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NOTE ON KNEŽEVIĆ-MILJANOVIĆ'S THEOREM IN A CLASS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday.

Abstract. In this paper, we show the existence and uniqueness of solutions of the Cauchy problem in a class of singular fractional differential equations. Let $1 < \alpha \leq 2$. We consider the Cauchy problem

$$\begin{cases} D^{\alpha}_{0+}u(t) = p(t)t^a u(t)^{\sigma}, \\ \lim_{t\to 0+} u(t) = 0, \ \lim_{t\to 0+} \dfrac{u'(t)}{t^{\alpha-2}} = (\alpha-1)\lambda \end{cases}$$

where p is continuous, $a, \sigma, \lambda \in \mathbf{R}$ with $\sigma < 0$, $\lambda > 0$ and D^{α}_{0+} is the Riemann-Liouville fractional derivative. If $\alpha = 2$, then this problem is the problem in [6].

1. Introduction

In [6], Knežević-Miljanović considered the Cauchy problem for singular differential equations

$$\begin{cases} u''(t) = p(t)t^a u(t)^{\sigma}, \\ \lim_{t\to 0+} u(t) = 0, \ \lim_{t\to 0+} u'(t) = \lambda, \end{cases} \tag{1.1}$$

where p is continuous, $a, \sigma, \lambda \in \mathbf{R}$ with $\sigma < 0$ and $\lambda > 0$. She proved that if p satisfies

$$\int_0^1 |p(t)|t^{a+\sigma}dt < \infty, \tag{1.2}$$

then the problem has a solution. For related results of the Cauchy problem for singular differential equations (1.1), see [2], [3] and [4].

On the other hand, fractional differential equations have been studied by many mathematicians. For instance, in [1] and [8], the authors considered the fractional differential equation

$$D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0$$

where $1 < \alpha \leq 2$ and D^{α}_{0+} is the Riemann-Liouville fractional derivative. The αth Riemann-Liouville fractional derivative of u is given by

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds$$

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where n is an integer with $n-1 \leq \alpha < n$ and $\Gamma(\cdot)$ is the gamma function. If $\alpha = 2$, then $n = 3$ and

$$D^2_{0+}u(t) = \frac{1}{\Gamma(1)}\frac{d^3}{dt^3}\int_0^t u(s)ds = u''(t).$$

Recently, there have been studies concerned with initial value problems for singular fractional differential equations, for instance, see [7], [9] and [10]. But in the obtained results, initial value problems (1.1) cannot be treated.

In this paper, we show the existence and uniqueness of solutions of the Cauchy problem (1.1) in a class of singluar fractional differential equations. Let $1 < \alpha \leq 2$. We consider the Cauchy problem

$$\text{(1.3)} \qquad \begin{cases} D^\alpha_{0+}u(t) = p(t)t^a u(t)^\sigma, \\ \lim_{t\to 0+} u(t) = 0, \ \lim_{t\to 0+} \dfrac{u'(t)}{t^{\alpha-2}} = (\alpha-1)\lambda \end{cases}$$

where p is continuous, $a, \sigma, \lambda \in \mathbf{R}$ with $\sigma < 0$ and $\lambda > 0$ If $\alpha = 2$, then the Cauchy problem (1.3) is the problem (1.1).

2. Main result

In this section, we derive first the integral equation which is equivalent to the problem (1.3) (Lemma 2.2). Next, by using the Banach fixed point theorem, we establish the existence and uniqueness result for solutions of the problem (1.3) (Theorem 2.3).

Let u be a continuous function of $(0,\infty)$ into $\mathbf{R}$ and α be a positive real number. The αth Riemman-Liouville fractional integral of u is defined by

$$I^\alpha_{0+}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds.$$

The following lemma can be found in [1] and [5]. We denote by $C(0,1)$ the space of all continuous functions of $(0,1)$ into $\mathbf{R}$. Moreover we denote by $L(0,1)$ the space of all integrable functions of $(0,1)$ into $\mathbf{R}$.

Lemma 2.1. *Let $\alpha > 0$. Let $u \in C(0,1)\cap L(0,1)$ satisfying $D^\alpha_{0+}u \in C(0,1)\cap L(0,1)$. Then*

$$I^\alpha_{0+}D^\alpha_{0+}u(t) = u(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_nt^{\alpha-n}$$

for some $C_1, C_2, \ldots, C_n \in \mathbf{R}$ and an integer n with $n-1 \leq \alpha < n$.

Next we derive the integral equation which is equivalent to the problem (1.3).

Lemma 2.2. *Let $1 < \alpha \leq 2$. Let p be a continuous function of $[0,1]$ into $\mathbf{R}$. Let $a \in \mathbf{R}$ and $\sigma < 0$. Let $\lambda > 0$. If u is a solution of the Cauchy problem (1.3), then u is a solution of the equation*

$$\text{(2.1)} \qquad u(t) = \lambda t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)s^a u(s)^\sigma ds.$$

Moreover if p satisfies

$$\text{(2.2)} \qquad \lim_{t\to 0+} t^{a+(\alpha-1)\sigma+1}\int_0^1 |p(ts)|s^{a+(\alpha-1)\sigma}(1-s)^{\alpha-2}ds = 0,$$

then u is a solution of the Cauchy problem (1.3) if and only if u is a solution of the equation (2.1) under the assumption that $\frac{\lambda}{2}t^{\alpha-1} \le u(t)$ for all $t \in (0,1]$.

Proof. Let u be a solution of (1.3). Since $D^{\alpha}_{0+}u(t) = p(t)t^a u(t)^{\sigma}$, we obtain the integral equation

$$u(t) = I^{\alpha}_{0+}p(t)t^a u(t)^{\sigma} + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}$$

for some C_1 and C_2 by Lemma 2.1. By the definition of the Riemman-Liouville fractional integral I^{α}_{0+}, we have

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)s^a u(s)^{\sigma}ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}.$$

The condition $\lim_{t\to 0} u(t) = 0$ implies $C_2 = 0$. Thus

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)s^a u(s)^{\sigma}ds + C_1 t^{\alpha-1}.$$

Since

$$\lim_{t\to 0}\frac{u'(t)}{t^{\alpha-2}} = (\alpha-1)C_1,$$

we have $C_1 = \lambda$. Therefore we obtain that u is a solution of the equation (2.1).

Let u be a solution of the equation (2.1) under the assumption that $\frac{\lambda}{2}t^{\alpha-1} \le u(t)$ for all $t \in (0,1]$. Suppose that p satisfies (2.2). We will show that u is a solution of the Cauchy problem (1.3). Since u satisfies (2.1), we have $D^{\alpha}_{0+}u(t) = p(t)t^a u(t)^{\sigma}$. Since

$$\begin{aligned}
|u(t)| &\le \lambda t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|p(s)|s^a u(s)^{\sigma}ds \\
&\le \lambda t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\left(\frac{\lambda}{2}\right)^{\sigma}\int_0^t (t-s)^{\alpha-1}|p(s)|s^{a+(\alpha-1)\sigma}ds \\
&= \lambda t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\left(\frac{\lambda}{2}\right)^{\sigma} t^{a+(\alpha-1)\sigma+\alpha}\int_0^1 |p(ts)|s^{a+(\alpha-1)\sigma}(1-s)^{\alpha-1}ds \\
&\le \lambda t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\left(\frac{\lambda}{2}\right)^{\sigma} t^{a+(\alpha-1)\sigma+1}\int_0^1 |p(ts)|s^{a+(\alpha-1)\sigma}(1-s)^{\alpha-2}ds
\end{aligned}$$

and (2.2), we obtain that $\lim_{t\to 0+} u(t) = 0$. Since

$$u'(t) = (\alpha-1)\lambda t^{\alpha-2} + \frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}p(s)s^a u(s)^{\sigma}ds,$$

we obtain that

$$\begin{aligned}
\left|\frac{u'(t)}{t^{\alpha-2}} - (\alpha-1)\lambda\right| &\le \frac{\alpha-1}{\Gamma(\alpha)}\int_0^t \left(1-\frac{s}{t}\right)^{\alpha-2}|p(s)|s^a u(s)^{\sigma}ds \\
&\le \frac{\alpha-1}{\Gamma(\alpha)}\left(\frac{\lambda}{2}\right)^{\sigma}\int_0^t \left(1-\frac{s}{t}\right)^{\alpha-2}|p(s)|s^{a+(\alpha-1)\sigma}ds \\
&= \frac{\alpha-1}{\Gamma(\alpha)}\left(\frac{\lambda}{2}\right)^{\sigma} t^{a+(\alpha-1)\sigma+1}\int_0^1 |p(ts)|s^{a+(\alpha-1)\sigma}(1-s)^{\alpha-2}ds.
\end{aligned}$$

By (2.2), we have $\lim_{t\to 0+}\frac{u'(t)}{t^{\alpha-2}}=(\alpha-1)\lambda$. Therefore u is a solution of the Cauchy problem (1.3). □

If a continuous function p satisfies the condition (1.2)

$$\int_0^1 |p(s)|s^{a+\sigma}ds<\infty,$$

where $a,\sigma,\lambda\in\mathbf{R}$ with $\sigma<0$ and $\lambda>0$, then p satisfies (2.2). Indeed, by (1.2), we have

$$\lim_{t\to 0+}\int_0^t |p(s)|s^{a+\sigma}ds=0.$$

Since

$$\int_0^t |p(s)|s^{a+\sigma}ds=t^{a+\sigma+1}\int_0^1 |p(ts)|s^{a+\sigma}ds,$$

we have

$$\lim_{t\to 0+} t^{a+\sigma+1}\int_0^1 |p(ts)|s^{a+\sigma}ds=0.$$

Therefore p in [6] satisfies the condition (2.2).

Using Lemma 2.2, we can show our main result.

Theorem 2.3. *Let $1<\alpha\le 2$. Let p be a continuous function of $[0,1]$ into $\mathbf{R}$ satisfying (2.2) for $a\in\mathbf{R}$ and $\sigma<0$. Let $\lambda>0$. Then there exists a unique solution $u:(0,h]\to\mathbf{R}$ of the Cauchy problem (1.3) satisfying $\frac{\lambda}{2}t^{\alpha-1}\le u(t)$ for all $t\in(0,h]$.*

Proof. By Lemma 2.2, we consider the integral equation

$$u(t)=\lambda t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)s^a u(s)^\sigma ds.$$

Let $K\in(0,1)$. Choose $0<h<1$ satisfying

$$t^{a+(\alpha-1)\sigma+1}\int_0^1 |p(ts)|s^{a+(\alpha-1)\sigma}(1-s)^{\alpha-2}ds\le\Gamma(\alpha)\left(\frac{\lambda}{2}\right)^{1-\sigma}$$

and

$$t^{a+(\alpha-1)\sigma+1}\int_0^1 |p(ts)|s^{a+(\alpha-1)\sigma}(1-s)^{\alpha-2}ds\le\frac{\Gamma(\alpha)}{|\sigma|}\left(\frac{\lambda}{2}\right)^{1-\sigma}K$$

for all $t\in(0,h]$. We denote by $C[0,h]$ the space of all continuous functions of $[0,h]$ into $\mathbf{R}$ with the maximum norm given by $\|u\|=\max_{0\le t\le h}|u(t)|$ for $u\in C[0,h]$. Let X be a subset of $C[0,h]$ defined by

$$X=\left\{u\in C[0,h]\mid u(0)=0,\lim_{t\to 0+}\frac{u'(t)}{t^{\alpha-2}}=(\alpha-1)\lambda,\frac{\lambda}{2}t^{\alpha-1}\le u(t),\forall t\in(0,h]\right\}.$$

Since a mapping $t\longmapsto\lambda t^{\alpha-1}$ belongs to X, we have $X\ne\emptyset$. Let A be an operator of X into $C[0,h]$ defined by

$$Au(t)=\lambda t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)s^a u(s)^\sigma ds.$$

Then $A(X) \subset X$. Indeed, let $u \in X$. Then we can show similarly as in the proof of Lemma 2.2 that $Au(0) = 0$ and $\lim_{t\to 0} \frac{(Au)'(t)}{t^{\alpha-2}} = (\alpha-1)\lambda$. Moreover we obtain that

$$\begin{aligned}
Au(t) &\geq \lambda t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |p(s)| s^a u(s)^\sigma ds \\
&\geq \lambda t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \left(\frac{\lambda}{2}\right)^\sigma \int_0^t (t-s)^{\alpha-1} |p(s)| s^{a+(\alpha-1)\sigma} ds \\
&= \lambda t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \left(\frac{\lambda}{2}\right)^\sigma t^{a+(\alpha-1)\sigma+\alpha} \int_0^1 |p(ts)| s^{a+(\alpha-1)\sigma} (1-s)^{\alpha-1} ds \\
&\geq \lambda t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \left(\frac{\lambda}{2}\right)^\sigma t^{a+(\alpha-1)\sigma+\alpha} \int_0^1 |p(ts)| s^{a+(\alpha-1)\sigma} (1-s)^{\alpha-2} ds \\
&= \lambda t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \left(\frac{\lambda}{2}\right)^\sigma \left(t^{a+(\alpha-1)\sigma+1} \int_0^1 |p(ts)| s^{a+(\alpha-1)\sigma} (1-s)^{\alpha-2} ds \right) t^{\alpha-1} \\
&\geq \lambda t^{\alpha-1} - \frac{\lambda}{2} t^{\alpha-1} \\
&= \frac{\lambda}{2} t^{\alpha-1}.
\end{aligned}$$

Therefore we have $Au \in X$.

We will find a fixed point of A. Let φ be an operator of X into $C[0,h]$ defined by

$$\varphi[u](t) = \begin{cases} \frac{u(t)}{t^{\alpha-1}}, & t \neq 0, \\ \lambda, & t = 0. \end{cases}$$

Then we have

$$\varphi[X] = \left\{ z \in C[0,h] \mid z(0) = \lambda,\ \frac{\lambda}{2} \leq z(t), \forall t \in [0,h] \right\}$$

and $\varphi[X]$ is a closed subset of $C[0,h]$. Hence it is a complete metric space. Let Φ_A be an operator of $\varphi[X]$ into $\varphi[X]$ defined by

$$\Phi_A \varphi[u] = \varphi[Au].$$

By the mean value theorem for all $u_1, u_2 \in X$ there exists a mapping ξ such that

$$\frac{u_1^\sigma(t) - u_2^\sigma(t)}{u_1(t) - u_2(t)} = \sigma \xi(t)^{\sigma-1},$$

where

$$\min\{u_1(t), u_2(t)\} \leq \xi(t) \leq \max\{u_1(t), u_2(t)\}$$

for almost every $t \in [0,h]$. For $t \in (0,h]$, we have

$$\begin{aligned}
|\Phi_A\varphi[u_1](t) - \Phi_A\varphi[u_2](t)| &= |\varphi[Au_1](t) - \varphi[Au_2](t)| \\
&= \left| \frac{1}{t^{\alpha-1}\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) s^a (u_1(s)^\sigma - u_2(s)^\sigma) ds \right| \\
&\leq \frac{1}{t^{\alpha-1}\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |p(s)| s^a |u_1(s)^\sigma - u_2(s)^\sigma| ds.
\end{aligned}$$

Since

$$\begin{aligned}|u_1(s)^\sigma - u_2(s)^\sigma| &= |\sigma||\xi(s)|^{\sigma-1}|u_1(s)-u_2(s)| \\ &\le |\sigma|\left(\frac{\lambda}{2}s^{\alpha-1}\right)^{\sigma-1}|u_1(s)-u_2(s)|\end{aligned}$$

for all $s \in (0,h]$, we have

$$\begin{aligned}&|\Phi_A\varphi[u_1](t) - \Phi_A\varphi[u_2](t)| \\ &\le \frac{1}{t^{\alpha-1}\Gamma(\alpha)}\left(\frac{\lambda}{2}\right)^{\sigma-1}\int_0^t (t-s)^{\alpha-1}|p(s)|s^{a+(\alpha-1)\sigma}\left|\frac{u_1(s)}{s^{\alpha-1}} - \frac{u_2(s)}{s^{\alpha-1}}\right| ds \\ &\le \frac{1}{t^{\alpha-1}\Gamma(\alpha)}\left(\frac{\lambda}{2}\right)^{\sigma-1}\left(\int_0^t (t-s)^{\alpha-1}|p(s)|s^{a+(\alpha-1)\sigma}ds\right)\|\varphi[u_1]-\varphi[u_2]\| \\ &\le \frac{1}{\Gamma(\alpha)}\left(\frac{\lambda}{2}\right)^{\sigma-1}\left(t^{a+(\alpha-1)\sigma+1}\int_0^1 |p(ts)|s^{a+(\alpha-1)\sigma}(1-s)^{\alpha-1}ds\right)\|\varphi[u_1]-\varphi[u_2]\| \\ &\le \frac{1}{\Gamma(\alpha)}\left(\frac{\lambda}{2}\right)^{\sigma-1}\left(t^{a+(\alpha-1)\sigma+1}\int_0^1 |p(ts)|s^{a+(\alpha-1)\sigma}(1-s)^{\alpha-2}ds\right)\|\varphi[u_1]-\varphi[u_2]\| \\ &\le K\|\varphi[u_1]-\varphi[u_2]\|\end{aligned}$$

for all $t \in [0,h]$. Therefore we obtain that

$$\|\Phi_A\varphi[u_1] - \Phi_A\varphi[u_2]\| \le K\|\varphi[u_1]-\varphi[u_2]\|.$$

Hence Φ_A is contractive. By the Banach fixed point theorem, there exists a unique mapping $\varphi[u] \in \varphi[X]$ of Φ_A. Since $\Phi_A\varphi[u] = \varphi[u]$, we have $Au = u$. By Lemma 2.2, u is a unique solution of the Cauchy problem (1.3). □

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