quasi-nonexpansive and $ST = TS$, we have

$$\begin{aligned} \left\|p - S^{k'}T^{l'}x\right\| &\le \left\|p - S^{k}T^{l'}x\right\| \\ &= \left\|p - T^{l'}S^{k}x\right\| \le \left\|p - T^{l}S^{k}x\right\| = \left\|p - S^{k}T^{l}x\right\| \end{aligned} \tag{4.2}$$

whenever $p \in F$ and $(k,l) \le (k',l')$. Thus the part (i) of Lemma 2.3 implies that

$$\left\|PS^{k'}T^{l'}x - S^{k'}T^{l'}x\right\| \le \left\|PS^{k}T^{l}x - S^{k}T^{l}x\right\| \tag{4.3}$$

whenever $(k,l) \le (k',l')$. By the part (ii) of Lemma 2.3, we also know that $\{PS^kT^lx\}_{(k,l)\in\mathbb{N}^2}$ converges strongly to an element u of F. Using this property, we can see that

$$\begin{aligned} 0 &\le \left\|\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} PS^kT^lx - u\right\| \\ &\le \frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n}\left\|PS^kT^lx - u\right\| \to 0 \end{aligned} \tag{4.4}$$

as $n \to \infty$.

We next show the part (ii). Since S and T are quasi-nonexpansive and F is nonempty, we know that $\{S^kT^lx\}$ is bounded and hence so is $\{V_nx\}$. Thus we have a subsequence $\{V_{n_i}x\}$ of $\{V_nx\}$ such that $V_{n_i}x \rightharpoonup v \in C$. By the part (ii) of Corollary 3.6, we know that $\{V_n\}$ satisfies the condition (S) with respect to F and hence v is an element of F. Then it follows from (2.7) that

$$\left\langle v - PS^kT^lx, S^kT^lx - PS^kT^lx\right\rangle \le 0 \tag{4.5}$$

for all $(k,l) \in \mathbb{N}^2$. By (4.3) and (4.5), we have

$$\begin{aligned} \left\langle v - u, S^kT^lx - PS^kT^lx\right\rangle &\le \left\langle PS^kT^lx - u, S^kT^lx - PS^kT^lx\right\rangle \\ &\le \left\|PS^kT^lx - u\right\|\left\|S^kT^lx - PS^kT^lx\right\| \\ &\le \left\|PS^kT^lx - u\right\|\|x - Px\| \end{aligned} \tag{4.6}$$

for all $(k,l) \in \mathbb{N}^2$. Hence we have

$$\begin{aligned} &\left\langle v - u, V_{n_i}x - \frac{1}{(n_i+1)^2}\sum_{k=0}^{n_i}\sum_{l=0}^{n_i} PS^kT^lx\right\rangle \\ &\le \frac{1}{(n_i+1)^2}\sum_{k=0}^{n_i}\sum_{l=0}^{n_i}\left\|PS^kT^lx - u\right\|\|x - Px\| \end{aligned} \tag{4.7}$$

for all $i \in \mathbb{N}$. Since (4.4) holds and $V_{n_i}x \rightharpoonup v$, by letting $i \to \infty$ in (4.7), we obtain $\|v-u\|^2 \le 0$ and hence $v = u$. Therefore, the sequence $\{x_n\}$ converges weakly to the strong limit of $\{PS^kT^lx\}_{(k,l)\in\mathbb{N}^2}$. □

As a direct consequence of Theorem 4.1, we obtain the following corollary:

Corollary 4.2 ([2, Theorem 5.2])**.** *Let H be a real Hilbert space, C a nonempty closed convex subset of H, $S\colon C \to C$ a λ-hybrid mapping for some $\lambda \in \mathbb{R}$ such that $\mathcal{F}(S)$ is nonempty, x an element of C, and $\{x_n\}$ a sequence defined by*

$$x_n = \frac{1}{n+1}\sum_{k=0}^{n} S^k x \quad (n = 0, 1, \dots). \tag{4.8}$$

Then $\{x_n\}$ converges weakly to the strong limit of $\{P_{\mathcal{F}(S)}S^n x\}$.

Proof. Letting $T = I$, we know that T is 1-hybrid, $ST = TS$, and $\mathcal{F}(S) \cap \mathcal{F}(T) = \mathcal{F}(S)$ is nonempty. Further, it holds that

$$\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l = \frac{1}{n+1}\sum_{k=0}^{n} S^k \tag{4.9}$$

for all $n \in \mathbb{N}$. Thus the result follows from Theorem 4.1. □

By Theorem 2.6 and Corollary 3.6, we obtain the following strong convergence theorem and its corollary:

Theorem 4.3. *Let H, C, S, T, and F be the same as in Theorem 4.1, u an element of C, and $\{x_n\}$ a sequence defined by $x_0 \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n \quad (n = 0, 1, \dots), \tag{4.10}$$

where $\{\alpha_n\}$ is a sequence of $[0,1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_F u$.

Remark 4.4. In the case where S and T are nonexpansive in Theorem 4.3, we obtain a result which is similar to [22, Theorem 1].

Corollary 4.5 ([1, Theorem 2])**.** *Let H, C, and S be the same as in Corollary 4.2, u an element of C, and $\{x_n\}$ a sequence defined by $x_0 \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\frac{1}{n+1}\sum_{k=0}^{n} S^k x_n \quad (n = 0, 1, \dots), \tag{4.11}$$

where $\{\alpha_n\}$ is a sequence of $[0,1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{\mathcal{F}(S)} u$.

By Theorem 2.7 and Corollary 3.6, we obtain the following weak convergence theorem and its corollary:

Theorem 4.6. *Let H, C, S, T, and F be the same as in Theorem 4.1 and $\{x_n\}$ a sequence defined by $x_0 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n \quad (n = 0, 1, \dots), \tag{4.12}$$

where $\{\alpha_n\}$ is a sequence of $[0,1]$ such that $\sup_n \alpha_n < 1$. Then $\{x_n\}$ converges weakly to the strong limit of $\{P_F x_n\}$.

Remark 4.7. In the case where S and T are nonexpansive in Theorem 4.6, we obtain a corresponding result [6, Theorem 1] in the Hilbert space setting.

Corollary 4.8 ([18, Corollary 5.2])**.** *Let H, C, and S be the same as in Corollary 4.2 and $\{x_n\}$ a sequence defined by $x_0 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \frac{1}{n+1} \sum_{k=0}^{n} S^k x_n \quad (n = 0, 1, \dots), \tag{4.13}$$

where $\{\alpha_n\}$ is a sequence of $[0, 1]$ such that $\sup_n \alpha_n < 1$. Then $\{x_n\}$ converges weakly to the strong limit of $\{P_{\mathcal{F}(S)} x_n\}$.

5. Appendix

In this final section, we give some examples of two commutative hybrid mappings in a real Hilbert space H.

Two commutative 1-hybrid mappings can be obtained by applying the nonexpansive semigroup $\{S(t)\}_{t>0}$ on $\overline{\mathcal{D}(A)}$ generated by $-A$, where $A\colon H \to 2^H$ is a maximal monotone operator and $\mathcal{D}(A)$ is the domain of A; see [8] for more details on the generation of nonexpansive semigroups. In this case, we know that $S(t_1)$ and $S(t_2)$ are commutative 1-hybrid mappings of $\overline{\mathcal{D}(A)}$ into itself for all positive real numbers t_1 and t_2.

We first show the following simple example:

Example 5.1. *Let $S\colon H \to H$ be a linear operator such that $\|Sx\| = \|x\|$ for all $x \in H$, and T the metric projection of H onto rB_H for some $r > 0$. Then S is 1-hybrid, T is firmly nonexpansive, and $ST = TS$.*

Proof. Since S is nonexpansive, it is 1-hybrid. Since T is the metric projection onto rB_H, it is firmly nonexpansive. If $x \in rB_H$, then $\|Sx\| = \|x\| \le r$ and hence $STx = Sx = TSx$. If $x \in H \setminus rB_H$, then we have $\|Sx\| = \|x\| > r$ and hence

$$STx = S\left(\frac{r}{\|x\|} x\right) = \frac{r}{\|x\|} Sx = \frac{r}{\|Sx\|} Sx = TSx. \tag{5.1}$$

Thus we obtain $ST = TS$. □

Using some results in [3, 4], we show the following two examples:

Example 5.2. *Let $S\colon H \to H$ be a linear operator such that $\|Sx\| = \|x\|$ for all $x \in H$, both U and V firmly nonexpansive mappings of H into itself such that $SU = US$, $SV = VS$, and $U(H) \cup V(H)$ is contained in rB_H for some $r > 0$, both λ and δ real numbers such that $0 \le \lambda < 1$ and*

$$\delta \ge \left(1 + 2\sqrt{\frac{2-\lambda}{1-\lambda}}\right) r, \tag{5.2}$$

and $T\colon H \to H$ the mapping defined by

$$Tx = \begin{cases} Ux & (x \in \delta B_H); \\ Vx & (\text{otherwise}). \end{cases} \tag{5.3}$$

Then S is 1-hybrid, T is λ-hybrid, and $ST = TS$.

Proof. It follows from [4, Proposition 2.2] and [4, Example 2.4] that T is λ-hybrid. If $x \in \delta B_H$, then $\|Sx\| = \|x\| \leq \delta$ and hence

$$STx = SUx = USx = TSx. \tag{5.4}$$

If $x \in H \setminus \delta B_H$, then $\|Sx\| = \|x\| > \delta$ and hence

$$STx = SVx = VSx = TSx. \tag{5.5}$$

Thus we obtain $ST = TS$. □

Example 5.3. *Let $S\colon H \to H$ be an affine and nonexpansive mapping, $N\colon H \to H$ a nonspreading mapping such that $SN = NS$, β a real number such that $0 \leq \beta < 1$, and T the mapping defined by*

$$T = \beta I + (1-\beta)N. \tag{5.6}$$

Then S is 1-hybrid, T is $-\beta/(1-\beta)$-hybrid, and $ST = TS$.

Proof. It follows from [3, Lemma 2.2] that T is $-\beta/(1-\beta)$-hybrid. Since S is affine and $SN = NS$, we know that

$$STx = \beta Sx + (1-\beta)SNx = \beta Sx + (1-\beta)NSx = TSx \tag{5.7}$$

for all $x \in H$. Thus we know that $ST = TS$. □

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