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THE EMBEDDING PROBLEM OF THE RANGES OF A HILBERT SPACE UNDER THE COMPACT POSITIVE OPERATORS

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

Abstract. Topological structure of the ranges of a Hilbert space under the compact positive operators is discussed. This result and the entropy theoretic classification of compact operators are applied to the embedding problem of Banach spaces which is constructed from these ranges with Minkowski norms.

1. Introduction

The concept of ε-entropy was developed by Kolmogorov (cf. [3]) in 1957 in connection with Hilbert's 13th problem, Prosser (cf. [5]) applied Kolmogorov's methods to the entropy theoretical classification of compact operators, and moreover, Gelfand and Vilenkin gives entropy theoretic characterization of nuclear spaces (cf. [2])

In this paper, topological structure of the ranges of a Hilbert space under the compact positive operators is discussed. This result and the entropy theoretic classification of compact operators are applied to the embedding problem of Banach spaces which are constructed from these ranges with Minkowski norms.

2. Preliminaries

Throughout this paper, N denotes the set of all positive integers. Let $\mathcal{H}$ be a separable Hilbert space with its inner product $< \cdot, \cdot >$ and norm $\|\cdot\|$.

For any compact positive operator T on $\mathcal{H}$, there exist the non-increasing sequence of eigenvalues $\{\lambda_i(T); i\in N\}$ and the orthonormal system of eigenvectors $\{e_i(T); i\in N\}$ satisfying

$$Te_i(T) = \lambda_i(T)e_i(T), \qquad i\in N.$$

Then, T can be represented by

$$Tx = \sum_{i=1}^{\infty} \lambda_i(T) < x, e_i(T) > e_i(T), \qquad x\in\mathcal{H}.$$

Here, the exponent of convergence $E(T)$ (cf. [4]) is defined as

$$E(T) = inf \left\{ r\geq 0; \sum_{i=1}^{\infty} \lambda_i(T)^r < \infty \right\}.$$

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Let $card(T,r)$ be the distribution of the eigenvalues of T which is defined as

$$card(T,r) \;=\; max\{i{\in}N;\lambda_i(T)>r\}, \quad r>0.$$

Moreover, let $G(T)$ and $g(T)$ be the upper growth order and the lower growth order of $\{\lambda_i; i{\in}N\}$ which are defined as

$$G(T) \;=\; \limsup_{r\to+0}\frac{card(T,r)}{log(1/r)}$$

and

$$g(T) \;=\; \liminf_{r\to+0}\frac{card(T,r)}{log(1/r)},$$

respectively. It is well known that the equality

$$E(T) \;=\; G(T)$$

holds (cf. [5]).

For any $\varepsilon>0$ and for any relatively compact subset $\mathcal{F}$ of $\mathcal{H}$, an ε-covering is defined as a family of open balls with radii ε and whose union can cover $\mathcal{F}$, and moreover, an ε-packing is defined as a family of open balls with centers in $\mathcal{F}$ and radii ε whose pairwise intersections are all empty. Here, the ε-entropy of $\mathcal{F}$, which is denoted by $S(\mathcal{F},\varepsilon)$, is defined as the base-2 logarithm of the minimum number of elements of all ε-covering of $\mathcal{F}$, and the ε-capacity of $\mathcal{F}$, which is denoted by $C(\mathcal{F},\varepsilon)$, is defined as the base-2 logarithm of the maximum number of elements of all ε-packing of $\mathcal{F}$.

For any positive number ε, the ε-entropy of T and the ε-capacity of T are defined as $S(T(\mathcal{U}),\varepsilon)$ and $C(T(\mathcal{U}),\varepsilon)$, respectively, where $\mathcal{U}$ is the closed unit ball of $\mathcal{H}$. Then, by Prosser (cf. [5]) and Akashi (cf. [1]), it is known that, for any positive number δ, there exists a positive number ε_δ satisfying

$$\left(\frac{1}{\varepsilon}\right)^{g(T)-\delta} \;\le\; S(T(U),\varepsilon) \;\le\; \left(\frac{1}{\varepsilon}\right)^{G(T)+\delta}, \qquad 0<\varepsilon<\varepsilon_\delta.$$

3. Topological structures of the ranges of $\mathcal{H}$ under the compact operators

In this section, we discuss the topological structure of the ranges of the closed unit ball under the compact positive operators.

Let T be a compact positive operator on $\mathcal{H}$. Then, the range of $\mathcal{H}$ under the operator T is exactly represented as the following:

$$T(\mathcal{H}) \;=\; \bigcup_{c>0} T(\mathcal{U}),$$

where $\mathcal{U}$ is the closed unit ball of $\mathcal{H}$. Here, let $q_T(\cdot)$ be the Minkowski norm on $T(\mathcal{H})$ which is defined as

$$q_T(y) \;=\; inf\left\{c>0;\, y\in cT(\mathcal{U})\right\}.$$

Then, we obtain the following:

Proposition 3.1. $(T(\mathcal{H}),q_T(\cdot))$ *is a Banach space.*

Proof. Since it is clear that $(T(\mathcal{H}), q_T(\cdot))$ is a normed space, we have only to prove that this space is complete. Let $\{y_n\}_{n=1}^{\infty}$ be a Cauchy sequence consisting of $T(\mathcal{H})$. Then, for any positive number ε, there exists a positive integer n_ε satisfying the following condition:

$$q_T(y_m - y_n) \, < \, \varepsilon, \quad m, n \geq n_\varepsilon.$$

This inequality implies that, for any n which is greater than n_ε, the following inequalities:

$$\begin{aligned} q_T(y_n) &\leq q_T(y_n - y_{n_\varepsilon}) + q_T(y_{n_\varepsilon}) \\ &\leq \varepsilon + q_T(y_{n_\varepsilon}) \end{aligned}$$

hold. This result implies that $\{y_n\}_{n=1}^{\infty}$ can be included by the range of a certain bounded subset of $\mathcal{H}$ under the mapping T. Namely, there exists a certain positive number M satisfying

$$y_n \in T(M\mathcal{U}), \quad n \geq 1.$$

Therefore, for any positive integer n, there exists x_n which belongs to $M\mathcal{U}$ and satisfies $y_n = Tx_n$. Since Banach-Alaoglu theorem assures that there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges weakly at a certain element z belonging to $\mathcal{H}$, $\{Tx_{n_k}\}_{k=1}^{\infty}$ also converges strongly at Tz. These results imply that the equality:

$$\lim_{n \to 0} q_T(Tx_n - Tz) \, = \, 0$$

holds, because, for any positive integer k, $||x_{n_k} - Tz|| = q_T(Tx_{n_k} - Tz)$ holds. These results conclude the proof. □

Now, we can prove the following:

Theorem 3.2. *Let T_1 and T_2 be two compact positive operators on $\mathcal{H}$ satisfying $G(T_1) \neq g(T_2)$. Then, $(T_1(\mathcal{H}), q_{T_1}(\cdot))$ (resp. $(T_2(\mathcal{H}), q_{T_2}(\cdot))$) is not embedded into $(T_2(\mathcal{H}), q_{T_2}(\cdot))$ (resp. $(T_1(\mathcal{H}), q_{T_1}(\cdot))$), that is, there exist neither any bijective continuous linear operator on $(T_1(\mathcal{H}), q_{T_1}(\cdot)))$ with values in $(T_2(\mathcal{H}), q_{T_2}(\cdot)))$ nor any bijective continuous linear operator on $(T_2(\mathcal{H}), q_{T_2}(\cdot)))$ with values in $(T_1(\mathcal{H}), q_{T_1}(\cdot)))$*

Proof. Without loss of generality, we can assume $G(T_1) < G(T_2)$. Assume that there exists a bijective continuous linear operator W satisfying $W(T_1(\mathcal{H})) = T_2(\mathcal{H})$. Then, we can assume that there exist two positive constants c and d such that the two inequalities:

$$q_{T_2}(Wx) \, \leq \, c q_{T_1}(x), \quad x \in T_1(\mathcal{H})$$

and

$$q_{T_1}(x) \, \leq \, d q_{T_2}(Wx), \quad x \in T_1(\mathcal{H})$$

hold. Here Proposition 1 and the assumption lead us to the following inclusions:

$$\begin{aligned} T_2(\mathcal{U}) &= \{y \in T_2(\mathcal{H});\, q_{T_2}(y) \leq 1\} \\ &\subset \{Wx;\, x \in T_1(\mathcal{H}), q_{T_1}(x) \leq d\} \\ &= W(dT_1(\mathcal{U})). \end{aligned}$$

But these inclusions imply that, for any positive number ε,

$$S(T_2(\mathcal{U}), \varepsilon) \leq S(W(dT_1(\mathcal{U})), \varepsilon).$$

Moreover, according to the relations between $q_{T_1}(\cdot)$ and $q_{T_2}(\cdot)$, we have

$$S(W(T_1(\mathcal{U})), \|W\|\varepsilon) \leq S(T_1(\mathcal{U}), \varepsilon).$$

Therefore, these two inequalities show that

$$G(T_2) \leq G(T_1).$$

But this inequality contradicts that $G(T_1) < G(T_2)$ holds. Therefore, it has been proved that there does not exist any bijective continuous linear operator W on $T_1(\mathcal{H})$ with values in $T_2(\mathcal{H})$. If we assume that there exists a bijective continuous linear operator on $T_2(\mathcal{H})$ with values in $T_1(\mathcal{H})$, Banach-Steinhaus theorem leads to the conclusion that this bijective continuous linear operator should play the role of a homeomorphic homomorphism on $T_1(\mathcal{H})$ with values in $T_2(\mathcal{H})$. Therefore, this assumption also contradicts the former part of this proof. □

Remarks. Theorem 3.2 shows that, for any two compact positive operators T_1 and T_2 satisfying $G(T_1) \neq G(T_2)$, there does not exist such a unitary operator U that $T_1 = U^*T_2U$ holds. Moreover, it can be proved that, for any compact positive operator T, there does not exist any Banach space where $T(\mathcal{H})$ can be embedded.

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