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WEAK CONVERGENCE THEOREMS FOR FIRMLY GENERALIZED NONEXPANSIVE MAPPINGS WITH BREGMAN DISTANCES IN BANACH SPACES

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday.

Abstract. In this paper, we study some properties of firmly generalized nonexpansive mappings with respect to a Bargeman distance in a Banach space. Next, we prove weak convergence theorems of Pazy's type [18] and Baillon's type [1] for finding a fixed point of such a mapping in a Banach space.

1. Introduction

Let C be a nonempty subset of a smooth Banach space E and let J be the normalized duality mapping on E. A mapping $T : C \to C$ is said to be firmly generalized nonexpansive type [11] if

$$V(x, Tx) + V(y, Ty) + V(Tx, Ty) + V(Ty, Tx) \leq V(x, Ty) + V(y, Tx)$$

for all $x, y \in C$, where $V(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. In 2009, Ibaraki and Takahashi [11] proved the following weak convergence theorem of Pazy's type for firmly generalized nonexpansive type mappings in a Banach space:

Theorem 1.1. *Let E be a uniformly convex Banach space with uniformly Gâteaux differentiable norm and let T be a firmly generalized nonexpansive type mapping from E into itself. If the duality mapping J is weakly sequentially continuous, then the following are equivalent:*

(1) *The set $F(T)$ of fixed points of T is nonempty;*
(2) *$\{T^n x\}$ is bounded for some $x \in E$.*

In this case, $\{T^n x\}$ converges weakly to an element of $F(T)$ for each $x \in E$.

They also characterized the convergent point by using nonlinear projections under suitable conditions. In 2010, Honda, Ibaraki and Takahashi [7] prove a weak convergence theorem of Baillon's type [1] (see also [6, 23]) for finding a fixed point of firmly generalized nonexpansive mappings in a Banach space.

On the other hand, using Bregman distances, Naraghirad, Takahashi and Yao [17] introduced a Bregman firmly generalized nonexpansive type mapping in a reflexive Banach space. They generalize Theorem 1.1 for Bregman firmly generalized nonexpansive type mappings under suitable conditions.

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In this paper, we first study Bregman firmly generalized nonexpansive type mappings [17] in a reflexive Banach space. Next, we prove a weak convergence theorem of Pazy's type [11] for Bregman firmly generalized nonexpansive type mappings in a reflexive Banach space. This result characterized the convergent point by using nonlinear projections. Finally, we also obtain weak convergence theorems of Baillon's type [7] in a reflexive Banach space.

2. Preliminaries

Let E be a real Banach space with its dual E^*. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote the weak* convergence of a sequence $\{x_n^*\}$ to x^* in E^* by $x_n^* \overset{*}{\rightharpoonup} x^*$. For $p \in (1, \infty)$, the duality mapping J_p from E into E^* corresponding to the weight function $\omega(t) = t^{p-1}$ is defined by

$$J_p x = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \ \|x^*\| = \omega(\|x\|)\}$$

for each $x \in E$. The mapping J_2 is called the normalized duality mapping from E into E^* and it is denoted by J (see [5, 24] for details). A Banach space E is said to be strictly convex if $\|(x+y)/2\| < 1$ whenever $x, y \in S := \{z \in E : \|z\| = 1\}$ and $x \neq y$. Also, E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $x, y \in S$ and $\|x - y\| \geq \varepsilon$ imply $\|(x+y)/2\| < 1 - \delta$. A Banach space E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S$. In this case, the norm of E is said to be Gâteaux differentiable. The space E is said to have a uniformly Gâteaux differentiable norm if for each $y \in S$, the limit (2.1) is attained uniformly for $x \in S$. The norm of E is said to be Fréchet differentiable if for each $x \in S$, the limit (2.1) is attained uniformly for $y \in S$. The norm of E is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit (2.1) is attained uniformly for $x, y \in S$.

An operator $A \subset E \times E^*$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \cup\{Ax : x \in D(A)\}$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ for any $(x, x^*), (y, y^*) \in A$. An operator A is said to be strictly monotone if $\langle x - y, x^* - y^* \rangle > 0$ for any $(x, x^*), (y, y^*) \in A$ $(x \neq y)$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then the set $A^{-1}0 = \{u \in E : 0 \in Au\}$ is closed and convex (see [5, 25] for more details). A mapping $A : E \to E^*$ is said to be weakly sequentially continuous if $z_n \rightharpoonup z$ implies $Az_n \overset{*}{\rightharpoonup} Az$.

A function $f : E \to (-\infty, \infty]$ is said to be proper if the domain $D(f) = \{x \in E : f(x) < \infty\}$ is nonempty. It is also called lower semicontinuous if $\{x \in E : f(x) \leq r\}$ is closed for all $r \in \mathbb{R}$. The function f is also said to be convex if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha) f(y) \tag{2.2}$$

for all $x, y \in E$ and $\alpha \in (0, 1)$. It is also said to be strictly convex if the strict inequality holds in (2.2) for all $x, y \in D(f)$ with $x \neq y$ and $\alpha \in (0, 1)$. For a proper lower semicontinuous convex function $f : E \to (-\infty, \infty]$, the subdifferential ∂f of f is defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \le f(y),\ \forall y \in E\}$$

for all $x \in E$. It is well known that $\partial f \subset E \times E^*$ is maximal monotone (see [21, 22] for more details). A mapping $g : E \to \mathbb{R}$ is said to be strongly coercive if $g(z_n)/\|z_n\| \to \infty$ whenever $\{z_n\}$ is a sequence of E such that $\|z_n\| \to \infty$. It is also said to be bounded on bounded sets if $g(U)$ is bounded for each bounded subset U of E. If $p \in (1, \infty)$ and g is defined by $g(x) = \|x\|^p/p$ for all $x \in E$, then $\partial g = J_p$. For a proper lower semicontinuous convex function $f : E \to (-\infty, \infty]$, the conjugate function f^* of f is defined by

$$f^*(x^*) = \sup_{x \in E}\{\langle x, x^* \rangle - f(x)\}$$

for all $x^* \in E^*$. It is well known that $f(x)+f^*(x^*) \ge \langle x, x^* \rangle$ for all $(x, x^*) \in E \times E^*$. It is also known that $(x, x^*) \in \partial f$ is equivalent to

$$f(x) + f^*(x^*) = \langle x, x^* \rangle.$$

We also know that if $f : E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, then $f^* : E^* \to (-\infty, \infty]$ is a proper weak* lower semicontinuous convex function (see [19, 25] for more details on convex analysis).

3. Bregman distance

Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex function. Then the directional derivative $d^+g(x)(y)$ of g at $x \in E$ with the direction $y \in E$ is defined by

$$d^+g(x)(y) = \lim_{t \downarrow 0} \frac{g(x + ty) - g(x)}{t}.$$

The function g is said to be Gâteaux differentiable at $x \in E$ if $d^+g(x) \in E^*$. In this case, we denote $d^+g(x)$ by $\nabla g(x)$. The function g is also said to be Fréchet differentiable at $x \in E$ if for $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x - y\| \le \delta$ implies that

$$|g(y) - g(x) - \langle y - x, \nabla g(x) \rangle| \le \varepsilon \|y - x\|.$$

A function $g : E \to \mathbb{R}$ is said to be Gâteaux differentiable (resp. Fréchet differentiable) if it is Gâteaux differentiable at everywhere (resp. Fréchet differentiable at everywhere). We know that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous and $\partial g = \nabla g$. We also know that if g is Fréchet differentiable, then ∇g is norm-to-norm continuous.

Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the Bregman distance [2, 4] corresponding to g is defined by

$$D(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y) \rangle$$

for all $x, y \in E$. It is obvious that $D(x, y) \ge 0$ for all $x, y \in E$. We also know that $D(\cdot, y)$ is convex for all $y \in E$. The following definition is in the sense of Kohsaka and Takahashi [15] (see also [3]).

Definition 3.1. Let E be a Banach space. Then a function $g : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(1) g is continuous, strictly convex and Gâteaux differentiable;
(2) the set $\{y \in E : D(x,y) \le r\}$ is bounded for all $x \in E$ and $r > 0$.

We know the following Lemma (see [3, 26] for more details).

Lemma 3.2. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then*

(1) *$\nabla g : E \to E^*$ is one-to-one, onto, norm-to-weak* continuous and monotone;*
(2) *$\langle x - y, \nabla g(x) - \nabla g(y)\rangle = 0$ if and only if $x = y$;*
(3) *the set $\{x \in E : D(x,y) \le r\}$ is bounded for all $y \in E$ and $r > 0$;*
(4) *$D(g^*) = E^*$, g^* is Gâteaux differentiable and $\nabla g^* = (\nabla g)^{-1}$.*

We also know the following result (see [3, 14, 15, 17] for more details).

Theorem 3.3. *Let C be a nonempty closed convex subset of a reflexive Banach space E and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then, for each $x \in E$, there exists a unique $x_0 \in C$ such that*

$$D(x_0, x) = \min_{y \in C} D(y, x).$$

Moreover, for the mapping P_C defined by $P_C x = x_0$ for all $x \in E$, the following conditions hold: For $x \in E$,

(1) *$x_0 = P_C x$ if and only if $\langle y - x_0, \nabla g(x_0) - \nabla g(x)\rangle \ge 0$ for all $y \in C$;*
(2) *$D(P_C x, x) + D(y, P_C x) \le D(y, x)$ for all $y \in C$.*

The mapping P_C from E onto C is called the Bregman projection of E onto C.

Let E be a Banach space. The closed unit ball and the unit sphere of E are denoted by B and S, respectively. We also denote rB the set $\{z \in E : \|z\| \le r\}$ for all $r > 0$. Then a function $g : E \to \mathbb{R}$ is said to be uniformly convex on bounded sets [26] if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0,\infty) \to [0,\infty]$ is defined by

$$\rho_r(t) = \inf_{x,y \in rB, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha) g(y) - g(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)} \tag{3.1}$$

for all $t \ge 0$. It is known that ρ_r is a nondecreasing function. The function g is also said to be uniformly smooth on bounded sets [26] if $\lim_{t \downarrow 0} \sigma_r(t)/t = 0$ for all $r > 0$, where $\sigma_r : [0,\infty) \to [0,\infty]$ is defined by

$$\sigma_r(t) = \inf_{x \in rB, y \in S, \alpha \in (0,1)} \frac{\alpha g(\alpha x + (1-\alpha) t y) + (1-\alpha) g(x - \alpha t y) - g(x)}{\alpha(1-\alpha)}$$

for all $t \ge 0$. We know the following results (see [8, 15, 20, 26] for more details).

Theorem 3.4. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following are equivalent:*

(1) *g is bounded on bounded sets and uniformly smooth on bounded sets;*
(2) *g is Fréchet differentiable and ∇g is uniformly norm-to-norm continuous on bounded sets;*
(3) *$D(g^*) = E^*$, g^* is strongly coercive and uniformly convex on bounded sets.*

Theorem 3.5. *Let E be a Banach space, let $p \in (1,\infty)$ and let $g = \|\cdot\|^p/p$. Then*

(1) *E is uniformly convex iff g is uniformly convex on bounded sets;*

(2) *E is uniformly smooth iff g is uniformly smooth on bounded sets.*

Lemma 3.6. *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded sets. Let $r > 0$ and let ρ_r be defined as in* (3.1)*. Then the following hold:*

(1) *$\rho_r(\|x - y\|) \leq D(x, y)$ for each $x, y \in rB$;*
(2) *if $\{x_n\}$ and $\{y_n\}$ are sequences in rB such that $\lim_n D(x_n, y_n) = 0$, then $\lim_n \|x_n - y_n\| = 0$;*
(3) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in rB$ and $\rho_r(\|x - y\|) < \delta$ then $\|x - y\| < \varepsilon$.*

4. Bregman firmly generalized nonexpansive mappings

Let C be a nonempty subset of a Banach space E and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. A mapping $T : C \to C$ is said to be Bregman firmly generalized nonexpansive type [11, 17] if

$$D(x, Tx) + D(y, Ty) + D(Tx, Ty) + D(Ty, Tx) \leq D(x, Ty) + D(y, Tx) \tag{4.1}$$

for each $x, y \in C$. A mapping $T : C \to C$ is said to be Bregman generalized nonexpansive type [11, 17] if

$$D(Tx, Ty) + D(Ty, Tx) \leq D(x, Ty) + D(y, Tx)$$

for each $x, y \in C$. A mapping $T : C \to C$ is said to be Bregman firmly generalized nonexpansive [12, 17] if $F(T) \neq \emptyset$ and

$$D(x, Tx) + D(Tx, p) \leq D(x, p)$$

for each $x \in C$ and $p \in F(T)$. A mapping $T : C \to C$ is said to be Bregman generalized nonexpansive [10, 17] if $F(T) \neq \emptyset$ and

$$D(Tx, p) \leq D(x, p)$$

for each $x \in C$ and $p \in F(T)$. It is clear that Bregman firmly generalized nonexpansive type (resp. a Bregman firmly generalized nonexpansive) is Bregman generalized nonexpansive type (resp. a Bregman generalized nonexpansive) in a Banach space (see also [11, 12, 17]).

A point z in C is said to be Bregman generalized asymptotic fixed point of T [13, 17] if C contains a sequence $\{x_n\}$ such that $\nabla g(x_n) \overset{*}{\rightharpoonup} \nabla g(z)$ and $\nabla g(x_n) - \nabla g(Tx_n) \to 0$. The set of all Bregman generalized asymptotic fixed points of T is denoted by $\check{F}(T)$.

Let C_0 be a subset of C. A mapping $R : C \to C_0$ is said to be sunny if $R(Rx + t(x - Rx)) = Rx$ whenever $Rx + t(x - Rx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $R : C \to C_0$ is said to be a retraction if $R^2 = R$. The following results were proved in [17] (see also [16]).

Lemma 4.1. *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Let C be a nonempty closed subset of E. If $T : C \to C$ is a Bregman firmly generalized nonexpansive type mapping (resp. a Bregman generalized nonexpansive type mapping) with $F(T) \neq \emptyset$, then T is Bregman firmly generalized nonexpansive (resp. a Bregman generalized nonexpansive).*

Lemma 4.2. *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Let C be a nonempty closed subset of E. Then, a mapping $T : C \to C$ is of Bregman firmly generalized nonexpansive type if and only if*

$$\langle (x - Tx) - (y - Ty), \nabla g(Tx) - \nabla g(Ty) \rangle \geq 0.$$

for each $x, y \in C$.

Lemma 4.3. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Let C be a nonempty closed subset of E and let R be a retraction of E onto C. Then the following are equivalent:*

(1) *R is sunny and Bregman generalized nonexpansive;*
(2) *$\langle x - Rx, \nabla g(Rx) - \nabla g(y) \rangle \geq 0$ for each $x \in E$ and $y \in C$.*

Furthermore, a sunny Bregman generalized nonexpansive retraction of E onto C is uniquely determined.

Lemma 4.4. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Let C be a nonempty closed subset of E and let R be a sunny Bregman generalized nonexpansive retraction of E onto C. Let $x \in E$ and $z \in C$. Then the following hold:*

(1) *$z = Rx$ if and only if $\langle x - z, \nabla g(z) - \nabla g(y) \rangle \geq 0$ for all $y \in C$;*
(2) *$D(x, Rx) + D(Rx, z) \leq D(x, z)$.*

Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. If a sunny Bregman generalized nonexpansive retraction of E onto C exists then it is uniquely determined (see Lemma 4.3). A nonempty subset C of E is said to be a sunny Bregman generalized nonexpansive retract (resp. a Bregman generalized nonexpansive retract) of E if there exists a sunny Bregman generalized nonexpansive retraction (resp. a Bregman generalized nonexpansive retraction) of E onto C. The set of all fixed points of such a sunny Bregman generalized nonexpansive retraction of E onto C is, of course, C (see [9,10,17] for more details). The following results have been proved in [17] (see also [16]).

Theorem 4.5. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let C be a nonempty closed subset of E. Then, the following conditions are equivalent:*

(1) *C is a sunny Bregman generalized nonexpansive retract of E;*
(2) *C is a Bregman generalized nonexpansive retract of E;*
(3) *∇gC is closed and convex.*

In this case, the unique sunny Bregman generalized nonexpansive retraction of E onto C is given by $(\nabla g)^{-1} P_{C_*} \nabla g$, where P_{C_*} is the Bregman projection of E^* onto $C_* = \nabla gC$.

Lemma 4.6. *Let E be a reflexive Banach space, let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets and let C be a nonempty closed subset of E such that ∇gC is closed and convex. Let T be a mapping from C into itself. Then the following hold:*

(1) *if T is a Bregman generalized nonexpansive mapping, then $F(T)$ is closed and $\nabla gF(T)$ is closed and convex. Moreover, $F(T)$ is sunny Bregman generalized nonexpansive retract of E;*
(2) *if T is a Bregman generalized nonexpansive type mapping with $F(T) \neq \emptyset$, then $F(T) = \check{F}(T)$.*

5. Weak convergence theorem of Pazy's type

In this section, we prove a weak convergence theorem of Pazy's type for Bregman firmly generalized nonexpansive type mappings in a Banach space. We first recall the following result for Bregman firmly generalized nonexpansive type mappings in a Banach space (see [17] for more details).

Theorem 5.1. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous and strongly coercive function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let C be a nonempty closed subset of E such that ∇gC is closed and convex and let $T : C \to C$ be a Bregman firmly generalized nonexpansive type mapping. If the mapping ∇g is weakly sequentially continuous, then the following are equivalent:*
(1) *$F(T)$ is nonempty;*
(2) *$\{T^n x\}$ is bounded for some $x \in C$.*
In this case, $\{T^n x\}$ is converges weakly to an element of $F(T)$ for each $x \in C$.

To prove our result, we need the following lemmas.

Lemma 5.2. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Let C be a nonempty closed subset of E such that ∇gC is closed and convex and let R_C be a sunny Bregman generalized nonexpansive retraction of E onto C. Then R_C is of Bregman firmly generalized nonexpansive type.*

Proof. Let $x, y \in C$. Then, by Lemma 4.4, we obtain that
$$\langle x - R_C x, \nabla g(R_C x) - \nabla g(R_C y)\rangle \geq 0$$
and
$$\langle y - R_C y, \nabla g(R_C y) - \nabla g(R_C x)\rangle \geq 0.$$
From these inequalities, we have
$$\langle x - R_C x, \nabla g(R_C x) - \nabla g(R_C y)\rangle + \langle y - R_C y, \nabla g(R_C y) - \nabla g(R_C x)\rangle \geq 0 + 0.$$
and hence
$$\langle (x - R_C x) - (y - R_C y), \nabla g(R_C x) - \nabla g(R_C y)\rangle \geq 0$$
for each $x, y \in C$. Therefore, by Lemma 4.2, we obtain that R_C is of Bregman firmly generalized nonexpansive type. □

Lemma 5.3. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let C be a nonempty closed subset of E such that ∇gC is closed and convex. Let $T : C \to C$ be a Bregman generalized nonexpansive mapping. Then $\{RT^n x\}$ converges strongly to some element of $F(T)$*

for each $x \in C$, where R is the sunny Bregman generalized nonexpansive retraction of E onto $F(T)$.

Proof. Let $x \in C$. Then we have from Lemma 4.4 that

$$\begin{aligned} &D(T^{n+1}x, RT^{n+1}x) \\ &\quad \leq \; D(T^{n+1}x, RT^{n+1}x) + D(RT^{n+1}x, RT^n x) \\ &\quad \leq \; D(T^{n+1}x, RT^n x) \\ &\quad \leq \; D(T^n x, RT^n x) \end{aligned}$$

for each $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} D(T^n x, RT^n x)$ exists. It follows from Lemma 4.4 that, for each $k \in \mathbb{N}$,

$$D(T^{n+k}x, RT^{n+k}x) + D(RT^{n+k}x, RT^n x) \leq D(T^{n+k}x, RT^n x)$$

and hence

$$\begin{aligned} D(RT^m x, RT^n x) &\leq D(T^m x, RT^n x) - D(T^m x, RT^m x) \\ &\leq D(T^n x, RT^n x) - D(T^m x, RT^m x) \end{aligned} \tag{5.1}$$

for each $m, n \in \mathbb{N}$ $(m > n)$. Then we show that $\{RT^n x\}$ is a Cauchy sequence. In fact, since $F(T) \neq \emptyset$, we also obtain

$$D(RT^n x, p) \leq D(x, p)$$

for each $p \in F(T)$ and hence, by Lemma 3.2 (3), $\{RT^n x\}$ is bounded. Let $r = \sup_{n\in\mathbb{N}}\{\|RT^n x\|\}$. Using Lemma 3.6 (1), we obtain that

$$\rho_r(\|RT^m x - RT^n x\|) \leq D(RT^m x, RT^n x)$$

for each $m, n \in \mathbb{N}$ $(m > n)$. By (5.1), the existence of $\lim_{n\to\infty} D(T^n x, RT^n x)$ and Lemma 3.6 (3), $\{RT^n x\}$ is a Cauchy sequence. Since E is complete and $F(T)$ is closed, $\{RT^n x\}$ converges strongly to some point u in $F(T)$. □

Now, we can prove the following weak convergence theorem of Pazy's type for Bregman firmly generalized nonexpansive type mappings in a Banach space.

Theorem 5.4. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded sets, uniformly convex and uniformly smooth on bounded sets. Let C be a nonempty closed subset of E such that ∇gC is closed and convex. Let $T : C \to C$ be a Bregman firmly generalized nonexpansive type mapping. If the mapping ∇g is weakly sequentially continuous, then the following are equivalent:*

(1) *$F(T)$ is nonempty;*
(2) *$\{T^n x\}$ is bounded for some $x \in C$.*

In this case, $\{T^n x\}$ converges weakly to $p \in F(T)$ for each $x \in C$, where $p = \lim_{n\to\infty} RT^n x$ and R is a sunny generalized nonexpansive retraction of E onto $F(T)$.

Proof. From Theorem 5.1, we know that the conditions (1) and (2) are equivalent. Moreover, in this case, we also know that, for each $x \in C$, $\{T^n x\}$ converges weakly to an element $p \in F(T)$. Since Lemma 4.3, we have that

$$\langle T^n x - RT^n x, \nabla g(RT^n x) - \nabla g(p)\rangle \geq 0 \tag{5.2}$$

for each $n \in \mathbb{N}$. From Lemma 5.3, we have that $\{RT^n x\}$ converges strongly to some point u in $F(T)$. By Theorem 3.4, the mapping ∇g is (uniformly) norm to norm continuous. Therefore, letting $n \to \infty$ in (5.2), we obtain from $T^n x \rightharpoonup p$ and $RT^n x \to u$ that

$$\langle p - u, \nabla g(u) - \nabla g(p) \rangle \geq 0.$$

By Lemma 3.2 (1) and (2), we obtain that $u = p$. Therefore, $\{T^n x\}$ converges weakly to $p = \lim_{n\to\infty} RT^n x$. This completes the proof. □

6. Weak convergence theorems of Baillon's type

In this section, we prove weak convergence theorems of Baillon's type in a Banach space. To obtain our result, we need the following lemma.

Lemma 6.1. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is Fréchet differentiable. Let C be a nonempty closed subset of E such that ∇gC is closed and convex and let R be a sunny Bregman generalized nonexpansive mapping of E onto C. Then R is demiclosed, i.e., $x_n \rightharpoonup x_0$ and $Rx_n \to y_0$ imply $Rx_0 = y_0$.*

Proof. Let $\{x_n\}$ be a sequence of E such that $x_n \rightharpoonup x_0$ and $Rx_n \to y_0$. Since g is Fréchet differentiable function, then the mapping ∇g is norm to norm continuous and hence $\nabla g(Rx_n) \to \nabla g(y_0)$. Using Lemma 4.4, we have that

$$\langle x_0 - Rx_0, \nabla g(Rx_0) - \nabla g(u) \rangle \geq 0 \tag{6.1}$$

and

$$\langle x_n - Rx_n, \nabla g(Rx_n) - \nabla g(u) \rangle \geq 0$$

for each $u \in C$. Letting $n \to \infty$, we get

$$\langle x_0 - y_0, \nabla g(y_0) - \nabla g(u) \rangle \geq 0 \tag{6.2}$$

for each $u \in C$. Since $\{Rx_n\} \subset C$ and $Rx_n \to y_0$, from closedness of C we have $y_0 \in C$. By (6.1) and (6.2), we have

$$\langle x_0 - Rx_0, \nabla g(Rx_0) - \nabla(y_0) \rangle \geq 0 \text{ and } \langle x_0 - y_0, \nabla g(y_0) - \nabla g(Rx_0) \rangle \geq 0$$

and hence

$$\langle y_0 - Rx_0, \nabla g(Rx_0) - \nabla g(y_0) \rangle \geq 0.$$

From Lemma 3.2 (1) and (2), we obtain that $y_0 = Rx_0$. This implies that R is demiclosed. □

Finally, we can prove the following weak convergence theorems of Baillon's type in a Banach space.

Theorem 6.2. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded sets, uniformly convex and uniformly smooth on bounded sets. Let C be a nonempty closed subset of E such that ∇gC is closed and convex. Let $T : C \to C$ be a Bregman firmly generalized nonexpansive mapping such that $\check{F}(T) = F(T)$. If the mapping ∇g is weakly sequentially continuous, then $\{T^n x\}$ converges weakly to some $u \in F(T)$ for each $x \in C$, where $u = \lim_{n\to\infty} RT^n x$ and R is a sunny Bregman generalized nonexpansive retraction of E onto $F(T)$.*

Further, if $R_0x = \text{w-lim}_{n\to\infty} T^n x$ *for each* $x \in C$*, then* R_0 *is a Bregman generalized nonexpansive retraction of* C *onto* $F(T)$ *such that* $R_0T^n = T^nR_0 = R_0$ *for each* $n \in \mathbb{N}$ *and*

$$R_0x \in \overline{co}\{T^n x : n \in \mathbb{N}\}$$

for each $x \in C$.

Proof. Let $x \in C$ and $p \in F(T)$. From the definition of T, we have that

$$D(T^{n+1}x, p) \le D(T^n x, T^{n+1}x) + D(T^{n+1}x, p) \le D(T^n x, p) \tag{6.3}$$

for each $n \in \mathbb{N}$ and hence $\lim_{n\to\infty} D(T^n x, p)$ exists. From (6.3), we obtain that

$$D(T^n x, T^{n+1}x) \le D(T^n x, p) - D(T^{n+1}x, p)$$

for each $n \in \mathbb{N}$. Since $\{D(T^n x, p)\}$ converges, it follows that

$$\lim_{n\to\infty} D(T^n x, T^{n+1}x) = 0. \tag{6.4}$$

Since $\lim_{n\to\infty} D(T^n x, p)$ exists, by Lemma 3.2 (3) $\{T^n x\}$ is bounded. Let $r = \sup_{n\in\mathbb{N}}\{\|T^n x\|\}$. From Lemma 3.6 (2) and (6.4), we obtain that

$$\lim_{n\to\infty} \|T^n x - T^{n+1}x\| = 0. \tag{6.5}$$

From Theorem 3.4, ∇g is uniformly norm to norm continuous, we have that

$$\lim_{n\to\infty} \Big(\nabla g(T^n x) - \nabla g(T^{n+1}x)\Big) = 0. \tag{6.6}$$

For a subsequence $\{T^{n_i}x\}$ of $\{T^n x\}$ such that $T^{n_i}x \rightharpoonup p$ for some $p \in E$, we have from weakly sequentially continuity of ∇g that $\nabla g(T^{n_i}x) \rightharpoonup \nabla g(p)$. Since $\{\nabla g(T^{n_i}x)\} \subset \nabla gC$ and ∇gC is closed and convex, ∇gC is weakly closed and hence we have that $\nabla g(p) \in \nabla gC$. This implies that $p \in C$. Since $\check{F}(T) = F(T)$, p is a fixed point of T.

On the other hand, from Lemma 5.3, $\{RT^n x\}$ converges strongly to some $u \in F(T)$. Since $T^{n_i}x \rightharpoonup p$, from Lemma 6.1 we have $Rp = u$. It follows from $p \in F(T) = F(R)$ that $u = p$. This implies that $T^n x \rightharpoonup u = \lim_{n\to\infty} RT^n x$.

Defining a mapping R_0 from C to itself by

$$R_0x := \underset{n\to\infty}{\text{w-lim}}\, T^n x$$

for each $x \in C$. It is obvious that $R(R_0) \subset F(T)$, which $R(R_0)$ is the range of R_0. Conversely, let $z \in F(T)$. Then we have

$$R_0z = \underset{n\to\infty}{\text{w-lim}}\, T^n z = \underset{n\to\infty}{\text{w-lim}}\, z = z.$$

So, we have $z \in R(R_0)$ and hence

$$R(R_0) \subset F(T) \subset F(R_0) \subset R(R_0).$$

Therefore, we get $F(T) = F(R_0) = R(R_0)$. This implies that R_0 is a retraction of C onto $F(T)$. Let $x \in C$ and $p \in F(R_0) = F(T)$. Since the function g is weakly lower semicontinuous, we have

$$D(R_0x, p) \le \liminf_{n\to\infty} D(T^n x, p) \le \liminf_{n\to\infty} D(x, p) = D(x, p).$$

This implies that R_0 is Bregman generalized nonexpansive. It is obvious from $F(T) = R(R_0)$ that $TR_0 = R_0$. Moreover, we have that for any $x \in C$,

$$R_0x = \operatorname*{w\text{-}lim}_{n\to\infty} T^n x = \operatorname*{w\text{-}lim}_{n\to\infty} T^{n+1}x = \operatorname*{w\text{-}lim}_{n\to\infty} T^n(Tx) = R_0Tx$$

and hence $R_0x = R_0Tx$. So, we have $TR_0 = R_0T = R_0$. This implies that $T^nR_0 = R_0T^n = R_0$ for each $n \in \mathbb{N}$. Finally, we show that

$$R_0x \in \overline{co}\{T^n x : n \in \mathbb{N}\}(=: D)$$

for each $x \in C$. Suppose that $R_0z \notin D$ for some $z \in C$. From the separation theorem, there exists $z^* \in E^*$ such that $\langle R_0z, z^*\rangle > \sup_{y\in D}\langle y, z^*\rangle$. So, we have that

$$\begin{aligned}\langle R_0z, z^*\rangle &> \sup_{y\in D}\langle y, z^*\rangle \\ &\geq \sup_{n\in\mathbb{N}}\langle T^n z, z^*\rangle \\ &\geq \lim_{n\to\infty}\langle T^n z, z^*\rangle = \langle R_0z, z^*\rangle.\end{aligned}$$

This is a contradiction. So, we have $R_0x \in \overline{co}\{T^n x : n \in \mathbb{N}\}$ for each $x \in C$. This completes the proof. □

As a direct direct consequence of Theorem 6.2, we obtain following result.

Theorem 6.3. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded sets, uniformly convex and uniformly smooth on bounded sets. Let C be a nonempty closed subset of E such that ∇gC is closed and convex. Let $T : C \to C$ be a Bregman firmly generalized nonexpansive type mapping such that $F(T) \neq \emptyset$. If the mapping ∇g is weakly sequentially continuous, then $\{T^n x\}$ converges weakly to some $u \in F(T)$ for each $x \in C$, where $u = \lim_{n\to\infty} RT^n x$ and R is a sunny Bregman generalized nonexpansive retraction of E onto $F(T)$.*

Further, if $R_0x = \text{w-lim}_{n\to\infty} T^n x$ for each $x \in C$, then R_0 is a Bregman generalized nonexpansive retraction of C onto $F(T)$ such that $R_0T^n = T^nR_0 = R_0$ for each $n \in \mathbb{N}$ and

$$R_0x \in \overline{co}\{T^n x : n \in \mathbb{N}\}$$

for each $x \in C$.

Proof. From Lemmas 4.1 and 4.6 we have that T is a Bregman firmly generalized nonexpansive mapping and $\check{F}(T) = F(T)$. As a direct consequence of Theorem 6.2, □

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