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ON PROJECTION REFLECTION METHOD IN HILBERT SPACES

SHIN-YA MATSUSHITA AND LI XU

Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday.

Abstract. We study the finite convergence of a variant of a projection reflection method for computing a point in the intersection of a closed convex set and an obtuse cone in a Hilbert space. We present a variant of the projection reflection method, and prove that the proposed variant converges to a solution to the problem in a finite number of iterations under certain assumptions.

1. Introduction

Let C and D be closed and convex sets in a Hilbert space H. This paper deals with the convex feasibility problem of the form

$$\text{find } u \in \mathrm{int}C \cap D, \tag{1.1}$$

where int$C$ is the interior of C. Problem (1.1) has applications in various areas. For instance, the *linear matrix inequality* (LMI) feasibility problem is a well-known type of convex feasibility problem that is a central issue in control theory and has found many applications in control system analysis and design [5, 6, 12, 15, 17–19].

The method of alternating projections [7, 16] is a standard algorithm for computing a point in the intersection of some sets. In the convex case, this method has the following form: given $x_0 \in H$,

$$x_n = P_D P_C(x_{n-1}) \ (n = 1, 2, \dots), \tag{1.2}$$

where P_C and P_D are metric projections onto C and D, respectively. When C and D are subspaces, von Neumann [16] proved that $\{x_n\}$ converges strongly to $P_{C\cap D}(x_0)$ in $C \cap D$. When C and D are closed and convex sets, Bregman [7] proved that $\{x_n\}$ converges weakly to some point in $C \cap D$. We refer to [2, 8–10, 14, 20, 21] for recent developments of (1.2).

Recently, Rami, Helmke and Moore [18] revealed the finite convergence of (1.2). They investigated the following variant of (1.2):

$$x_n = P_D P_{e+C}(x_{n-1}) \ (n = 1, 2, \dots), \tag{1.3}$$

where $e \in H$. When C and D are closed and convex cones, $e \in \mathrm{int}C$ and $(e+C)\cap D \neq \emptyset$, the sequence generated by (1.3) can solve (1.1) in a finite number of iterations. Moreover, an explicit upper bound for the required number of iterations is also obtained [18, Theorem 2.3]. An interesting application of (1.3) is the LMI feasibility problem [6, 12, 17, 18]. Reference [17] proved that (1.3) performs favorably with a

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large number of constraints, relative to the number of variables. However, the sequence generated by the method of alternating projections may suffer from slow convergence (for example, consider the case where two sets are hyperplanes with small angles between them) and oscillate for many iterations between the two sets before it converges to a solution (see [10, 17, 20] for related results). This motivates the development of practically efficient methods for solving (1.1).

From a computational point of view, several methods have been proposed to accelerate the convergence of the method of alternating projections [1–4,8,10,13–15]. Bauschke and Kruk [4] investigated a reflection projection method when C is an obtuse cone (i.e., C is cone and includes the positive polar cone of C). Cegielski [8] investigated a method which is based on the reflection projection method, and, to be distinguishable, called it a projection reflection method. The projection reflection method in [8] has the following form:

$$x_n = R_C P_D^{\mu}(x_{n-1}) \ (n = 1, 2, \dots), \tag{1.4}$$

where $x_0 \in H$, $R_C = 2P_C - I$, $P_D^{\mu} = (1-\mu)I + \mu P_C$, $\mu \in (0, 2)$ and I is the identity mapping on H. The sequence generated by (1.4) converges weakly to some point in $C \cap D$ [8, Corollary 5.5.2]. However, the projection reflection method can not be applied directly to solve (1.1) because the projection of the set $\mathrm{int}C$ does not exist for points exterior to C.

The main purpose of this paper is to investigate the finite convergence of a variant of (1.4). We propose a variant of (1.4) that can be applied to solve (1.1), and establish the finite convergence of the sequence generated by the proposed variant. Moreover, we establish an explicit upper bound for the required number of iterations.

The paper is organized as follows. Section 2 introduces the main definitions and some necessary preliminaries. Section 3 presents a variant of the projection reflection method and proves that the sequence generated by the proposed variant has the finite convergence property. The conclusion will be shown in Section 4.

2. Basic definitions and preliminaries

The following notations will be used in this paper: H denotes a real Hilbert space; for any $x, y \in H$, $\langle x, y\rangle$ denotes the inner product; for any $z \in H$, $\|z\|$ denotes the norm of z, i.e., $\|z\| = \sqrt{\langle z, z\rangle}$; $\mathrm{int}A$ denotes the interior of set A; A^c denotes the complement of A; for any $w \in H$ and $B \subset H$, $w + B$ is a parallel translation, i.e., $w + B = \{w + b : b \in B\}$; for any $C \subset H$ and mapping $U : C \to C$, $\mathrm{Fix}(U)$ denotes the fixed point set of U, i.e., $\mathrm{Fix}(U) = \{x \in C : U(x) = x\}$; for any $E, F \subset H$, $\mathrm{dist}(E, F)$ denotes the distance between two sets E and F, i.e., $\mathrm{dist}(E, F) = \inf\{\|x - y\| : x \in E, y \in F\}$.

Let C be a closed and convex subset of H. A mapping $U : C \to C$ is said to be firmly nonexpansive if

$$\|U(x) - U(y)\| \le \langle x - y, U(x) - U(y)\rangle \ (x, y \in C),$$

and is said to be nonexpansive if

$$\|U(x) - U(y)\| \le \|x - y\| \ (x, y \in C).$$

Then, by Lemma 1.1 in [11] (see also [2, Proposition 4.2]), U is firmly nonexpansive if and only if $2U - I$ is nonexpansive.

The metric projection of a point $x \in H$ onto C, denoted by $P_C(x)$, is defined as a unique solution of the problem

$$\text{minimize } \|x - y\| \text{ subject to } y \in C.$$

R_C denotes the reflector with respect to C, i.e., $R_C = 2P_C - I$ [4]. We list the following useful properties of the metric projection.

(i) P_C is firmly nonexpansive;
(ii) R_C is nonexpansive;
(iii) $P_C^\lambda = (1 - \lambda)I + \lambda P_C$ is nonexpansive for all $\lambda \in [0, 2]$.

The proofs of these results can be found in [2, 8, 9, 20, 21].

The positive polar cone K^* of a closed convex cone K in H is defined by

$$K^* = \{y \in H : \langle y, x\rangle \geq 0, \text{ for all } x \in K\}.$$

K is said to be obtuse if $K^* \subset K$ [4]. The following result can be found in [15, Lemma 3.1].

Lemma 2.1. *Let C be a closed convex and obtuse cone in H, let $\lambda \in [1, 2]$ and let $e \in H$. Then, for all $x \in H$, $P_{e+C}^\lambda(x) \in e + C$.*

3. Finite convergence of a variant of projection reflection method

In this section, we consider the finite convergence of a variant of the projection reflection method. We consider the following iterative method:

$$\begin{cases} y_{n-1} = P_D(x_{n-1}) \\ x_n = P_{e+C}^\lambda P_D^\mu(x_{n-1}) \ (n = 1, 2, \dots), \end{cases} \tag{3.1}$$

where $x_0 \in H$, $\mu \in (0, 2)$ and $\lambda \in [1, 2]$. Note that if $e = 0$ and $\lambda = 2$, then $\{x_n\}$ in (3.1) is equivalent to (1.4), and thus $\{x_n\}$ can be viewed as a more general modified version of the one of (1.4).

To establish the finite convergence of (3.1), we need the following assumption.

$$\text{int}C \neq \emptyset, \ e \in \text{int}C \text{ and } (e + C) \cap D \neq \emptyset. \tag{3.2}$$

The following results are useful for establishing the finite convergence of (3.1).

Proposition 3.1 ([18, Lemma 2.3])**.** *Let C be a closed convex and cone in H such that* $\text{int}C \neq \emptyset$*. For any element $e \in \text{int}C$ we have that*

$$\text{dist}(e + C, (\text{int}C)^c) > 0.$$

Proposition 3.2 ([8, Theorem 2.1.51])**.** *Let C and D be closed convex sets in H, $S : H \to C$ be nonexpansive, and $\mu \in (0, 2)$. If* $\text{Fix}(S) \cap D \neq \emptyset$*, then,*

$$\|SP_D^\mu(x) - z\|^2 \leq \|x - z\|^2 - \mu(2 - \mu)\|P_D(x) - x\|^2,$$

for all $x \in \text{Fix}(S)$ and $z \in \text{Fix}(S) \cap D$,

We next prove the following result.

Lemma 3.3. *Let $\{y_n\}$ be a sequence generated by (3.1). Then, for any $m \in \mathbb{N}$, we have*

$$\operatorname{dist}(x_0, (e+C)\cap D)^2 \geq \mu(2-\mu)\sum_{k=1}^{m}\|y_k - x_{k-1}\|^2. \tag{3.3}$$

Proof. Let $u \in (e+C)\cap D$. Since the assumptions of Proposition 3.2 are satisfied at this theorem by taking $S = P^{\lambda}_{e+C}$, we have that

$$\begin{aligned}\|x_n - u\|^2 &= \|P^{\lambda}_{e+C}P^{\mu}_{D}(x_{n-1}) - u\|^2\\ &\leq \|x_{n-1} - u\|^2 - \mu(2-\mu)\|P_D(x_{n-1}) - x_{n-1}\|^2\\ &\leq \|x_{n-1} - u\|^2 - \mu(2-\mu)\|y_n - x_{n-1}\|^2.\end{aligned} \tag{3.4}$$

From (3.4), we get

$$\begin{aligned}\|x_0 - u\|^2 &\geq \|x_1 - u\|^2 + \mu(2-\mu)\|y_1 - x_0\|^2\\ &\geq \|x_2 - u\|^2 + \mu(2-\mu)\|y_1 - x_0\|^2 + \mu(2-\mu)\|y_2 - x_1\|^2\\ &\geq \cdots\\ &\geq \|x_m - u\|^2 + \mu(2-\mu)\sum_{k=1}^{m}\|y_k - x_{k-1}\|^2.\end{aligned}$$

Since $u \in (e+C)\cap D$ is arbitrary, we get

$$\operatorname{dist}(x_0, (e+C)\cap D)^2 = \inf_{u\in(e+C)\cap D}\|x_0 - u\|^2 \geq \mu(2-\mu)\sum_{k=1}^{m}\|y_k - x_{k-1}\|^2.$$

□

The main theorem can now be given as follows.

Theorem 3.4. *Let C be a closed convex and obtuse cone in H, let D be a closed convex set in H such that (3.2) holds. Then, the sequence $\{y_n\}$ generated by (3.1) converges at a point in $\operatorname{int}C \cap D$ at most l iterations with*

$$l \leq \frac{\operatorname{dist}(x_0, (e+C)\cap D)^2}{\mu(2-\mu)\gamma(e)^2} + 1, \tag{3.5}$$

where $\gamma(e) = \operatorname{dist}(e+C, D\cap(\operatorname{int}C)^c)$.

Proof. Proposition 3.1 guarantees that $\gamma(e) > 0$. By Lemma 3.3, we have $\lim_{n\to\infty}\|y_n - x_{n-1}\| = 0$. In this case, there exists the smallest integer l such that $\|y_l - x_{l-1}\| < \gamma(e)$. We consider two cases: $y_l \in \operatorname{int}C$ and $y_l \notin \operatorname{int}C$. In the case of $y_l \in \operatorname{int}C$, using Lemma 3.3, $\operatorname{dist}(x_0, (e+C)\cap D)^2 \geq (l-1)\mu(2-\mu)\gamma(e)^2$ and we found a solution in at most l iterations. This implies that inequality (3.5) holds. In the case of $y_l \notin \operatorname{int}C$, we have

$$\|y_l - x_{l-1}\| \geq \operatorname{dist}(e+C, D\cap(\operatorname{int}C)^c) = \gamma(e),$$

which is a contradiction. □

4. Conclusion

In this paper we have presented the iterative method (3.1) for solving (1.1). We proved that (3.1) has the finite convergence property under the assumption of (3.2). Our method is closely related to the reflection projection method investigated in [4, 8]. Related results for the LMI problems can be found in [15, 17, 18].

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Shin-ya Matsushita
Department of Electronics and Information Systems, Akita Prefectural University, 84-4 Yuri-Honjo, Akita, Japan
E-mail address: matsushita@akita-pu.ac.jp

Li Xu
Department of Electronics and Information Systems, Akita Prefectural University, 84-4 Yuri-Honjo, Akita, Japan
E-mail address: xuli@akita-pu.ac.jp