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ITERATIVE METHODS FOR SPLIT FEASIBILITY PROBLEMS IN CERTAIN BANACH SPACES

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday with love and respect

Abstract. In this paper, our aim is to introduce an iterative algorithm for solving split feasibility problems and prove the strong convergence of the sequence generated by our iterative scheme in p-uniformly convex and uniformly smooth Banach spaces. Our result complements many recent and important results in this direction.

1. Introduction

Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. The *split feasibility problem* (SFP) is to find a point

$$x \in C \text{ such that } Ax \in Q. \tag{1.1}$$

We assume that SFP (1.1) has a nonempty solution set $\Omega := \{y \in C : Ay \in Q\} = C \cap A^{-1}(Q)$. Then, we have that Ω is a closed and convex subset of E_1.

The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [5] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [3]. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms have been invented to solve it (see, for example, [4, 13, 16, 31–34] and references therein).

In solving SFP (1.1) in p-uniformly convex real Banach spaces which are also uniformly smooth, Schöpfer *et al.* [21] proposed the following algorithm: For $x_1 \in E_1$ and $n \geq 1$, set

$$x_{n+1} = \Pi_C J_{E_1}^* [J_{E_1}(x_n) - t_n A^* J_{E_2}(Ax_n - P_Q(Ax_n))], \tag{1.2}$$

where Π_C denotes the Bregman projection and J the duality mapping. Clearly the above algorithm covers the Byrne's CQ algorithm [3]

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), n \geq 1,$$

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which is found to be a gradient-projection method (GPM) in convex minimization as a special case. They established the *weak convergence* of algorithm (1.2) under the condition that E_1 is p-uniformly convex, uniformly smooth and the duality mapping of E_1 is sequentially weak-to-weak-continuous.

We remark here that the condition that the duality mapping of E_1 is sequentially weak-to-weak-continuous assumed in [21] excludes some important Banach spaces, such as the classical $L_p(2 < p < \infty)$ spaces.

Recently, Wang [29] modified the above algorithm (1.2) and proved strong convergence for the following multiple-sets split feasibility problem (MSSFP): find $x \in E_1$ satisfying

$$x \in \bigcap_{i=1}^{r} C_i, Ax \in \bigcap_{j=1+r}^{r+s} Q_j, \tag{1.3}$$

where r, s are two given integers, $C_i, i = 1, \dots, r$ is a closed convex subset in E_1, and $Q_j, j = r+1, \dots, r+s$, is a closed convex subset in E_2. He defined for each $n \in \mathbb{N}$,

$$T_n(x) = \begin{cases} \Pi_{C_{i(n)}}(x), \ 1 \leq i(n) \leq r, \\ J_{E_1}^*[J_{E_1}(x) - t_n A^* J_{E_2}(Ax - P_{Q_{j(n)}}(Ax))], \ r+1 \leq i(n) \leq r+s, \end{cases}$$

where $i : \mathbb{N} \to I$ is the cyclic control mapping

$$i(n) = n \bmod (r+s) + 1,$$

and t_n satisfies

$$0 < t \leq t_n \leq \Big(\frac{q}{C_q||A||^q}\Big)^{\frac{1}{q-1}}, \tag{1.4}$$

with C_q a constant defined as in Lemma 2.1 and proposed the following algorithm: For any initial guess $x_1 = \bar{x}$, define $\{x_n\}$ recursively by

$$\begin{cases} y_n = T_n x_n \\ D_n = \{w \in E_1 : \Delta_p(y_n, w) \leq \Delta_p(x_n, w)\} \\ E_n = \{w \in E_1 : \langle x_n - w, J_p(\bar{x}) - J_p(x_n) \geq 0\} \\ x_{n+1} = \Pi_{D_n \cap E_n}(\bar{x}). \end{cases} \tag{1.5}$$

Using the idea in the work of Nakajo and Takahashi [14], he proved the following strong convergence theorem in p-uniformly convex Banach spaces which is also uniformly smooth.

Theorem 1.1. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP (1.3) has a nonempty solution set Ω. Let the sequence $\{x_n\}_{n=1}^\infty$ be generated by (1.5). Then $\{x_n\}_{n=1}^\infty$ converges strongly to the Bregman projection of $\bar{x}$ onto the solution set Ω.*

The main advantage of result of Wang [29] is that the weak-to-weak continuity of the duality mapping, assumed in [21] is dispensed with and strong convergence result was achieved. On the other hand, to implement the algorithm (1.5) of Wang [29], one has to calculate, at each iteration, the Bregman projection onto the intersection of two half spaces D_n and E_n. Recently, some researchers have considered SFP in Banach spaces (see, for example, [2, 22–25, 28]).

Our aim in this paper is to construct another iterative scheme for solving problem (1.1) for which its implementation does not involve calculation of Bregman projection onto the intersection of two half spaces at each step of the iteration for which strong convergence is achieved in p-uniformly convex real Banach spaces which are also uniformly smooth.

2. Preliminaries

Let E_1 and E_2 be real Banach spaces and let $A : E_1 \to E_2$ be a bounded linear operator. The *dual (adjoint)* operator of A, denoted by A^*, is a bounded linear operator defined by $A^* : E_2^* \to E_1^*$

$$\langle A^*\bar{y}, x\rangle := \langle \bar{y}, Ax\rangle,\ \forall x \in E_1, \bar{y} \in E_2^*$$

and the equalities $||A^*|| = ||A||$ and $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$ are valid, where $\mathcal{R}(A)^\perp := \{x^* \in E_2^* : \langle x^*, u\rangle = 0,\ \forall u \in \mathcal{R}(A)\}$. For more details on bounded linear operators and their duals, please see [8, 26, 27].

Let $1 < q \le 2 \le p$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let E be a real Banach space. The *modulus of convexity* $\delta_E : [0, 2] \to [0, 1]$ is defined as

$$\delta_E(\epsilon) = \inf\Big\{1 - \frac{||x + y||}{2} : ||x|| = 1 = ||y||, ||x - y|| \ge \epsilon\Big\}.$$

E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$; *p-uniformly convex* if there is a $c_p > 0$ so that $\delta_E(\epsilon) \ge c_p\epsilon^p$ for any $\epsilon \in (0, 2]$. The *modulus of smoothness* $\rho_E(\tau) : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_E(\tau) = \Big\{\frac{||x + \tau y|| + ||x - \tau y||}{2} - 1 : ||x|| = ||y|| = 1\Big\}.$$

E is called *uniformly smooth* if $\lim\limits_{n\to\infty} \frac{\rho_E(\tau)}{\tau} = 0$; *$q$-uniformly smooth* if there is a $C_q > 0$ so that $\rho_E(\tau) \le C_q\tau^q$ for any $\tau > 0$. The L_p space is 2-uniformly convex for $1 < p \le 2$ and p-uniformly convex for $p \ge 2$. It is known that E is p-uniformly convex if and only if its dual E^* is q-uniformly smooth (see [12]).

The q-uniformly smooth spaces have the following estimate [30].

Lemma 2.1 (Xu, [30]). *Let $x, y \in E$. If E is q-uniformly smooth, then there is a $C_q > 0$ so that*

$$||x - y||^q \le ||x||^q - q\langle y, J_E^q(x)\rangle + C_q||y||^q.$$

Here and hereafter, we assume that E is a p-uniformly convex and uniformly smooth, which implies that its dual space, E^*, is q-uniformly smooth and uniformly convex. In this situation, it is known that the duality mapping J_E^p is one-to-one, single-valued and satisfies $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ is the duality mapping of E^* (see [1,7]). Here the *duality mapping* $J_E^p : E \to 2^{E^*}$ is defined by

$$J_E^p(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = ||x||^p, ||\bar{x}|| = ||x||^{p-1}\}.$$

The duality mapping J_E^p is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \Rightarrow \langle J_E^p x_n, y\rangle \to \langle J_E^p x, y\rangle$$

holds true for any $y \in E$. It is worth noting that the $\ell_p(p > 1)$ space has such a property, but the $J_E^p(p > 2)$ space does not share this property.

Given a Gâteaux differentiable convex function $f : E \to \mathbb{R}$, the *Bregman distance* with respect to f is defined as:

$$\Delta_f(x, y) = f(y) - f(x) - \langle f'(x), y - x\rangle, \ x, y \in E$$

It is worth noting that the duality mapping J_p is in fact the derivative of the function $f_p(x) = (\frac{1}{p})||x||^p$. Then the Bregman distance with respect to f_p is given by

$$\begin{aligned}\Delta_p(x, y) &= \frac{1}{q}||x||^p - \langle J_E^p x, y\rangle + \frac{1}{p}||y||^p \\ &= \frac{1}{p}(||y||^p - ||x||^p) + \langle J_E^p x, x - y\rangle \\ &= \frac{1}{q}(||x||^p - ||y||^p) - \langle J_E^p x - J_E^p y, x\rangle.\end{aligned}$$

Given $x, y, z \in E$, one can easily get

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_E^p x - J_E^p z\rangle, \tag{2.1}$$

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_E^p x - J_E^p y\rangle. \tag{2.2}$$

Generally speaking, the Bregman distance is not a metric due to the absence of symmetry, but it has some distance-like properties. For the p-uniformly convex space, the metric and Bregman distance has the following relation (see [21]):

$$\tau||x - y||^p \le \Delta_p(x, y) \le \langle x - y, J_E^p x - J_E^p y\rangle, \tag{2.3}$$

where $\tau > 0$ is some fixed number.

It is easy to see that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences of a p-uniformly convex and uniformly smooth E, then $x_n - y_n \to 0, \ n \to \infty$ implies that $\Delta_p(x_n, y_n) \to 0, \ n \to \infty$.

Let C be a nonempty, closed and convex subset of E. The metric projection

$$P_C x = \text{argmin}_{y \in C}||x - y||, \ x \in E,$$

is the unique minimizer of the norm distance, which can be characterized by a variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \le 0, \ \forall z \in C. \tag{2.4}$$

Likewise, one can define the Bregman projection:

$$\Pi_C x = \text{argmin}_{y\in C}\Delta_p(x,y),\ \ x\in E,$$

as the unique minimizer of the Bregman distance (see [20]). The Bregman projection can also be characterized by a variational inequality:

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x\rangle \le 0,\ \forall z\in C, \tag{2.5}$$

from which one has

$$\Delta_p(\Pi_C x, z) \le \Delta_p(x,z) - \Delta_p(x, \Pi_C x),\ \forall z \in C. \tag{2.6}$$

In Hilbert spaces, the metric projection and the Bregman projection with respect to f_2 are coincident, but in general they are different. More importantly, the metric projection can not share the decent property (2.6) as the Bregman projection in Banach spaces.

Following [1, 6], we make use of the function $V_p : E^* \times E \to [0, +\infty)$ associated with f_p, which is defined by

$$V_p(\bar{x}, x) = \frac{1}{q}||\bar{x}||^q - \langle \bar{x}, x\rangle + \frac{1}{p}||x||^p, \forall x\in E, \bar{x}\in E^*.$$

Then V_p is nonnegative and

$$V_p(\bar{x}, x) = \Delta_p(J_E^*(\bar{x}), x) = \Delta_p(J_E^q(\bar{x}), x) \tag{2.7}$$

for all $x\in E$ and $\bar{x}\in E^*$. Moreover, by the subdifferential inequality,

$$V_p(\bar{x}, x) + \langle \bar{y}, J_E^*(\bar{x}) - x\rangle \le V_p(\bar{x}+\bar{y}, x) \tag{2.8}$$

for all $x\in E$ and $\bar{x}, \bar{y}\in E^*$ (see also [11], Lemmas 3.2 and 3.3; [15]). In addition, V_p is convex in the first variable. Thus, for all $z\in E$,

$$\begin{aligned} \Delta_p\Big(J_E^q\Big(\sum_{i=1}^N t_i J_E^p(x_i)\Big), z\Big) &= \Delta_p\Big(J_E^*\Big(\sum_{i=1}^N t_i J_E^p(x_i)\Big), z\Big) \\ &\le \sum_{i=1}^N t_i \Delta_p(x_i, z), \end{aligned} \tag{2.9}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$. For more details, please see [22].

We next state the following lemma which will be used in the sequel.

Lemma 2.2. (Xu [30]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n\ge 1,$$

where, (i) $\{\alpha_n\}\subset[0,1]$, $\sum\alpha_n = \infty$; (ii) $\limsup\ \sigma_n \le 0$; (iii) $\gamma_n\ge 0$; $(n\ge 1)$, $\sum\gamma_n < \infty$. Then, $a_n\to 0$ as $n\to\infty$.

We shall adopt the following notations in this paper:

- $x_n \to x$ means that $x_n\to x$ strongly;
- $x_n \rightharpoonup x$ means that $x_n \to x$ weakly;

- $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ is the weak w-limit set of the sequence $\{x_n\}_{n=1}^{\infty}$.

In this paper, we assume that E_1 and E_2 are p-uniformly convex real Banach spaces which are also uniformly smooth, E_1^* is q-uniformly smooth real Banach space which is also uniformly convex where $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We further assume that $J_{E_1}^p$ and $J_{E_2}^p$ represent the duality mappings of E_1 and E_2 respectively and $J_{E_1}^p = (J_{E_1^*}^q)^{-1}$, where $J_{E_1^*}^q$ is the duality mapping of ${E_1}^*$.

3. Main results

Theorem 3.1. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP* (1.1) *has a nonempty solution set Ω. Let $\{\alpha_n\}$ be a sequence in* $(0,1)$*. For a fixed $u \in C$, let sequences $\{y_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be generated by $x_1 \in C$,*

$$(3.1) \qquad \begin{cases} y_n = J_{E_1^*}^q[J_{E_1}^p(x_n) - t_n A^* J_{E_2}^p(Ax_n - P_Q(Ax_n))] \\ x_{n+1} = \Pi_C J_{E_1^*}^q(\alpha_n J_{E_1}^p(u) + (1-\alpha_n)J_{E_1}^p(y_n)),\ n \geq 1. \end{cases}$$

Suppose the following conditions are satisfied:

(a) $\lim\limits_{n\to\infty} \alpha_n = 0$*;*

(b) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ *and*

(c) $0 < t \leq t_n \leq k < \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element $\bar{x} \in \Omega$, where $\bar{x} = \Pi_\Omega u$.

Proof. Let $x^* \in \Omega$. Suppose $w_n := Ax_n - P_Q(Ax_n),\ \forall n \geq 1$. Then we have $y_n = J_{E_1^*}^q[J_{E_1}^p(x_n) - t_n A^* J_{E_2}^p(w_n)],\ \forall n \geq 1$. It follows from (2.4) that

$$\begin{aligned} \langle J_{E_2}^p(w_n), Ax_n - Ax^*\rangle &= ||Ax_n - P_Q(Ax_n)||^p + \langle J_{E_2}^p(w_n), P_Q(Ax_n) - Ax^*\rangle \\ &\geq ||Ax_n - P_Q(Ax_n)||^p = ||w_n||^p, \end{aligned} \tag{3.2}$$

which, with Lemma 2.1, yields

$$\begin{aligned} \Delta_p(y_n, x^*) &= \Delta_p(J_{E_1^*}^q[J_{E_1}^p(x_n) - t_n A^* J_{E_2}^p(w_n)], x^*) \\ &= \frac{1}{q}||J_{E_1}^p(x_n) - t_n A^* J_{E_2}^p(w_n)||^q - \langle J_{E_1}^p(x_n), x^*\rangle \\ &\quad + t_n\langle J_{E_2}^p(w_n), Ax^*\rangle + \frac{1}{p}||x^*||^p \\ &\leq \frac{1}{q}||J_{E_1}^p(x_n)||^q - t_n\langle Ax_n, J_{E_2}^p(w_n)\rangle + \frac{C_q(t_n||A||)^q}{q}||J_{E_2}^p(w_n)||^q \\ &\quad - \langle J_{E_1}^p(x_n), x^*\rangle + t_n\langle J_{E_2}^p(w_n), Ax^*\rangle + \frac{1}{p}||x^*||^p \\ &= \frac{1}{q}||x_n||^p - \langle J_{E_1}^p(x_n), x^*\rangle + \frac{1}{p}||x^*||^p + t_n\langle J_{E_2}^p(w_n), Ax^* - Ax_n\rangle \end{aligned}$$

$$
\begin{aligned}
& +\frac{C_q(t_n||A||)^q}{q}||J^p_{E_2}(w_n)||^q \\
&= \Delta_p(x_n,x^*)+t_n\langle J^p_{E_2}(w_n),Ax^*-Ax_n\rangle+\frac{C_q(t_n||A||)^q}{q}||J^p_{E_2}(w_n)||^q \\
&\le \Delta_p(x_n,x^*)-\Big(t_n-\frac{C_q(t_n||A||)^q}{q}\Big)||w_n||^p. \qquad (3.3)
\end{aligned}
$$

Using the condition (c), we have

$$\Delta_p(y_n,x^*)\le\Delta_p(x_n,x^*),\ \forall n\ge 1.$$

Now, using (3.1), we have

$$
\begin{aligned}
\Delta_p(x_{n+1},x^*) &\le \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\Delta_p(y_n,x^*) \qquad (3.4)\\
&\le \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\Delta_p(x_n,x^*)\\
&\le \max\{\Delta_p(u,x^*),\Delta_p(x_n,x^*)\}\\
&\ \vdots\\
&\le \max\{\Delta_p(u,x^*),\Delta_p(x_1,x^*)\}.
\end{aligned}
$$

Hence, $\{x_n\}_{n=1}^\infty$ is bounded.

Let $\bar{x}=\Pi_\Omega u$. The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\Delta_p(x_n,\bar{x})\}_{n=n_0}^\infty$ is non-increasing. Then $\{\Delta_p(x_n,\bar{x})\}_{n=1}^\infty$ converges and $\Delta_p(x_n,\bar{x})-\Delta_p(x_{n+1},\bar{x})\to 0,\ n\to\infty$. Then from (3.3) and (3.4), we obtain

$$
\begin{aligned}
\Big(t_n-\frac{C_q(t_n||A||)^q}{q}\Big)||Ax_n-P_Q(Ax_n)||^p &\le \Delta_p(x_n,\bar{x})-\Delta_p(y_n,\bar{x})\\
&\le \Delta_p(x_n,\bar{x})-\Delta_p(x_{n+1},\bar{x})\\
&\quad+\alpha_n[\Delta_p(u,\bar{x})-\Delta_p(y_n,\bar{x})]. \qquad (3.5)
\end{aligned}
$$

By condition (c) and (3.5), we have

$$
\begin{aligned}
0 &< t\Big(1-\frac{C_qk^{q-1}||A||^q}{q}\Big)||Ax_n-P_Q(Ax_n)||^p\\
&\le \Big(t_n-\frac{C_q(t_n||A||)^q}{q}\Big)||Ax_n-P_Q(Ax_n)||^p\\
&\le \Delta_p(x_n,\bar{x})-\Delta_p(x_{n+1},\bar{x})\\
&\quad+\alpha_n[\Delta_p(u,\bar{x})-\Delta_p(y_n,\bar{x})]\to 0, n\to\infty.
\end{aligned}
$$

Hence, we obtain

$$\lim_{n\to\infty}||Ax_n-P_Q(Ax_n)||=0. \qquad (3.6)$$

Since $y_n=J^q_{E_1^*}[J^p_{E_1}(x_n)-t_nA^*J^p_{E_2}(Ax_n-P_Q(Ax_n))]$, then we have

$$
\begin{aligned}
0\le||J^p_{E_1}(y_n)-J^p_{E_1}(x_n)|| &\le t_n||A^*||||J^p_{E_2}(Ax_n-P_Q(Ax_n))||\\
&\le \Big(\frac{q}{C_q||A||^q}\Big)^{\frac{1}{q-1}}||A^*||||Ax_n-P_Q(Ax_n)||\to 0, n\to\infty. \qquad (3.7)
\end{aligned}
$$

Therefore, we obtain

$$\lim_{n\to\infty}||J^p_{E_1}(y_n) - J^p_{E_1}(x_n)|| = 0.$$

Since $J^q_{E_1^*}$ is also norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty}||y_n - x_n|| = 0.$$

Furthermore, we have from (3.1) that

$$\begin{aligned}\Delta_p(x_{n+1}, y_n) &\leq \alpha_n\Delta_p(u, y_n) + (1-\alpha_n)\Delta_p(y_n, y_n)\\ &= \alpha_n\Delta_p(u, y_n) \to 0, n\to\infty.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}||x_{n+1} - y_n|| = 0$$

and this implies that

$$||x_{n+1} - x_n|| \leq ||y_n - x_n|| + ||x_{n+1} - y_n|| \to 0, n\to\infty.$$

Similarly,

$$||J^q_{E_1^*}[J^p_{E_1}(x_n) - t_nA^*J^p_{E_2}(Ax_n - P_Q(Ax_n))] - x_n|| = ||y_n - x_n|| \to 0, n\to\infty.$$

Since $J^p_{E_1}$ is norm-to-norm uniformly continuous on bounded sets, then

$$\begin{aligned}t||A^*J^p_{E_2}(Ax_n - P_Q(Ax_n))|| &\leq t_n||A^*J^p_{E_2}(Ax_n - P_Q(Ax_n))||\\ &= ||J^p_{E_1}(x_n) - t_nA^*J^p_{E_2}(Ax_n - P_Q(Ax_n)) - J^p_{E_1}(x_n)||\\ &\quad \to 0, n\to\infty.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}||A^*J^p_{E_2}(Ax_n - P_Q(Ax_n))|| = 0. \tag{3.8}$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z \in \omega_w(x_n)$. From (2.2), (2.5) and (2.3), we have that

$$\begin{aligned}\Delta_p(z, \Pi_C z) &\leq \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_C z), z - \Pi_C z\rangle\\ &= \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_C z), z - x_{n_j}\rangle + \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_C z), x_{n_j} - \Pi_C x_{n_j}\rangle\\ &\quad +\langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_C z), \Pi_C x_{n_j} - \Pi_C z\rangle\\ &\leq \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_C z), z - x_{n_j}\rangle + \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_C z), x_{n_j} - \Pi_C x_{n_j}\rangle.\end{aligned}$$

As $j\to\infty$, we obtain that $\Delta_p(z, \Pi_C z) = 0$. Thus, $z \in C$. Let us now fix $x \in C$. Then, $Ax \in Q$ and

$$\begin{aligned}||(I - P_Q)Ax_{n_j}||^p &= \langle J^p_{E_2}(Ax_n - P_Q(Ax_{n_j})), Ax_n - P_Q(Ax_{n_j})\rangle\\ &= \langle J^p_{E_2}(Ax_n - P_Q(Ax_{n_j})), Ax_{n_j} - Ax\rangle\\ &\quad +\langle J^p_{E_2}(Ax_n - P_Q(Ax_{n_j})), Ax - P_Q(Ax_{n_j})\rangle\\ &\leq \langle J^p_{E_2}(Ax_n - P_Q(Ax_{n_j})), Ax_{n_j} - Ax\rangle\\ &\leq M||A^*J^p_{E_2}(I - P_Q)Ax_{n_j}|| \to 0, n\to\infty,\end{aligned}$$

where $M > 0$ is sufficiently large number. It then follows from (2.4) that

$$\begin{aligned} ||(I-P_Q)Az||^p &= \langle J^p_{E_2}(Az-P_Q(Az)), Az-P_Q(Az)\rangle \\ &= \langle J^p_{E_2}(Az-P_Q(Az)), Az-Ax_{n_j}\rangle \\ &\quad +\langle J^p_{E_2}(Az-P_Q(Az)), Ax_{n_j}-P_Q(Ax_{n_j})\rangle \\ &\quad +\langle J^p_{E_2}(Az-P_Q(Az)), P_Q(Ax_{n_j})-P_Q(Az)\rangle \\ &\leq \langle J^p_{E_2}(Az-P_Q(Az)), Az-Ax_{n_j}\rangle \\ &\quad +\langle J^p_{E_2}(Az-P_Q(Az)), Ax_{n_j}-P_Q(Ax_{n_j})\rangle. \end{aligned} \tag{3.9}$$

Let $w_n = J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(y_n)),\ n \geq 1$. Then

$$\begin{aligned} \Delta_p(w_n, y_n) &= \Delta_p(J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(y_n)), y_n) \\ &\leq \alpha_n\Delta_p(u, y_n) + (1-\alpha_n)\Delta_p(y_n, y_n) \\ &= \alpha_n\Delta_p(u, y_n) \to 0,\ n\to\infty. \end{aligned}$$

Hence, by (2.3) we have $\lim_{n\to\infty} ||w_n - y_n|| = 0$. Furthermore

$$||w_n - x_n|| \leq ||x_n - y_n|| + ||w_n - y_n|| \to 0,\ n \to \infty.$$

Since $x_{n_j} \rightharpoonup z$ and $||w_n - x_n|| \to 0$, we have that $w_{n_j} \rightharpoonup z$. Also, since $Ax_{n_j} \rightharpoonup Az$, we have from (3.9) that

$$||(I-P_Q)Az|| = 0.$$

Thus, $Az \in Q$. Furthermore, we observe that

$$\begin{aligned} \limsup_{n\to\infty}\langle w_n - \bar{x}, J^p_{E_1}(u) - J^p_{E_1}(\bar{x})\rangle &= \lim_{j\to\infty}\langle w_{n_j} - \bar{x}, J^p_{E_1}(u) - J^p_{E_1}(\bar{x})\rangle \\ &= \langle z - \bar{x}, J^p_{E_1}(u) - J^p_{E_1}(\bar{x})\rangle \leq 0. \end{aligned} \tag{3.10}$$

Furthermore, by (2.8) and (2.7) we have

$$\begin{aligned} \Delta_p(x_{n+1}, \bar{x}) &\leq \Delta_p(J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(y_n)), \bar{x}) \\ &= V_p(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(y_n), \bar{x}) \\ &\leq V_p(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(y_n) - \alpha_n(J^p_{E_1}(u) - J^p_{E_1}(\bar{x})), \bar{x}) \\ &\quad -\langle J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(y_n)) - \bar{x}, -\alpha_n(J^p_{E_1}(u) - J^p_{E_1}(\bar{x}))\rangle \\ &= V_p(\alpha_n J^p_{E_1}(\bar{x}) + (1-\alpha_n)J^p_{E_1}(y_n), \bar{x}) \\ &\quad +\alpha_n\langle w_n - \bar{x}, J^p_{E_1}(u) - J^p_{E_1}(\bar{x})\rangle \\ &= \Delta_p(J^q_{E_1^*}(\alpha_n J^p_{E_1}(\bar{x}) + (1-\alpha_n)J^p_{E_1}(y_n)), \bar{x}) \\ &\quad +\alpha_n\langle w_n - \bar{x}, J^p_{E_1}(u) - J^p_{E_1}(\bar{x})\rangle \\ &\leq \alpha_n\Delta_p(\bar{x}, \bar{x}) + (1-\alpha_n)\Delta_p(y_n, \bar{x}) \\ &\quad +\alpha_n\langle w_n - \bar{x}, J^p_{E_1}(u) - J^p_{E_1}(\bar{x})\rangle \\ &= (1-\alpha_n)\Delta_p(y_n, \bar{x}) + \alpha_n\langle w_n - \bar{x}, J^p_{E_1}(u) - J^p_{E_1}(\bar{x})\rangle \\ &\leq (1-\alpha_n)\Delta_p(x_n, \bar{x}) + \alpha_n\langle w_n - \bar{x}, J^p_{E_1}(u) - J^p_{E_1}(\bar{x})\rangle. \end{aligned} \tag{3.11}$$

Using Lemma 2.2 and (3.10) in (3.11), we obtain

$$\lim_{n\to\infty}\Delta_p(x_n, \bar{x}) = 0.$$

Thus, $x_n \to \bar{x}, n \to \infty$.

Case 2

Assume that $\{\Delta_p(x_n, \bar{x})\}_{n=1}^{\infty}$ is not monotonically decreasing sequence. Set $\Gamma_n = \Delta_p(x_n, \bar{x}), \ \forall n \geq 1$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \forall n \geq n_0.$$

After a similar conclusion from (3.6), it is easy to see that

$$||Ax_{\tau(n)} - P_Q x_{\tau(n)}|| \to 0, n \to \infty.$$

By the similar argument as above in Case 1, we conclude immediately that

$$\lim_{n\to\infty} ||A^* J_{E_2}^p (Ax_{\tau(n)} - P_Q(Ax_{\tau(n)}))|| = 0,$$

$$\lim_{n\to\infty} ||x_{\tau(n)+1} - x_{\tau(n)}|| = 0$$

and

$$\limsup_{n\to\infty} \langle w_{\tau(n)} - \bar{x}, J_{E_1}^p(u) - J_{E_1}^p(\bar{x})\rangle \leq 0.$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$ which converges weakly to $z \in C$ and $Az \in Q$. From (3.11) we have that

$$\Delta_p(x_{\tau(n)+1}, \bar{x}) \leq (1 - \alpha_{\tau(n)})\Delta_p(x_{\tau(n)}, \bar{x}) + \alpha_{\tau(n)}\langle w_{\tau(n)} - \bar{x}, J_{E_1}^p(u) - J_{E_1}^p(\bar{x})\rangle$$

which implies that (noting that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)} > 0$)

$$\Delta_p(x_{\tau(n)}, \bar{x}) \leq \langle w_{\tau(n)} - \bar{x}, J_{E_1}^p(u) - J_{E_1}^p(\bar{x})\rangle.$$

This implies that

$$\limsup_{n\to\infty} \Delta_p(x_{\tau(n)}, \bar{x}) \leq 0.$$

Thus, $\lim_{n\to\infty} \Delta_p(x_{\tau(n)}, \bar{x}) = 0$. So,

$$\lim_{n\to\infty} ||x_{\tau(n)} - \bar{x}|| = 0. \tag{3.12}$$

Since $\lim_{n\to\infty} ||x_{\tau(n)+1} - x_{\tau(n)}|| = 0$, we have that $\lim_{n\to\infty} ||x_{\tau(n)+1} - \bar{x}|| = 0$. Now, by (2.3), we have that

$$\begin{aligned} 0 \leq \Delta_p(x_{\tau(n)+1}, \bar{x}) &\leq \langle x_{\tau(n)+1} - \bar{x}, J_{E_1}^p(x_{\tau(n)+1}) - J_{E_1}^p(\bar{x})\rangle \\ &\leq ||x_{\tau(n)+1} - \bar{x}|| ||J_{E_1}^p(x_{\tau(n)+1}) - J_{E_1}^p(\bar{x})|| \to 0, n \to \infty. \end{aligned}$$

Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim \Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof. □

Corollary 3.2. *Let E_1 and E_2 be two L_p spaces with $2 \leq p < \infty$. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP* (1.1) *has a nonempty solution set Ω. Let $\{\alpha_n\}$ be a sequence in* $(0,1)$. *For a fixed $u \in C$, let sequences $\{y_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ be generated by $x_1 \in C$,*

$$\begin{cases} y_n = J_{E_1^*}^q [J_{E_1}^p(x_n) - t_n A^* J_{E_2}^p(Ax_n - P_Q(Ax_n))] \\ x_{n+1} = \Pi_C J_{E_1^*}^q(\alpha_n J_{E_1}^p(u) + (1-\alpha_n) J_{E_1}^p(y_n)), \ n \geq 1. \end{cases}$$

Suppose the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*

(c) $0 < t \leq t_n \leq k < \left(\frac{q}{C_q ||A||^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to an element $\bar{x} \in \Omega$, where $\bar{x} = \Pi_\Omega u$.

Corollary 3.3. *Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively, $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Suppose that SFP* (1.1) *has a nonempty solution set Ω. Let $\{\alpha_n\}$ be a sequence in* $(0,1)$. *For a fixed $u \in C$, let sequences $\{y_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ be generated by $x_1 \in C$,*

$$\begin{cases} y_n = x_n - t_n A^*(Ax_n - P_Q(Ax_n)) \\ x_{n+1} = P_C(\alpha_n u + (1-\alpha_n) y_n), \ n \geq 1. \end{cases}$$

Suppose the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*

(c) $0 < t \leq t_n \leq k < \frac{2}{||A||^2}$.

Then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to an element $\bar{x} \in \Omega$, where $\bar{x} = P_\Omega u$.

4. An application

In this section, we give an application of Theorem 3.1 to the convexly constrained linear inverse problem in p-uniformly convex real Banach spaces which are also uniformly smooth.

Consider the convexly constrained linear inverse problem (cf [9])

$$\begin{cases} Ax = b, \\ x \in C, \end{cases} \tag{4.1}$$

where E_1 and E_2 are two p-uniformly convex real Banach spaces which are also uniformly smooth and $A : E_1 \to E_2$ is a bounded linear mapping and $b \in E_2$.
It is well known that the projected Landweber method (see, [10]) given by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C[x_n - \lambda A^*(Ax_n - b)], n \geq 1, \end{cases}$$

where A^* is the adjoint of A and $0 < \lambda < 2\alpha$ with $\alpha = \frac{1}{||A||^2}$, converges weakly to a solution of (4.1). In what follows, we present an algorithm with strong convergence for solving (4.1).

Corollary 4.1. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that the convexly constrained linear inverse problem (4.1) is consistent and let Ω denote its solution set. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. For a fixed $u \in E_1$, let sequences $\{y_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be generated by $x_1 \in E_1$,*

$$(4.2) \qquad \begin{cases} y_n = J_{E_1^*}^q [J_{E_1}^p(x_n) - t_n A^* J_{E_2}^p (Ax_n - b)] \\ x_{n+1} = \Pi_C J_{E_1^*}^q (\alpha_n J_{E_1}^p(u) + (1-\alpha_n) J_{E_1}^p(y_n)), \ n \geq 1. \end{cases}$$

Suppose the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*

(c) $0 < t \leq t_n \leq k < \left(\frac{q}{C_q||A||^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element $\bar{x} \in \Omega$, where $\bar{x} = \Pi_{\Omega} u$.

Proof. For each $n \geq 1$, replacing $b = P_Q(Ax_n), x_n \in E_1$ implies that (3.1) reduces to (4.2). Thus, by Theorem 3.1 we obtain the desired conclusion. □

Remark 4.2. We make the following remark concerning our contributions in this paper.

1. The weak-to-weak continuity of the duality mapping assumed in [21] is dispensed with in this paper and strong convergence is achieved.
2. In implementing the algorithm (1.5), one has to calculate, at each iteration, the Bregman projection onto the intersection of two half spaces but in this our iterative algorithm (3.1), one does not have to calculate, at each iteration, the Bregman projection onto the intersection of two half spaces. Hence, our algorithm (3.1) appears more efficient and implementable than the algorithm of Wang [29].
3. Our result in this paper complement the recent results of [2, 22–25, 28] on split feasibility problems in Banach spaces.

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