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WEAKLY CONVERGENT ITERATIVE METHOD FOR THE SPLIT COMMON NULL POINT PROBLEM IN BANACH SPACES

SAUD M. ALSULAMI, NAWAB HUSSAIN, AND WATARU TAKAHASHI

Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

Abstract. In this paper, we consider the split common null point problem in Banach spaces. Then using the idea of Mann's iteration, we prove a weak convergence theorem for finding a solution of the split common null point problem in Banach spaces. Furthermore, using the result, we get a new weak convergence theorem which is connected with the split common null point problem and an equilibrium problem in Banach spaces. It seems that these results are first in Banach spaces.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\kappa > 0$ such that

$$\langle x - y, Ux - Uy \rangle \geq \kappa \|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [8] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Recently, Byrne, Censor, Gibali and Reich [7] considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2$, $1 \leq j \leq n$, the *split common null point problem* [7] is to find a point $z \in H_1$ such that

$$z \in \left(\cap_{i=1}^m A_i^{-1}0 \right) \cap \left(\cap_{j=1}^n T_j^{-1}(B_j^{-1}0) \right),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [1], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split

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feasibility peoblem and generalized split feasibility peoblems including the split common null point problem in Hilbert spaces; see, for instance, [7, 9, 12, 21].

Very recently, using the idea of Mann's iteration, Alsulami and Takahashi [2] proved the following weak convergence theorem for finding a solution of the split feasibility problem in Banach spaces.

Theorem 1.1. *Let H be a Hilbert space and let F be a strictly convex, reflexive and smooth Banach space. Let J_F be the duality mapping on F. Let C and D be nonempty, closed and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. For any $x_1 = x \in H$, define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) P_C\big(I - rA^* J_F(A - P_D A)\big)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset [0,1]$ and $r \in (0,\infty)$ satisfy the following:

$$0 < a \leq \beta_n \leq b < 1 \quad \textit{and} \quad 0 < r\|A\|^2 < 2$$

for some $a, b \in \mathbb{R}$. Then $x_n \rightharpoonup z_0 \in C \cap A^{-1}D$, where $z_0 = \lim_{n\to\infty} P_{C\cap A^{-1}D} x_n$.

In this paper, motivated by these problems and results, we consider the split common null point problem in Banach spaces. Then using the idea of Mann's iteration, we prove a weak convergence theorem for finding a solution of the split common null point problem in Banach spaces. Furthermore, using the result, we get a new weak convergence theorem which is connected with the split common null point problem and an equilibrium problem in Banach spaces. It seems that these results are first in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot \cdot \rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [19] that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle; \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Putting $U = I - T$, where T is nonexpansive, we have that U is $\frac{1}{2}$-inverse strongly monotone; see [19]. For a mapping $T : C \to H$, we denote by $F(T)$ the set of fixed points of T. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x \rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [17].

Lemma 2.1 ([20])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - x\| \leq \|x_n - x\|$ for all $n \in \mathbb{N}$ and $x \in C$, then $\{P_C x_n\}$ converges strongly to some $z \in C$, where P_C is the metric projection on H onto C.*

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n \to \infty} \|x_n\| = \lim_{n \to \infty} \|y_n\| = 1 \text{ and } \lim_{n \to \infty} \|x_n + y_n\| = 2,$$

$\lim_{n \to \infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$.

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.5}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [17] and [18]. We know the following result.

Lemma 2.2 ([17])**.** *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy \rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 2.3 ([17]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E, and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent*

(1) $z = P_C x_1$;
(2) $\langle z - y, J(x_1 - z)\rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let A be a mapping of of E into 2^{E^*}. The effective domain of A is denoted by $\mathrm{dom}(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in \mathrm{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [6]; see also [18, Theorem 3.5.4].

Theorem 2.4 ([6]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r > 0$,*

$$R(J + rA) = E^*,$$

where $R(J + rA)$ is the range of $J + rA$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of A. In a Hilbert space H, the metric resolvent J_r of A is simply called the resolvent of A. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [18].

3. Main result

In this section, using the idea of Mann's iteration, we prove a weak convergence theorem for finding a solution of the split common null point problem in Banach spaces.

Theorem 3.1. *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let A and B be maximal monotone operators of H into 2^H and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let J_λ be the resolvent of A for $\lambda > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T : H \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. For any $x_1 = x \in H$, define*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) J_{\lambda_n}\big(I - \lambda_n T^* J_F (T - Q_{\mu_n} T)\big) x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\}$, $\{\mu_n\} \subset (0, \infty)$ satisfy the following conditions:

$$0 < a \leq \beta_n \leq b < 1, \; 0 < c \leq \lambda_n \|T\|^2 \leq d < 2 \;\; \text{and} \;\; 0 < k \leq \mu_n, \quad \forall n \in \mathbb{N}$$

for some $a, b, c, d, k \in \mathbb{R}$. *Then* $\{x_n\}$ *converges weakly to* $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, *where* $z_0 = \lim_{n\to\infty} P_{A^{-1}0\cap T^{-1}(B^{-1}0)} x_n$.

Proof. Let $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$. Then we have that $z = J_{\lambda_n} z$ and $Tz = Q_{\mu_n} Tz$ for all $n \in \mathbb{N}$. Put $y_n = J_{\lambda_n}\big(x_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\big)$ for all $n \in \mathbb{N}$. Since J_{λ_n} is nonexpansive, we have that

$$\begin{aligned}
\|y_n - z\|^2 &= \big\|J_{\lambda_n}\big(x_n - \lambda_n A^* J_F(Tx_n - Q_{\mu_n} Tx_n)\big) - J_{\lambda_n} z\big\|^2 \\
&\le \|x_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n) - z\|^2 \\
&= \|x_n - z - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\|^2 \\
&= \|x_n - z\|^2 - 2\langle x_n - z, \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\rangle \\
&\quad + \|\lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\|^2 \\
&\le \|x_n - z\|^2 - 2\lambda_n \langle Tx_n - Tz, J_F(Tx_n - Q_{\mu_n} Tx_n)\rangle \qquad (3.1) \\
&\quad + \lambda_n^2 \|T\|^2 \|J_F(Tx_n - Q_{\mu_n} Tx_n)\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n \langle Tx_n - Q_{\mu_n} Tx_n + Q_{\mu_n} Tx_n - Tz, J_F(Tx_n - Q_{\mu_n} Tx_n)\rangle \\
&\quad + \lambda_n^2 \|T\|^2 \|Tx_n - Q_{\mu_n} Tx_n\|^2 \\
&\le \|x_n - z\|^2 - 2\lambda_n \|Tx_n - Q_{\mu_n} Tx_n\|^2 + \lambda_n^2 \|T\|^2 \|Tx_n - Q_{\mu_n} Tx_n\|^2 \\
&= \|x_n - z\|^2 + \lambda_n(\lambda_n \|T\|^2 - 2)\|Tx_n - Q_{\mu_n} Tx_n\|^2.
\end{aligned}$$

From $0 < \lambda_n \|T\|^2 < 2$ we have that $\|y_n - z\| \le \|x_n - z\|$ and hence

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\beta_n x_n + (1-\beta_n) y_n - z\| \\
&\le \beta_n \|x_n - z\| + (1-\beta_n)\|y_n - z\| \\
&\le \beta_n \|x_n - z\| + (1-\beta_n)\|x_n - z\| \\
&\le \|x_n - z\|.
\end{aligned}$$

Then $\lim_{n\to\infty} \|x_n - z\|$ exists. Thus $\{x_n\}$, $\{Tx_n\}$ and $\{y_n\}$ are bounded. Using the equality (2.2), we have that for $n \in \mathbb{N}$ and $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$,

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\beta_n x_n + (1-\beta_n) y_n - z\|^2 \\
&= \beta_n \|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n)\|x_n - z\|^2 \\
&\quad + (1-\beta_n)\lambda_n(\lambda_n \|T\|^2 - 2)\|Tx_n - Q_{\mu_n} Tx_n\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 \\
&= \|x_n - z\|^2 + (1-\beta_n)\lambda_n(\lambda_n \|T\|^2 - 2)\|Tx_n - Q_{\mu_n} Tx_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - y_n\|^2.
\end{aligned}$$

Therefore, we have that $\beta_n(1-\beta_n)\|x_n - y_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2$ and

$$(1-\beta_n)\lambda_n(\lambda_n \|T\|^2 - 2)\|Tx_n - Q_{\mu_n} Tx_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Thus we have from $0 < a \le \beta_n \le b < 1$ and $0 < c \le \lambda_n \|T\|^2 \le d < 2$ that

$$\lim_{n\to\infty} \|x_n - y_n\|^2 = 0 \text{ and } \lim_{n\to\infty} \|Tx_n - Q_{\mu_n} Tx_n\|^2 = 0. \qquad (3.2)$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. From (3.2), $\{y_{n_i}\}$ converges weakly to w. Since T is bounded and linear, we also have that $\{Tx_{n_i}\}$ converges weakly to Tw. Using this and $\lim_{n\to\infty}\|Tx_n - Q_{\mu_n}Tx_n\| = 0$, we have that $Q_{\mu_{n_i}}Tx_{n_i} \rightharpoonup Tw$. Since Q_{μ_n} is the metric resolvent of B for $\mu_n > 0$, we have that $\frac{J_F(Tx_n - Q_{\mu_n}Tx_n)}{\mu_n} \in BQ_{\mu_n}Tx_n$ for all $n \in \mathbb{N}$. From the monotonicity of B we have that

$$0 \leq \left\langle u - Q_{\mu_{n_i}}Tx_{n_i}, v^* - \frac{J_F(Tx_{n_i} - Q_{\mu_{n_i}}Tx_{n_i})}{\mu_{n_i}} \right\rangle$$

for all $(u, v^*) \in B$. Taking $i \to \infty$, we have from $\|J_F(Tx_{n_i} - Q_{\mu_{n_i}}Tx_{n_i})\| = \|Tx_{n_i} - Q_{\mu_{n_i}}Tx_{n_i}\| \to 0$ and $0 < k \leq \mu_{n_i}$ that $0 \leq \langle u - Tw, v^* - 0\rangle$ for all $(u, v^*) \in B$. Since B is maximal monotone, we have that $Tw \in B^{-1}0$. This implies that $w \in T^{-1}(B^{-1}0)$. Since $y_n = J_{\lambda_n}\big(x_n - \lambda_n T^*J_F(Tx_n - Q_{\mu_n}Tx_n)\big)$, we have that

$$\begin{aligned} y_n = {} & J_{\lambda_n}\big(x_n - \lambda_n T^*J_F(Tx_n - Q_{\mu_n}Tx_n)\big) \\ & \Leftrightarrow x_n - \lambda_n T^*J_F(Tx_n - Q_{\mu_n}Tx_n) \in y_n + \lambda_n Ay_n \\ & \Leftrightarrow x_n - y_n - \lambda_n T^*J_F(Tx_n - Q_{\mu_n}Tx_n) \in \lambda_n Ay_n \\ & \Leftrightarrow \frac{1}{\lambda_n}\big(x_n - y_n - \lambda_n T^*J_F(Tx_n - Q_{\mu_n}Tx_n)\big) \in Ay_n. \end{aligned}$$

Since A is monotone, we have that for $(u, v) \in A$,

$$\left\langle y_n - u, \frac{1}{\lambda_n}\big(x_n - y_n - \lambda_n T^*J_F(Tx_n - Q_{\mu_n}Tx_n)\big) - v \right\rangle \geq 0$$

and hence

$$\left\langle y_n - u, \frac{x_n - y_n}{\lambda_n} - T^*J_F(Tx_n - Q_{\mu_n}Tx_n) - v \right\rangle \geq 0.$$

Replacing n by n_i, we have that

$$\left\langle y_{n_i} - u, \frac{x_{n_i} - y_{n_i}}{\lambda_{n_i}} - T^*J_F(Tx_{n_i} - Q_{\mu_{n_i}}Tx_{n_i}) - v \right\rangle \geq 0.$$

Since $x_{n_i} - y_{n_i} \to 0$, $0 < c \leq \lambda_{n_i}\|T\|^2$, $y_{n_i} \rightharpoonup w$ and $T^*J_F(Tx_n - Q_{\mu_{n_i}}Tx_{n_i}) \to 0$, we have that $\langle w - u, -v\rangle \geq 0$. Since A is maximal monotone, we have that $0 \in Aw$. Therefore, $w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$.

We next show that if $x_{n_i} \rightharpoonup w_1$ and $x_{n_j} \rightharpoonup w_2$, then $w_1 = w_2$. We know $w_1, w_2 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$ and hence $\lim_{n\to\infty}\|x_n - w_1\|$ and $\lim_{n\to\infty}\|x_n - w_2\|$ exist. Suppose $w_1 \neq w_2$. Since H satisfies Opial's condition [14], we have that

$$\begin{aligned} \lim_{n\to\infty}\|x_n - w_1\| &= \lim_{i\to\infty}\|x_{n_i} - w_1\| < \lim_{i\to\infty}\|x_{n_i} - w_2\| \\ &= \lim_{n\to\infty}\|x_n - w_2\| = \lim_{j\to\infty}\|x_{n_j} - w_2\| \\ &< \lim_{j\to\infty}\|x_{n_j} - w_1\| = \lim_{n\to\infty}\|x_n - w_1\|. \end{aligned}$$

This is a contradiction. Then $w_1 = w_2$. Therefore, $x_n \rightharpoonup w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$. Moreover, since for any $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \quad \forall n \in \mathbb{N},$$

we have from Lemma 2.1 that $\{P_{A^{-1}0\cap T^{-1}(B^{-1}0)}x_n\}$ converges strongly to z_0 for some $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$. The property of metric projection implies that

$$\langle w - P_{A^{-1}0\cap T^{-1}(B^{-1}0)}x_n, x_n - P_{A^{-1}0\cap T^{-1}(B^{-1}0)}x_n\rangle \le 0.$$

Therefore, we have that

$$\|w - z_0\|^2 = \langle w - z_0, w - z_0\rangle \le 0.$$

This means that $w = z_0$, i.e., $x_n \rightharpoonup z_0$. □

4. Application

In this section, using Theorem 3.1, we get a new weak convergence theorem which is connected with the split common null point problem and an equilibrium problem in Banach spaces.

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a bifunction. Then an equilibrium problem (with respect to C) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \ge 0, \quad \forall y \in C. \tag{4.1}$$

The set of such solutions $\hat{x}$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \ge 0,\ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction $f : C\times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;
(A3) for any $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \le f(x, y);$$

(A4) for any $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

We know the following lemma which appears in Blum and Oettli [5].

Lemma 4.1 ([5]). *Let C be a nonempty, closed and convex subset of H and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1) − (A4). Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C.$$

The following lemma was also given in Combettes and Hirstoaga [10].

Lemma 4.2 ([10]). *Assume that $f : C \times C \to \mathbb{R}$ satisfies (A1) − (A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C \right\}.$$

Then the following hold:

(1) *T_r is single-valued;*
(2) *T_r is a firmly nonexpansive mapping, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

We call such T_r the resolvent of f for $r > 0$. Using Lemmas 4.1 and 4.2, Takahashi, Takahashi and Toyoda [16] obtained the following lemma. See [3] for a more general result.

Lemma 4.3 ([16]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy (A1) − (A4). Let A_f be a set-valued mapping of H into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \forall x \in C, \\ \emptyset, \quad \forall x \notin C. \end{cases}$$

Then $EP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $\operatorname{dom}(A_f) \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f, i.e.,

$$T_r x = (I + rA_f)^{-1} x.$$

We obtain the following theorem from Theorem 3.1.

Theorem 4.4. *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let C be a nonempty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let B be a maximal monotone operator of F into 2^{F^*}. Let T_λ denote the resolvent of A_f (as defined in Lemma 4.3) for $\lambda > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T : H \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $EP(f) \cap T^{-1}(B^{-1}0) \neq \emptyset$. For any $x_1 = x \in H$, define*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T_{\lambda_n}\big(I - \lambda_n T^* J_F(T - Q_{\mu_n}T)x_n\big), \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset (0, 1)$ and $\{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$ satisfy the following conditions:

$$0 < a \leq \beta_n \leq b < 1, \ 0 < c \leq \lambda_n \|T\|^2 \leq d < 2 \ \text{ and } \ 0 < k \leq \mu_n, \quad \forall n \in \mathbb{N}$$

for some $a, b, c, d, k \in \mathbb{R}$. Then $x_n \rightharpoonup z_0 \in EP(f) \cap T^{-1}(B^{-1}0)$, where $z_0 = \lim_{n\to\infty} P_{EP(f)\cap T^{-1}(B^{-1}0)} x_n$.

Proof. For the bifunction $f : C \times C \to \mathbb{R}$, we can define A_f in Lemma 4.3. Putting $A = A_f$ in Theorem 3.1, we obtain from Lemma 4.3 that $J_{\lambda_n} = T_{\lambda_n} = (I + \lambda_n A_f)^{-1}$ for all $\lambda_n > 0$. Thus we obtain the desired result by Theorem 3.1. □

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Saud M. Alsulami
Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: alsulami@kau.edu.sa

Nawab Hussain
Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: nhusain@kau.edu.sa

W. Takahashi
Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung 80702, Taiwan; Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: wataru@is.titech.ac.jp; wataru@a00.itscom.net