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ON THE CONVEXITY OF VON NEUMANN-JORDAN CONSTANT

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This paper is dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday.

Abstract. For a vector space X, we denote the set of all norms on X by N_X and define a function on N_X by $f(\|\cdot\|) = C_{\mathrm{NJ}}((X, \|\cdot\|))$ where $C_{\mathrm{NJ}}((X, \|\cdot\|))$ is the von Neumann-Jordan constant of normed linear space $(X, \|\cdot\|)$. We examine the convexity of f on N_X and on some subsets of N_X.

The notion of the von Neumann-Jordan constant of normed linear spaces was introduced by Clarkson in [1] and recently it has been studied by several authors (cf. [2, 3, 4, 5]). The von Neumann-Jordan constant $C_{\mathrm{NJ}}((X, \|\cdot\|))$ of normed linear space $(X, \|\cdot\|)$ is defined by

$$C_{\mathrm{NJ}}((X, \|\cdot\|)) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : (x, y) \neq (0, 0) \right\}.$$

It is well-known that $1 \leq C_{\mathrm{NJ}}((X, \|\cdot\|)) \leq 2$ for any normed linear space $(X, \|\cdot\|)$, and $(X, \|\cdot\|)$ is an inner product space if and only if $C_{\mathrm{NJ}}((X, \|\cdot\|)) = 1$. If $1 \leq p \leq \infty$, $\dim L_p \geq 2$, then $C_{\mathrm{NJ}}(L_p) = 2^{2/\min\{p,p'\}-1}$, where $1/p + 1/p' = 1$. We can describe several geometrical and topological properties of normed linear spaces by means of the von Neumann-Jordan constant (see [3, 4]).

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x, y)\| = \|(|x|, |y|)\|$ for all $(x, y) \in \mathbb{R}^2$, and normalized if $\|(1, 0)\| = \|(0, 1)\| = 1$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by AN_2. Let Ψ_2 be the set of all convex functions ψ on $[0, 1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $t \in [0, 1]$. Ψ_2 and AN_2 can be identified by a one to one correspondence $\psi \to \|\cdot\|_\psi$ with the relation $\psi(t) = \|(1-t, t)\|_\psi$ for $t \in [0, 1]$ (see [5]). For $1 \leq p \leq \infty$, we denote

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{\frac{1}{p}} & (1 \leq p < \infty) \\ \max\{1-t, t\} & (p = \infty). \end{cases}$$

Then $\psi_p \in \Psi_2$, and ψ_p corresponds to the l_p-norms $\|\cdot\|_p$ on $\mathbb{R}^2$ defined by

$$\|(x, y)\|_p = \begin{cases} (|x|^p + |y|^p)^{\frac{1}{p}} & (1 \leq p < \infty) \\ \max\{|x|, |y|\} & (p = \infty). \end{cases}$$

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The set Ψ_2 has a convex structure, that is,

$$\psi,\ \psi' \in \Psi_2 \Longrightarrow (1-\lambda)\psi + \lambda\psi' \in \Psi_2$$

for $\lambda \in [0,1]$. The correspondence $\psi \to \|\cdot\|_\psi$ preserves the operation to take a convex combination, that is,

$$(1-\lambda)\|\cdot\|_\psi + \lambda\|\cdot\|_{\psi'} = \|\cdot\|_{(1-\lambda)\psi+\lambda\psi'}$$

for $\psi,\ \psi' \in \Psi_2$ and $\lambda \in [0,1]$. Hence, Ψ_2 and AN_2 are isomorphic with respect to this convex structure.

In [4], we considered the convex property of von Neumann-Jordan constant on AN_2. More generally, let X be a real (or complex) vector space and let N_X be the set of all norms on X. We define a function on N_X by $f(\|\cdot\|) = C_{\rm NJ}((X, \|\cdot\|))$. We showed that f is a convex function on N_X(Theorem 3.1 in [4]). However, this result is not correct.

Our aim in this note is to present a partial result that f is convex on certain subsets of AN_2 and also present counterexamples of Theorem 3.1 and Corollary 3.1 in [4].

In this paper, we only consider a *real* vector space. For a *complex* vector space, we can similarly prove the same statements and so omit them.

We denote

$$\Psi_2^+ = \{\psi \in \Psi_2 : \psi(t) \geq \psi_2(t)\ (t \in [0,1])\}$$

and

$$\Psi_2^- = \{\psi \in \Psi_2 : \psi(t) \leq \psi_2(t)\ (t \in [0,1])\}.$$

Then Ψ_2^+ and Ψ_2^- are obviously convex subsets of Ψ_2.

Theorem 1. *Keep the notations as above. Then the function $\psi \to C_{\rm NJ}((\mathbb{R}^2, \|\cdot\|_\psi))$ is convex on each of Ψ_2^+ and Ψ_2^-. That is, if $\psi, \psi' \in \Psi_2^+$ or $\psi, \psi' \in \Psi_2^-$, then for any λ with $0 \leq \lambda \leq 1$,*

$$C_{\rm NJ}((\mathbb{R}^2, \|\cdot\|_{(1-\lambda)\psi+\lambda\psi'})) \leq (1-\lambda)C_{\rm NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) + \lambda C_{\rm NJ}((\mathbb{R}^2, \|\cdot\|_{\psi'})).$$

Proof. For each $t \in [0,1]$ we define a function $g_t : \Psi_2^+ \to [1,2]$ by

$$g_t(\psi) = \frac{\psi(t)^2}{\psi_2(t)^2}.$$

By the convexity of the function $f_1(x) = x^2$, we have $((1-\lambda)x+\lambda y)^2 \leq (1-\lambda)x^2+\lambda y^2$ for $x, y \in \mathbb{R}, \lambda \in [0,1]$. Hence

$$\begin{aligned} g_t((1-\lambda)\psi + \lambda\psi') &= ((1-\lambda)\psi(t) + \lambda\psi'(t))^2/\psi_2(t)^2 \\ &\leq (1-\lambda)\psi(t)^2/\psi_2(t)^2 + \lambda\psi'(t)^2/\psi_2(t)^2 \\ &= (1-\lambda)g_t(\psi) + \lambda g_t(\psi') \end{aligned}$$

holds for $\psi, \psi' \in \Psi_2^+, \lambda \in [0,1]$, and this means that each g_t is convex on Ψ_2^+. By [5, Theorem 1], we have

$$C_{\rm NJ}((\mathbb{R}^2, \|\cdot\|_\psi)) = \sup_{t\in[0,1]} g_t(\psi)$$

for $\psi \in \Psi_2^+$. Since the convexity preserves under taking supremum, we can conclude that $\psi \to C_{\rm NJ}((\mathbb{R}^2, \|\cdot\|_\psi))$ is a convex function on Ψ_2^+.

Since the function $f_2(x) = \frac{1}{x^2}$ is also convex on $(0, \infty)$, we can conclude similarly that for each $t \in [0,1]$ the function $h_t(\psi) = \psi_2(t)^2/\psi(t)^2$ is convex on Ψ_2^-. Thus the convexity of $\psi \to C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_\psi))$ on Ψ_2^- can be shown. This completes the proof. $\square$

We showed that f is a convex function on N_X(Theorem 3.1 in [4]). However, this result is not correct. We first present a counterexample of Theorem 3.1 in [4].

Theorem 2. *Let X be a vector space with $\dim X \geq 2$. Then f is not convex on N_X.*

Proof. We first consider that $X = \mathbb{R}^2$. Let $\|\cdot\|_2$ be the ℓ_2-norm on $\mathbb{R}^2$. Since $(\mathbb{R}^2, \|\cdot\|_2)$ is an inner product space, we have $C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_2)) = 1$. We define the norm $\|\cdot\|$ on $\mathbb{R}^2$ by $\|(x,y)\| = \|(2x, y)\|_2$. Since $\|\cdot\|$ is induced from the inner product

$$\langle (x_1, y_1), (x_2, y_2)\rangle' = 4x_1x_2 + y_1y_2,$$

we have $C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|)) = 1$. We now define a norm $\|\cdot\|_0$ on $\mathbb{R}^2$ by

$$\|\cdot\|_0 = \frac{\|\cdot\| + \|\cdot\|_2}{2}.$$

We take $e_1 = (1,0)$ and $e_2 = (0,1)$. Then we have

$$\begin{aligned}\|e_1 + e_2\|_0^2 + \|e_1 - e_2\|_0^2 &= \frac{7 + 2\sqrt{10}}{2} \\ &\neq \frac{13}{2} = 2(\|e_1\|_0^2 + \|e_2\|_0^2).\end{aligned}$$

This implies that $(\mathbb{R}^2, \|\cdot\|_0)$ is not an inner product space. Hence,

$$f(\|\cdot\|_0) > 1 = \frac{f(\|\cdot\|) + f(\|\cdot\|_2)}{2}.$$

Therefore f is not convex on $N_{\mathbb{R}^2}$.

We next consider that X is a real vector space with $\dim X \geq 2$. Let $\{e_i\}_{i \in I}$ be a Hamel basis of X. For any $x \in X$, we can write $x = \sum_{i\in I} a_i e_i$ $(a_i \in \mathbb{R})$, where $\{i \in I : a_i \neq 0\}$ is at most finite. For any $x = \sum_{i \in I} a_i e_i,\ y = \sum_{i\in I} b_i e_i \in X$, we define an inner product on X by

$$\langle x, y\rangle = \sum_{i \in I} a_i b_i.$$

Then the induced norm $\|\cdot\|_2$ is defined by $\|x\|_2 = (\sum_{i\in I} |a_i|^2)^{1/2}$. We choose an $i_0 \in I$ and define another norm $\|\cdot\|$ by

$$\|x\| = \big(|2a_{i_0}|^2 + \sum_{i \neq i_0} |a_i|^2\big)^{1/2}$$

for any $x = \sum_{i \in I} a_i e_i \in X$. Then $(X, \|\cdot\|)$ is also an inner product space, because $\|\cdot\|$ is induced from the inner product

$$\langle x, y\rangle' = 4a_{i_0}b_{i_0} + \sum_{i \neq i_0} a_i b_i.$$

We now put $\|\cdot\|_0 = \frac{\|\cdot\|+\|\cdot\|_2}{2}$. As in the case of $\mathbb{R}^2$, $(X, \|\cdot\|_0)$ does not satisfy the parallelogram law and so $(X, \|\cdot\|_0)$ is not an inner product space. Hence we have

$$f(\|\cdot\|_0) > 1 = \frac{f(\|\cdot\|) + f(\|\cdot\|_2)}{2}.$$

Therefore, f is not convex on N_X. This completes the proof. $\square$

Next we consider the convexity of f on AN_2. To do this, we only consider the function g on Ψ_2 defined by $g(\psi) = C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_\psi))$ for $\psi \in \Psi_2$. Let $\Psi_2^s = \{\psi \in \Psi_2 : \psi(1-t) = \psi(t)$ for any $t \in [0,1]\}$. Then Ψ_2^s is a convex subset of Ψ_2. First we have the following theorem.

Theorem 3. *Keep the notations as above. Then the function g is not convex on Ψ_2^s.*

Proof. We define a convex function $\varphi_1 \in \Psi_2^s$ by

$$\varphi_1(t) = \max\left\{\frac{3}{\sqrt{10}}\psi_2(t), \psi_\infty(t), \frac{1}{\sqrt{2}}\right\},$$

that is,

$$\varphi_1(t) = \begin{cases} 1-t & (0 \le t \le 1/4), \\ \frac{3}{\sqrt{10}}\psi_2(t) & (1/4 \le t \le 1/3), \\ \frac{1}{\sqrt{2}} & (1/3 \le t \le 2/3), \\ \frac{3}{\sqrt{10}}\psi_2(t) & (2/3 \le t \le 3/4), \\ t & (3/4 \le t \le 1). \end{cases}$$

Since $\varphi_1 \le \psi_2$ and $\frac{\psi_2}{\varphi_1}$ has the maximum at $t = 1/4$, by [5, Theorem 1], we have

$$g(\varphi_1) = C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_{\varphi_1})) = \frac{\psi_2(1/4)^2}{\varphi_1(1/4)^2} = \frac{10}{9}.$$

We next define a convex function $\varphi_2 \in \Psi_2^s$ by $\varphi_2(t) = \max\{\psi_2(t), \frac{3}{4}\}$. Since $\varphi_2 \ge \psi_2$ and $\frac{\varphi_2}{\psi_2}$ has the maximum at $t = 1/2$, by [5, Theorem 1], we have

$$g(\varphi_2) = C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_{\varphi_2})) = \frac{\varphi_2(1/2)^2}{\psi_2(1/2)^2} = \frac{9}{8}.$$

We now put the convex function $\varphi = \frac{\varphi_1+\varphi_2}{2}$. Then $\varphi \in \Psi_2^s$ and

$$\varphi(t) = \begin{cases} \frac{1-t+\psi_2(t)}{2} & (0 \le t \le 1/4), \\ \frac{3\sqrt{10}+10}{20}\psi_2(t) & (1/4 \le t \le \frac{4-\sqrt{2}}{8}), \\ \frac{3}{8} + \frac{3}{2\sqrt{10}}\psi_2(t) & (\frac{4-\sqrt{2}}{8} \le t \le 1/3), \\ \frac{3+2\sqrt{2}}{8} & (1/3 \le t \le 2/3), \\ \frac{3}{8} + \frac{3}{2\sqrt{10}}\psi_2(t) & (2/3 \le t \le \frac{4+\sqrt{2}}{8}), \\ \frac{3\sqrt{10}+10}{20}\psi_2(t) & (2/3 \le t \le 3/4), \\ \frac{t+\psi_2(t)}{2} & (3/4 \le t \le 1). \end{cases}$$

Since $\frac{\varphi}{\psi_2}$ has the maximum at $t = 1/2$, we have

$$M_1 = \max_{0\le t\le 1} \frac{\varphi(t)}{\psi_2(t)} = \frac{\varphi(1/2)}{\psi_2(1/2)} = \frac{4+3\sqrt{2}}{8}.$$

Since $\frac{\psi_2}{\varphi}$ has the maximum at $t = 1/4$, we have

$$M_2 = \max_{0\le t\le 1} \frac{\psi_2(t)}{\varphi(t)} = \frac{\psi_2(1/4)}{\varphi(1/4)} = 2\sqrt{10}(\sqrt{10}-3).$$

By [5, Theorem 1], we have

$$\begin{aligned} g(\varphi) &= C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_\varphi)) = M_1^2 M_2^2 \\ &= \frac{5}{4}(17+12\sqrt{2})(19-6\sqrt{10}) \risingdotseq 1.1182276549\cdots. \end{aligned}$$

On the other hand,

$$\frac{g(\varphi_1)+g(\varphi_2)}{2} = \frac{\frac{10}{9}+\frac{9}{8}}{2} = \frac{161}{144} \risingdotseq 1.1180555555\cdots.$$

Therefore we have

$$g(\varphi) > \frac{g(\varphi_1)+g(\varphi_2)}{2}.$$

This completes the proof. □

Corollary 3.1 in [4] asserts the function g is convex on Ψ_2. However, this is not correct. Since Ψ_2^s is a convex subset of Ψ_2, we immediately have the following by Theorem 3.

Corollary 4. *Keep the notations as above. Then the function g is not convex on Ψ_2.*

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