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THE SPLIT COMMON NULL POINT PROBLEM IN TWO BANACH SPACES

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Abstract. In this paper, we consider the split common null point problem in two Banach spaces. Then using the metric resolvents of maximal monotone operators and the metric projections, we prove a strong convergence theorem for finding a solution of the split common null point problem in two Banach spaces.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [5] is to find $z \in H_1$ such that $z \in D \cap T^{-1}Q$. Byrne, Censor, Gibali and Reich [4] also considered the following problem: Given set-valued mappings $A : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$, and a bounded linear operator $T : H_1 \to H_2$, the *split common null point problem* [4] is to find a point $z \in H_1$ such that

$$z \in A^{-1}0 \cap T^{-1}(B^{-1}0),$$

where $A^{-1}0$ and $B^{-1}0$ are null point sets of A and B, respectively. Defining $U = T^*(I - P_Q)T$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [1], where T^* is the adjoint operator of T and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap T^{-1}Q$ is nonempty, then $z \in D \cap T^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda T^*(I - P_Q)T)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem and the split common null point problem; see, for instance, [1, 4, 6, 10, 11, 22].

Recently, using the methods of [12, 13, 15], Takahashi [20] proved the following theorem; see also [19].

Theorem 1.1. *Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let A and B be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint*

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operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - \mu_n J_E^{-1} T^* J_F(Tx_n - Q_{\mu_n} Tx_n), \\ C_n = \{z \in A^{-1}0 : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{z \in A^{-1}0 : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\mu_n\} \subset (0,\infty)$ satisfies that for some $a, b \in \mathbb{R}$,

$$0 < a \leq \mu_n \leq b < \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $w_1 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$.

In this paper, motivated by Takahashi's theorem (Theorem 1.1), we consider the split common null point problem with metric resolvents of maximal monotone operators in two Banach spaces. Then using the metric resolvents of maximal monotone operators and the metric projections, we prove a strong convergence theorem for finding a solution of the split null point problem in two Banach spaces.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1$$

and

$$\lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, that is, $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [7, 14].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [16] and [17]. We know the following result:

Lemma 2.1. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x-z\| \leq \|x-y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 2.2 ([16])**.** *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x_1$;
(2) $\langle z-y, J(x_1-z)\rangle \geq 0, \quad \forall y \in C.$

Let E be a Banach space and let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by $\mathrm{dom}(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be monotone if $\langle x-y, u^*-v^*\rangle \geq 0$ for all $x, y \in \mathrm{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [3]; see also [17, Theorem 3.5.4].

Theorem 2.3 ([3])**.** *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r > 0$,*

$$R(J + rA) = E^*,$$

where $R(J + rA)$ is the range of $J + rA$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of A. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [17].

3. Main result

In this section, using the metric resolvents of maximal monotone operators and the metric projections, we prove a strong convergence theorem for finding a solution of the split common null point problem in two Banach spaces. We follow [20] for the proof.

Theorem 3.1. *Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let A and B be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let J_λ and Q_μ be the metric resolvents of A for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - \mu_n J_E^{-1} T^* J_F(Tx_n - Q_{\mu_n} Tx_n), \\ y_n = J_{\lambda_n} z_n, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ D_n = \{z \in E : \langle y_n - z, J_E(z_n - y_n)\rangle \geq 0\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$ satisfy that for some $a, b, c \in \mathbb{R}$,

$$0 < a \leq \mu_n \leq b < \frac{1}{\|T\|^2} \text{ and } 0 < c \leq \lambda_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $w_1 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$.

Proof. It is obvious that $C_n \cap D_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. To show that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n$ for all $n \in \mathbb{N}$, let us show that $\langle z_n - z, J_E(x_n - z_n)\rangle \geq 0$ for all $z \in T^{-1}(B^{-1}0)$ and $n \in \mathbb{N}$. In fact, we have that for all $z \in T^{-1}(B^{-1}0)$ and $n \in \mathbb{N}$,

$$\begin{aligned} \langle z_n - z, J_E(x_n - z_n)\rangle &= \langle z_n - x_n + x_n - z, J_E(x_n - z_n)\rangle \\ &= \langle -\mu_n J_E^{-1} T^* J_F(Tx_n - Q_{\mu_n} Tx_n) \\ &\qquad + x_n - z, J_E(\mu_n J_E^{-1} T^* J_F(Tx_n - Q_{\mu_n} Tx_n))\rangle \\ &= \langle -\mu_n J_E^{-1} T^* J_F(Tx_n - Q_{\mu_n} Tx_n) + x_n - z, \mu_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\rangle \\ &= -\mu_n^2 \|T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\|^2 + \langle x_n - z, \mu_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\rangle \\ &= -\mu_n^2 \|T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\|^2 + \mu_n \langle Tx_n - Tz, J_F(Tx_n - Q_{\mu_n} Tx_n)\rangle \qquad (3.1) \\ &= -\mu_n^2 \|T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\|^2 \\ &\qquad + \mu_n \langle Tx_n - Q_{\mu_n} Tx_n + Q_{\mu_n} Tx_n - Tz, J_F(Tx_n - Q_{\mu_n} Tx_n)\rangle \\ &= -\mu_n^2 \|T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\|^2 + \mu_n \|Tx_n - Q_{\mu_n} Tx_n\|^2 \\ &\qquad + \mu_n \langle Q_{\mu_n} Tx_n - Tz, J_F(Tx_n - Q_{\mu_n} Tx_n)\rangle \\ &\geq -\mu_n^2 \|T\|^2 \|Tx_n - Q_{\mu_n} Tx_n\|^2 + \mu_n \|Tx_n - Q_{\mu_n} Tx_n\|^2 \end{aligned}$$

$$= \mu_n(1 - \mu_n\|T\|^2)\|Tx_n - Q_{\mu_n}Tx_n\|^2 \geq 0.$$

Then we have that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n$ for all $n \in \mathbb{N}$. Next, to show that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset D_n$ for all $n \in \mathbb{N}$, let us show that $\langle y_n - z, J_E(z_n - y_n)\rangle \geq 0$ for all $z \in A^{-1}0$ and $n \in \mathbb{N}$. In fact, we have that for all $z \in A^{-1}0$ and $n \in \mathbb{N}$,

$$\langle y_n - z, J_E(z_n - y_n)\rangle = \langle J_{\lambda_n}z_n - z, J_E(z_n - J_{\lambda_n}z_n)\rangle \geq 0. \tag{3.2}$$

Then we have that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset D_n$ for all $n \in \mathbb{N}$. We shall show that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset Q_n$ for all $n \in \mathbb{N}$. Since $\langle x_1 - z, J_E(x_1 - x_1)\rangle \geq 0$ for all $z \in E$, it is obvious that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset Q_1$. Suppose that, for some $k \in \mathbb{N}$, $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset Q_k$. Then $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_k \cap D_k \cap Q_k$. From $x_{k+1} = P_{C_k\cap D_k\cap Q_k}x_1$, we have that

$$\langle x_{k+1} - z, J_E(x_1 - x_{k+1})\rangle \geq 0, \quad \forall z \in C_k \cap D_k \cap Q_k$$

and hence

$$\langle x_{k+1} - z, J_E(x_1 - x_{k+1})\rangle \geq 0, \quad \forall z \in A^{-1}0 \cap T^{-1}(B^{-1}0).$$

Then we get that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset Q_{k+1}$. By mathematical induction, we have that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset Q_n$ for all $n \in \mathbb{N}$. Thus, we have that

$$A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n \cap D_n \cap Q_n$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $A^{-1}0 \cap T^{-1}(B^{-1}0)$ is a nonempty, closed and convex subset of E, there exists $w_1 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$ such that $w_1 = P_{A^{-1}0\cap T^{-1}(B^{-1}0)}x_1$. We have from $x_{n+1} = P_{C_n\cap D_n\cap Q_n}x_1$ that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - y\|$$

for all $y \in C_n \cap D_n \cap Q_n$. Since $w_1 \in A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n \cap D_n \cap Q_n$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - w_1\|. \tag{3.3}$$

This means that $\{x_n\}$ is bounded.

Next we show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From $x_{n+1} = P_{C_n\cap D_n\cap Q_n}x_1$ we have that $x_{n+1} \in Q_n$ and hence

$$\langle x_n - x_{n+1}, J_E(x_1 - x_n)\rangle \geq 0.$$

From this, we have that

$$\langle x_n - x_1 + x_1 - x_{n+1}, J_E(x_1 - x_n)\rangle \geq 0.$$

This implies that $\langle x_1 - x_{n+1}, J_E(x_1 - x_n)\rangle \geq \|x_n - x_1\|^2$ and hence

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|.$$

Therefore, $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. Put $\lim_{n\to\infty} \|x_n - x_1\| = c$. If $c = 0$, then $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Assume that $c > 0$. Since $x_n \in Q_n$, $x_{n+1} \in Q_n$ and $\frac{x_n+x_{n+1}}{2} \in Q_n$, we have that

$$\begin{aligned}\|x_1 - x_n\| &\leq \left\|x_1 - \frac{x_n + x_{n+1}}{2}\right\| \\ &\leq \frac{1}{2}(\|x_1 - x_n\| + \|x_1 - x_{n+1}\|)\end{aligned}$$

and hence

$$\lim_{n\to\infty}\left\|x_1-\frac{x_n+x_{n+1}}{2}\right\|=c.$$

Since E is uniformly convex, we get that $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$.

We have from $x_{n+1}\in C_n$ that

$$\langle z_n-x_{n+1},J_E(x_n-z_n)\rangle\geq 0$$

and hence

$$\langle z_n-x_n+x_n-x_{n+1},J_E(x_n-z_n)\rangle\geq 0.$$

This implies that

$$\|x_n-z_n\|\leq\|x_n-x_{n+1}\|.$$

From $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$ we have that $\lim_{n\to\infty}\|x_n-z_n\|=0$. On the other hand, we know that

$$\begin{aligned}\|x_n-z_n\|&=\|J_E(x_n-z_n)\|\\&=\|\mu_nT^*J_F(Tx_n-Q_{\mu_n}Tx_n)\|.\end{aligned}$$

Since $0<a\leq\mu_n\leq b<\frac{1}{\|T\|^2}$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}\|x_n-z_n\|=0$, we have that $\lim_{n\to\infty}\|T^*J_F(Tx_n-Q_{\mu_n}Tx_n)\|=0$. Then we get from (3.1) that

$$\lim_{n\to\infty}\|Tx_n-Q_{\mu_n}Tx_n\|=0. \tag{3.4}$$

Furthermore, We have from $x_{n+1}\in D_n$ that

$$\langle y_n-x_{n+1},J_E(z_n-y_n)\rangle\geq 0$$

and hence

$$\langle y_n-z_n+z_n-x_n+x_n-x_{n+1},J_E(z_n-y_n)\rangle\geq 0.$$

This implies that

$$\langle z_n-x_n+x_n-x_{n+1},J_E(z_n-y_n)\rangle\geq\|z_n-y_n\|^2.$$

From $\|x_n-x_{n+1}\|\to 0$ and $\|x_n-z_n\|\to 0$, we have that $\lim_{n\to\infty}\|y_n-z_n\|=0$. Then we get that

$$\lim_{n\to\infty}\|z_n-J_{\lambda_n}z_n\|=0. \tag{3.5}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. We have from $\lim_{n\to\infty}\|x_n-z_n\|=0$ that $\{z_{n_i}\}$ converges weakly to w. We also have from (3.5) that $\{J_{\lambda_{n_i}}z_{n_i}\}$ converges weakly to w. Since J_{λ_n} is the metric resolvent of A, we have that $\frac{J_E(z_n-J_{\lambda_n}z_n)}{\lambda_n}\in AJ_{\lambda_n}z_n$ for all $n\in\mathbb{N}$. From the monotonicity of A we have that

$$0\leq\left\langle s-J_{\lambda_{n_i}}z_{n_i},t^*-\frac{J_E(z_{n_i}-J_{\lambda_{n_i}}z_{n_i})}{\lambda_{n_i}}\right\rangle$$

for all $(s,t^*)\in A$. We have from $\|J_E(z_{n_i}-J_{\lambda_{n_i}}z_{n_i})\|=\|z_{n_i}-J_{\lambda_{n_i}}z_{n_i}\|\to 0$ and $0<c\leq\lambda_{n_i}$ that $0\leq\langle s-w,t^*-0\rangle$ for all $(s,t^*)\in A$. Since A is maximal monotone, we have that $w\in A^{-1}0$. Furthermore, since T is bounded and linear, we also have that $\{Tx_{n_i}\}$ converges weakly to Tw. From (3.4) we have that $\{Q_{\mu_{n_i}}Tx_{n_i}\}$ converges weakly to Tw. Since Q_{μ_n} is the metric resolvent of B, we have that

$\frac{J_F(Tx_n - Q_{\mu_n}Tx_n)}{\mu_n} \in BQ_{\mu_n}Tx_n$ for all $n \in \mathbb{N}$. From the monotonicity of B we have that

$$0 \leq \left\langle u - Q_{\mu_{n_i}}Tx_{n_i}, v^* - \frac{J_F(Tx_{n_i} - Q_{\mu_{n_i}}Tx_{n_i})}{\mu_{n_i}} \right\rangle$$

for all $(u, v^*) \in B$. From $\|J_F(Tx_{n_i} - Q_{\mu_{n_i}}Tx_{n_i})\| = \|Tx_{n_i} - Q_{\mu_{n_i}}Tx_{n_i}\| \to 0$ and $0 < a \leq \mu_{n_i}$, we have that $0 \leq \langle u - Tw, v^* - 0 \rangle$ for all $(u, v^*) \in B$. Since B is maximal monotone, we have that $Tw \in B^{-1}0$. Therefore, $w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$.

From $w_1 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)}x_1$ and $w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, we have from (3.3) that

$$\begin{aligned} \|x_1 - w_1\| \leq \|x_1 - w\| &\leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \|x_1 - w_1\|. \end{aligned}$$

Then we get that

$$\begin{aligned} \lim_{i\to\infty} \|x_1 - x_{n_i}\| &= \|x_1 - w\| \\ &= \|x_1 - w_1\|. \end{aligned}$$

From the Kadec-Klee property of E, we have that $x_1 - x_{n_i} \to x_1 - w$ and hence

$$x_{n_i} \to w = w_1.$$

Therefore, we have $x_n \to w = w_1$. This completes the proof. □

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