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THE SUPPORTING HYPERPLANE AND AN ALTERNATIVE TO SOLUTIONS OF VARIATIONAL INEQUALITIES

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This paper is dedicated to Professor Wataru Takahashi on the occasion of his 70th Birthday

Abstract. A theorem of supporting hyperplanes for a class of convex level sets in a Hilbert space is obtained. As an application of this result, we prove an alternative theorem on solutions of variational inequalities defined on convex level sets. Two examples are given to demonstrate the usefulness and advantages of our alternative theorem.

1. Introduction

Variational inequalities (VIs) were initially introduced and studied by Stampacchia [19] in 1964. Since then, VIs have extensively been studied and applied to a large variety of applied problems arising from structural analysis, economics, optimization, management science, operations research and engineering sciences, see [1, 3, 4, 6-12, 13, 14, 16, 17, 20-23] and the references therein. Typically, a variational inequality, denoted as $VI(C,P)$, is stated as follows:

$$\text{find an } x^* \in C \text{ such that } \langle Px^*, x - x^* \rangle \geq 0 \quad \forall x \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a Hilbert space H and $P : C \to H$ is a nonlinear mapping.

The complexity of $VI(C,P)$ depends on the nonlinear mapping P as well as the convex set C. It is trivial that x^* is a solution to $VI(C,P)$ if $x^* \in C$ is such that $Px^* = 0$. However, if $x^* \in C$ fails to satisfy the equation $Px^* = 0$, what can characterize a solution x^* to the variational inequality $VI(C,P)$? This is a broad question and there is no general answer to it. We aim in this paper to provide an affirmative answer to this question in the case where the convex set C is the level set of finitely many continuous convex functions $\varphi_i : H \to R$, that is,

$$C = \{x \in H : \varphi_i(x) \leq 0, \quad i = 1, 2, \cdots, n\}. \tag{1.2}$$

More precisely, we shall prove an alternative result for $VI(C,P)$; namely, a given $x^* \in C$ is a solution to $VI(C,P)$ if and only if either $Px^* = 0$ or $x^* \in \partial C$ and there exists $v \in \text{conv}\{\varphi_i'(x^*) : i \in I^*\}$ such that $Px^* = -tv$ for some positive constant t, where $I^* = \{i : \varphi_i(x^*) = 0$. The contribution of this result lies in that we convert $VI(C,F)$ equivalently to a nonlinear equation

$$Px^* = -tv, \tag{1.3}$$

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which would be easier to solve. We will show this by two examples in Sect. 4.

We find that our alternative result has close connection with supporting hyperplanes to convex sets. As a matter of fact, we shall prove our alternative result by proving a theorem on supporting hyperplanes to C at its boundary points. We therefore discuss some basics about supporting hyperplanes in the next section. A property of supporting hyperplanes will be proved in Sect. 3 which is then applied it to prove our alternative result on $VI(C, P)$. Finally in Sect. 4, we use two examples to illustrate advantages of solving the Eq. (1.3) over solving $VI(C, P)$.

2. Preliminaries

Let X be a linear space. It is well known that a proper linear subspace F of X is maximal if any linear subspace F_1 of X which properly contains F coincides with the entire space X. Let $x_0 \in X$ and let F be a maximal linear subspace of X, then $L = x_0 + F$, a translation of F, is called a hyperplane. It is also well known that if X is a normed linear space and $L \subset X$, then L is a closed hyperplane if and only if there exists a non-zero continuous linear functional l on X and a constant r such that $L = F_l^r$, where $F_l^r = \{x \in X : l(x) = r\}$ is the level set of l at level r. In particular, the level set F_l^0 is the null space of l. For a given hyperplane H_l^r, it is easy to see that $F_l^r = x_0 + F_l^0$, where x_0 is an arbitrary point in F_l^r. (See [2, 5] for more details about Banach space theory and its geometry). Recall that a hyperplane $L = F_l^r$ is called a supporting hyperplane to a convex set C in X at a boundary point $x_0 \in \partial C$ if

$$l(x) \leq r = l(x_0), \quad x \in C$$

or

$$l(x) \geq r = l(x_0), \quad x \in C.$$

Below is a classical result on supporting hyperplanes; see [2, 5] for details.

Theorem 2.1. *Let X be a normed linear space. If E is a closed convex set of X with interior points, then there exists a supporting hyperplane to E at each of the boundary points of E.*

In particular, if $X = H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and F a hyperplane pass through x_0 then $F = \{x \in H : \langle v, x - x_0 \rangle = 0\}$ for some $v \neq 0$.

From now on, we always assume that H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and normal $\| \cdot \|$. Let φ be a real continuous convex function which is Gateaux differentiable at each $x \in H$, that is, for an arbitrary $x \in H$, there exists an element $\varphi'(x) \in H$, called Gateaux differential of φ at x, such that

$$\lim_{t \to 0} \frac{\varphi(x + tv) - \varphi(x)}{t} = \langle v, \varphi'(x) \rangle, \quad v \in H.$$

Let $\varphi_1, \varphi_2, \cdots, \varphi_m$ be continuous convex functions on H and let C be defined by

$$C = \{x \in H : \varphi_i(x) \leq 0, \quad i = 1, \cdots, m\}.$$

Recall that the variational inequality, $VI(C, P)$, is started below:

$$\text{find an } x^* \in C \text{ such that } \langle Px^*, x - x^* \rangle \geq 0,$$

where $P : C \to H$ is a nonlinear mapping.

Remark 2.2. We will assume that C has interior points since, otherwise, if C has empty interior in H, we then restrict C to a closed subspace H' of H in which C has nonempty interior.

At a boundary point x^* of C, we use the following notation:

$$I^* := \{i : \varphi_i(x^*) = 0\},$$
$$C^* := \{x \in H : \langle \varphi_i'(x^*), x - x^* \rangle \le 0, \quad i \in I^*\}.$$

Lemma 2.3. *Assume that x^* is a boundary point of C. Then, for any interior point x of C^*, there exists $t > 0$ such that $x^* + s(x - x^*)$ is an interior point of C for all $s \in (0, t)$.*

Proof. Let $i \in I^*$, from the Gateaux differentiability of φ_i, we have:

$$\lim_{s \to 0} \frac{\varphi_i(x^* + s(x - x^*)) - \varphi_i(x^*)}{s} = \langle x - x^*, \varphi_i'(x^*) \rangle.$$

Further, since $\varphi_i(x^*) = 0$, $\langle x - x^*, \varphi_i'(x^*) \rangle < 0$ (by the fact that x is an interior point of C^*). These imply that $\varphi_i(x^* + s(x - x^*)) < 0$ for all $s \in (0, t_i)$ such that t_i is sufficiently small. Let $t = \min\{t_i : i \in I^*\}$; then $\varphi_i(x^* + s(x - x^*)) < 0$ for all $s \in (0, t)$ for $i \in I^*$. Further, as $x^* \in C \Rightarrow \varphi_i(x^*) < 0$ for all $i \in \{1, 2, \cdots, m\} \setminus I^*$. So with t sufficiently small, $\varphi_i(x^* + s(x - x^*)) < 0$ for all $s \in (0, t)$ for every $i \in \{1, 2, \cdots, m\}$. □

Lemma 2.4. *Assume that x^* is a boundary point of C. Then a hyperplane F supports C at x^* if and only if F supports C^* at x^*.*

Proof. We first prove the sufficiency part. If $x \in C$, then by the subdifferential inequality, we have

$$\langle \varphi_i'(x^*), x - x^* \rangle = \varphi_(x^*) + \langle \varphi_i'(x^*), x - x^* \rangle \le \varphi(x) \le 0 \ \forall x \in C, \ \forall i \in I^*. \tag{2.1}$$

It turns out that $x \in C^*$, yielding $C \subset C^*$. Hence, if a hyperplane F supports C^* at x^*, then F also supports C at x^*.

Next we prove the necessity part. If a hyperplane F is a supporting plane to C at x^*, then F is also a supporting plane to C^* at x^*. Indeed, if we assume the contrary that F is not supporting to C^* at x^*, then $\exists x_1, x_2 \in C^*$ such that $\langle f, x_1 - x^* \rangle < 0 < \langle f, x_2 - x^* \rangle$, where f is a normal vector of F. Now, if $x_1 \in \partial(C^*)$, by the fact that $\langle f, x_1 - x^* \rangle < 0$, there exists a ball $B(x_1, r)$ such that $\langle f, x - x^* \rangle < 0$ for all $x \in B(x_1, r)$. Since C^* is convex with nonempty interior point set and $x_1 \in \partial(C^*)$, we can take an interior point a_1 of $B(x_1, r) \cap C^*$. Hence there exists an interior point a_1 of C^* such that $\langle f, a_1 - x^* \rangle < 0$. By the same argument, there exists an interior point a_2 of C^* such that $\langle f, a_2 - x^* \rangle > 0$.

Thank to Lemma 2.3, there exist $t_1 > 0, t_2 > 0$ such that $x^* + s(a_1 - x^*)$ and $x^* + k(a_2 - x^*)$ are interior points of C for all $s \in (0, t_1)$, $k \in (0, t_2)$.

Take $b_1 = x^* + \frac{t_1}{2}(a_1 - x^*)$ and $b_2 = x^* + \frac{t_2}{2}(a_2 - x^*)$. Then $b_1, b_2 \in C$, $\langle f, b_1 - x^* \rangle = \frac{t_1}{2}\langle f, a_1 - x^* \rangle < 0$ and $\langle f, b_2 - x^* \rangle = \frac{t_2}{2}\langle f, a_2 - x^* \rangle > 0$. Thus, F is not a supporting plane to C at x^*. This is a contradiction.

□

3. Main results

Theorem 3.1. *A Hyperplane F is a supporting plane to C at a boundary point x^* of C if and only if there exists α belonging to conv$\{\varphi_i'(x^*) : i \in I^*\}$ such that $F = \{x \in H : \langle \alpha, x - x^* \rangle = 0\}$.*

Proof. We first prove the necessity part. Denote

$$a^i := \varphi_i'(x^*),$$
$$F_i := \{x \in H : \langle a^i, x - x^* \rangle = 0\},$$
$$C_i^* := \{x \in H : \langle a^i, x - x^* \rangle \leq 0\}.$$

Without any loss of generality, we assume $x^* = 0$. Take an interior point a of C^*. Then $a \neq 0$ and $\langle a, a^i \rangle < 0$ for all $i \in I^*$.

Denote $v := \frac{a}{\|a\|}$ and if F_0 is a hyperplane with normal vector v, then $\langle v, a^i \rangle < 0$ for all $i \in I^*$.

For each $x \in H$, let x_0 be the point of F_0 which is closest to x. We have:

$$x = x_0 + \langle x - x_0, v \rangle v = x_0 + \langle x, v \rangle v \equiv (x_0; x_v),$$

where $x_v = \langle x, v \rangle$, and

$$\langle x, y \rangle = \langle x_0, y_0 \rangle + x_v y_v.$$

We have:

$$a^i = a_0^i + \langle a^i, v \rangle v = r_i\Big(\frac{a_0^i}{r_i} - v\Big) = r_i\Big(\frac{a_0^i}{r_i}; -1\Big),$$

where $r_i = |\langle a^i, v \rangle|$ and $a_0^i = \text{proj}_{F_0}(a^i)$, and

$$C_i^* = \Big\{(x_0; x_v) \in H : \Big\langle \Big(\frac{a_0^i}{r_i}; -1\Big), (x_0; x_v) \Big\rangle \leq 0\Big\} = \Big\{(x_0; x_v) \in H : \Big\langle \frac{a_0^i}{r_i}, x_0 \Big\rangle \leq x_v\Big\}.$$

Since v is an interior point of C^* and the hyperplane F is supporting C at $x^* = 0$, by Lemma 2.4, F is also supporting C^* at $x^* = 0$. So we can choose a normal vector to F, f, such that $\langle f, v \rangle < 0$. We have

$$f = f_0 + \langle f, v \rangle v = r\Big(\frac{f_0}{r} - v\Big) = r\Big(\frac{f_0}{r}; -1\Big) \quad \text{with } r = |\langle f, v \rangle|, \ f_0 = \text{proj}_{F_0}(f)$$

and

$$F = \Big\{(x_0; x_v) \in H : \Big\langle \Big(\frac{f_0}{r}; -1\Big), (x_0; x_v) \Big\rangle = 0\Big\} = \Big\{(x_0; x_v) \in H : \Big\langle \frac{f_0}{r}, x_0 \Big\rangle = x_v\Big\}.$$

We will prove that $\frac{f_0}{r} \in \text{conv}\{\frac{a_0^i}{r_i} : i \in I^*\}$. Suppose to the contrary that $\frac{f_0}{r}$ is not belong to conv$\{\frac{a_0^i}{r_i} : i \in I^*\}$. Let z_0 be the point of conv$\{\frac{a_0^i}{r_i} : i \in I^*\}$ which is closest to $\frac{f_0}{r}$.

We have $\langle \frac{f_0}{r} - z_0, y_0 - z_0 \rangle \leq 0$ for all $y_0 \in \text{conv}\{\frac{a_0^i}{r_i} : i \in I^*\}$. Let $w_0 = \frac{f_0}{r} - z_0$; then $\langle w_0, y_0 \rangle \leq \langle w_0, z_0 \rangle = \langle w_0, z_0 - \frac{f_0}{r} \rangle + \langle w_0, \frac{f_0}{r} \rangle = \langle w_0, \frac{f_0}{r} \rangle - \|w_0\|^2$ for all $y_0 \in$ conv$\{\frac{a_0^i}{r_i} : i \in I^*\}$. It turns out that

$$\langle w_0, y_0 \rangle \leq \Big\langle w_0, \frac{f_0}{r} \Big\rangle - \|w_0\|^2 \quad \text{for all} \quad y_0 \in \text{conv}\Big\{\frac{a_0^i}{r_i} : i \in I^*\Big\}. \tag{3.1}$$

As $0 \in F_0$, F_0 is a subspace of H. Moreover $\frac{f_0}{r}, \frac{a_0^i}{r_i} \in F_0$, so $z_0 \in F_0$, implying $w_0 \in F_0$. Now, because F is a supporting plane to C^* at $x^* = 0$ (by Lemma 2.4), there exists $i_0 \in I^*$ such that

$$\Big\langle \frac{a_0^{i_0}}{r_{i_0}}, w_0 \Big\rangle \geq \Big\langle \frac{f_0}{r}, w_0 \Big\rangle. \tag{3.2}$$

Indeed, by contradiction, suppose that $\big\langle \frac{a_0^i}{r_i}, w_0 \big\rangle < \big\langle \frac{f_0}{r}, w_0 \big\rangle$, for all $i \in I^*$. Let $s = \big\langle \frac{f_0}{r}, w_0 \big\rangle$. Then $(w_0, s) \in F$ since $\big\langle (\frac{f_0}{r}; -1), (w_0, s) \big\rangle = 0$, (w_0, s) is a interior point of C^* since $\big\langle (\frac{a_0^i}{r_i}; -1)$. Since $(w_0, s) \big\rangle = \big\langle \frac{a_0^i}{r_i}, w_0 \big\rangle - s = \big\langle \frac{a_0^i}{r_i}, w_0 \big\rangle - \big\langle \frac{f_0}{r}, w_0 \big\rangle < 0$ for all $i \in I^*$, F contains an interior point of C^* and this is a contradiction to the fact that F is a supporting plane to C^* at $x^* = 0$. Hence, inequality (3.2) holds.

Combining (3.1) and (3.2), we obtain $\big\langle \frac{a_0^{i_0}}{r_{i_0}}, w_0 \big\rangle \geq \big\langle \frac{f_0}{r}, w_0 \big\rangle \geq \big\langle y_0, w_0 \big\rangle + \|w_0\|^2$, for all $y_0 \in \text{conv}\{\frac{a_0^i}{r_i} : i \in I^*\}$, which turns out to be a contradiction if we select $y_0 = \frac{a_0^{i_0}}{r_{i_0}}$. Hence $\frac{f_0}{r} \in \text{conv}\{\frac{a_0^i}{r_i} : i \in I^*\}$, implying $(\frac{f_0}{r}, -1) \in \text{conv}\{(\frac{a_0^i}{r_i}, -1) : i \in I^*\}$. In other words, there exists α belonging to $\text{conv}\{\varphi_i'(x^*) : i \in I^*\}$ such that $F = \{x \in H : \langle \alpha, x - x^* \rangle = 0\}$.

The sufficiency part is trivial. □

Theorem 3.2. *Consider VI(C, P) and assume its solution set $S \neq \emptyset$. Let $x^* \in C$ be given. Then we have the following alternative to the solution of VI(C, P). Namely, $x^* \in S$ if and only if there holds either:*

- $Px^* = 0$, *or*
- $x^* \in \partial C$ *and* $\exists\, t > 0$, $\exists\, v \in$ *conv*$\{\varphi_i'(x^*) : i \in I^*\}$, *such that* $Px^* = -tv$.

Proof. (The proof of this theorem is similar to that of [18, Theorem 4.1].)
($\Rightarrow$): Suppose $x^* \in S$. If $Px^* \neq 0$, we shall prove $x^* \in \partial C$. Suppose to the contrary that x^* is an interior point of C. Let $r > 0$ be such that the open ball $B(x^*, r) \subset C$. Then

$$\langle Px^*, x - x^* \rangle \geq 0 \quad \forall x \in B(x^*, r).$$

Take $y = x^* - \frac{rPx^*}{2\|Px^*\|}$ to get $y \in B(x^*, r)$ and $0 \leq \langle Px^*, y - x^* \rangle = -\frac{r\|Px^*\|}{2}$. Consequently, $Px^* = 0$, which is a contradiction to the assumption that $Px^* \neq 0$. Hence $x^* \in \partial C$.

It remains to show that there exist $t > 0$ and $v \in \text{conv}\{\varphi_i'(x^*) : i \in I^*\}$ such that $Px^* = -tv$. Let F be a hyperplane with normal vector Px^* and contain the point x^*; then F is a supporting plane to C at x^*.

Thanks to Theorem 3.1, there exists $s \neq 0$ such that $sPx^* \in \text{conv}\{\varphi_i'(x^*) : i \in I^*\}$. On the other hand, $C \subset C^* = \{x \in H : \langle \varphi_i'(x^*), x - x^* \rangle \leq 0, i \in I^*\}$. Thus, $\langle sPx^*, x - x^* \rangle \leq 0$ for all $x \in C$; consequently, $s < 0$, setting $t = -s$ then proves the the necessity part.
($\Leftarrow$): It is trivial that the conclusion of the theorem holds if $Px^* = 0$. We next consider the case where we assume that $x^* \in \partial C$ and there exist $t > 0$

and $v \in \operatorname{conv}\{\varphi_i'(x^*) : i \in I^*\}$ such that $Px^* = -tv = \sum_{i\in I^*} \lambda_i \varphi_i'(x^*)$ with $\lambda_i \geq 0, \sum_{i\in I^*} \lambda_i = 1$. It follows from (2.1)that $\langle Px^*, x - x^*\rangle \geq 0$ for all $x \in C$. □

Remark 3.3. Although we always assume C is a closed convex set, it is true that if C is not convex, VI(C, P) can still be resolved by using the same arguments in Lemma 2.3, Lemma 2.4, and Theorems 3.1 and 3.2. We will illuminate this by an example in Section 4.

4. Applications

Example 4.1. Let $H = \mathbb{R}^2$, $\varphi_1(x, y) = x^2 - y$; $\varphi_2(x, y) = x^2 + y^2 - 2$; $\varphi_3(x, y) = -x$ and $P(x, y) = (3e^x, 2x + y)$. Set $C_i = \{x \in H : \varphi_i(x) \leq 0\}$, $i = 1, 2, 3$. Let S be the solution set of the associating VI. It is easy to see that $P(x, y) \neq 0$ on $C = \bigcap_{i=1}^{3} C_i$. Hence, the solutions must occur in ∂C. By Theorem 3.2:

- $X = (0, 0) \in S \Leftrightarrow P(X) = s\varphi_1'(X) + t\varphi_3(X), t \geq 0, s \geq 0, s + t \neq 0 \Leftrightarrow (3, 0) = -s(0, -1) - t(-1, 0), t \geq 0, s \geq 0, s + t \neq 0 \Leftrightarrow \{t = 0, s = 3\}$, so $(0, 0) \in S$.
- $X = (1, 1) \in S \Leftrightarrow P(X) = -s\varphi_1^{'}(X) - t\varphi_2(X), t \geq 0, s \geq 0, s + t \neq 0 \Leftrightarrow (3e, 3) = -s(2, -1) - t(2, 2), t \geq 0, s \geq 0, s + t \neq 0 \Rightarrow 3e = -2s - 2t, t \geq 0, s \geq 0$, so $(1, 1) \notin S$.
- $X = (0, \sqrt{2}) \in S \Leftrightarrow P(X) = -s\varphi_2'(X) - t\varphi_2(X), t \geq 0, s \geq 0, s + t \neq 0 \Leftrightarrow (3, \sqrt{2}) = -s(0, 2\sqrt{2}) - t(-1, 0), t \geq 0, s \geq 0, s + t \neq 0 \Rightarrow -2s = 1, s \geq 0$, so $(0, \sqrt{2}) \notin S$.
- $X = (x, y) \in \partial C_1$, with $x \in (0, 1)$, $X \in S \Leftrightarrow P(x, y) = -t\varphi_1'(x, y),\ t > 0$ which is equivalently rewritten as

$$\begin{cases} (3e^x, 2x + y) = -t(2x, -1) \\ y = x^2,\ x \in (0, 1),\ t > 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 3e^x = -2tx,\ 2x + y = t \\ y = x^2,\ x \in (0, 1),\ t > 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 3e^x + 2x^3 + 4x = 0 \\ t = 2x + y,\ y = x^2,\ x \in (0, 1),\ t > 0 \end{cases}$$

for $f(x) = 3e^x + 2x^3 + 4x$ is an increasing function on $(0, 1)$ and $f(0) = 3 > 0$; so the above system has no solution.
- $X = (x, y) \in \partial C_2$, with $x \in (0, 1)$, $X \in S \Leftrightarrow P(x, y) = -t\varphi_2'(x, y)$, $t > 0$ which is equivalently rewritten as

$$\begin{cases} (3e^x, 2x + y) = -t(2x, 2y) \\ x = \sqrt{2}\cos z,\ y = \sqrt{2}\sin z,\ z \in (\pi/4, \pi/2),\ t > 0 \end{cases}$$

$$\Rightarrow \begin{cases} 3e^{\sqrt{2}\cos z} = -t.2\sqrt{2}\cos z \\ z \in (\pi/4, \pi/2),\ t > 0 \end{cases}$$

since $\cos z > 0$ on $(\pi/4, \pi/2)$; hence, $X \notin S$.

- $X = (0, y) \in \partial C_3$, with $y \in (0, \sqrt{2})$, $X \in S \Leftrightarrow P(0, y) = -t\varphi_3'(0, y),\ t > 0$ which is equivalently rewritten as

$$\begin{cases} (3, y) = -t(0, 2y) \\ y \in (0, \sqrt{2}),\ t > 0 \end{cases}$$

$\Rightarrow 3 = 0$ impossible. Therefore, $X \notin S$, and $X = (0, 0)$ is the unique solution of VI(C, P).

Example 4.2. Let $H = \mathbb{R}^2$, $\varphi_1(x, y) = 6x^2+4x-3y+2$, $\varphi_2(x, y) = 4^x-y$, $P(x, y) = (48x^2 - 8, 9y)$, and $C = \{(x, y) \in \mathbb{R}^2 : \varphi_1(x, y) \le 0,\ \varphi_2(x, y) \ge 0\}$. Again let S denote the solution set of the VI VI(C, P). Note that this C is not a convex set. However, by making use of the arguments in Section 3, we can resolve the above problem.

Indeed, it is easy to see that $P(x, y) = 0 \Leftrightarrow (x, y) = (\pm\frac{1}{\sqrt{6}}, 0)$. Now, $(-\frac{1}{\sqrt{6}}, 0) \in C$ and $(\frac{1}{\sqrt{6}}, 0) \notin C$. It turns out that $(-\frac{1}{\sqrt{6}}, 0) \in S$. Let $X \ne (-\frac{1}{\sqrt{6}}, 0)$. Then $X \in S$ implies $X \in \partial C$. We have $\varphi_1(x, y) = \varphi_2(x, y) = 0$ if and only if $(x, y) = A(-\frac{1}{2}, \frac{1}{2})$ or $(x, y) = B(1, 4)$. Thus, $\partial C = \{A, B\} \cup \{(x, y) : \varphi_1(x, y) = 0,\ x \in (-\frac{1}{2}, 1)\} \cup \{(x, y) : \varphi_2(x, y) = 0,\ x \in (-\frac{1}{2}, 1)\}$. Set $\varphi_3(x, y) = -7x + 3y - 5$, $(\varphi_3(A) = \varphi_3(B) = 0)$. By the same arguments as Lemam 2.3, Lemma 2.4, Theorem 3.1 and Theorem 3.2, it is easy to see that:

- $X^* = (x, y) \in C_1 = \{(x, y) : \varphi_1(x, y) = 0,\ x \in (-\frac{1}{2}, 1)\}$, $X^* \in S \Leftrightarrow P(X^*) = -t\varphi_1'(X^*)$, where $t > 0$.
which is equivalently rewritten as

$$\begin{cases} (48x^2 - 8, 9y) = -t(12x + 4, -3),\ t > 0 \\ 3y = 6x^2 + 4x + 2,\ x \in (-\frac{1}{2}, 1) \end{cases}$$

$$\Leftrightarrow \begin{cases} 48x^2 - 8 = -t(12x + 4),\ t > 0 \\ t = 3y = 6x^2 + 4x + 2,\ x \in (-\frac{1}{2}, 1) \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0, y = \frac{2}{3}, t = 2 \\ x = \frac{-5+\sqrt{5}}{6}, y = \frac{11-3\sqrt{5}}{9}, t = \frac{11-3\sqrt{5}}{3}. \end{cases}$$

- $X^* = (x, y) \in C_2 = \{(x, y) : \varphi_2(x, y) = 0,\ x \in (-\frac{1}{2}, 1)\}$, $X^* \in S \Leftrightarrow P(X^*) = 0$. Thus, $P(X) = 0 \Leftrightarrow X = (-\frac{1}{\sqrt{6}}, 0)$. We find that VI(C, P) has no solution in C_2.
- $X^* = A(-\frac{1}{2}, \frac{1}{2}) \in S \Leftrightarrow P(A) = -t\varphi_1'(A) - s\varphi_3'(A),\ t \ge 0, s \ge 0, s + t > 0$. These are equivalent to:

$$\begin{cases} (4, \frac{9}{2}) = -t(-2, -3) - s(-7, 3) \\ t \ge 0,\ s \ge 0,\ s + t > 0 \end{cases} \Leftrightarrow t = \frac{29}{18},\ s = \frac{1}{9}.$$

- $X^* = B(1, 4) \in S \Leftrightarrow P(B) = -t\varphi_1'(B) - s\varphi_3'(B),\ t \ge 0, s \ge 0, s + t > 0$. These are equivalently rewritten as the system:

$$\begin{cases} (40, 36) = -t(16, -3) - s(-7, 3) \\ t \ge 0,\ s \ge 0,\ s + t > 0. \end{cases}$$

It is easy to find that this system has no solution. Hence,

$$S = \Big\{ \Big(-\frac{1}{\sqrt{6}}, 0 \Big), \Big(0, \frac{2}{3} \Big), \Big(\frac{-5+\sqrt{5}}{6}, \frac{11-3\sqrt{5}}{9} \Big), \Big(-\frac{1}{2}, \frac{1}{2} \Big) \Big\}.$$

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