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ON THE JAMES CONSTANTS OF TWO-DIMENSIONAL LORENTZ SEQUENCE SPACES AND ITS DUAL

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This paper is dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday.

Abstract. Let $J(X)$ denote the James constant of a Banach space X. Then, for a Banach space X and its dual X^*, it is known that $J(X) \neq J(X^*)$ in general. In this paper, we show that $J(d^{(2)}(\omega, q)) = J(d^{(2)}(\omega, q)^*)$ for all $1 < q < \infty$ and $0 < \omega < 1$, where $d^{(2)}(\omega, q)$ is a two-dimensional Lorentz sequence space. We also give some remarks on the James constant of $J(d^{(2)}(\omega, q))$.

1. Introduction and preliminaries

Let S_X be the unit sphere of a Banach space X. The *James constant* $J(X)$ of a Banach space X is defined by

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}$$

(Gao and Lau [2]). It is well-known that $\sqrt{2} \leq J(X) \leq 2$ for any Banach space X, and $J(X) < 2$ if and only if X is uniformly non-square ([2, 4]).

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(z,w)\| = \|(|z|,|w|)\|$ for all $(z,\, w) \in \mathbb{R}^2$, and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. The ℓ_p-norms $\|\cdot\|_p$ are such examples;

$$\|(z,w)\|_p = \begin{cases} (|z|^p + |w|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|z|,|w|\} & \text{if } p = \infty. \end{cases}$$

Let AN_2 be the set of all absolute normalized norms on $\mathbb{R}^2$, and Ψ_2 the set of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ $(0 \leq t \leq 1)$. As in Bonsall and Duncan [1] (cf. [10]), AN_2 and Ψ_2 are in 1-1 correspondence under the equation

$$\psi(t) = \|(1-t,\, t)\| \ (0 \leq t \leq 1). \tag{1.1}$$

Indeed, for all $\psi \in \Psi_2$ let

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\dfrac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \neq (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_2$, and $\|\cdot\|_\psi$ satisfies (1.1). From this result, we can consider many non ℓ_p-type norms easily. Now let $\psi_p(t) = \{(1-t)^p + t^p\}^{1/p} \in \Psi_2$. As is easily seen, the ℓ_p-norm $\|\cdot\|_p$ is associated with ψ_p.

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An important example of absolute normed spaces is the following 2-dimensional Lorentz sequence space $d^{(2)}(\omega, q)$. Let $0 < \omega < 1$ and $1 \leq q < \infty$. The space $d^{(2)}(\omega, q)$ is $\mathbb{R}^2$ with the norm

$$\|(x, y)\|_{\omega,q} = (x^{*q} + \omega y^{*q})^{1/q},$$

where (x^*, y^*) is the non-increasing rearrangement of $(|x|, |y|)$, that is, $x^* \geq y^*$ (cf. [3]).

In [3], Kato and Maligranda considered the James constant of $d^{(2)}(\omega, q)$ and calculated it in the case where $q \geq 2$. For $1 \leq q < 2$ it was completely determined by Mitani, Saito and Suzuki [8]. Furthermore, we completely computed the James constant of the dual space $d^{(2)}(\omega, q)^*$ of $d^{(2)}(\omega, q)$ as in Mitani and Saito [6]; see also [11, 12].

As an important remark on James constants, it is known that $J(X) \neq J(X^*)$ in general (cf. [4]). However, the equality holds if $X = \ell_p$, that is, $J(\ell^p) = J(\ell^q)$ for each $1 < p, q < \infty$ satisfying $1/p + 1/q = 1$. Motivated by these observation, we consider the following problem in this paper.

Problem. *Does the equality* $J(d^{(2)}(\omega, q)) = J(d^{(2)}(\omega, q)^*)$ *hold for all* $1 < q < \infty$ *and* $0 < \omega < 1$*?*

In fact, this problem is partially solved by some results in [3, 7, 8], that is, we have the following theorem.

Theorem 1.1. *Let* $1 < q < \infty$*. Then*

$$J(d^{(2)}(\omega, q)) = 2\left(\frac{1}{1+\omega}\right)^{1/q} = J(d^{(2)}(\omega, q)^*)$$

if either $q \geq 2$ *or* $1 < q < 2$ *and* $0 < \omega \leq (\sqrt{2} - 1)^{2-q}$.

Thus, throughout this paper, we consider the case of $1 < q < 2$ and $(\sqrt{2}-1)^{2-q} < \omega < 1$ unless otherwise stated.

2. James constant of absolute norms on $\mathbb{R}^2$

Let X be a Banach space and $x, y \in X$. We say that x is isosceles orthogonal to y, denoted by $x \perp_I y$, if $\|x + y\| = \|x - y\|$. We define the function $\beta(x)$ on X by

$$\beta(x) = \sup\{\min\{\|x + y\|, \|x - y\|\} : y \in S_X\}.$$

To calculate the constant $J((\mathbb{R}^2, \|\cdot\|_\psi))$, we need the following lemma given in Gao and Lau [2].

Lemma 2.1 ([2])**.** *Let* $\psi \in \Psi_2$ *and* $x \in S_{(\mathbb{R}^2,\|\cdot\|_\psi)}$*. Then there exists a unique (up to the sign) vector* $y_0 \in S_{(\mathbb{R}^2,\|\cdot\|_\psi)}$ *with* $x \perp_I y_0$*. Moreover,* $\beta(x) = \|x + y_0\|_\psi$*.*

From Lemma 2.1 we can write

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \sup\left\{\|x + y\|_\psi : x, y \in S_{(\mathbb{R}^2,\|\cdot\|_\psi)} \text{ with } x \perp_I y\right\}.$$

We recall that an absolute normalized norm $\|\cdot\|$ on $\mathbb{R}^2$ is symmetric in the sense that $\|(x, y)\| = \|(y, x)\|$ for all $(x, y) \in \mathbb{R}^2$ if and only if the corresponding function ψ is symmetric with respect to $t = 1/2$, that is, $\psi(1 - t) = \psi(t)$ for every $t \in [0, 1]$.

Using Lemma 2.1 we gave the following formula for the case where ψ is symmetric with respect to $t = 1/2$.

Theorem 2.2 ([6]). *Let $\psi \in \Psi_2$. If ψ is symmetric with respect to $t = 1/2$, then*

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \max_{0\le t\le 1/2} \frac{2-2t}{\psi(t)}\psi\left(\frac{1}{2-2t}\right).$$

We consider a function in the theorem above:

$$f(t) = \frac{2-2t}{\psi(t)}\psi\left(\frac{1}{2-2t}\right) \quad \text{for all } t \in [0, 1/2].$$

Since $f(0) = 2\psi(1/2)$ and $f(1/2) = 1/\psi(1/2)$, we always have the inequality

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) \ge \max\{2\psi(1/2), 1/\psi(1/2)\}.$$

As a direct consequence of Theorem 2.2, we have

Proposition 2.3 ([6]). *Let $\psi \in \Psi_2$. Assume that ψ is symmetric with respect to $t = 1/2$.*

(i) *If $\psi \ge \psi_2$ and $M_1 = \max_{0\le t\le 1} \psi(t)/\psi_2(t)$ is taken at $t = 1/2$, then*

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = 2\psi\left(\frac{1}{2}\right).$$

(ii) *If $\psi \le \psi_2$ and $M_2 = \max_{0\le t\le 1} \psi_2(t)/\psi(t)$ is taken at $t = 1/2$, then*

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \frac{1}{\psi(1/2)}.$$

Example 2.4. Let $1 \le p \le \infty$ and $1/p + 1/p' = 1$.

(i) If $1 \le p \le 2$, then $J((\mathbb{R}^2, \|\cdot\|_p)) = 2\psi_p(1/2) = 2^{1/p}$.

(ii) If $2 \le p \le \infty$, then $J((\mathbb{R}^2, \|\cdot\|_p)) = 1/\psi_p(1/2) = 2^{1/p'}$.

Example 2.5. Let $1/2 \le \beta \le 1$, and let $\psi_\beta(t) = \max\{1-t, t, \beta\}$. Then

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) = \begin{cases} 1/\beta = 1/\psi_\beta(1/2) & \text{if } 1/2 \le \beta \le 1/\sqrt{2}, \\ 2\beta = 2\psi_\beta(1/2) & \text{if } 1/\sqrt{2} \le \beta \le 1. \end{cases}$$

Here, we propose the following problem: Does there exist a function $\psi \in \Psi_2$ satisfying

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) > \max\{2\psi(1/2), 1/\psi(1/2)\}?$$

In Section 4, we discuss the case of two dimensional Lorentz sequence spaces.

3. James constant of $d^{(2)}(w, q)$

Let X be a normed space. As in [2], the Schäffer constant $g(X)$ of X is defined by

$$g(X) = \inf\{\max\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

By [2, Theorem 2.5], we have

Proposition 3.1. *Let X be a normed space. Then $1 \le g(X) \le \sqrt{2} \le J(X) \le 2$ and $g(X)J(X) = 2$.*

Note here that the norm $\|\cdot\|_{\omega,q}$ of $d^{(2)}(\omega,q)$ is a symmetric absolute normalized norm on $\mathbb{R}^2$, and the corresponding convex function is given by

$$\psi_{\omega,q}(t)=\begin{cases}((1-t)^q+\omega t^q)^{1/q} & \text{if } 0\le t\le 1/2,\\ (t^q+\omega(1-t)^q)^{1/q} & \text{if } 1/2\le t\le 1.\end{cases}$$

Theorem 3.2 ([8]). *Let $1<q<2$ and $1/p+1/q=1$. If $(\sqrt{2}-1)^{2-q}<\omega<1$, then there exists a unique pair of real numbers s_0, s_1 such that*

$$\left(\frac{1-\omega}{\omega(1+\omega)}\right)^{p-1}<s_0<\omega^{1/(2-q)}<s_1<1$$

and $(1+s_i)^{q-1}(1-\omega s_i^{q-1})=\omega(1-s_i)^{q-1}(1+\omega s_i^{q-1})$ for $i=0,1$.

(i) *If $(\sqrt{2}-1)^{2-q}<\omega\le\sqrt{2}^q-1$, then*

$$J(d^{(2)}(\omega,q))=\max\left\{\left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q}, 2\left(\frac{1}{1+\omega}\right)^{1/q}\right\}$$

and

$$g(d^{(2)}(\omega,q))=\min\left\{\left(\frac{2(1+s_1)^{q-1}}{1+\omega s_1^{q-1}}\right)^{1/q}, (1+\omega)^{1/q}\right\}.$$

(ii) *If $\sqrt{2}^q-1<\omega<1$, then*

$$J(d^{(2)}(\omega,q))=\left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q}$$

and

$$g(d^{(2)}(\omega,q))=\left(\frac{2(1+s_1)^{q-1}}{1+\omega s_1^{q-1}}\right)^{1/q}.$$

Proof. Let f be a real-valued function on $[0,1/2]$ given by

$$f(t)=\frac{2-2t}{\psi_{\omega,q}(t)}\psi_{\omega,q}\left(\frac{1}{2-2t}\right)=\left(\frac{\omega(1-2t)^q+1}{(1-t)^q+\omega t^q}\right)^{1/q}$$

for all $t\in[0,1/2]$. We also put

$$g(s)=f\left(\frac{s}{1+s}\right)=\left(\frac{(1+s)^q+\omega(1-s)^q}{1+\omega s^q}\right)^{1/q}$$

for all $s\in[0,1]$. Then the derivative of g is

$$g'(s)=\frac{((1+s)^q+\omega(1-s)^q)^{1/q-1}}{(1+\omega s^q)^{1/q+1}} \times\{(1+s)^{q-1}(1-\omega s^{q-1})-\omega(1-s)^{q-1}(1+\omega s^{q-1})\}.$$

We put $\alpha=q-1$ and define a function g_1 from $[0,1]$ into $\mathbb{R}$ by

$$g_1(s)=(1+s)^\alpha(1-\omega s^\alpha)-\omega(1-s)^\alpha(1+\omega s^\alpha)$$

for s with $0 \le s \le 1$. If $(\sqrt{2}-1)^{2-q} < \omega < 1$, then $\omega^{1/(2-q)} > \sqrt{2}-1$. Thus we have

$$\omega^{1/(2-q)} > \frac{1-\omega^{1/(2-q)}}{1+\omega^{1/(2-q)}} > \frac{1-\omega}{1+\omega}.$$

We consider a function $h(s) = \omega^{1-p}s^{p-1}$ on $[0,1]$. Since h is an increasing function on $[0,1]$,

$$\left(\frac{1-\omega}{\omega(1+\omega)}\right)^{p-1} = h\left(\frac{1-\omega}{1+\omega}\right) < h(\omega^{1/(2-q)}) = \omega^{1/(2-q)}.$$

Thus we have

$$\left(\frac{1-\omega}{\omega(1+\omega)}\right)^{p-1} < \omega^{1/(2-q)}.$$

Now, since $(p-1)(q-1) = 1$, one has that

$$\begin{aligned} &g_1\left(\left(\frac{1-\omega}{\omega(1+\omega)}\right)^{p-1}\right) \\ &= \frac{2\omega}{1+\omega}\left[\left\{1+\left(\frac{1-\omega}{\omega(1+\omega)}\right)^{p-1}\right\}^{q-1} - \left\{1-\left(\frac{1-\omega}{\omega(1+\omega)}\right)^{p-1}\right\}^{q-1}\right] \\ &> 0. \end{aligned}$$

These observation and [8, Table 1] together show that there uniquely exists a pair of real numbers s_0, s_1 such that

$$\left(\frac{1-\omega}{\omega(1+\omega)}\right)^{p-1} < s_0 < \omega^{1/(2-q)} < s_1 < 1$$

and $g_1(s_0) = g_1(s_1) = 0$. For $i = 0, 1$, we have

$$f\left(\frac{s_i}{1+s_i}\right) = g(s_i) = \left(\frac{2(1+s_i)^{q-1}}{1+\omega s_i^{q-1}}\right)^{1/q}.$$

We now remark that

$$f\left(\frac{s}{1+s}\right) f\left(\frac{1-s}{2}\right) = 2$$

for any $s \in [0,1]$. Then, as was shown in [8], the function $g(s) = f(s/(1+s))$ has a maximal at s_0 and a minimal at s_1, and so, the function $f((1-s)/2) = 2/f(s/(1+s))$ has a minimal at s_0 and a maximal at s_1.

s	0		s_0		s_1		1
$f\left(\frac{s}{1+s}\right)$		$\nearrow$		$\searrow$		$\nearrow$	
$f\left(\frac{1-s}{2}\right)$		$\searrow$		$\nearrow$		$\searrow$	

This implies that $f((1-s_0)/2) = f(s_1/(1+s_1))$ and $f((1-s_1)/2) = f(s_0/(1+s_0))$. Hence it must be $s_1/(1+s_1) = (1-s_0)/2$, which implies that

$$f\left(\frac{s_0}{1+s_0}\right) f\left(\frac{s_1}{1+s_1}\right) = 2.$$

Thus we finally have this theorem by Proposition 3.1. This completes the proof. □

4. James constant of $d^{(2)}(w,q)^*$

For $\psi \in \Psi_2$ let $\|\cdot\|_\psi^*$ be the dual of the norm $\|\cdot\|_\psi$. Namely,

$$\|x\|_\psi^* = \sup\left\{|\langle x,y\rangle| : y \in S_{(\mathbb{R}^2,\|\cdot\|_\psi)}\right\}$$

for any $x \in \mathbb{R}^2$. From [5] we have $\|\cdot\|_\psi^* \in AN_2$ and the corresponding convex function ψ^* in Ψ_2 is

$$\psi^*(t) = \sup_{0\le s\le 1} \frac{(1-s)(1-t)+st}{\psi(s)}$$

for t with $0 \le t \le 1$.

To obtain the dual norm of $\|\cdot\|_{\omega,q}$ we first determine the function $\psi^*_{\omega,q}$.

Theorem 4.1 ([6]). *Let $0<\omega<1$. If $1<q<\infty$, then*

$$\psi^*_{\omega,q}(t) = \begin{cases} ((1-t)^p + \omega^{1-p}t^p)^{1/p} & \text{if } 0 \le t < \omega/(1+\omega), \\ (1+\omega)^{1/p-1} & \text{if } \omega/(1+\omega) \le t < 1/(1+\omega), \\ (t^p + \omega^{1-p}(1-t)^p)^{1/p} & \text{if } 1/(1+\omega) \le t \le 1, \end{cases}$$

where $1/p+1/q=1$.

Hence $d^{(2)}(\omega,q)^*$ is isometrically isomorphic to the space $\mathbb{R}^2$ endowed with the norm $\|\cdot\|^*_{\omega,q}$ defined by

$$\|(x,y)\|^*_{\omega,q} = \begin{cases} (|x|^p + \omega^{1-p}|y|^p)^{1/p} & \text{if } |y| \le \omega|x|, \\ (1+\omega)^{1/p-1}(|x|+|y|) & \text{if } \omega|x| \le |y| \le \omega^{-1}|x|, \\ (\omega^{1-p}|x|^p + |y|^p)^{1/p} & \text{if } \omega^{-1}|x| \le |y|, \end{cases}$$

where $1/p+1/q=1$.

We now suppose that $1<q<2$ and $(\sqrt{2}-1)^{2-q} < \omega < 1$. Since $\psi^*_{\omega,q}$ is symmetric with respect to $t=1/2$, we define a function f^* from $[0,1/2]$ into $\mathbb{R}$ by

$$f^*(t) = \frac{2-2t}{\psi^*_{\omega,q}(t)}\psi^*_{\omega,q}\left(\frac{1}{2-2t}\right) = \frac{\left((1+s)^p + w^{1-p}(1-s)^p\right)^{1/p}}{(1+w^{1-p}s^p)^{1/p}}.$$

As in the proof of [6, Theorem 13], we only calculate the maximum of f^* on $[(1-\omega)/2, \omega/(1+\omega)]$ to calculate the maximum of f^* on $[0,1]$. To do this, we define a function g from $[(1-\omega)/(1+\omega),\omega]$ into $\mathbb{R}$ by

$$g^*(s) = f^*\left(\frac{s}{1+s}\right) = \frac{\left((1+s)^p + w^{1-p}(1-s)^p\right)^{1/p}}{(1+w^{1-p}s^p)^{1/p}}.$$

Since

$$\max\left\{g^*(s) : \frac{1-\omega}{1+\omega} \le s \le \omega\right\} = \max\left\{f^*(t) : \frac{1-\omega}{2} \le t \le \frac{\omega}{1+\omega}\right\},$$

it is enough to calculate the maximum of g^* on $[(1-\omega)/(1+\omega),\omega]$. The derivative of g^* is

$$\begin{aligned}(g^*)'(s) &\\ &= \frac{((1+s)^p + \omega^{1-p}(1-s)^p)^{1/p-1}}{(1+\omega^{1-p}s^p)^{1/p+1}} \\ &\qquad \times \{(1+s)^{p-1}(1-\omega^{1-p}s^{p-1}) - \omega^{1-p}(1-s)^{p-1}(1+\omega^{1-p}s^{p-1})\}.\end{aligned}$$

We define

$$g_1^*(s) = (1+s)^{p-1}(1-\omega^{1-p}s^{p-1}) - \omega^{1-p}(1-s)^{p-1}(1+\omega^{1-p}s^{p-1}).$$

Since $\omega > (\sqrt{2}-1)^{2-q}$, it follows that

$$\omega^{1/(2-q)} > \frac{1-\omega^{1/(2-q)}}{1+\omega^{1/(2-q)}} > \frac{1-\omega}{1+\omega}$$

and so $(1-\omega)/(1+\omega) < \omega^{1/(2-q)} < \omega$. As in the Table 3 of [6], there uniquely exists a pair of real numbers s_0^*, s_1^* such that

$$\frac{1-\omega}{1+\omega} < s_0^* < \omega^{1/(2-q)} < s_1^* < \omega$$

and $g_1^*(s_0^*) = g_1^*(s_1^*) = 0$. By Theorem 3.2, s_0 and s_1 satisfy the equation

$$(1+s_i)^{q-1}(1-\omega s_i^{q-1}) = \omega(1-s_i)^{q-1}(1+\omega s_i^{q-1}) \ (i=0,1).$$

We put $s_i' = \omega s_i^{q-1}$ $(i=0,1)$. Then we obtain

$$\frac{1-\omega}{1+\omega} < s_0' < \omega^{1/(2-q)} < s_1' < \omega.$$

Furthermore, we have for $i = 0,1$,

$$\begin{aligned}g_1^*(s_i') &= (1+\omega s_i^{q-1})^{p-1}(1-s_i) - \omega^{1-p}(1-\omega s_i^{q-1})^{p-1}(1+s_i) \\ &= \{(1-s_i)^{q-1}(1+\omega s_i^{q-1})\}^{p-1} - \{\omega^{-1}(1+s_i)^{q-1}(1-\omega s_i^{q-1})\}^{p-1} \\ &= 0\end{aligned}$$

since $(p-1)(q-1) = 1$. By the uniqueness of $\{s_0^*, s_1^*\}$, one has $s_i^* = s_i' = w s_i^{q-1}$ for $i = 0,1$. Since $s_2^* = \omega s_2^{q-1}$, we have

$$\left(\frac{2(1+s_1)^{q-1}}{1+\omega s_1^{q-1}}\right)^{1/q} \left(\frac{2(1+s_1^*)^{p-1}}{1+\omega^{1-p}s_1^{*p-1}}\right)^{1/p} = 2.$$

On the other hand, as in the proof of Theorem 3.2,

$$\left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q} \left(\frac{2(1+s_1)^{q-1}}{1+\omega s_1^{q-1}}\right)^{1/q} = 2.$$

Therefore, it follows that

$$\left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q} = \left(\frac{2(1+s_1^*)^{p-1}}{1+\omega^{1-p}s_1^{*p-1}}\right)^{1/p}.$$

By the argument in above and [6, Theorem 13], we have the following result on the James and Schäffer constants of $d^{(2)}(\omega,q)^*$.

Theorem 4.2. *Let $1 < q < 2$ and $1/p + 1/q = 1$. If $(\sqrt{2}-1)^{2-q} < \omega < 1$, then there exists a unique pair of real numbers s_0, s_1 such that*

$$\left(\frac{1-\omega}{\omega(1+\omega)}\right)^{p-1} < s_0 < \omega^{1/(2-q)} < s_1 < 1$$

and $(1+s_i)^{q-1}(1-\omega s_i^{q-1}) = \omega(1-s_i)^{q-1}(1+\omega s_i^{q-1})$ for $i = 0, 1$.

(a) *If $(\sqrt{2}-1)^{2-q} < \omega \le \sqrt{2}^q - 1$, then*

$$J(d^{(2)}(\omega,q)^*) = \max\left\{\left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q}, 2\left(\frac{1}{1+\omega}\right)^{1/q}\right\}$$

and

$$g(d^{(2)}(\omega,q)^*) = \min\left\{\left(\frac{2(1+s_1)^{q-1}}{1+\omega s_1^{q-1}}\right)^{1/q}, (1+\omega)^{1/q}\right\}.$$

(b) *If $\sqrt{2}^q - 1 < \omega < 1$, then*

$$J(d^{(2)}(\omega,q)^*) = \left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q}$$

and

$$g(d^{(2)}(\omega,q)^*) = \left(\frac{2(1+s_1)^{q-1}}{1+\omega s_1^{q-1}}\right)^{1/q}.$$

As an immediate consequence of Theorems 3.2 and 4.2, one has

$$J(d^{(2)}(\omega,q)^*) = J(d^{(2)}(\omega,q))$$

and

$$g(d^{(2)}(\omega,q)^*) = g(d^{(2)}(\omega,q))$$

for all $1 < q < 2$ and $(\sqrt{2}-1)^{2-q} < \omega < 1$. Thus we finally have the following result.

Theorem 4.3. *Let $1 < q < 2$ and $0 < \omega < 1$. Then*

$$J(d^{(2)}(\omega,q)^*) = J(d^{(2)}(\omega,q)) \quad \text{and} \quad g(d^{(2)}(\omega,q)^*) = g(d^{(2)}(\omega,q)).$$

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