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ON MINIMAX VARIATIONAL INEQUALITY PROBLEMS FOR RELAXED α-PSEUDOMONOTONE AND GENERALIZED LIPSCHITZ MAPPINGS

KAN BURANAKORN AND SOMYOT PLUBTIENG*

Dedicated to Wataru Takahashi on the occasion of his 70th birth day

Abstract. In this paper, we obtain some sufficient conditions the solution existence of minimax variational inequality problems for relaxed α-pseudomonotone mappings in reflexive Banach spaces, and generalized Lipschiz mappings in Hilbert spaces.

1. Introduction

Let A be nonempty closed convex subset in Banach spaces X and $f : A \to \mathbb{R}$ be arbitrary function. The *variational inequality problem* (VIP) is to find a point $\bar{x} \in A$ such that

$$\langle f(\bar{x}), x - \bar{x} \rangle \geq 0, \ \forall x \in A.$$

Variational inequality problems plays a significant role in economics, engineering mechanics, mathematical programming, transportation, etc; (see e.g.[2, 4, 6, 13]). In recent years, many authors obtained some variational inequality problems for generalizations of monotonicity such as quasi-monotonicity, pseudo-monotonicity, relaxed monotonicity, α-monotonicity, semi-monotonicity, etc; (see e.g.[5, 8, 9, 16, 17]).

The notion of minimax variational inequality problems (MVIP) is a new mathematical model by Huy and Yen ([10]) in 2011. They studied some sufficient condition for the solution existence of MVIP for nonmonotone in Euclidean spaces, pseudomonotone in reflexive Banach spaces and strongly monotone in Hilbert spaces. They also shown that MVIP can serve as a good tool study minimax problems given by differentiable functions and convex set.

Let A, B be nonempty closed convex sets in Banach spaces X and Y, respectively. Suppose that $f : \Omega \to \mathbb{R}$ is a Fréchet continuously differentiable function defined on an open subset Ω of $X \times Y$ with $A \times B \subset \Omega$.

The *minimax problem* given by the convex sets A, B and the function f, which is written formally as

$$\max_{y\in B} \min_{x\in A} f(x, y), \tag{1.1}$$

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*Corresponding author.

is that one of finding a point $(\bar{x},\bar{y}) \in A \times B$ such that

$$f(\bar{x},y) \le f(\bar{x},\bar{y}) \le f(x,\bar{y}) \ \forall x \in A,\ \forall y \in B. \tag{1.2}$$

If $(\bar{x},\bar{y}) \in A \times B$ satisfies (1.2), then one says that it is a *saddle point* of the minimax problem (1.1).

Definition 1.1 ([10])**.** Let X, Y be Banach spaces with the dual spaces denoted respectively by X^* and Y^*. Let $A \subset X, B \subset Y$ be nonempty closed convex sets, and let $F_1 : A \times B \to X^*$, $F_2 : A \times B \to Y^*$ be arbitrary given functions. The *minimax variational inequality problem* $(MVIP)$ defined by data set $\{A, B, F_1, F_2\}$ is the problem for finding a point $(\bar{x},\bar{y}) \in A \times B$ such that

$$\langle F_2(\bar{x},\bar{y}), y - \bar{y}\rangle \le 0 \le \langle F_1(\bar{x},\bar{y}), x - \bar{x}\rangle \ \forall x \in A,\ \forall y \in B.$$

The solution set of $(MVIP)$ is denoted by $Sol(MVIP)$.

Remark 1.2. According to Theorem 1.1 of [10], if the solution set of (1.1) is denoted by S then it holds $S \subset Sol(MVIP)$, provided that we put $F_1 = \nabla_x f$ and $F_2 = \nabla_y f$. Moreover, if $f(\cdot, y)$ is pseudo-convex on A and $f(x, \cdot)$ is pseudo-concave on B for every $(x, y) \in A \times B$, then $S = Sol(MVIP)$ by Theorem 1.2 of [10]. Thus, $(MVIP)$ can be used in studying the minimax problem (1.1).

Consider an operator $G : A \times B \to X^* \times Y^*$ defined by $G(x, y) = (F_1(x, y), -F_2(x, y))$ for all $(x, y) \in A \times B$. Thus, the value of the functional $G(x, y) \in A^* \times B^*$ at $(u, v) \in A \times B$ is given by

$$\langle G(x,y), (u,v)\rangle = \langle F_1(x,y), u\rangle - \langle F_2(x,y), v\rangle. \tag{1.3}$$

Unless otherwise stated, the norm in the product space $X \times Y$ is defined by setting $\|(x, y)\| = \|x\| + \|y\|$. We are interested in the variational inequality defined by the closed convex set $A \times B \subset X \times Y$ and the operator $G : A \times B \to X^* \times Y^*$.

(1.4) *Find* $(\bar{x},\bar{y}) \in A \times B$ *such that*

$$\langle G(\bar{x},\bar{y}), (x,y) - (\bar{x},\bar{y})\rangle \ge 0 \ \forall (x,y) \in A \times B.$$

Proposition 1.3 ([10])**.** *The inclusion* $(\bar{x},\bar{y}) \in Sol(MVIP)$ *holds if and only if* $(\bar{x},\bar{y})$ *is a solution of* (1.4)*.*

In this paper, we will study some sufficient conditions the solution existence of minimax variational inequality problems for relaxed α-pseudomonotone mappings in reflexive Banach spaces, and generalized Lipschiz mappings in Hilbert spaces. The remainder of this paper has 3 sections. In section 2, we recall and state preliminaries various operator in form MVIP. In section 3, we present the solution existence of MVIP for relaxed α-pseudomonotone in reflexive Banach spaces and in the last section we investigate the solution existence and uniqueness of MVIP for Generalized Lipschitz in Hilbert spaces.

2. Preliminaries

In this paper the variational inequalities, weakly coercivity, monotonicity, pseudomonotonicity, relaxed α-pseudomonotonicity and strong monotonicity are fundamental concepts; see e.g.[1, 7, 11, 12, 19]. It is widely known that strong monotonicity $\Rightarrow$ monotonicity $\Rightarrow$ pseudomonotonicity $\Rightarrow$ relaxed α-pseudomonotonicity and strong monotonicity $\Rightarrow$ weakly coercivity.

Applied to the operator $G = (F_1, -F_2) : A \times B \to X^* \times Y^*$ given in (1.3) and the variational inequality (1.4), weakly coercivity, monotonicity, pseudomonotonicity, relaxed α-pseudomonotonicity and strong monotonicity in theory of VIP mean the following.

Definition 2.1. Problem (1.4) is said to satisfy the weakly coercivity condition if there exists a point $(u, v) \in A \times B$ such that

$$\lim_{\|(x,y)\|\to\infty,\ (x,y)\in A\times B} \langle G(x, y) - G(u, v), (x, y) - (u, v)\rangle = \infty. \tag{2.1}$$

Definition 2.2 ([10])**.** Problem (1.4) is said to be monotone if

$$\langle G(x, y) - G(u, v), (x, y) - (u, v)\rangle \geq 0, \tag{2.2}$$

$\forall (x, y), (u, v) \in A \times B$.

Definition 2.3 ([10])**.** Problem (1.4) is said to be pseudomonotone if

$$\langle G(u, v), (x, y) - (u, v)\rangle \geq 0 \Rightarrow \langle G(x, y), (x, y) - (u, v)\rangle \geq 0, \tag{2.3}$$

$\forall (x, y), (u, v) \in A \times B$.

Definition 2.4. Problem (1.4) is said to be relaxed α-pseudomonotone if

$$\begin{aligned} &\langle G(u, v), (x, y) - (u, v)\rangle \geq 0 \Rightarrow \\ &\qquad\qquad \langle G(x, y), (x, y) - (u, v)\rangle \geq \alpha(x - u, y - v), \end{aligned} \tag{2.4}$$

$\forall (x, y), (u, v) \in A \times B$ where $\alpha : A \times B \to \mathbb{R}$ with

$$\alpha(tz) = t^p \alpha(z), \forall t > 0,\ \forall p > 1,\ \forall z \in A \times B.$$

Definition 2.5. [10] Problem (1.4) is said to be strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle G(x, y) - G(u, v), (x, y) - (u, v)\rangle \geq \alpha(\|x - u\|^2 + \|y - v\|^2), \tag{2.5}$$

$\forall (x, y), (u, v) \in A \times B$.

Example 2.6. Let $X = Y = A = \mathbb{R},\ P = \mathbb{R}_+$ and let $T : X \to Y$ be defined by

$$T(x) = \begin{cases} \frac{3}{2}x & \text{if } x \geq 0, \\ -\frac{1}{2}x & \text{if } x < 0, \end{cases}$$

and

$$\alpha(x) = -\frac{1}{2}x^2,\ \forall x \in X.$$

This is an example to show that there is a mapping T such that T is relaxed α-pseudomonotone but not pseudomonotone.

Definition 2.7. The operator T is hemicontinuous on A if for every pair of points $x, y \in A$, the following function is continuous

$$t \to \langle T(tx + (1-t)y), x - u\rangle, \quad 0 \le t \le 1.$$

3. MVIP for relaxed α-pseudomonotone mapping in Banach Spaces

In this section, we assumed that X, Y are reflexive Banach spaces. The norm in the product space $X \times Y$ is given by setting $\|(x,y)\| = \|x\| + \|y\|$. Then $X \times Y$ is also a reflexive Banach space. Besides, $(X \times Y)^* \equiv X^* \times Y^*$ and the value of $(x^*, y^*) \in X^* \times Y^*$ at $(x,y) \in X \times Y$ is given by $\langle (x^*, y^*), (x,y)\rangle = \langle x^*, x\rangle + \langle y^*, y\rangle$. This conventions imply that $\|(x^*, y^*)\| = max\{\|x^*\|, \|y^*\|\}$.

Lemma 3.1 ([1],Theorem 3.1))**.** *Let X be a reflexive Banach space, A a nonempty closed convex subset of X and $F : A \to X^*$ a relaxed α-pseudomonotone operator which is hemicontinuous on finite-dimensional subspace. Then vector u is a solution of the variational inequaity*

$$u \in A,\ \langle F(u), x - u\rangle \ge 0,\ \forall x \in A \tag{3.1}$$

if and only if

$$u \in A,\ \langle F(x), x - u\rangle \ge \alpha(x-u),\ \forall x \in A. \tag{3.2}$$

Moreover, the solution set of (3.1) *is closed and convex.*

Theorem 3.2. *Let X be a reflexive Banach space, A a nonempty closed bounded and convex subset of X and $F : A \to X^*$ a relaxed α-pseudomonotone operator which is continuous on finite-dimensional subspace. Then there exists $u \in A$ such that*

$$\langle F(u), x - u\rangle \ge 0,\ \forall x \in A.$$

Moreover, the solution set is nonempty closed bounded and convex.

Proof. We consider a finite-dimensional subspace M of X such that $A \cap M$ is nonempty. Since $A \cap M$ is nonempty closed bounded and convex in M and $F : A \cap M \to X^*$ is continuous, it follows from Stampacchia ([14], Lemma 3.1) that there exists $x_M \in A \cap M$ such that

$$\langle F(x_M), x - x_M\rangle \ge 0,\ \forall x \in A \cap M.$$

By Theorem 3.1, we have

$$\langle F(x), x - x_M\rangle \ge \alpha(x - x_M),\ \forall x \in A \cap M. \tag{3.3}$$

For each $x \in A$ define

$$S(x) = \{u \in A : \langle F(x), x - u\rangle \ge \alpha(x-u)\}.$$

Then the family $\{S(x) : x \in A\}$ has a finite intersection property. Indeed, for any finite subset $\{x_i : 1 \le i \le m\}$ of K, let M be the finite-dimensional subspace spanned by $\{x_i : 1 \le i \le m\}$. By the finite-dimensional case (3.3) that we have shown, there exist $x_M \in A \cap M$ such that

$$\langle F(x), x - x_M\rangle \ge \alpha(x - x_M),\ \forall x \in A \cap M.$$

In particular, we have $\langle F(x_i), x_i - x_M\rangle \ge 0$ for all $1 \le i \le m$. This implies that $x_M \in \bigcap_{i=1}^m S(x_i)$. Hence $S(x)$ is not empty for all $x \in A$. Since $S(x)$ is closed for all $x \in A$ and A is closed and bounded, it follows that $\bigcap_{x\in A} S(x)$ is nonempty. Let $u \in \bigcap_{x\in A} S(x)$. Then $\langle F(x), x-u\rangle \geq \alpha(x-u)$ for all $x \in A$. By Theorem 3.1, we have $\langle F(u), x-u\rangle \geq 0,\ \forall x \in A$. Finally, by Theorem 3.1 again and bounded of A, the solution set is nonempty closed bounded and convex. □

Lemma 3.3 ([19],Theorem 4.4)). *Let X be a reflexive Banach space, A a nonempty closed convex subset of X, $F : A \to X^*$ and there exists a closed bounded and convex subset C of A with $int_A(C)$ nonempty satisfying the following condition: for each $x \in \partial_A(C)$ there exists $u \in int_A(C)$ such that $\langle F(x), x-u\rangle \geq 0$. If $u \in A$ such that $\langle F(u), x-u\rangle \geq 0$ for all $x \in C$, then $\langle F(u), x-u\rangle \geq 0$ for all $x \in A$.*

Lemma 3.4. *Let X be a reflexive Banach space, A a nonempty closed convex subset of X and $F : A \to X^*$ a relaxed α-pseudomonotone operator which is continuous on finite-dimensional subspace. If there exists a closed bounded and convex subset C of A with $int_A(C)$ nonempty satisfying the following condition: for each $x \in \partial_A(C)$ there exists $u \in int_A(C)$ such that $\langle F(x), x-u\rangle \geq 0$. Then there exists $u \in A$ such that*

$$\langle F(u), x-u\rangle \geq 0,\ \forall x \in A.$$

Proof. The result is follows from Theorem 3.2 and Lemma 3.3. □

Theorem 3.5. *Let X be a reflexive Banach space. Suppose that A is a closed convex subset of X with $0 \in A$ and $F : A \to X^*$ is a weakly coercive relaxed α-pseudomonotone operator which is continuous on finite-dimensional subspace. Then there exists $u \in A$ such that*

$$\langle F(u), x-u\rangle \geq 0, \forall x \in A.$$

Proof. Since F is weakly coercive there is an $r > 0$ such that $\langle F(x), x\rangle \geq 0$ for all $x \in A$ and $\|x\| \geq r$. Let $C = A \cap \overline{B}_r$. Then the set C is closed bounded and convex with $int_A(C)$ nonempty. For each $x \in \partial_A(C)$, there exists $0 \in int_A(C)$ such that $\langle F(x), x-0\rangle \geq 0$. Therefore, by Lemma 3.4, we can complete the proof. □

An analogue of Theorem 3.1, 3.2 and 3.5, MVIP for relaxed α-pseudomonotone can be formulated as follows.

Theorem 3.6. *Let X, Y be reflexive Banach spaces. Suppose that $A \subset X, B \subset Y$ are nonempty closed convex subsets and $F_1 : A \times B \to X^*, F_2 : A \times B \to Y^*$ are hemicontinuous on finite-dimensional subspaces and $(MVIP)$ is relaxed α-pseudomonotone. Then vector (u, v) is a solution of the $(MVIP)$*

$$\langle F_1(u, v), x-u\rangle - \langle F_2(u, v), y-v\rangle \geq 0, \forall (x, y) \in A \times B.$$

if and only if

$$\langle F_1(x, y), x-u\rangle - \langle F_2(x, y), y-v\rangle \geq \alpha(x-u, y-v), \forall (x, y) \in A \times B.$$

Moreover, the solution set of (3.1) *is closed and convex.*

Proof. Let $G = (F_1, -F_2)$. Since F_1, F_2 are hemicontinuous on finite-dimentional subspaces, we obtains that $G : A \times B \to X^* \times Y^*$ is hemicontinuous on finite-dimentional subspaces. Hence, by Theorem 3.1, we can complete the proof. □

Theorem 3.7. *Let X, Y be reflexive Banach spaces. Suppose that $A \subset X, B \subset Y$ are nonempty closed bounded and convex subsets and $F_1 : A \times B \to X^*, F_2 : A \times B \to Y^*$ are continuous on finite-dimensional subspaces and $(MVIP)$ is relaxed α-pseudomonotone. Then there exists $(u, v) \in A \times B$ such that*

$$\langle F_1(u, v), x - u\rangle - \langle F_2(u, v), y - v\rangle \geq 0, \forall (x, y) \in A \times B.$$

Moreover, the solution set is nonempty closed bounded and convex.

Proof. We note that, if F_1, F_2 are continuous on finite-dimentional subspaces, then $G = (F_1, -F_2)$ is also continuous on finite-dimentional subspaces. Hence by our assumption, we can get the assertion directly from Theorem 3.2. □

Theorem 3.8. *Let X, Y be reflexive Banach spaces. Suppose that $A \subset X, B \subset Y$ are nonempty closed convex subsets of X with $(0, 0) \in A \times B$ and $F_1 : A \times B \to X^*, F_2 : A \times B \to Y^*$ are continuous on finite-dimensional subspaces and $(MVIP)$ is weakly coercive relaxed α-pseudomonotone. Then there exists $(u, v) \in A \times B$ such that*

$$\langle F_1(u, v), x - u\rangle - \langle F_2(u, v), y - v\rangle \geq 0, \forall (x, y) \in A \times B.$$

Proof. The proof is similar with previous theorem. Hence we can also get the assertion directly from Theorem 3.5. □

4. MVIP for generalized Lipschitz mapping in Hilbert Spaces

In this section, we assumed that X, Y are Hilbert spaces. Then X^* and Y^* can be identified with X and Y, respectively. The value of $x^* \in X^*$ at $x \in X$ is identified with the inner product $\langle x^*, x\rangle$ of two vectors in X. A similar interpretation is given for the inner product $\langle y^*, y\rangle$ of two vectors in Y. Setting $\langle (x, y), (u, v)\rangle = \langle x, u\rangle + \langle y, v\rangle$ for all $(x, y), (u, v) \in X \times Y$, we define an innerproduct in $X \times Y$. Note that $X \times Y$ is also a Hilbert space with the norm $\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$.

We call $z \in A$ the metric projection of a point $x \in X$ onto a closed convex subset $A \subset X$ and write $z = P_A(x)$ if $\|x - z\| = inf\{\|x - u\| : u \in A\}$.

It is well known that the metric projection $z = P_A(x)$ exists and is uniquely defined by x ([12],Lemma 2.1 p.8). Besides, $z = P_A(x)$ if and only if $z \in A$ and $\langle x - z, u - z\rangle \leq 0$ for every $u \in A$ ([12],Theorem 2.3 p.9). We also know that $P_K(\cdot) : X \to K$ is a nonexpansive mapping, that is, $\|P_K(x') - P_K(x)\| \leq \|x' - x\|$, $\forall x, x' \in X$ ([12],Corollary 2.4 p.10).

Theorem 4.1. *Let X be a Hilbert space. Suppose that A is a nonempty closed convex subset of X and $F : A \to X$ is a generalized Lipschitz, strongly monotone mapping, i.e., there exist constant $L > 0$ and $\alpha \geq 0$ such that*

$\|F(x) - F(u)\| \leq L(1 + \|x - u\|)\ \ \forall x, u \in A$

$\langle F(x) - F(u), x - u\rangle \geq \alpha\|x - u\|^2\ \ \forall x, u \in A$

Then the variational inequality problem;

Find $\bar{x} \in A$ such that $\langle F(\bar{z}), x - \bar{x}\rangle \geq 0$ $\forall x \in A$,

has a unique solution.

Proof. If A is a singleton set, it obviously that the solution has unique. Suppose that A has at least two elements, then by estimation

$$|\langle F(x) - F(u), x - u\rangle| \leq \|F(x) - F(u)\|\|x - u\|$$

and by assumption we note that

$$\alpha\|x-u\|^2 \leq |\langle F(x)-F(u), x-u\rangle|$$

and

$$|\langle F(x)-F(u), x-u\rangle| \leq L(1+\|x-u\|)\|x-u\|,$$

this implies that $0 < \alpha \leq LC$ where $C = \frac{1+\|x-u\|}{\|x-u\|}$. With loss of generality we may assume that $\alpha \leq LC$. Take any $\rho \in (0, \frac{\alpha}{L^2C^2}]$ and we define a map $g : A \to A$ by setting $g(x) = P_A(x - \rho F(x))$ for all $x \in A$. Thus,

$$\begin{aligned}
\|(g(x)-g(u)\|^2 &= \|P_A(x-\rho F(x)) - P_A(u-\rho F(u))\|^2 \\
&\leq \|(x-\rho F(x)) - (u-\rho F(u))\|^2 \\
&= \|-(\rho F(x)-\rho F(u)) + (x-u)\|^2 \\
&= \|\rho F(x)-\rho F(u)\|^2 - 2\langle \rho F(x)-\rho F(u), x-u\rangle + \|x-u\|^2 \\
&= \|F(x)-F(u)\|^2 - 2\rho\langle F(x)-F(u), x-u\rangle + \|x-u\|^2 \\
&\leq \rho^2L^2C^2\|x-u\|^2 - 2\rho\alpha\|x-u\|^2 + \|x-u\|^2 \\
&= (\rho^2L^2C^2 - 2\rho\alpha + 1)\|x-u\|^2, \ \forall x, u \in A.
\end{aligned}$$

Since $0 < \rho^2 \leq \frac{\rho\alpha}{L^2C^2}$, we have $\rho^2L^2C^2 \leq \rho\alpha$. Hence
$\rho^2L^2C^2 - 2\rho\alpha + 1 = (\rho^2L^2C^2 - \rho\alpha) + (1-\rho\alpha) \leq 1 - \rho\alpha.$
And since $0 < \rho\alpha \leq \frac{\alpha^2}{L^2C^2} \leq 1$, we see that $0 \leq 1 - \rho\alpha < 1$.
It follows that

$$\|g(x)-g(u)\| \quad \leq \quad r\|x-u\|$$

where $r := \sqrt{1-\rho\alpha} \in [0, 1)$.

By the Banach contractive mapping principle, there is a unique point $\bar{x} \in A$ satisfying $g(\bar{x}) = \bar{x}$. Thus $P_A(\bar{x} - \rho F(\bar{x})) = \bar{x}$. By using the characterization of metric projection, we obtain that

$$\begin{aligned}
\bar{x} = P_A(\bar{x}-\rho F(\bar{x})) &\Leftrightarrow \bar{x} \in A \textit{ and } \langle \bar{x}-\rho F(\bar{x}) - \bar{x}, x-\bar{x}\rangle \leq 0, \ \forall x \in A \\
&\Leftrightarrow \bar{x} \in A \textit{ and } \langle -\rho F(\bar{x}), x-\bar{x}\rangle \leq 0, \ \forall x \in A \\
&\Leftrightarrow \bar{x} \in A \textit{ and } \langle \rho F(\bar{x}), x-\bar{x}\rangle \geq 0, \ \forall x \in A \\
&\Leftrightarrow \bar{x} \in A \textit{ and } \langle F(\bar{x}), x-\bar{x}\rangle \geq 0, \ \forall x \in A.
\end{aligned}$$

Since $g(\bar{x}) = \bar{x}$ has a unique fixed point, we can conclude that $\langle F(\bar{x}), x-\bar{x}\rangle \geq 0$ for all $x \in A$ has a unique solution. □

Theorem 4.2. *Let X, Y be a Hilbert spaces. Suppose that $A \subset X$, $B \subset Y$ are nonempty closed convex subsets and $F_1 : A \times B \to X$, $F_2 : A \times B \to Y$ are generalized Lipschitz. i.e., there exist constant $L_i > 0$ $(i = 1, 2)$ such that*

$$\|F_i(x,y) - F_i(u,v)\| \leq L_i(1 + \|(x,y)-(u,v)\|) \tag{4.1}$$

for all $(x,y), (u,v) \in A \times B$, $i = 1, 2$. If the (MVIP) is strongly monotone, then it has a unique solution.

Proof. Consider the mapping $G(F_1, -F_2) : A \times B \to X \times Y$.Thus,

$$\begin{aligned}
\|G(x,y) - G(u,v)\|^2 &= \|(F_1(x,y), -F_2(x,y)) - (F_1(u,v), -F_2(u,v))\|^2 \\
&= \|(F_1(x,y) - F_1(u,v), -F_2(x,y) + F_2(u,v))\|^2
\end{aligned}$$

$$\begin{aligned}
&= \|F_1(x,y) - F_1(u,v)\|^2 + \| - F_2(x,y) + F_2(u,v)\|^2 \\
&\leq L_1^2(1 + \|(x,y) - (u,v)\|)^2 + L_2^2(1 + \|(x,y) - (u,v)\|)^2 \\
&= (L_1^2 + L_2^2)(1 + \|(x,y) - (u,v)\|)^2, \ \forall (x,y),(u,v) \in A \times B.
\end{aligned}$$

Hence

$$\|G(x,y) - G(u,v)\| \leq \sqrt{L_1^2 + L_2^2}(1 + \|(x,y) - (u,v)\|),$$

for all $(x,y),(u,v) \in A \times B$. We obtain that $G = (F_1, -F_2)$ is a generalized Lipschitz mapping. By Theorem 4.1, we can conclude the theorem. □

Theorem 4.3. *Let X be a Hilbert space. Suppose that A is a nonempty bounded closed and convex subset of X and $F : A \to 2^X$ is a generalized Lipschitz, strongly monotone mapping, i.e., there exist constant $L > 0$ and $\alpha \geq 0$ such that*
$H(F(x), F(u)) \leq L(1 + \|x - u\|)$ $\forall x, u \in A$, where H is the Hausdorff metric,
$\langle y - v, x - u\rangle \geq \alpha\|x - u\|^2$ $\forall F(x), F(u) \subset A$.
Then the variational inequality problem;
Find $\bar{x} \in A$ and $\bar{z} \in F(\bar{x})$ such that $\langle \bar{z}, x - \bar{x}\rangle \geq 0$ $\forall x \in A$,
has a solution.

Proof. Suppose that A has at least two elements, then by estimation

$$|\langle y - v, x - u\rangle| \leq \|y - v\|\|x - u\|$$

and by assumption we note that

$$\alpha\|x - u\|^2 \leq |\langle y - v, x - u\rangle|$$

and

$$|\langle y - v, x - u\rangle| \leq L(1 + \|x - u\|)\|x - u\|,$$

this implies that $0 < \alpha \leq LC$ where $C = \frac{1+\|x-u\|}{\|x-u\|}$. With loss of generality we may assume that $\alpha \leq LC$. Take any $\rho \in (0, \frac{\alpha}{L^2C^2}]$ and we define a map $g : A \to 2^A$ by setting $g(x) = \{P_A(x - \rho z) : z \in F(x)\}$. Suppose that

$$H(g(x), g(u)) = \sup_{z\in g(x)} \inf_{w\in g(u)} \|z - w\|.$$

Thus,

$$\begin{aligned}
[H(g(x), g(u))]^2 &= \sup_{z\in g(x)} \inf_{w\in g(u)} \|z - w\|^2 \\
&= \sup_{y\in F(x)} \inf_{v\in F(u)} \|P_A(x - \rho y) - P_A(u - \rho v)\|^2 \\
&\leq \sup_{y\in F(x)} \inf_{v\in F(u)} \|(x - \rho y) - (u - \rho v)\|^2 \\
&= \sup_{y\in F(x)} \inf_{v\in F(u)} \| - (\rho y - \rho v) + (x - u)\|^2 \\
&= \sup_{y\in F(x)} \inf_{v\in F(u)} (\|\rho y - \rho v\|^2 - 2\langle \rho y - \rho v, x - u\rangle + \|x - u\|)^2 \\
&= \rho^2 \sup_{y\in F(x)} \inf_{v\in F(u)} \|y - v\|^2 - 2\rho \sup_{y\in F(x)} \inf_{v\in F(u)} \langle y - v, x - u\rangle \\
&\quad + \|x - u\|^2
\end{aligned}$$

$$\begin{aligned} &\leq \rho^2L^2C^2\|x-u\|^2 - 2\rho\alpha\|x-u\|^2 + \|x-u\|^2 \\ &= (\rho^2L^2C^2 - 2\rho\alpha + 1)\|x-u\|^2. \end{aligned}$$

Since $0 < \rho^2 \leq \frac{\rho\alpha}{L^2C^2}$, we have $\rho^2L^2C^2 \leq \rho\alpha$. Hence
$\rho^2L^2C^2 - 2\rho\alpha + 1 = (\rho^2L^2C^2 - \rho\alpha) + (1-\rho\alpha) \leq 1-\rho\alpha.$
And since $0 < \rho\alpha \leq \frac{\alpha^2}{L^2C^2} \leq 1$, we see that $0 \leq 1-\rho\alpha < 1$.
It follows that

$$H(g(x), g(u)) \leq r\|x-u\|$$

where $r := \sqrt{1-\rho\alpha} \in [0,1)$.

Similarly, if $H(g(x), g(u)) = \sup_{w\in g(u)} \inf_{z\in g(x)} \|w-z\|$, we also get the same result. By Nadler's fixed point theorem, there exists a point $\bar{x} \in A$ and $\bar{z} \in F(\bar{x})$ satisfying $\bar{x} \in g(\bar{x})$. Thus $\bar{x} = P_A(\bar{x}-\rho\bar{z})$. By using the characterization of metric projection, we obtain that

$$\begin{aligned} \bar{x} = P_A(\bar{x}-\rho\bar{z}) &\Leftrightarrow \bar{x}\in A,\ \bar{z}\in F(\bar{x}) \textit{ and } \langle \bar{x}-\rho\bar{z}-\bar{x}, x-\bar{x}\rangle \leq 0,\ \forall x\in A \\ &\Leftrightarrow \bar{x}\in A,\ \bar{z}\in F(\bar{x}) \textit{ and } \langle -\rho\bar{z}, x-\bar{x}\rangle \leq 0,\ \forall x\in A \\ &\Leftrightarrow \bar{x}\in A,\ \bar{z}\in F(\bar{x}) \textit{ and } \langle \rho\bar{z}, x-\bar{x}\rangle \geq 0,\ \forall x\in A \\ &\Leftrightarrow \bar{x}\in A,\ \bar{z}\in F(\bar{x}) \textit{ and } \langle \bar{z}, x-\bar{x}\rangle \geq 0,\ \forall x\in A. \end{aligned}$$

Therefore, we can conclude that $\langle \bar{z}, x-\bar{x}\rangle \geq 0$ for all $x \in A$ has a solution. □

Theorem 4.4. *Let X, Y be a Hilbert spaces. Suppose that $A \subset X$, $B \subset Y$ are nonempty bounded closed and convex subsets and $F_1 : A\times B \to 2^X$, $F_2 : A\times B \to 2^Y$ are generalized Lipschitz. i.e., there exist constant $L_i > 0$ $(i = 1, 2)$ such that*

$$H(F_i(x,y), F_i(u,v)) \leq L_i(1 + \|(x,y)-(u,v)\|)$$

for all $(x,y),(u,v) \in A\times B$, $i = 1, 2$. If the (MVIP) is strongly monotone, then it has a solution.

Proof. Consider the mapping $G(F_1, -F_2) : A\times B \to 2^X \times 2^Y$. Suppose that

$$H(G(x,y), G(u,v)) = \sup_{(z,w)\in G(x,y)} \inf_{(s,t)\in G(u,v)} \|(z,w)-(s,t)\|,$$

we have

$$\begin{aligned} [H(G(x,y), G(u,v))]^2 &= \sup_{(z,w)\in G(x,y)} \inf_{(s,t)\in G(u,v)} \|(z-s, w-t)\|^2 \\ &= \sup_{(z,w)\in G(x,y)} \inf_{(s,t)\in G(u,v)} (\|z-s\|^2 + \|w-t\|^2) \\ &= \sup_{(z,w)\in G(x,y)} \inf_{(s,t)\in G(u,v)} \|z-s\|^2 \\ &\quad + \sup_{(z,w)\in G(x,y)} \inf_{(s,t)\in G(u,v)} \|w-t\|^2 \\ &\leq (H(F_1(x,y), F_1(u,v)))^2 + (H(F_2(x,y), F_2(u,v)))^2 \\ &\leq L_1^2(1+\|(x,y)-(u,v)\|)^2 + L_2^2(1+\|(x,y)-(u,v)\|)^2 \\ &= (L_1^2+L_2^2)(1+\|(x,y)-(u,v)\|)^2,\ \forall (x,y),(u,v)\in K\times L. \end{aligned}$$

Hence

$$H(G(x,y),G(u,v)) \leq \sqrt{L_1^2+L_2^2}(1+\|(x,y)-(u,v)\|),$$

for all $(x,y),(u,v) \in A\times B$. Similarly, if

$$H(G(x,y),G(u,v)) = \sup_{(s,t)\in G(u,v)} \inf_{(z,w)\in G(x,y)} \|(s,t)-(z,w)\|,$$

we also get that $G=(F_1,-F_2)$ is a generalized Lipschitz mapping. By Theorem 4.3, we can conclude the theorem. □

Recall that a function $\varphi : X \to \mathbb{R}$ is said to be strongly convex on a convex set $A \subset X$ if there exists $\rho > 0$ such that

$$\varphi((1-t)x+tu) \leq (1-t)\varphi(x)+t\varphi(u)-\rho t(1-t)\|x-u\|^2,$$

$\forall x,u \in A$, $\forall t \in (0,1)$. The number ρ is called a coefficient of strong convexity of φ on K. If $-\varphi$ is strongly convex on A with a coefficient of strong convexity $\rho > 0$, then φ is said to be strongly concave on A with the coefficient of strong concavity $\rho > 0$. It is well known that φ is strongly convex on A with a coefficient of strong convexity ρ if and only if the function $\overline{\varphi}(x) := \varphi(x)-\rho\|x\|^2$ is convex on A (see [15]). Moreover, if φ is continuously differentiable in an open set containing A, then this strong convexity property holds if and only if

$$\langle \nabla\varphi(x)-\nabla\varphi(u), x-u\rangle \geq 2\rho\|x-u\|^2,\ \forall x,u \in A.$$

A proof of the fact can be found in [15] for the case $X=\mathbb{R}^n$. Observe that the method of proof works also for the case where X is an arbitrary Hilbert space (see also [18]).

Theorem 4.2 gives us the following result on the existence and uniqueness of a saddle point.

Theorem 4.5. *Let X,Y be a Hilbert spaces. Suppose that $A \subset X$, $B \subset Y$ are nonempty closed convex subsets and there exist constants $\rho > 0$ and $L_i > 0$ $(i=1,2)$ such that such that the conditions* (2.5) *and* (4.1) *are satisfied for $G(x,y) = (F_1(x,y),-F_2(x,y))$ where $F_1(x,y) = \nabla_x f(x,y)$ and $F_2(x,y) = \nabla_y f(x,y)$, then* (1.1) *has a unique saddle point.*

Proof. It suffices to apply Theorems 4.2 and Theorem 1.2 in [10], observing that the assumptions made imply that, for any $(x,y)\in A\times B$, $f(\cdot,y)$ is strongly-convex on A and $f(x,\cdot)$ is strongly-concave on B with the coefficient of strong convexity $\frac{\rho}{2}$ and the coefficient of strong concavity $\frac{\rho}{2}$. □

Example 4.6 ([3])**.** Let $E=\mathbb{R}$ and let $T : E \to E$ be defined by

$$Tx = \begin{cases} x-1 & \text{if } x \in (-\infty,-1), \\ x-\sqrt{1-(x+1)^2} & \text{if } x\in[-1,0). \\ x+\sqrt{1-(x-1)^2} & \text{if } x \in [0,1]. \\ x+1 & \text{if } x\in(1,\infty). \end{cases}$$

This is an easy example to show that T is a generalized Lipschitz mapping but not Lipschitz.

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Kan Buranakorn
Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000 Thailand
E-mail address: `kanburanakorn@hotmail.com`

Somyot Plubtieng
Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000 Thailand and;, Research center for Academic Excellence in Mathematics, Naresuan University
E-mail address: `somyotp@nu.ac.th`