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ON THE UNIFORM NON-ℓ_1^n-NESS AND NEW CLASSES OF CONVEX FUNCTIONS

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Dedicated to Professor Wataru Takahashi on the ocasion of his 70th birthday

Abstract. Let $X_1, \dots, X_N$ be uniformly non-square Banach spaces. We shall characterize the uniform non-ℓ_1^n-ness of the ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ with a strictly monotone norm by means of the convex function ψ, which extends some recent results on the uniform non-squareness. In the course of doing this we shall introduce subclasses $\Psi_N^{(1,n)}$ of the class $\Psi_N^{(1)}$ of convex functions which yields partiall ℓ_1-norms; this enables us to have precise observations on the structure of the class of $\Psi_N^{(1)}$. Our results hold true for A-direct sums, more general direct sums with the norm induced from an arbitrary norm on $\mathbb{R}^N$, a fortiori, for the Z-direct sums. As a corollary the uniform non-ℓ_1^n-ness will be characterized for $\mathbb{C}^N$, as well.

1. Introduction

Recently the uniform non-squareness and non-ℓ_1^n-ness have been discussed for direct sums of Banach spaces ([3–7, 9, 11–13, 15, 16], etc.) in connection with the fixed point property for nonexpansive mappings, super-reflexivity, and various estimates of geometric constants, etc. Our concern in this paper is originated from the following result ([9]): *A ψ-direct sum $X \oplus_\psi Y$ is uniformly non-square if and only if X and Y are uniformly non-square and the convex function ψ is neither ψ_1 nor ψ_∞, where ψ_1 and ψ_∞ are the corresponding convex functions to the ℓ_1- and ℓ_∞-norms on $\mathbb{C}^2$, respectively.*

In the same paper [9] it was asked to extend this result to the finitely many Banach spaces case. Dowling-Saejung [7] presented a partial answer in terms of properties T_1^N and T_∞^N under the condition that the ψ-norm $\|\cdot\|_\psi$ is strictly monotone. In the recent paper [5] of Dompongsa, Kato and Tamura an equivalent result was presented by means of a class of convex functions $\Psi_N^{(1)}$ which yield ℓ_1-like norms: *Let $X_1, \dots, X_N$ be uniformly non-square Banach spaces and $\|\cdot\|_\psi$ is strictly monotone. Then, the ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly non-square if and only if $\psi \notin \Psi_N^{(1)}$.*

In this paper we shall extend this result to characterize the uniform non-ℓ_1^n-ness of the ψ-direct sum of uniformly non-square Banach spaces $X_1, \dots, X_N$ by means of the convex function ψ (Section 4, Theorem 4.8). In the course of doing this we

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shall introduce and investigate subclasses $\Psi_N^{(1,n)}$ of the class $\Psi_N^{(1)}$, which enables us to make precise observations on the structure of the class $\Psi_N^{(1)}$ (Section 3). In particular a norm on $\mathbb{C}^N$ which is uniformly non-ℓ_1^{n+1} but not uniformly non-ℓ_1^n will be easily constructed (Section 4, Example 4.10). As a corollary the uniform non-ℓ_1^n-ness of $\mathbb{C}^N$ will be characterized (Corollary 4.9).

Finally, in Section 5 we shall observe that some of our main results hold true for more general A-direct sums, those with the norm induced from an arbitrary norm on $\mathbb{R}^N$ ([5]), a fortiori, for the Z-direct sums.

2. Preliminaries

A norm $\|\cdot\|$ on $\mathbb{C}^N$ is called *absolute* if $\|(z_1,\dots,z_N)\| = \|(|z_1|,\dots,|z_N|)\|$ for all $(z_1,\dots,z_N)\in\mathbb{C}^N$, and *normalized* if $\|(1,0,\dots,0)\| = \cdots = \|(0,\dots,0,1)\| = 1$. The collection of all absolute normalized norms on $\mathbb{C}^N$ is denoted by AN_N. A norm $\|\cdot\|$ on $\mathbb{C}^N$ is called *monotone* provided that

$$\|(z_1,\dots,z_N)\| \le \|(w_1,\dots,w_N)\| \text{ if } |z_j|\le|w_j| \ (1\le j\le N), \tag{2.1}$$

and is called *strictly monotone* if it is monotone and the inequality (2.1) is strict if $|z_j|<|w_j|$ for some j. The following fact is known.

Proposition 2.1 ([1])**.** *A norm $\|\cdot\|$ on $\mathbb{C}^N$ is absolute if and only if it is monotone.*

For strict monotonicity we have the following.

Proposition 2.2 ([14])**.** *Let $\|\cdot\|$ be an absolute norm on $\mathbb{C}^N$. Let $(z_1,\dots,z_N)\in\mathbb{C}^N$ and let $0<|z_j|<|w_j|$ with $w_j\in\mathbb{C}$ for some $1\le j\le N$. Then the following are equivalent.*

(i) $\|(z_1,\dots,\overset{j}{\overbrace{z_j}},\dots,z_N)\| < \|(z_1,\dots,\overset{j}{\overbrace{w_j}},\dots,z_N)\|$

(ii) $\|(z_1,\dots,\overset{j}{\overbrace{0}},\dots,z_N)\| < \|(z_1,\dots,\overset{j}{\overbrace{w_j}},\dots,z_N)\|$.

For every absolute normalized norm on $\mathbb{C}^N$ there corresponds a unique convex function ψ on the standard N-simplex $\Delta_N\subset\mathbb{R}^{N-1}$ and vice versa ([17]; cf. [2] for the case $N=2$): Let $\|\cdot\|\in AN_N$ and let

$$\psi(s) = \left\|\left(1-\sum_{i=1}^{N-1}s_i, s_1,\dots,s_{N-1}\right)\right\| \text{ for } s=(s_1,\dots,s_{N-1})\in\Delta_N, \tag{2.2}$$

where

$$\Delta_N = \left\{s=(s_1,\dots,s_{N-1})\in\mathbb{R}^{N-1} : \sum_{i=1}^{N-1}s_i\le 1,\ s_i\ge 0\right\}.$$

Then, ψ is convex (continuous) on the convex set Δ_N and satisfies the following.

(A_0) $\psi(0,\dots,0)=\psi(1,0,\dots,0)=\cdots=\psi(0,\dots,0,1)=1$,

(A_1) $\psi(s_1,\dots,s_{N-1}) \ge \Big(\sum_{i=1}^{N-1}s_i\Big)\psi\Big(\frac{s_1}{\sum_{i=1}^{N-1}s_i},\dots,\frac{s_{N-1}}{\sum_{i=1}^{N-1}s_i}\Big)$ if $0<\sum_{i=1}^{N-1}s_i\le 1$,

$$(A_2)\ \psi(s_1,\dots,s_{N-1}) \geq (1-s_1)\psi\left(0, \frac{s_2}{1-s_1},\dots,\frac{s_{N-1}}{1-s_1}\right) \qquad \text{if } 0 \leq s_1 < 1,$$

....

$$(A_N)\ \psi(s_1,\dots,s_{N-1}) \geq (1-s_{N-1})\psi\left(\frac{s_1}{1-s_{N-1}},\dots,\frac{s_{N-2}}{1-s_{N-1}},0\right) \qquad \text{if } 0 \leq s_{N-1} < 1.$$

The converse holds true: Denote by Ψ_N the class of all convex functions ψ on Δ_N satisfying $(A_0)-(A_N)$. For any $\psi \in \Psi_N$ define

$$\|(z_1,\dots,z_N)\|_\psi = \begin{cases} \left(\sum_{j=1}^N |z_j|\right)\psi\left(\frac{|z_2|}{\sum_{j=1}^N |z_j|},\dots,\frac{|z_N|}{\sum_{j=1}^N |z_j|}\right) & \text{if } (z_1,\dots,z_N) \neq (0,\dots,0), \\ 0 & \text{if } (z_1,\dots,z_N) = (0,\dots,0) \end{cases} \tag{2.3}$$

Then $\|\cdot\|_\psi \in AN_N$ and $\|\cdot\|_\psi$ satisfies (2.2).

For the ℓ_p-norm $\|\cdot\|_p \in AN_N$:

$$\|(z_1,\dots,z_N)\|_p = \begin{cases} \{|z_1|^p + \cdots + |z_N|^p\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|z_1|,\dots,|z_N|\} & \text{if } p = \infty, \end{cases}$$

the corresponding convex function ψ_p is given by

$$\psi_p(s) = \begin{cases} \left\{\left(1-\sum_{i=1}^{N-1} s_i\right)^p + s_1^p + \cdots + s_{N-1}^p\right\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{1-\sum_{i=1}^{N-1} s_i, s_1,\dots,s_{N-1}\} & \text{if } p = \infty \end{cases}$$

for $s = (s_1,\dots,s_{N-1}) \in \Delta_N$. In particular, the convex function corresponding to the ℓ_1-norm is $\psi_1(s) = 1$. For any $\psi \in \Psi_N$ we have $\|\cdot\|_\infty \leq \|\cdot\|_\psi \leq \|\cdot\|_1$ ([17]).

Let $X_1,\dots,X_N$ be Banach spaces and let $\psi \in \Psi_N$. The ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ is their direct sum equipped with the norm

$$\|(x_1,\dots,x_N)\|_\psi := \|(\|x_1\|,\dots,\|x_N\|)\|_\psi \text{ for } (x_1,\dots,x_N) \in X_1 \oplus \cdots \oplus X_N$$

([8, 18]).

3. Subclasses $\Psi_N^{(1,n)}$ of $\Psi_N^{(1)}$

Recently, the present authors [14] (cf. [15]) introduced a subclass $\Psi_N^{(1)}$ of Ψ_N consisting of those functions which yield ℓ_1-like norms (or "partial ℓ_1-norms") as follows:

Definition 3.1 (cf. [14–16]). Let $\psi \in \Psi_N$. We say $\psi \in \Psi_N^{(1)}$ if there exists $(a_1,\dots,a_N) \in \mathbb{R}_+^N$ (with non-negative entries) and some nonempty proper subset T of $\{1,\dots,N\}$

$$\|(a_1,\dots,a_N)\|_\psi = \|(\chi_T(1)a_1,\dots,\chi_T(N)a_N)\|_\psi + \|(\chi_{T^c}(1)a_1,\dots,\chi_{T^c}(N)a_N)\|_\psi,$$

where $(\chi_T(1)a_1,\dots,\chi_T(N)a_N)$ and $(\chi_{T^c}(1)a_1,\dots,\chi_{T^c}(N)a_N)$ are nonzero.

It is obvious that the ℓ_1-norm satisfies this property, and in the case $N = 2$, $\Psi_2^{(1)} = \{\psi_1\}$. We refer the reader to [14, 16] for several examples. The functions ψ in $\Psi_N^{(1)}$ are characterized as follows:

Theorem 3.2 ([14])**.** *Let $\psi \in \Psi_N$. Then the following are equivalent.*

(i) $\psi \in \Psi_N^{(1)}$.

(ii) *There exists an element $(s_1, \dots, s_{N-1}) \in \Delta_N$ and some nonempty subset S of $\{1, \dots, N-1\}$ with $0 < M := \sum_{i=1}^{N-1} \chi_S(i) s_i < 1$,*

$$\begin{aligned} \psi(s_1, \dots, s_{N-1}) &= M\psi\left(\frac{\chi_S(1)s_1}{M}, \dots, \frac{\chi_S(N-1)s_{N-1}}{M}\right) \\ &\quad + (1-M)\,\psi\left(\frac{\chi_{S^c}(1)s_1}{1-M}, \dots, \frac{\chi_{S^c}(N-1)s_{N-1}}{1-M}\right). \end{aligned}$$

Now, we shall introduce subclasses $\Psi_N^{(1,n)}$ of $\Psi_N^{(1)}$, which will enable us to make more precise investigation on $\Psi_N^{(1)}$.

Definition 3.3. Let $\psi \in \Psi_N$ and let $2 \le n \le N$. We say $\psi \in \Psi_N^{(1,n)}$ if there exist $(a_1, \dots, a_N) \in \mathbb{R}_+^N$ and mutually disjoint nonempty proper subsets $T_1, \dots, T_n$ of $\{1, \dots, N\}$ with $\cup_{k=1}^n T_k = \{1, \dots, N\}$ such that

$$\|(a_1, \dots, a_N)\|_\psi = \sum_{k=1}^n \|(\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N)\|_\psi, \tag{3.1}$$

where $(\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N)$ are nonzero.

The next inequalities are useful.

Lemma 3.4 (Sharp triangle inequality: [10])**.** *Let X be a Banach space. Then, for nonzero $x_1, \dots, x_N \in X$*

$$\begin{aligned} &\left\|\sum_{j=1}^N x_j\right\| + \left(N - \left\|\sum_{j=1}^N \frac{x_j}{\|x_j\|}\right\|\right) \min_{1\le j\le N} \|x_j\| \\ &\qquad \le \sum_{j=1}^N \|x_j\| \le \left\|\sum_{j=1}^N x_j\right\| + \left(N - \left\|\sum_{j=1}^N \frac{x_j}{\|x_j\|}\right\|\right) \max_{1\le j\le N} \|x_j\|. \end{aligned}$$

Theorem 3.5 (cf. [16, Theorem 7])**.** *Let $\psi \in \Psi_N$ and let $2 \le n \le N$. Then, the following are equivalent.*

(i) $\psi \in \Psi_N^{(1,n)}$.

(ii) *There exist $(a_1, \dots, a_N) \in \mathbb{R}_+^N$ and mutually disjoint nonempty proper subsets $T_1, \dots, T_n$ of $\{1, \dots, N\}$ with $\cup_{k=1}^n T_k = \{1, \dots, N\}$ such that*

$$\|(a_1, \dots, a_N)\|_\psi = \sum_{k=1}^n \|(\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N)\|_\psi,$$

where $\|(\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N)\| = 1$ for $1 \le k \le n$.

(iii) *There exist* $(a_1, \ldots, a_N) \in \mathbb{R}_+^N$ *and mutually disjoint nonempty proper subsets* $T_1, \ldots, T_n$ *of* $\{1, \ldots, N\}$ *with* $\cup_{k=1}^n T_k = \{1, \ldots, N\}$ *such that for every* $1 \le k \le n$

$$\begin{aligned}\|(a_1, \ldots, a_N)\|_\psi &= \|(\chi_{T_k}(1)a_1, \ldots, \chi_{T_k}(N)a_N)\|_\psi \\ &\quad + \|(\chi_{T_k^c}(1)a_1, \ldots, \chi_{T_k^c}(N)a_N)\|_\psi,\end{aligned}$$

where $(\chi_{T_k}(1)a_1, \ldots, \chi_{T_k}(N)a_N)$ *are nonzero.*

(iv) *There exist* $(s_1, \ldots, s_{N-1}) \in \Delta_N$ *and nonempty subsets* $S_1, \ldots, S_n$ *of* $\{1, \ldots, N-1\}$ *with* $0 < M_k := \sum_{i=1}^{N-1} \chi_{S_k}(i)s_i < 1$ *such that*

$$\begin{aligned}\psi(s_1, \ldots, s_{N-1}) &= M_k \psi\left(\frac{\chi_{S_k}(1)s_1}{M_k}, \ldots, \frac{\chi_{S_k}(N-1)s_{N-1}}{M_k}\right) \\ &\quad + (1 - M_k)\,\psi\left(\frac{\chi_{S_k^c}(1)s_1}{1 - M_k}, \ldots, \frac{\chi_{S_k^c}(N-1)s_{N-1}}{1 - M_k}\right)\end{aligned} \tag{3.2}$$

for every $1 \le k \le n$*, where* $S_1^c, S_2, \ldots, S_n$ *are mutually disjoint and* $S_1^c \cup (\cup_{k=2}^n S_k) = \{1, \ldots, N-1\}$ *(*S_1^c *can be empty).*

Proof. The implication (ii) $\Rightarrow$ (i) is trivial. Conversely, assume that $\psi \in \Psi_N^{(1,n)}$, and let $(a_1, \ldots, a_N) \in \mathbb{R}_+^N$ and $T_1, \ldots, T_n$ be as in Definition 3.3. Let $\boldsymbol{v}_k = (\chi_{T_k}(1)a_1, \ldots, \chi_{T_k}(N)a_N)$, $1 \le k \le n$. Then, by Lemma 3.4

$$\left\|\sum_{k=1}^n \boldsymbol{v}_k\right\|_\psi + \left(n - \left\|\sum_{k=1}^n \frac{\boldsymbol{v}_k}{\|\boldsymbol{v}_k\|_\psi}\right\|_\psi\right) \min_{1 \le k \le n} \|\boldsymbol{v}_k\|_\psi \le \sum_{k=1}^n \|\boldsymbol{v}_k\|_\psi.$$

Hence by (3.1) we have

$$\left\|\sum_{k=1}^n \frac{\boldsymbol{v}_k}{\|\boldsymbol{v}_k\|_\psi}\right\|_\psi = n,$$

which implies (ii). The implications (ii) $\Leftrightarrow$ (iii) is obtained in [16, Theorem 7].

(iii) $\Rightarrow$ (iv). Let $(a_1, \ldots, a_N) \in \mathbb{R}_+^N$ and $T_1, \ldots, T_n$ be as in (iii). Without loss of generality we may assume that $T_1 \ni 1$. Let

$$S_k = T_k - 1 \text{ for } 2 \le k \le n \text{ and } S_1 = T_1^c - 1.$$

Then, $S_1, S_2, \ldots, S_n$ are nonempty and $S_1^c, S_2, \ldots, S_n$ are mutually disjoint. In fact, $S_2, \ldots, S_n$ are evidently mutually disjoint. If we have $j \in S_1^c \cap S_k$ for some $2 \le k \le n$, then $j + 1 \notin T_1^c$, while $j + 1 \in T_k$, a contradiction. Also it is immediate to see that $S_1^c \cup (\cup_{k=2}^n S_k) = \{1, \ldots, N-1\}$. Let

$$s_i = \frac{a_{i+1}}{\sum_{j=1}^N a_j} \text{ for } 1 \le i \le N-1.$$

Let $k \ge 2$. Then, we have

$$\begin{aligned}\|(a_1, \ldots, a_N)\|_\psi &= \|(0, \chi_{T_k}(2)a_2 \ldots, \chi_{T_k}(N)a_N)\|_\psi \\ &\quad + \|(a_1, \chi_{T_k^c}(2)a_2, \ldots, \chi_{T_k^c}(N)a_N)\|_\psi,\end{aligned}$$

from which it follows that

$$\psi(s_1, \ldots, s_{N-1}) = \left(\sum_{i=1}^{N-1} \chi_{S_k}(i)s_i\right)\psi\left(\frac{\chi_{S_k}(1)s_1}{\sum_{i=1}^{N-1} \chi_{S_k}(i)s_i}, \ldots, \frac{\chi_{S_k}(N-1)s_{N-1}}{\sum_{i=1}^{N-1} \chi_{S_k}(i)s_i}\right)$$

$$+\left(1-\sum_{i=1}^{N-1}\chi_{S_k}(i)s_i\right)\psi\left(\frac{\chi_{S_k^c}(1)s_1}{1-\sum_{i=1}^{N-1}\chi_{S_k}(i)s_i},\dots,\frac{\chi_{S_k^c}(N-1)s_{N-1}}{1-\sum_{i=1}^{N-1}\chi_{S_k}(i)s_i}\right),$$

or (3.3) with $M_k=\sum_{i=1}^{N-1}\chi_{S_k}(i)s_i$ (see [14,16] for the same discussion). Let $k=1$. Then, since

$$\|(a_1,\dots,a_N)\|_\psi=\|(\chi_{T_1}(1)a_1,\dots,\chi_{T_1}(N)a_N)\|_\psi+\|(\chi_{T_1^c}(1)a_1,\dots,\chi_{T_1^c}(N)a_N)\|_\psi,$$

we have

$$\left\|\left(1-\sum_{i=1}^{N-1}s_i,s_1,\dots,s_{N-1}\right)\right\|_\psi=\left\|\left(1-\sum_{i=1}^{N-1}s_i,\ \chi_{S_1^c}(1)s_1,\ \dots,\ \chi_{S_1^c}(N-1)s_{N-1}\right)\right\|_\psi$$
$$+\|(0,\chi_{S_1}(1)s_1,\dots,\chi_{S_1}(N-1)s_{N-1})\|_\psi.$$

Here the both terms of the right side term are not zero (see Remark 3.6). Therefore we have $0<M_1=\sum_{i=1}^{N-1}\chi_{S_1}(i)s_i<1$. From the foregoing formula it follows that

$$\begin{aligned}&\psi(s_1,\dots,s_{N-1})\\&=\left(1-\sum_{i=1}^{N-1}\chi_{S_1}(i)s_i\right)\psi\left(\frac{\chi_{S_1^c}(1)s_1}{1-\sum_{i=1}^{N-1}\chi_{S_1}(i)s_i},\dots,\frac{\chi_{S_1^c}(N-1)s_{N-1}}{1-\sum_{i=1}^{N-1}\chi_{S_1}(i)s_i}\right)\\&\quad+\left(\sum_{i=1}^{N-1}\chi_{S_1}(i)s_i\right)\psi\left(\frac{\chi_{S_1}(1)s_1}{\sum_{i=1}^{N-1}\chi_{S_1}(i)s_i},\dots,\frac{\chi_{S_1}(N-1)s_{N-1}}{\sum_{i=1}^{N-1}\chi_{S_1}(i)s_i}\right),\end{aligned}$$

or (3.3) with $M_1=\sum_{i=1}^{N-1}\chi_{S_1}(i)s_i$. We note that in the above argument, if $T_1=\{1\}$, we have $S_1=\{1,\dots,N-1\}$, or $S_1^c=\emptyset$.

(iv) $\Rightarrow$ (iii). Let $(s_1,\dots,s_{N-1})\in\Delta_N$, $S_1,\dots,S_n$ and M_k be as in (iv). Then we have

$$\left\|\left(1-\sum_{i-1}^{N-1}s_i,s_1,\dots,s_{N-1}\right)\right\|_\psi=\|(0,\chi_{S_k}(1)s_1,\dots,\chi_{S_k}(N-1)s_{N-1})\|_\psi$$
$$+\left\|\left(1-\sum_{i=1}^{N}s_i,\ \chi_{S_k^c}(1)s_1,\ \dots,\ \chi_{S_k^c}(N-1)s_{N-1}\right)\right\|_\psi, \tag{3.3}$$

where $(0,\chi_{S_k}(1)s_1,\dots,\chi_{S_k}(N-1)s_{N-1})$, $(1-\sum_{i=1}^{N}s_i,\chi_{S_k^c}(1)s_1,\dots,\chi_{S_k^c}(N-1)s_{N-1})$ are nonzero. Let

$$(a_1,a_2,\dots,a_N)=\left(1-\sum_{i-1}^{N-1}s_i,s_1,\dots,s_{N-1}\right)$$

and let

$$T_k=S_k+1\text{ for }k\ge 2\text{ and }T_1=(S_1^c+1)\cup\{1\}.$$

Then, T_k's are mutually disjoint and their union is $\{1,\dots,N\}$. From the formula (3.3) we have for $k\ge 2$

$$\begin{aligned}\|(a_1,a_2,\dots,a_N)\|_\psi&=\|(0,\chi_{T_k}(2)a_2,\dots,\chi_{T_k}(N)a_N)\|_\psi\\&\quad+\|(a_1,\chi_{T_k^c}(2)a_2,\dots,\chi_{T_k^c}(N)a_N)\|_\psi,\end{aligned}$$

where $(0, \chi_{T_k}(2)a_2, \dots, \chi_{T_k}(N)a_N), (a_1, \chi_{T_k^c}(2)a_2, \dots, \chi_{T_k^c}(N)a_N)$ are nonzero. If $k = 1$, since $\chi_{S_1}(i) = \chi_{T_1^c}(i+1)$, we have by (3.3)

$$\begin{aligned}\|(a_1, a_2, \dots, a_N)\|_\psi &= \|(0, \chi_{T_1^c}(2)a_2, \dots, \chi_{T_1^c}(N)a_N)\|_\psi \\ &\quad + \|(a_1, \chi_{T_1}(2)a_2, \dots, \chi_{T_1}(N)a_N)\|_\psi.\end{aligned}$$

This completes the proof. □

Remark 3.6 ([16, Remark 2])**.** We note that in Theorem 3.5 (iii) we have that $(\chi_{T_k^c}(1)a_1, ..., \chi_{T_k^c}(N)a_N)$ are nonzero from the assumption $(\chi_{T_k}(1)a_1, ..., \chi_{T_k}(N)a_N)$ are nonzero.

Proposition 3.7. $\Psi_N^{(1)} = \Psi_N^{(1,2)} \supset \Psi_N^{(1,3)} \supset \cdots \supset \Psi_N^{(1,N)} = \{\psi_1\}$.

Proof. Let $2 \le n < N$. It is obvious that $\Psi_N^{(1)} = \Psi_N^{(1,2)}$. So we shall show that $\Psi_N^{(1,n)} \supset \Psi_N^{(1,n+1)}$ and $\Psi_N^{(1,N)} = \{\psi_1\}$. Assume that $\psi \in \Psi_N^{(1,n+1)}$. Then there exist $(a_1, \dots, a_N) \in \mathbb{R}_+^N$ and mutually disjoint nonempty subsets $T_1, \dots, T_{n+1}$ of $\{1, \dots, N\}$ with $\cup_{k=1}^{n+1} T_k = \{1, \dots, N\}$ such that

$$\|(a_1, \dots, a_N)\|_\psi = \sum_{k=1}^{n+1} \|(\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N)\|_\psi,$$

where $(\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N)$ are nonzero. Then we have

$$\begin{aligned}\|(a_1, \dots, a_N)\|_\psi &= \Big\| \sum_{k=1}^{n+1} (\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N) \Big\|_\psi \\ &= \Big\| \sum_{k=1}^{n-1} (\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N) + (\chi_{T_n \cup T_{n+1}}(1)a_1, \dots, \chi_{T_n \cup T_{n+1}}(N)a_N) \Big\|_\psi \\ &\le \sum_{k=1}^{n-1} \|(\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N)\|_\psi + \|(\chi_{T_n \cup T_{n+1}}(1)a_1, \dots, \chi_{T_n \cup T_{n+1}}(N)a_N)\|_\psi \\ &\le \sum_{k=1}^{n+1} \|(\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N)\|_\psi \\ &= \|(a_1, \dots, a_N)\|_\psi,\end{aligned}$$

from which it follows that

$$\begin{aligned}\|(a_1, \dots, a_N)\|_\psi &= \sum_{k=1}^{n-1} \|(\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N)\|_\psi \\ &\quad + \|(\chi_{T_n \cup T_{n+1}}(1)a_1, \dots, \chi_{T_n \cup T_{n+1}}(N)a_N)\|_\psi\end{aligned}$$

Since $(\cup_{k=1}^{n-1} T_k) \cup (T_n \cup T_{n+1}) = \{1, \dots, N\}$ and $(\chi_{T_n \cup T_{n+1}}(1)a_1, \dots, \chi_{T_n \cup T_{n+1}}(N)a_N)$ is nonzero, we have $\psi \in \Psi_N^{(1,n)}$.

Next, assume that $\psi \in \Psi_N^{(1,N)}$. Then, we can take $(a_1, \dots, a_N) \in \mathbb{R}_+^N$ so that

$$\|(a_1, \dots, a_N)\|_\psi = \sum_{k=1}^{N} \|(0, \dots, 0, \overset{k}{\breve{a_k}}, 0, \dots, 0)\|_\psi = \sum_{k=1}^{N} a_k,$$

where $a_k > 0$. Let $M = \sum_{k=1}^{N} a_k$. Then

$$\psi\Big(\frac{a_2}{M}, \dots, \frac{a_N}{M}\Big) = \|(a_1/M, \dots, a_N/M)\| = 1,$$

which implies that $\psi = \psi_1$, as ψ is convex (cf. [16, Theorem 1]). This completes the proof. □

Example 3.8. Let $2 \le n < N$. The inclusion $\Psi_N^{(1,n)} \supset \Psi_N^{(1,n+1)}$ is strict. In fact, define an absolute normalized norm on $\mathbb{C}^N$ by

$$\|(a_1, \dots, a_N)\| = \max\left\{ \sum_{j=1}^{n} a_j,\ a_{n+1}, \dots, a_N \right\} \text{ for } (a_1, \dots, a_N) \in \mathbb{C}^N.$$

The corresponding convex function ψ is given by

$$\psi(s) = \max\left\{ 1 - \sum_{i=n}^{N-1} s_i,\ s_n, \dots, s_{N-1} \right\} \text{ for } s = (s_1, \dots, s_{N-1}) \in \Delta_N.$$

Then, $\psi \in \Psi_N^{(1,n)} \backslash \Psi_N^{(1,n+1)}$. In fact,

$$\|(\overbrace{1, \dots, 1}^{n}, 0, \dots, 0)\|_\psi = n = \sum_{k=1}^{n} \|(0, \dots 0, \overset{k}{\smile}{1}, 0, \dots, 0)\|_\psi.$$

Therefore, $\psi \in \Psi_N^{(1,n)}$. Next, suppose that $\psi \in \Psi_N^{(1,n+1)}$. Then there exist $(a_1, \dots, a_N) \in \mathbb{R}_+^N$ and mutually disjoint nonempty proper subsets $T_1, \dots, T_{n+1}$ of $\{1, \dots, N\}$ with $\cup_{k=1}^{n+1} T_k = \{1, \dots, N\}$ such that

$$\|(a_1, \dots, a_N)\|_\psi = \sum_{k=1}^{n+1} \|(\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N)\|_\psi,$$

where $(\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N)$ are nonzero. Then, for some $1 \le k \le N$, $T_k \subset \{n+1, \dots, N\}$. Without loss of generality we may assume that $T_k = T_{n+1}$. Since

$$\begin{aligned}
\|(a_1, \dots, a_N)\|_\psi &= \left\| \left(\sum_{k=1}^{n+1} \chi_{T_k}(1)a_1, \dots, \sum_{k=1}^{n+1} \chi_{T_k}(N)a_N \right) \right\|_\psi \\
&\le \left\| \left(\sum_{k=1}^{n} \chi_{T_k}(1)a_1, \dots, \sum_{k=1}^{n} \chi_{T_k}(N)a_N \right) \right\|_\psi \\
&\quad + \|(\chi_{T_{n+1}}(1)a_1, \dots, \chi_{T_{n+1}}(N)a_N)\|_\psi \\
&\le \sum_{k=1}^{n+1} \|(\chi_{T_k}(1)a_1, \dots, \chi_{T_k}(N)a_N)\|_\psi,
\end{aligned}$$

we have

$$\begin{aligned}
\|(a_1, \dots, a_N)\|_\psi &= \left\| \left(\sum_{k=1}^{n} \chi_{T_k}(1)a_1, \dots, \sum_{k=1}^{n} \chi_{T_k}(N)a_N \right) \right\|_\psi \\
&\quad + \|(\chi_{T_{n+1}}(1)a_1, \dots, \chi_{T_{n+1}}(N)a_N)\|_\psi.
\end{aligned}$$

Here, we obtain that

$$\|(a_1,\dots,a_N)\|_\psi = \left\|\left(\sum_{k=1}^n \chi_{T_k}(1)a_1,\dots,\sum_{k=1}^n \chi_{T_k}(N)a_N\right)\right\|_\psi \tag{3.4}$$

or

$$\|(a_1,\dots,a_N)\|_\psi = \|(\chi_{T_{n+1}}(1)a_1,\dots,\chi_{T_{n+1}}(N)a_N)\|_\psi. \tag{3.5}$$

Indeed,

$$\|(a_1,\dots,a_N)\|_\psi = \max\{\|(a_1,\dots,a_n)\|_1,\ \|(a_{n+1},\dots,a_N)\|_\infty\}$$

and

$$\begin{aligned}&\left\|\left(\sum_{k=1}^n \chi_{T_k}(1)a_1,\dots,\sum_{k=1}^n \chi_{T_k}(N)a_N\right)\right\|_\psi\\ &\qquad = \max\{\|(a_1,\dots,a_n)\|_1,\ \|(\chi_{T_{N+1}^c}(n+1)a_{n+1},\dots,\chi_{T_{N+1}^c}(n+1)a_N)\|_\infty\},\end{aligned}$$

$$\begin{aligned}&\|(\chi_{T_{n+1}}(1)a_1,\dots,\chi_{T_{n+1}}(N)a_N)\|_\psi\\ &\qquad = \max\{\|(\chi_{T_{n+1}}(1)a_1,\dots,\chi_{T_{n+1}}(n)a_n)\|_1,\\ &\qquad\qquad \|(\chi_{T_{n+1}}(n+1)a_{n+1},\dots,\chi_{T_{n+1}(N)}a_N)\|_\infty\}\\ &\qquad = \|(\chi_{T_{n+1}}(n+1)a_{n+1},\dots,\chi_{T_{n+1}(N)}a_N)\|_\infty.\end{aligned}$$

Now, let $\|(a_1,\dots,a_n)\|_1 \ge \|(a_{n+1},\dots,a_N)\|_\infty$. Then

$$\|(a_1,\dots,a_N)\|_\psi = \|(a_1,\dots,a_n)\|_1 = \left\|\left(\sum_{k=1}^n \chi_{T_k}(1)a_1,\dots,\sum_{k=1}^n \chi_{T_k}(N)a_N\right)\right\|_\psi,$$

or (3.4). Let $\|(a_1,\dots,a_n)\|_1 \le \|(a_{n+1},\dots,a_N)\|_\infty$. Then, if $\|(a_{n+1},\dots,a_N)\|_\infty = \|(\chi_{T_{N+1}}(n+1)a_{n+1},\dots,\chi_{T_{N+1}}(n+1)a_N)\|_\infty$, we have

$$\|(a_1,\dots,a_N)\|_\psi = \|(a_{n+1},\dots,a_N)\|_\infty = \|(\chi_{T_{n+1}}(1)a_1,\dots,\chi_{T_{n+1}}(N)a_N)\|_\psi,$$

or (3.5). If $\|(a_{n+1},\dots,a_N)\|_\infty = \|(\chi_{T_{N+1}^c}(n+1)a_{n+1},\dots,\chi_{T_{N+1}^c}(n+1)a_N)\|_\infty$, we have

$$\|(a_1,\dots,a_N)\|_\psi = \|(a_{n+1},\dots,a_N)\|_\infty = \left\|\left(\sum_{k=1}^n \chi_{T_k}(1)a_1,\dots,\sum_{k=1}^n \chi_{T_k}(N)a_N\right)\right\|_\psi,$$

or (3.4). Since $(\chi_{T_k}(1)a_1,\dots,\chi_{T_k}(N)a_N)$ are nonzero and $T_1,\dots,T_{n+1}$ are mutually disjoint, $(\sum_{k=1}^n \chi_{T_k}(1)a_1,\dots,\sum_{k=1}^n \chi_{T_k}(N)a_N)$ is nonzero. Therefore we have

$$\begin{aligned}\|(a_1,\dots,a_N)\|_\psi \ &< \ \left\|\left(\sum_{k=1}^n \chi_{T_k}(1)a_1,\dots,\sum_{k=1}^n \chi_{T_k}(N)a_N\right)\right\|_\psi\\ &\quad + \|(\chi_{T_{n+1}}(1)a_1,\dots,\chi_{T_{n+1}}(N)a_N)\|_\psi,\end{aligned}$$

which is a contradiction. Thus, we have $\psi \notin \Psi_N^{(1,n+1)}$.

4. Uniform non-ℓ_1^n-ness

A Banach space X is called *uniformly non-ℓ_1^n*, $n \geq 2$, if there exists a constant $\varepsilon > 0$ such that

$$\min\left\{\left\|\sum_{j=1}^{n}\theta_j x_j\right\| : \theta_j = \pm 1\right\} \leq N(1-\varepsilon) \text{ for all } x_1, \dots, x_n \in S_X,$$

where S_X is the unit sphere of X. When $n = 2$, X is called *uniformly non-square.* It is known that every uniformly non-ℓ_1^n space is uniformly non-ℓ_1^{n+1}. In fact, we have the following.

Lemma 4.1. *Let X be a Banach space and let $\{x_k^{(1)}\}, \dots, \{x_k^{(n)}\}$ be n sequences in the unit sphere of X. Let $1 \leq m < n$. Then,*

$$\textit{If } \lim_{k\to\infty}\left\|\sum_{j=1}^{n} x_k^{(j)}\right\| = n, \textit{ then } \lim_{k\to\infty}\left\|\sum_{j=1}^{m} x_k^{(j)}\right\| = m.$$

Proof. Assume that $\lim_{k\to\infty}\|\sum_{j=1}^{n} x_k^{(j)}\| = n$. Then

$$\begin{aligned} m &\geq \left\|\sum_{j=1}^{m} x_k^{(j)}\right\| = \left\|\sum_{j=1}^{n} x_k^{(j)} - \sum_{j=m+1}^{n} x_k^{(j)}\right\| \\ &\geq \left\|\sum_{j=1}^{n} x_k^{(j)}\right\| - \sum_{j=m+1}^{n}\|x_k^{(j)}\| = \left\|\sum_{j=1}^{n} x_k^{(j)}\right\| - (n-m) \end{aligned}$$

for all k, from which it follows that $\lim_{k\to\infty}\|\sum_{j=1}^{m} x_j^{(k)}\| = m$. □

Recently Kato and Tamura [16] showed the following result which is equivalent to Dowling and Saejung's result in [7].

Theorem 4.2 ([16, Theorem 10]; cf. [7, Theorem 13])**.** *Let $X_1, \dots, X_N$ be Banach spaces and let $\psi \in \Psi_N$. Assume that $\|\cdot\|_\psi$ is strictly monotone. Then, the following are equivalent.*

(i) *$(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly non-square.*
(ii) *$X_1, \dots, X_N$ are uniformly non-square and $\psi \notin \Psi_N^{(1)}$.*

In the case $N = 2$, $\Psi_2^{(1)} = \{\psi_1\}$. Therefore we obtain the next result.

Corollary 4.3. *Let X and Y be Banach spaces and let $\psi \in \Psi_2$. Assume that $\|\cdot\|_\psi$ is strictly monotone. Then, the following are equivalent.*

(i) *$X \oplus_\psi Y$ is uniformly non-square.*
(ii) *X and Y are uniformly non-square and $\psi \neq \psi_1$.*

Remark 4.4. This result should be compared with the previous result in [9] without the assumption on strict monotonicity of $\|\cdot\|_\psi$: *$X \oplus_\psi Y$ is uniformly non-square if and only if X and Y are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$.*

Theorem 4.2 especially asserts the following.

Corollary 4.5. *Let $X_1,\dots,X_N$ be uniformly non-square Banach spaces. Let $\psi \in \Psi_N$ and assume that the norm $\|\cdot\|_\psi$ is strictly monotone. Then the following are equivalent.*

(i) $(X_1 \oplus \cdots \oplus X_N)_\psi$ *is uniformly non-square.*

(ii) $\psi \notin \Psi_N^{(1)}$.

In the following, we shall characterize the uniform non-ℓ_1^n-ness of $(X_1 \oplus \cdots \oplus X_N)_\psi$ *for uniformly non-square spaces* $X_1,\dots,X_N$. First we note that the ℓ_1-sum $(X_1 \oplus \cdots \oplus X_N)_1$ cannot be uniformly non-ℓ_1^N for any spaces $X_1,\dots,X_N$ (recall that ℓ_1^n is embedded into the direct sum of X_j's). To the contrary we have the following.

Theorem 4.6 ([12])**.** *Let $X_1,\dots,X_N$ be uniformly non-square. Then $(X_1 \oplus \cdots \oplus X_N)_1$ is uniformly non-ℓ_1^{N+1}.*

We are now in a position to present the main result, which extends Corollary 4.5.

Theorem 4.7. *Let $X_1,\dots,X_N$ be uniformly non-square Banach spaces. Let $\psi \in \Psi_N$ and let $\|\cdot\|_\psi$ be strictly monotone. Let $2 \le n \le N$. Then, the following are equivalent.*

(i) $(X_1 \oplus \cdots \oplus X_N)_\psi$ *is uniformly non-ℓ_1^n.*

(ii) $\psi \notin \Psi_N^{(1,n)}$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $\psi \in \Psi_N^{(1,n)}$. Then, by Theorem 3.5 there exist $(a_1,\dots,a_N) \in \mathbb{R}_+^N$ and mutually disjoint nonempty proper subsets $T_1,\dots,T_n$ of $\{1,\dots,N\}$ with $\cup_{k=1}^n T_k = \{1,\dots,N\}$ such that

$$\|(a_1,\dots,a_N)\|_\psi = \sum_{k=1}^n \|(\chi_{T_k}(1)a_1,\dots,\chi_{T_k}(N)a_N)\|_\psi,$$

and $\|(\chi_{T_k}(1)a_1,\dots,\chi_{T_k}(N)a_N)\| = 1$. Then, for any signs $\theta_k = \pm 1$

$$\begin{aligned}\left\|\sum_{k=1}^n \theta_k(\chi_{T_k}(1)a_1,\dots,\chi_{T_k}(N)a_N)\right\|_\psi &= \left\|\left(\sum_{k=1}^n \theta_k\chi_{T_k}(1)a_1,\dots,\sum_{k=1}^n \theta_k\chi_{T_k}(N)a_N\right)\right\|_\psi \\ &= \|(\theta_{k_1}a_1,\dots,\theta_{k_N}a_N)\|_\psi \\ &= \|(a_1,\dots,a_N)\|_\psi = n,\end{aligned}$$

where $k_j, 1 \le j \le N$, is chosen so that $j \in T_{k_j}$ for $j \in \{1,\dots,N\}$. This implies that $(X_1 \oplus \cdots \oplus X_N)_\psi$ is not uniformly non-ℓ_1^n.

(ii) $\Rightarrow$ (i). Suppose that $(X_1 \oplus \cdots \oplus X_N)_\psi$ is not uniformly non-ℓ_1^n. Then there exist n sequences $\{(x_{11}^{(\ell)},\dots,x_{N1}^{(\ell)})\}_\ell,\dots,\{(x_{1n}^{(\ell)},\dots,x_{Nn}^{(\ell)})\}_\ell$ in the unit sphere of $(X_1 \oplus \cdots \oplus X_N)_\psi$ such that

$$\lim_{\ell\to\infty}\left\|\sum_{k=1}^n \theta_k(x_{1k}^{(\ell)},\dots,x_{Nk}^{(\ell)})\right\|_\psi = n \text{ for all } \theta_k = \pm 1. \tag{4.1}$$

By taking subsequences if necessary, we may assume that

$$\lim_{\ell\to\infty}\|x_{1k}^{(\ell)}\| = a_{1k},\dots,\lim_{\ell\to\infty}\|x_{Nk}^{(\ell)}\| = a_{Nk}.$$

Then

$$\|(a_{1k}, \ldots, a_{Nk})\|_\psi = 1 \text{ for } 1 \le k \le n. \tag{4.2}$$

By the formula (4.1) and Lemma 4.1, we have for any $1 \le p < q \le n$

$$\lim_{\ell\to\infty} \|(x_{1p}^{(\ell)}, \ldots, x_{Np}^{(\ell)}) \pm (x_{1q}^{(\ell)}, \ldots, x_{Nq}^{(\ell)})\|_\psi = 2.$$

Then

$$\begin{aligned}
2 &= \lim_{\ell\to\infty} \|(x_{1p}^{(\ell)}, \ldots, x_{Np}^{(\ell)}) \pm (x_{1q}^{(\ell)}, \ldots, x_{Nq}^{(\ell)})\|_\psi \\
&= \|(\lim_{\ell\to\infty} \|x_{1p}^{(\ell)} \pm x_{1q}^{(\ell)}\|, \ldots, \lim_{\ell\to\infty} \|x_{Np}^{(\ell)} \pm x_{Nq}^{(\ell)}\|)\|_\psi \\
&\le \|(\lim_{\ell\to\infty} \|x_{1p}^{(\ell)}\| + \lim_{\ell\to\infty} \|x_{1q}^{(\ell)}\|, \ldots, \lim_{\ell\to\infty} \|x_{Np}^{(\ell)}\| + \lim_{\ell\to\infty} \|x_{Nq}^{(\ell)}\|)\|_\psi \\
&\le \|(\lim_{\ell\to\infty} \|x_{1p}^{(\ell)}\|, \ldots, \lim_{\ell\to\infty} \|x_{Np}^{(\ell)}\|)\|_\psi + \|(\lim_{\ell\to\infty} \|x_{1q}^{(\ell)}\|, \ldots, \lim_{\ell\to\infty} \|x_{Nq}^{(\ell)}\|)\|_\psi \\
&= \lim_{\ell\to\infty} \|(\|x_{1p}^{(\ell)}\|, \ldots, \|x_{Np}^{(\ell)}\|)\|_\psi + \lim_{\ell\to\infty} \|(\|x_{1q}^{(\ell)}\|, \ldots, \|x_{Nq}^{(\ell)}\|)\|_\psi = 2,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\|(\lim_{\ell\to\infty} \|x_{1p}^{(\ell)} \pm x_{1q}^{(\ell)}\|, \ldots, \lim_{\ell\to\infty} \|x_{Np}^{(\ell)} \pm x_{Nq}^{(\ell)}\|)\|_\psi \\
&= \|(\lim_{\ell\to\infty} \|x_{1p}^{(\ell)}\| + \lim_{\ell\to\infty} \|x_{1q}^{(\ell)}\|, \ldots, \lim_{\ell\to\infty} \|x_{Np}^{(\ell)}\| + \lim_{\ell\to\infty} \|x_{Nq}^{(\ell)}\|)\|_\psi.
\end{aligned}$$

By strict monotonicity of $\|\cdot\|_\psi$ we have

$$\lim_{\ell\to\infty} \|x_{jp}^{(\ell)} \pm x_{jq}^{(\ell)}\| = \lim_{\ell\to\infty} \|x_{jp}^{(\ell)}\| + \lim_{\ell\to\infty} \|x_{jq}^{(\ell)}\| = a_{jp} + a_{jq} \text{ for all } 1 \le j \le N. \tag{4.3}$$

Then, we have for every $1 \le j \le N$

$$\min\{a_{jp}, a_{jq}\} = 0 \text{ for all } 1 \le p < q \le n, \tag{4.4}$$

which implies that $\{k : a_{jk} > 0\}$ is at most a singleton for every $1 \le j \le N$. Indeed, suppose that for some $1 \le j \le N$ and some $1 \le p < q \le n$

$$\min\{a_{jp}, a_{jq}\} = \min\{\lim_{\ell\to\infty} \|x_{jp}^{(\ell)}\|, \lim_{\ell\to\infty} \|x_{jq}^{(\ell)}\|\} > 0.$$

By Lemma 3.4, for sufficiently large ℓ,

$$\|x_{jp}^{(\ell)} \pm x_{jq}^{(\ell)}\| + \left(2 - \left\|\frac{x_{jp}^{(\ell)}}{\|x_{jp}^{(\ell)}\|} \pm \frac{x_{jq}^{(\ell)}}{\|x_{jq}^{(\ell)}\|}\right\|\right) \min\{\|x_{jp}^{(\ell)}\|, \|x_{jp}^{(\ell)}\|\} \le \|x_{jp}^{(\ell)}\| + \|x_{jq}^{(\ell)}\|.$$

Letting $\ell \to \infty$, we have

$$\lim_{\ell\to\infty} \left\|\frac{x_{jp}^{(\ell)}}{\|x_{jp}^{(\ell)}\|} \pm \frac{x_{jq}^{(\ell)}}{\|x_{jq}^{(\ell)}\|}\right\| = 2$$

by (4.3), which contradicts to the uniform non-squareness of X_j. Therefore, we have (4.4).

Next let

$$T_k = \{j : a_{jk} > 0\} \text{ for } 1 \le k < n$$

and

$$T_n = \{j : a_{jn} > 0\} \cup \{j : a_{jk} = 0 \text{ for all } 1 \le k \le n\}.$$

Then, by (4.2) the sets $T_1, \dots, T_n$ are nonempty and clearly $\cup_{k=1}^n T_k = \{1, \dots, N\}$. Since the set $\{k : a_{jk} > 0\}$ is at most a singleton for every $1 \le j \le N$, the sets $T_1, \dots, T_{n-1}, \{j : a_{jn} > 0\}$ are mutually disjoint, and therefore $T_1, \dots, T_n$ are mutually disjoint. To obtain the conclusion we shall first see that

$$\sum_{k=1}^n a_{jk} = \sum_{k=1}^n \chi_{T_k}(j) a_{jk} \text{ for all } 1 \le j \le N. \tag{4.5}$$

Indeed, since $\{k : a_{jk} > 0\}$ is at most a singleton, if $\{k : a_{jk} > 0\}$ is nonempty for some $1 \le j \le N$, there exists a unique $1 \le k_j \le n$ such that $\{k : a_{jk} > 0\} = \{k_j\}$. Hence, $a_{jk_j} > 0$, or $j \in T_{k_j}$. Therefore, in this case, we have

$$\sum_{k=1}^n a_{jk} = a_{jk_j} = \chi_{T_{k_j}}(j) a_{jk_j} = \sum_{k=1}^n \chi_{T_k}(j) a_{jk},$$

or (4.5). If $a_{jk} = 0$ for all $1 \le k \le n$, the equation (4.5) is obvious. Next, by the formulae (4.1) and (4.5) we have

$$\begin{aligned}
n &= \lim_{\ell \to \infty} \left\| \sum_{k=1}^n \theta_k (x_{1k}^{(\ell)}, \dots, x_{Nk}^{(\ell)}) \right\|_\psi \\
&= \left\| \left(\lim_{\ell \to \infty} \left\| \sum_{k=1}^n \theta_k x_{1k}^{(\ell)} \right\|, \dots, \lim_{\ell \to \infty} \left\| \sum_{k=1}^n \theta_k x_{Nk}^{(\ell)} \right\| \right) \right\|_\psi \\
&\le \left\| \left(\lim_{\ell \to \infty} \sum_{k=1}^n \|x_{1k}^{(\ell)}\|, \dots, \lim_{\ell \to \infty} \sum_{k=1}^n \|x_{Nk}^{(\ell)}\| \right) \right\|_\psi \\
&= \left\| \left(\sum_{k=1}^n a_{1k}, \dots, \sum_{k=1}^n a_{Nk} \right) \right\|_\psi \\
&= \left\| \left(\sum_{k=1}^n \chi_{T_k}(1) a_{1k}, \dots, \sum_{k=1}^n \chi_{T_k}(N) a_{Nk} \right) \right\|_\psi \\
&\le \sum_{k=1}^n \|(\chi_{T_k}(1) a_{1k}, \dots, \chi_{T_k}(N) a_{Nk})\|_\psi \\
&\le \sum_{k=1}^n \|(a_{1k}, \dots, a_{Nk})\|_\psi = n,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\left\| \sum_{k=1}^n (\chi_{T_k}(1) a_{1k}, \dots, \chi_{T_k}(N) a_{Nk}) \right\|_\psi &= \sum_{k=1}^n \|(\chi_{T_k}(1) a_{1k}, \dots, \chi_{T_k}(N) a_{Nk})\|_\psi \\
&= n.
\end{aligned} \tag{4.6}$$

Noting that

$$\left\| \sum_{k=1}^{n} (\chi_{T_k}(1)a_{1k}, \ldots, \chi_{T_k}(N)a_{Nk}) \right\|_\psi = (a_{1k_1}, \ldots, a_{Nk_N}),$$

where $j \in T_{k_j},\ 1 \le j \le N$, we have

$$\begin{aligned}(a_{1k_1}, \ldots, a_{Nk_N}) &= \sum_{k=1}^{n} \|(\chi_{T_k}(1)a_{1k}, \ldots, \chi_{T_k}(N)a_{Nk})\|_\psi \\ &= \sum_{k\in\{k_1,\ldots,k_N\}} \|(\chi_{T_k}(1)a_{1k}, \ldots, \chi_{T_k}(N)a_{Nk})\|_\psi \\ &= \sum_{k\in\{k_1,\ldots,k_N\}} \|(\chi_{T_k}(1)a_{1k_1}, \ldots, \chi_{T_k}(N)a_{Nk_N})\|_\psi.\end{aligned}$$

Here, the last identity is valid since

$$\begin{pmatrix} \chi_{T_{k_1}}(1)a_{1k_1} \\ \chi_{T_{k_2}}(1)a_{1k_2} \\ \vdots \\ \chi_{T_{k_N}}(1)a_{1k_N} \end{pmatrix} = \begin{pmatrix} a_{1k_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \chi_{T_{k_1}}(1)a_{1k_1} \\ \chi_{T_{k_2}}(1)a_{1k_1} \\ \vdots \\ \chi_{T_{k_N}}(1)a_{1k_1} \end{pmatrix}$$

and the same is true for the rest columns. By (**??**), $(\chi_{T_k}(1)a_{1k_1}, \ldots, \chi_{T_k}(N)a_{Nk_N})$ are nonzero. Consequently, we have $\psi \in \Psi_N^{(1,n)}$. This completes the proof. $\square$

In the case $n = 2$, $\Psi_N^{(1,2)} = \Psi_N^{(1)}$. Thus, Theorem 4.7 includes Corollary 4.5. Also, if $n = N$, $\Psi_N^{(1,N)} = \{\psi_1\}$. Therefore, Theorem 4.7, combining Theorem 4.2, is reformulated as follows.

Theorem 4.8. *Let $X_1, \ldots, X_N$ be uniformly non-square Banach spaces. Let $\psi \in \Psi_N$ and let $\|\cdot\|_\psi$ be strictly monotone. Let $n \ge 2$.*

(i) *When $n = 2$, $(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly non-square if and only if $\psi \notin \Psi_N^{(1)}$.*
(ii) *When $2 < n < N$, $(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly non-ℓ_1^n if and only if $\psi \notin \Psi_N^{(1,n)}$.*
(iii) *When $n = N$, $(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly non-ℓ_1^N if and only if $\psi \ne \psi_1$.*
(iv) *When $n \ge N+1$, $(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly non-ℓ_1^{N+1}, and hence uniformly non-ℓ_1^n.*

For the assertion (iv), if $\psi \ne \psi_1$, $(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly non-ℓ_1^N by (iii) and hence uniformly non-ℓ_1^n for $n \ge N+1$. If $\psi = \psi_1$, we have the conclusion by Theorem 4.6.

As a direct consequence we have the following characterization of the uniform non-ℓ_1^n-ness for $\mathbb{C}^N$.

Corollary 4.9. *Let $\psi \in \Psi_N$. Assume that $\|\cdot\|_\psi$ is strictly monotone on $\mathbb{C}^N$. Let $n \ge 2$.*

(i) *When $n = 2$, $(\mathbb{C}^N, \|\cdot\|_\psi)$ is uniformly non-square if and only if $\psi \notin \Psi_N^{(1)}$.*

(ii) *When* $2 < n < N$, $(\mathbb{C}^N, \|\cdot\|_\psi)$ *is uniformly non-*ℓ_1^n *if and only if* $\psi \notin \Psi_N^{(1,n)}$.

(iii) *When* $n = N$, $(\mathbb{C}^N, \|\cdot\|_\psi)$ *is uniformly non-*ℓ_1^N *if and only if* $\psi \neq \psi_1$.

(iv) *When* $n \geq N+1$, $(\mathbb{C}^N, \|\cdot\|_\psi)$ *is uniformly non-*ℓ_1^{N+1}*, and hence uniformly non-*ℓ_1^n.

Example 4.10.

(i) Let $1 < p < \infty$. The ℓ_p-norm $\|\cdot\|_p$ is strictly monotone and $\psi_p \notin \Psi_N^{(1)}$ (cite[Example 5.10]KT3). Therefore, the ℓ_p-sum $(\mathbb{C}^N, \|\cdot\|_p)$ is uniformly non-square.

(ii) Let $2 \leq n < N$. Let $\psi \in \Psi_N$ be as in Example 3.8. Then, since $\psi \in \Psi_N^{(1,n)} \backslash \Psi_N^{(1,n+1)}$, $(\mathbb{C}^N, \|\cdot\|_\psi)$ is uniformly non-ℓ_1^{n+1}, but not uniformly non ℓ_1^n.

5. Concluding remarks

According to Dhompongsa, Kato and Tamura [5] an *A-direct sum* $(X_1 \oplus \cdots \oplus X_N)_A$ is defined to be a direct sum equipped with the norm

$$\|(x_1, \ldots, x_N)\|_A := \|(\|x_1\|, \ldots, \|x_N\|)\|_A \text{ for } (x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N,$$

where the norm $\|\cdot\|_A$ in the right side is an arbitrary norm on $\mathbb{R}^N$. A Z-direct sum is defined in the same way from a Z-norm $\|\cdot\|_Z$ which is a norm on $\mathbb{R}^N$ with monotonicity property on $\mathbb{R}_+^N$. Thus a ψ-direct sum is a Z-direct sum, and a Z-direct sum is an A-direct sum, while all of these notions are equivalent. In fact they showed the following.

Theorem 5.1 ([5, Theorem 5.2])**.** *For any A-direct sum there exists a $\psi \in \Psi_N$ for which the A-direct sum $(X_1 \oplus \cdots \oplus X_N)_A$ is isometrically isomorphic to the ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$.*

Now recall that a norm $\|\cdot\|$ on $\mathbb{C}^N$ is said to have *Property* T_1^N ([7]) if for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^N$ with $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \frac{1}{2}\|\boldsymbol{a}+\boldsymbol{b}\| = 1$ one has $\operatorname{supp} \boldsymbol{a} \cap \operatorname{supp} \boldsymbol{b} \neq \emptyset$, where $\operatorname{supp} \boldsymbol{a} = \{j : a_j \neq 0\}$. A recent reuslt [5, Theorem 4.3] says that this property is equivalent to $\psi \notin \Psi_N^{(1)}$ for a ψ-norm. Thus, owing to Theorem 5.1 we shall immediately obtain some consequences for A-direct sums from our preceeding results.

Corollary 5.2. *Let $X_1, \ldots, X_N$ be uniformly non-square. Let $\|\cdot\|$ be an arbitrary strictly monotone norm on $\mathbb{R}^N$. Then:*

(i) $(X_1 \oplus \cdots \oplus X_N)_A$ *is uniformly non-square if and only if the norm* $\|\cdot\|$ *has Property* T_1^N.

(ii) $(X_1 \oplus \cdots \oplus X_N)_A$ *is uniformly non-*ℓ_1^N *if and only if* $\|\cdot\| \neq \|\cdot\|_1$.

(iii) *For all* $n \geq N+1$, $(X_1 \oplus \cdots \oplus X_N)_A$ *is uniformly non-*ℓ_1^n.

In particular we have

Corollary 5.3. *Let $\|\cdot\|$ be an arbitrary strictly monotone norm on $\mathbb{C}^N$. Then:*

(i) $(\mathbb{C}^N, \|\cdot\|)$ *is uniformly non-square if and only if* $\|\cdot\|$ *has Property* T_1^N.

(ii) $(\mathbb{C}^N, \|\cdot\|)$ *is uniformly non-*ℓ_1^N *if and only if* $\|\cdot\| \neq \|\cdot\|_1$.

(iii) *For all* $n \geq N+1$, $(\mathbb{C}^N, \|\cdot\|)$ *is uniformly non-*ℓ_1^n.

We note that in the above two corollaries the corresponding results for the case $2 < n < N$ are not known because any equivalent property to $\psi \notin \Psi_N^{(1,n)}$ (free form ψ) is not known, although we have results using ψ for which $(X_1 \oplus \cdots \oplus X_N)_A$ is isometrically isomorphic to $(X_1 \oplus \cdots \oplus X_N)_\psi$ from Theorem 4.8 and Corollary 4.9.

We shall close our discussion with an observation on a relation between the uniform non-suareness for direct sums and for the scalar case $\mathbb{C}^N$. Betiuk-Pilarska and Prus [3] showed that *a Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ is uniformly non-square if and only if all the underlying spaces $X_1, \ldots, X_N$ and the Z-norm $\|\cdot\|_Z$ on $\mathbb{R}^N$ are uniformly non-square.* By virtue of Theorem 5.1 this holds true for the A-direct sum $(X_1 \oplus \cdots \oplus X_N)_A$, while it remains still unknown when a given norm on $\mathbb{R}^N$ is uniformly non-square. The forgoing result in Corollary 5.3 is a partial answer.

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