Journal of Nonlinear and Convex Analysis
Volume 16, Number 12, 2015, 2365–2400

ISSN 1880-5221 ONLINE JOURNAL

GENERALIZED MIXED EQUILIBRIA, VARIATIONAL INEQUALITIES AND COMMON FIXED POINT PROBLEMS

LU-CHUAN CENG*, CHI-MING CHEN, AND CHING-FENG WEN†

Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

Abstract. In this paper, we introduce two iterative algorithms for finding a common element of the set of solutions of finite generalized mixed equilibrium problems, the set of solutions of finite variational inequalities for inverse strong monotone mappings and the set of common fixed points of infinite nonexpansive mappings and an asymptotically κ-strict pseudocontractive mapping in the intermediate sense in a real Hilbert space. We prove some strong and weak convergence theorems for the proposed iterative algorithms under suitable conditions.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. Let $S : C \to C$ be a self-mapping on C. We denote by $\text{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $A : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$ then A is called a nonexpansive mapping [1]; if $L \in [0, 1)$ then A is called a contraction.

Let $A : C \to H$ be a nonlinear mapping on C. We consider the following variational inequality problem (VIP): find a point $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $\text{VI}(C, A)$.

The VIP (1.1) was first discussed by Lions [25] and has many applications in computational mathematics, mathematical physics, operations research, mathematical economics, optimization theory, and other fields; see, e.g., [21, 30, 38, 45].

2010 *Mathematics Subject Classification.* 49J30, 47H09, 47J20, 49M05.

Key words and phrases. Generalized mixed equilibrium, variational inequality, nonexpansive mapping, asymptotically strict pseudocontractive mapping in the intermediate sense, fixed point, inverse strong monotone mapping.

*This research was partially supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), Ph.D. Program Foundation of Ministry of Education of China (20123127110002), and Program for Outstanding Academic Leaders in Shanghai City (15XD1503100).

†Corresponding author. This reach was partially supported by a grant from MOST 104 2115-M-037-001, and a grant from Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Taiwan.

In 1976, Korpelevich [24] proposed an iterative algorithm for solving the VIP (1.1) in Euclidean space $\mathbf{R}^n$:

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ x_{n+1} = P_C(x_n - \tau A y_n), \quad \forall n \geq 0, \end{cases}$$

with $\tau > 0$ a given number, which is known as the extragradient method (see also [17]). The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see e.g., [3–5, 7, 9, 11, 12, 14, 15, 20, 28, 29, 34] and references therein, to name but a few. In particular, motivated by the idea of Korpelevich's extragradient method [24], Nadezhkina and Takahashi [28] introduced an extragradient iterative scheme:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \quad \forall n \geq 0, \end{cases}$$

where $A : C \to H$ is a monotone, L-Lipschitz continuous mapping, $S : C \to C$ is a nonexpansive mapping and $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/L)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. They proved the weak convergence of $\{x_n\}$ to an element of $\text{Fix}(S) \cap \text{VI}(C, A)$. Recently, inspired by Nadezhkina and Takahashi's iterative scheme [28], Zeng and Yao [11] introduced another iterative scheme for finding an element of $\text{Fix}(S) \cap \text{VI}(C, A)$ and derived the weak convergence result. Furthermore, by combining the CQ method and extragradient method, Nadezhkina and Takahashi [29] introduced an iterative process:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \geq 0. \end{cases}$$

They proved the strong convergence of $\{x_n\}$ to an element of $\text{Fix}(S) \cap \text{VI}(C, A)$ under appropriate conditions. Later on, Ceng and Yao [12] introduced an extragradient-like approximation method which is based on the above extragradient method and viscosity approximation method, and derived a strong convergence result as well.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, $A : H \to H$ be a nonlinear mapping and $\Theta : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [34] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

We denote the set of solutions of GMEP (1.2) by $\text{GMEP}(\Theta, \varphi, A)$. The GMEP (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others. The GMEP is further considered and studied; see e.g., [3, 4, 6, 8, 9].

We present some special cases of GMEP (1.2) as follows.

If $\varphi = 0$, then GMEP (1.2) reduces to the generalized equilibrium problem (GEP) which is to find $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C.$$

It is introduced and studied by Takahashi and Takahashi [37]. The set of solutions of GEP is denoted by GEP(Θ, A).

If $A = 0$, then GMEP (1.2) reduces to the mixed equilibrium problem (MEP) which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.$$

It is considered and studied in [13]. The set of solutions of MEP is denoted by MEP(Θ, φ).

If $\varphi = 0,\ A = 0$, then GMEP (1.2) reduces to the equilibrium problem (EP) which is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C.$$

It is considered and studied in [10]. The set of solutions of EP is denoted by EP(Θ). It is worth to mention that the EP is an unified model of several problems, namely, variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems, etc.

Throughout this paper, it is assumed as in [34] that $\Theta : C\times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)-(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;

(A3) Θ is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y);$$

(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) C is a bounded set.

Next we list some elementary conclusions for the MEP.

Proposition 1.1 (see [13]). *Assume that $\Theta : C \times C \to \mathbf{R}$ satisfies (A1)-(A4) and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$T_r^{(\Theta,\varphi)}(x) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $x \in H$. Then the following hold:

(i) *for each $x \in H$, $T_r^{(\Theta,\varphi)}(x) \neq \emptyset$;*

(ii) $T_r^{(\Theta,\varphi)}$ *is single-valued;*

(iii) $T_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\|^2 \leq \langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\rangle;$$

(iv) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$*;*

(v) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex.*

Let $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N} \in (0, 1]$, $n \geq 1$. Given the nonexpansive self-mappings $S_1, S_2, \ldots, S_N$ on C, for each $n \geq 1$, the mappings $U_{n,1}, U_{n,2}, \ldots, U_{n,N}$ are defined by

$$(1.3)\quad \begin{cases} U_{n,1} = \lambda_{n,1}S_1 + (1 - \lambda_{n,1})I, \\ U_{n,2} = \lambda_{n,2}S_nU_{n,1} + (1 - \lambda_{n,2})I, \\ U_{n,n-1} = \lambda_{n-1}T_{n-1}U_{n,n} + (1 - \lambda_{n-1})I, \\ \cdots \\ U_{n,N-1} = \lambda_{n,N-1}S_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})I, \\ W_n := U_{n,N} = \lambda_{n,N}S_NU_{n,N-1} + (1 - \lambda_{n,N})I. \end{cases}$$

The W_n is called the W-mapping generated by $S_1, \ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Note that the nonexpansivity of S_i implies the nonexpansivity of W_n.

In 2012, combining the hybrid steepest-descent method in [42] and hybrid viscosity approximation method in [16], Ceng, Guu and Yao [6] proposed and analyzed the following hybrid iterative method for finding a common element of the set of solutions of GMEP (1.2) and the set of fixed points of a finite family of nonexpansive mappings $\{S_i\}_{i=1}^N$.

Theorem 1.2 (see [6, Theorem 3.1])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying assumptions (A1)-(A4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with restriction (B1) or (B2). Let the mapping $A : H \to H$ be δ-inverse strongly monotone, and $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H such that $\cap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, \varphi, A) \neq \emptyset$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$ and $f : H \to H$ a l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$, $\{\gamma_n\}$ is a sequence in $(0, 2\delta]$ and $\{\lambda_{n,i}\}_{i=1}^N$ is a sequence in $[a, b]$ with $0 < a \leq b < 1$. For every $n \geq 1$, let W_n be the W-mapping generated by $S_1, \ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Given $x_1 \in H$ arbitrarily, suppose the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by*

$$(1.4)\quad \begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle \\ \qquad\qquad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_nx_n + ((1 - \beta_n)I - \alpha_n\mu F)W_nu_n, \quad \forall n \geq 1, \end{cases}$$

where the sequences $\{\alpha_n\}, \{\beta_n\}, \{r_n\}$ and the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^N$ satisfy the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iii) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\delta$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$;

(iv) $\lim_{n\to\infty}(\lambda_{n+1,i} - \lambda_{n,i}) = 0$ *for all* $i = 1, 2, \dots, N$.

Then both $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $x^* \in \cap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, \varphi, A)$, *where* $x^* = P_{\cap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta,\varphi,A)}(I - \mu F + \gamma f)x^*$ *is a unique solution of the variational inequality problem (VIP):*

$$\langle (\mu F - \gamma V)x^*, x^* - x\rangle \le 0, \quad \forall x \in \cap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, \varphi, A). \tag{1.5}$$

Next, recall some concepts. Let C be a nonempty subset of a normed space X and $S : C \to C$ be a self-mapping on C.

(i) S is asymptotically nonexpansive (see [18]) if there exists a sequence $\{k_n\}$ of positive numbers satisfying the property $\lim_{n\to\infty} k_n = 1$ and

$$\|S^n x - S^n y\| \le k_n \|x - y\|, \quad \forall n \ge 1, \ \forall x, y \in C;$$

(ii) S is asymptotically nonexpansive in the intermediate sense (see [2]) provided S is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|S^n x - S^n y\| - \|x - y\|) \le 0;$$

(iii) S is uniformly Lipschitzian if there exists a constant $\mathcal{L} > 0$ such that

$$\|S^n x - S^n y\| \le \mathcal{L}\|x - y\|, \quad \forall n \ge 1, \ \forall x, y \in C.$$

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly Lipschitzian. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [18] as an important generalization of the class of nonexpansive mappings. The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck, Kuczumow and Reich [2]. Recently, Kim and Xu [23] introduced the concept of asymptotically k-strict pseudocontractive mappings in a Hilbert space as below:

Definition 1.3. Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically k-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ if there exists a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\|S^n x - S^n y\|^2 \le (1 + \gamma_n)\|x - y\|^2 + k\|x - S^n x - (y - S^n y)\|^2, \quad \forall n \ge 1, \ \forall x, y \in C. \tag{1.6}$$

They studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically k-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ is a uniformly $\mathcal{L}$-Lipschitzian mapping with $\mathcal{L} = \sup\{\frac{k+\sqrt{1+(1-k)\gamma_n}}{1+k} : n \ge 1\}$.

Recently, Sahu, Xu and Yao [36] considered the concept of asymptotically k-strict pseudocontractive mappings in the intermediate sense, which are not necessarily Lipschitzian.

Definition 1.4. Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$(1.7)\quad \limsup_{n\to\infty} \sup_{x,y\in C} (\|S^n x - S^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - k\|x - S^n x - (y - S^n y)\|^2) \le 0.$$

Put $c_n := \max\{0, \sup_{x,y\in C}(\|S^n x - S^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - k\|x - S^n x - (y - S^n y)\|^2)\}$. Then $c_n \ge 0$ $(\forall n \ge 1)$, $c_n \to 0$ $(n \to \infty)$ and (1.4) reduces to the relation

$$(1.8)\quad \|S^n x - S^n y\|^2 \le (1 + \gamma_n)\|x - y\|^2 + k\|x - S^n x - (y - S^n y)\|^2 + c_n, \quad \forall n \ge 1,\ \forall x, y \in C.$$

Whenever $c_n = 0$ for all $n \ge 1$ in (1.5) then S is an asymptotically k-strict pseudocontractive mapping with sequence $\{\gamma_n\}$. In 2009, Sahu, Xu and Yao [36] derived the weak and strong convergence of the modified Mann iteration processes for an asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. More precisely, they first established one weak convergence theorem for the following iterative scheme

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n x_n, \quad \forall n \ge 1, \end{cases}$$

where $0 < \delta \le \alpha_n \le 1 - k - \delta$, $\sum_{n=1}^{\infty} \alpha_n c_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$; and then obtained another strong convergence theorem for the following iterative scheme

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ y_n = (1 - \alpha_n)x_n + \alpha_n S^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n} x,\ \forall n \ge 1, \end{cases}$$

where $0 < \delta \le \alpha_n \le 1-k$, $\theta_n = c_n + \gamma_n \Delta_n$ and $\Delta_n = \sup\{\|x_n - z\|^2 : z \in \mathrm{Fix}(S)\} < \infty$. Subsequently, the above iterative schemes are extended to develop new iterative algorithms for finding a common solution of the VIP and the fixed point problem of an asymptotically strict pseudocontractive mapping in the intermediate sense; see, e.g., [8, 22, 35,44].

Motivated and inspired by the above facts, we in this paper introduce two iterative algorithms for finding a common element of the set of solutions of finite generalized mixed equilibrium problems, the set of solutions of finite variational inequalities for inverse strong monotone mappings and the set of common fixed points of infinite nonexpansive mappings and an asymptotically k-strict pseudocontractive mapping in the intermediate sense in a real Hilbert space. We prove some strong and weak convergence theorems for the proposed iterative algorithms under mild conditions. Our results improve and extend the corresponding results announced by many others.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Recall that a mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that if A is α-inverse-strongly monotone, then A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_Cx \in C$ satisfying the property

$$\|x - P_Cx\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1. *For given $x \in H$ and $z \in C$:*

(i) $z = P_Cx \;\Leftrightarrow\; \langle x - z, y - z\rangle \leq 0,\ \forall y \in C$;
(ii) $z = P_Cx \;\Leftrightarrow\; \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2,\ \forall y \in C$;
(iii) $\langle P_Cx - P_Cy, x - y\rangle \geq \|P_Cx - P_Cy\|^2,\ \forall y \in H$.

Consequently, P_C is nonexpansive and monotone.

If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle Au - Av, u - v\rangle + \lambda^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \end{aligned} \tag{2.1}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H. It is easy to see that the projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.2. *Let X be a real inner product space. Then there holds the following inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in X.$$

Lemma 2.3. *Let H be a real Hilbert space. Then the following hold:*

(a) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x, y \in H$;

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x-y\|^2$ *for all* $x, y \in H$ *and* $\lambda, \mu \in [0,1]$ *with* $\lambda + \mu = 1$;

(c) *If* $\{x_n\}$ *is a sequence in* H *such that* $x_n \rightharpoonup x$, *it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x-y\|^2, \quad \forall y \in H.$$

Lemma 2.4 ([36, Lemma 2.5])**.** *Let H be a real Hilbert space. Given a nonempty closed convex subset of H and points $x, y, z \in H$ and given also a real number $a \in \mathbf{R}$, the set*

$$\{v \in C : \|y-v\|^2 \leq \|x-v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed).

Lemma 2.5 ([36, Lemma 2.6])**.** *Let C be a nonempty subset of a Hilbert space H and $S : C \to C$ be an asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then*

$$\|S^n x - S^n y\| \leq \frac{1}{1-k}\left(k\|x-y\| + \sqrt{(1+(1-k)\gamma_n)\|x-y\|^2 + (1-k)c_n}\right)$$

for all $x, y \in C$ and $n \geq 1$.

Lemma 2.6 ([36, Lemma 2.7])**.** *Let C be a nonempty subset of a Hilbert space H and $S : C \to C$ be a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - S^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Sx_n\| \to 0$ as $n \to \infty$.*

Lemma 2.7 (Demiclosedness principle [36, Proposition 3.1])**.** *Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \to C$ be a continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then $I - S$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - S^m x_n\| = 0$, then $(I-S)x = 0$.*

Lemma 2.8 ([36, Proposition 3.2])**.** . *Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \to C$ be a continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that* $\mathrm{Fix}(S) \neq \emptyset$. *Then* $\mathrm{Fix}(S)$ *is closed and convex.*

Remark 2.9. Lemmas 2.7 and 2.8 give some basic properties of an asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Moreover, Lemma 2.7 extends the demiclosedness principles studied for certain classes of nonlinear mappings in Kim and Xu [23], Gornicki [22], Marino and Xu [26] and Xu [40].

Lemma 2.10 ([33, p.80]). *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+\delta_n)a_n + b_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If, in addition, $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Corollary 2.11 ([39, p. 303]). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n, \quad \forall n \ge 0.$$

If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

Recall that a Banach space X is said to satisfy the Opial condition [32] if for any given sequence $\{x_n\} \subset X$ which converges weakly to an element $x \in X$, there holds the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X,\ y \ne x.$$

It is well known in [32] that every Hilbert space H satisfies the Opial condition.

Lemma 2.12 (see [20, Proposition 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{x_n\}$ be a sequence in H. Suppose that*

$$\|x_{n+1} - p\|^2 \le (1+\lambda_n)\|x_n - p\|^2 + \delta_n, \quad \forall p \in C,\ n \ge 1,$$

where $\{\lambda_n\}$ and $\{\delta_n\}$ are sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$. Then $\{P_C x_n\}$ converges strongly in C.

Lemma 2.13 (see [27]). *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \le \|u - q\|, \quad \text{for all } n,$$

then $x_n \to q$ as $n \to \infty$.

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \ge 1$, define a self-mapping W_n on C as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ \cdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ \cdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I. \end{cases} \tag{2.2}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$.

Lemma 2.14 (see [31, Lemma 3.2])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$ the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Lemma 2.15 (see [31, Lemma 3.3])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$, and let $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0,1)$. Then, $\mathrm{Fix}(W) = \cap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.*

Lemma 2.16 (see [19, Demiclosedness principle])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C with $\mathrm{Fix}(T) \neq \emptyset$. Then $I-T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

Lemma 2.17. *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see Proposition 2.1 (i)) implies*

$$u \in \mathrm{VI}(C, A) \quad \Leftrightarrow \quad u = P_C(u - \lambda Au), \quad \forall \lambda > 0.$$

The following lemma can be easily proven, and therefore, we omit the proof.

Lemma 2.18. *Let $f : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$, and $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Then for $0 \leq \gamma l < \mu\eta$,*

$$\langle(\mu F - \gamma f)x - (\mu F - \gamma f)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x, y \in H.$$

That is, $\mu F - \gamma f$ is strongly monotone with constant $\mu\eta - \gamma l$.

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0, 1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to C$, we define the mapping $T^{\lambda} : C \to H$ by

$$T^{\lambda}x := Tx - \lambda\mu F(Tx), \quad \forall x \in C,$$

where $F : H \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on H; that is, F satisfies the conditions:

$$\|Fx - Fy\| \leq \kappa\|x - y\| \quad \text{and} \quad \langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2$$

for all $x, y \in H$.

Lemma 2.19 (see [41, Lemma 3.1])**.** *T^{λ} is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,*

$$\|T^{\lambda}x - T^{\lambda}y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Remark 2.20. (i) Since F is κ-Lipschitzian and η-strongly monotone on H, we get $0 < \eta \leq \kappa$. Hence, whenever $0 < \mu < \frac{2\eta}{\kappa^2}$, we have $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

(ii) In Lemma 2.19, put $F = \frac{1}{2}I$ and $\mu = 2$. Then we know that $\kappa = \eta = \frac{1}{2}$, $0 < \mu = 2 < \frac{2\eta}{\kappa^2} = 4$ and $\tau = 1$.

Finally, recall that a set-valued mapping $T : D(T) \subset H \to 2^H$ is called monotone if for all $x, y \in D(T)$, $f \in Tx$ and $g \in Ty$ imply

$$\langle f - g, x - y\rangle \geq 0.$$

A set-valued mapping T is called maximal monotone if T is monotone and $(I + \lambda T)D(T) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(T)$ the graph of T. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$; see [5].

3. Strong convergence theorem

In this section, we will prove a strong convergence theorem for an iterative algorithm for finding a common element of the set of solutions of the set of solutions of finite generalized mixed equilibrium problems, the set of solutions of finite variational inequalities for inverse strong monotone mappings and the set of common fixed points of infinite nonexpansive mappings and asymptotically κ-strict pseudocontractive mapping $S : C \to C$ in the intermediate sense in a real Hilbert space. This iterative algorithm is based on the extragradient method, viscosity approximation method, Mann-type iterative method, shrinking projection method and hybrid steepest-descent method.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let M, N be two integers. Let Θ_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \dots, M\}$. Let $B_k : H \to H$ and $A_i : C \to H$ be μ_k-inverse strongly monotone and η_i-inverse-strongly monotone, respectively, where $k \in \{1, 2, \dots, M\}$, $i \in \{1, 2, \dots, N\}$. Let $S : C \to C$ be a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense for some $0 \leq k < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$ and $\{c_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} c_n = 0$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $f : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \cap_{k=1}^{M}\mathrm{GMEP}(\Theta_k, \varphi_k, B_k) \cap \cap_{i=1}^{N}\mathrm{VI}(C, A_i) \cap \mathrm{Fix}(S)$ is nonempty and bounded and that either (B1) or (B2) holds. Let W_n be the W-mapping defined by (2.2), and $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$ such that*

$\alpha_n + \beta_n \leq 1$ $(\forall n \geq 1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $k \leq \delta_n \leq d < 1$. *Pick any* $x_0 \in H$ *and set* $C_1 = H$, $x_1 = P_{C_1} x_0$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$(3.1)\qquad \begin{cases} u_n = T^{(\Theta_M,\varphi_M)}_{r_{M,n}}(I - r_{M,n}B_M)T^{(\Theta_{M-1},\varphi_{M-1})}_{r_{M-1,n}}(I - r_{M-1,n}B_{M-1}) \\ \qquad \cdots T^{(\Theta_1,\varphi_1)}_{r_{1,n}}(I - r_{1,n}B_1)x_n, \\ z_n = P_C(I - \lambda_{N,n}A_N)P_C(I - \lambda_{N-1,n}A_{N-1}) \\ \qquad \cdots P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1)u_n, \\ k_n = \delta_n z_n + (1 - \delta_n)S^n z_n, \\ y_n = \alpha_n \gamma f(x_n) + \beta_n k_n + [(1 - \beta_n)I - \alpha_n \mu F]W_n k_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where $\theta_n = (\alpha_n + \gamma_n)\Gamma_n \varrho + c_n \varrho$, $\Gamma_n = \sup\{\|x_n - p\|^2 + \|(\gamma f - \mu F)p\|^2 : p \in \Omega\} < \infty$, *and* $\varrho = \frac{1}{1-\sup_{n\geq 1}\alpha_n} < \infty$. *Assume that the following conditions are satisfied:*

(i) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(ii) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\forall i \in \{1, 2, \ldots, N\}$;
(iii) $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, $\forall k \in \{1, 2, \ldots, M\}$.

Then the following statements hold:

(I) $\{x_n\}$ *converges strongly to* $v = P_\Omega x_0$;
(II) $\{x_n\}$ *converges strongly to* $v = P_\Omega x_0$, *which is a unique solution in* Ω *to the VIP*

$$\langle(\mu F - \gamma f)v, u - v\rangle \geq 0, \quad \forall u \in \Omega,$$

provided $\gamma_n + c_n = o(\alpha_n)$ *and* $\|x_n - y_n\| = o(\alpha_n)$.

Proof. We divide the proof into four steps.

Step 1. We show that $\{x_n\}$ is well defined. It is obvious that C_n is closed and convex. As the defining inequality in C_n is equivalent to the inequality

$$\langle 2(x_n - y_n), z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + \theta_n,$$

by Lemma 2.4 we know that C_n is convex for every $n \geq 1$.

First of all, we show that $\Omega \subset C_n$ for all $n \geq 1$. Put

$$\Delta^k_n = T^{(\Theta_k,\varphi_k)}_{r_{k,n}}(I - r_{k,n}B_k)T^{(\Theta_{k-1},\varphi_{k-1})}_{r_{k-1,n}}(I - r_{k-1,n}B_{k-1}) \cdots T^{(\Theta_1,\varphi_1)}_{r_{1,n}}(I - r_{1,n}B_1)x_n$$

for all $k \in \{1, 2, \ldots, M\}$ and $n \geq 1$,

$$\Lambda^i_n = P_C(I - \lambda_{i,n}A_i)P_C(I - \lambda_{i-1,n}A_{i-1}) \cdots P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1)$$

for all $i \in \{1, 2, \ldots, N\}$ and $n \geq 1$, and $\Delta^0_n = \Lambda^0_n = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta^M_n x_n$ and $z_n = \Lambda^N_n u_n$. Suppose that $\Omega \subset C_n$ for some $n \geq 1$. Take $p \in \Omega$ arbitrarily. Then from (2.1) and Proposition 1.1 (iii) we have

$$\begin{aligned} \|u_n - p\| &= \|T^{(\Theta_M,\varphi_M)}_{r_{M,n}}(I - r_{M,n}B_M)\Delta^{M-1}_n x_n - T^{(\Theta_M,\varphi_M)}_{r_{M,n}}(I - r_{M,n}B_M)\Delta^{M-1}_n p\| \\ &\leq \|(I - r_{M,n}B_M)\Delta^{M-1}_n x_n - (I - r_{M,n}B_M)\Delta^{M-1}_n p\| \end{aligned}$$

$$
\begin{aligned}
&\leq \|\varDelta_n^{M-1}x_n - \varDelta_n^{M-1}p\| \\
&\leq \cdots \qquad (3.2)\\
&\leq \|\varDelta_n^{0}x_n - \varDelta_n^{0}p\| \\
&= \|x_n - p\|.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|z_n - p\| &= \|P_C(I-\lambda_{N,n}A_N)\varLambda_n^{N-1}u_n - P_C(I-\lambda_{N,n}A_N)\varLambda_n^{N-1}p\| \\
&\leq \|(I-\lambda_{N,n}A_N)\varLambda_n^{N-1}u_n - (I-\lambda_{N,n}A_N)\varLambda_n^{N-1}p\| \\
&\leq \|\varLambda_n^{N-1}u_n - \varLambda_n^{N-1}p\| \\
&\leq \cdots \qquad (3.3)\\
&\leq \|\varLambda_n^{0}x_n - \varLambda_n^{0}p\| \\
&= \|u_n - p\|.
\end{aligned}
$$

Combining (3.2) and (3.3), we have

$$
\|z_n - p\| \leq \|x_n - p\|. \tag{3.4}
$$

By Lemma 2.3 (b), we deduce from (3.1) and (3.4) that

$$
\begin{aligned}
\|k_n - p\|^2 &= \|\delta_n(z_n - p) + (1-\delta_n)(S^n z_n - p)\|^2 \\
&= \delta_n\|z_n - p\|^2 + (1-\delta_n)\|S^n z_n - p\|^2 - \delta_n(1-\delta_n)\|z_n - S^n z_n\|^2 \\
&\leq \delta_n\|z_n - p\|^2 + (1-\delta_n)[(1+\gamma_n)\|z_n - p\|^2 + k\|z_n - S^n z_n\|^2 + c_n] \\
&\quad -\delta_n(1-\delta_n)\|z_n - S^n z_n\|^2 \qquad (3.5)\\
&= [1+\gamma_n(1-\delta_n)]\|z_n - p\|^2 + (1-\delta_n)(k-\delta_n)\|z_n - S^n z_n\|^2 \\
&\quad +(1-\delta_n)c_n \\
&\leq (1+\gamma_n)\|z_n - p\|^2 + (1-\delta_n)(k-\delta_n)\|z_n - S^n z_n\|^2 + c_n \\
&\leq (1+\gamma_n)\|z_n - p\|^2 + c_n.
\end{aligned}
$$

Taking into account that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\alpha_n + \beta_n \leq 1$, we get $\alpha_n/(1-\beta_n) \leq 1$ for all $n \geq 1$. Then by Lemmas 2.2 and 2.19 we deduce from (3.4), (3.5) and $0 \leq \gamma l \leq \tau$ that

$$
\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n\gamma(f(x_n) - \mu Fp) + \beta_n(k_n - p) + ((1-\beta_n)I - \alpha_n\mu F)W_n k_n \\
&\quad -((1-\beta_n)I - \alpha_n\mu F)W_n p\|^2 \\
&= \|\alpha_n\gamma(f(x_n) - f(p)) + \beta_n(k_n - p) + ((1-\beta_n)I - \alpha_n\mu F)W_n k_n \\
&\quad -((1-\beta_n)I - \alpha_n\mu F)W_n p + \alpha_n(\gamma f(p) - \mu Fp)\|^2 \\
&\leq \|\alpha_n\gamma(f(x_n) - f(p)) + \beta_n(k_n - p) + ((1-\beta_n)I - \alpha_n\mu F)W_n k_n \\
&\quad -((1-\beta_n)I - \alpha_n\mu F)W_n p\|^2 + 2\alpha_n\langle\gamma f(p) - \mu Fp, y_n - p\rangle \\
&\leq [\alpha_n\gamma\|f(x_n) - f(p)\| + \beta_n\|k_n - p\| + (1-\beta_n)\|\Big(I - \frac{\alpha_n}{1-\beta_n}\mu F\Big)W_n k_n \\
&\quad -\Big(I - \frac{\alpha_n}{1-\beta_n}\mu F\Big)W_n p\|]^2 + 2\alpha_n\|(\gamma f - \mu F)p\|\|y_n - p\| \\
&\leq [\alpha_n\gamma l\|x_n - p\| + \beta_n\|k_n - p\| + (1-\beta_n)\Big(1 - \frac{\alpha_n\tau}{1-\beta_n}\Big)\|k_n - p\|]^2
\end{aligned}
$$

$$
\begin{aligned}
& +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|y_n-p\|^2) \\
\le\ & [\alpha_n\tau\|x_n-p\|+\beta_n\|k_n-p\|+(1-\beta_n-\alpha_n\tau)\|k_n-p\|]^2 \\
& +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|y_n-p\|^2) \\
=\ & [\alpha_n\tau\|x_n-p\|+(1-\alpha_n\tau)\|k_n-p\|]^2 \\
& +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|y_n-p\|^2) \\
\le\ & \alpha_n\tau\|x_n-p\|^2+(1-\alpha_n\tau)\|k_n-p\|^2 \\
& +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|y_n-p\|^2) \\
\le\ & \alpha_n\tau\|x_n-p\|^2+(1-\alpha_n\tau)((1+\gamma_n)\|z_n-p\|^2+c_n) \\
& +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|y_n-p\|^2) \\
\le\ & \alpha_n\tau\|x_n-p\|^2+(1-\alpha_n\tau)((1+\gamma_n)\|x_n-p\|^2+c_n) \\
& +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|y_n-p\|^2) \\
=\ & \|x_n-p\|^2+(1-\alpha_n\tau)\gamma_n\|x_n-p\|^2+(1-\alpha_n\tau)c_n \\
& +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|y_n-p\|^2) \\
\le\ & (1+\gamma_n)\|x_n-p\|^2+c_n+\alpha_n(\|(\gamma f-\mu F)p\|^2+\|y_n-p\|^2),
\end{aligned}
$$

which hence yields

$$
\begin{aligned}
\|y_n-p\|^2 \le\ & \frac{1+\gamma_n}{1-\alpha_n}\|x_n-p\|^2+\frac{\alpha_n}{1-\alpha_n}\|(\gamma f-\mu F)p\|^2+\frac{1}{1-\alpha_n}c_n \\
=\ & \Big(1+\frac{\alpha_n+\gamma_n}{1-\alpha_n}\Big)\|x_n-p\|^2+\frac{\alpha_n}{1-\alpha_n}\|(\gamma f-\mu F)p\|^2+\frac{1}{1-\alpha_n}c_n \\
\le\ & \Big(1+\frac{\alpha_n+\gamma_n}{1-\alpha_n}\Big)\|x_n-p\|^2+\frac{\alpha_n+\gamma_n}{1-\alpha_n}\|(\gamma f-\mu F)p\|^2+\frac{1}{1-\alpha_n}c_n \\
=\ & \|x_n-p\|^2+\frac{\alpha_n+\gamma_n}{1-\alpha_n}(\|x_n-p\|^2+\|(\gamma f-\mu F)p\|^2)+\frac{1}{1-\alpha_n}c_n \qquad (3.6)\\
\le\ & \|x_n-p\|^2+(\alpha_n+\gamma_n)\varrho(\|x_n-p\|^2+\|(\gamma f-\mu F)p\|^2)+\varrho c_n \\
\le\ & \|x_n-p\|^2+(\alpha_n+\gamma_n)\Gamma_n\varrho+c_n\varrho \\
=\ & \|x_n-p\|^2+\theta_n,
\end{aligned}
$$

where $\theta_n=(\alpha_n+\gamma_n)\Gamma_n\varrho+c_n\varrho$, $\Gamma_n=\sup\{\|x_n-p\|^2+\|(\gamma f-\mu F)p\|^2 : p\in\Omega\}<\infty$, and $\varrho=\frac{1}{1-\sup_{n\ge1}\alpha_n}<\infty$ (due to $\{\alpha_n\}\subset(0,1)$ and $\lim_{n\to\infty}\alpha_n=0$). Hence $p\in C_{n+1}$. This implies that $\Omega\subset C_n$ for all $n\ge1$. Therefore, $\{x_n\}$ is well defined.

Step 2. We prove that $\|x_n-k_n\|\to0$ as $n\to\infty$.

Indeed, let $v=P_\Omega x_0$. From $x_n=P_{C_n}x_0$ and $v\in\Omega\subset C_n$, we obtain

$$\|x_n-x_0\|\le\|v-x_0\|. \tag{3.7}$$

This implies that $\{x_n\}$ is bounded and hence $\{u_n\},\{z_n\},\{k_n\}$ and $\{y_n\}$ are also bounded. Since $x_{n+1}\in C_{n+1}\subset C_n$ and $x_n=P_{C_n}x_0$, we have

$$\|x_n-x_0\|\le\|x_{n+1}-x_0\|,\quad \forall n\ge1.$$

Therefore $\lim_{n\to\infty}\|x_n - x_0\|$ exists. From $x_n = P_{C_n}x_0,\ x_{n+1} \in C_{n+1} \subset C_n$, by Proposition 2.1 (ii) we obtain

$$\|x_{n+1} - x_n\|^2 \le \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2,$$

which implies

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.8}$$

It follows from $x_{n+1} \in C_{n+1}$ that $\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n$ and hence

$$\begin{aligned}
\|x_n - y_n\|^2 &\le 2(\|x_n - x_{n+1}\|^2 + \|x_{n+1} - y_n\|^2) \\
&\le 2(\|x_n - x_{n+1}\|^2 + \|x_n - x_{n+1}\|^2 + \theta_n) \\
&= 2(2\|x_n - x_{n+1}\|^2 + \theta_n).
\end{aligned}$$

From (3.8) and $\lim_{n\to\infty}\theta_n = 0$, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.9}$$

Also, utilizing Lemmas 2.2 and 2.3 (b) we obtain from (3.1), (3.4) and (3.5) that

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(\gamma f(x_n) - \mu F W_n k_n) + \beta_n(k_n - p) + (1-\beta_n)(W_n k_n - p)\|^2 \\
&\le \|\beta_n(k_n - p) + (1-\beta_n)(W_n k_n - p)\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - \mu F W_n k_n, y_n - p\rangle \\
&= \beta_n\|k_n - p\|^2 + (1-\beta_n)\|W_n k_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_n k_n\|^2 \\
&\quad + 2\alpha_n\|\gamma f(x_n) - \mu F W_n k_n\|\|y_n - p\| \\
&\le \beta_n\|k_n - p\|^2 + (1-\beta_n)\|k_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_n k_n\|^2 \\
&\quad + 2\alpha_n\|\gamma f(x_n) - \mu F W_n k_n\|\|y_n - p\| \\
&= \|k_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_n k_n\|^2 \\
&\quad + 2\alpha_n\|\gamma f(x_n) - \mu F W_n k_n\|\|y_n - p\| \\
&\le (1+\gamma_n)\|z_n - p\|^2 + c_n - \beta_n(1-\beta_n)\|k_n - W_n k_n\|^2 \\
&\quad + 2\alpha_n\|\gamma f(x_n) - \mu F W_n k_n\|\|y_n - p\| \\
&\le (1+\gamma_n)\|x_n - p\|^2 + c_n - \beta_n(1-\beta_n)\|k_n - W_n k_n\|^2 \\
&\quad + 2\alpha_n\|\gamma f(x_n) - \mu F W_n k_n\|\|y_n - p\|,
\end{aligned}$$

which leads to

$$\begin{aligned}
\beta_n(1-\beta_n)\|k_n - W_n k_n\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 + \gamma_n\|x_n - p\|^2 \\
&\quad + c_n + 2\alpha_n\|\gamma f(x_n) - \mu F W_n k_n\|\|y_n - p\| \\
&\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \gamma_n\|x_n - p\|^2 \\
&\quad + c_n + 2\alpha_n\|\gamma f(x_n) - \mu F W_n k_n\|\|y_n - p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0,\ \lim_{n\to\infty}\gamma_n = 0$ and $\lim_{n\to\infty}c_n = 0$, it follows from (3.9) and condition (i) that

$$\lim_{n\to\infty}\|k_n - W_n k_n\| = 0. \tag{3.10}$$

Note that

$$y_n - k_n = \alpha_n(\gamma f(x_n) - \mu F W_n k_n) + (1-\beta_n)(W_n k_n - k_n),$$

which yields

$$\begin{aligned}\|x_n-k_n\| &\le \|x_n-y_n\|+\|y_n-k_n\|\\ &\le \|x_n-y_n\|+\|\alpha_n(\gamma f(x_n)-\mu FW_nk_n)+(1-\beta_n)(W_nk_n-k_n)\|\\ &\le \|x_n-y_n\|+\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|+(1-\beta_n)\|W_nk_n-k_n\|\\ &\le \|x_n-y_n\|+\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|+\|W_nk_n-k_n\|.\end{aligned}$$

So, from (3.9), (3.10) and $\lim_{n\to\infty}\alpha_n=0$, we get

$$\lim_{n\to\infty}\|x_n-k_n\|=0. \tag{3.11}$$

Step 3. We prove that $\|x_n-u_n\|\to 0$, $\|u_n-z_n\|\to 0$, $\|z_n-Wz_n\|\to 0$ and $\|z_n-S^nz_n\|\to 0$ as $n\to\infty$.

Indeed, from (3.4) and (3.5) it follows that

$$\begin{aligned}\|k_n-p\|^2 &\le [1+\gamma_n(1-\delta_n)]\|z_n-p\|^2\\ &\quad +(1-\delta_n)(k-\delta_n)\|z_n-S^nz_n\|^2+(1-\delta_n)c_n\\ &\le \|z_n-p\|^2+\gamma_n\|z_n-p\|^2+c_n\\ &\le \|z_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n.\end{aligned}\tag{3.12}$$

Next we prove that

$$\lim_{n\to\infty}\|\Delta_n^kx_n-\Delta_n^{k-1}x_n\|=0,\quad k=1,2,\ldots,M. \tag{3.13}$$

For $p\in F$, it follows from (2.1) that

$$\begin{aligned}\|\Delta_n^kx_n-p\|^2 &= \|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}B_k)\Delta_n^{k-1}x_n-T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}B_k)p\|^2\\ &\le \|(I-r_{k,n}B_k)\Delta_n^{k-1}x_n-(I-r_{k,n}B_k)p\|^2\\ &\le \|\Delta_n^{k-1}x_n-p\|^2+r_{k,n}(r_{k,n}-2\mu_k)\|B_k\Delta_n^{k-1}x_n-B_kp\|^2\\ &\le \|x_n-p\|^2+r_{k,n}(r_{k,n}-2\mu_k)\|B_k\Delta_n^{k-1}x_n-B_kp\|^2.\end{aligned}\tag{3.14}$$

By (3.2), (3.3), (3.12) and (3.14), we obtain

$$\begin{aligned}\|k_n-p\|^2 &\le \|z_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n\\ &\le \|u_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n\\ &\le \|\Delta_n^kx_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n\\ &\le \|x_n-p\|^2+r_{k,n}(r_{k,n}-2\mu_k)\|B_k\Delta_n^{k-1}x_n-B_kp\|^2\\ &\quad +\gamma_n\|x_n-p\|^2+c_n,\end{aligned}$$

which implies that

$$\begin{aligned}r_{k,n}(2\mu_k-r_{k,n})\|B_k\Delta_n^{k-1}x_n-B_kp\|^2 &\le \|x_n-p\|^2-\|k_n-p\|^2\\ &\quad +\gamma_n\|x_n-p\|^2+c_n\\ &\le \|x_n-k_n\|(\|x_n-p\|+\|k_n-p\|)\\ &\quad +\gamma_n\|x_n-p\|^2+c_n.\end{aligned}$$

Since $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k), k \in \{1, 2, \ldots, M\}$, $\lim_{n\to\infty} \gamma_n = 0$, $\lim_{n\to\infty} c_n = 0$ and (3.11), we have

$$\lim_{n\to\infty} \|B_k \varDelta_n^{k-1} x_n - B_k p\| = 0, \quad k = 1, 2, \ldots, M. \tag{3.15}$$

By Proposition 1.1 (iii) and Lemma 2.3 (a) we have

$$\begin{aligned}
\|\varDelta_n^k x_n - p\|^2 &= \|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}B_k)\varDelta_n^{k-1}x_n - T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}B_k)p\|^2 \\
&\leq \langle (I - r_{k,n}B_k)\varDelta_n^{k-1}x_n - (I - r_{k,n}B_k)p, \varDelta_n^k x_n - p\rangle \\
&= \frac{1}{2}(\|(I - r_{k,n}B_k)\varDelta_n^{k-1}x_n - (I - r_{k,n}B_k)p\|^2 + \|\varDelta_n^k x_n - p\|^2 \\
&\quad -\|(I - r_{k,n}B_k)\varDelta_n^{k-1}x_n - (I - r_{k,n}B_k)p - (\varDelta_n^k x_n - p)\|^2) \\
&\leq \frac{1}{2}(\|\varDelta_n^{k-1}x_n - p\|^2 + \|\varDelta_n^k x_n - p\|^2 \\
&\quad -\|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n - r_{k,n}(B_k\varDelta_n^{k-1}x_n - B_k p)\|^2),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\varDelta_n^k x_n - p\|^2 &\leq \|\varDelta_n^{k-1}x_n - p\|^2 \\
&\quad -\|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n - r_{k,n}(B_k\varDelta_n^{k-1}x_n - B_k p)\|^2 \\
&= \|\varDelta_n^{k-1}x_n - p\|^2 - \|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n\|^2 - r_{k,n}^2\|B_k\varDelta_n^{k-1}x_n - B_k p\|^2 \\
&\quad +2r_{k,n}\langle \varDelta_n^{k-1}x_n - \varDelta_n^k x_n, B_k\varDelta_n^{k-1}x_n - B_k p\rangle \\
&\leq \|\varDelta_n^{k-1}x_n - p\|^2 - \|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n\|^2 \\
&\quad +2r_{k,n}\|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n\|\|B_k\varDelta_n^{k-1}x_n - B_k p\| \\
&\leq \|x_n - p\|^2 - \|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n\|^2 \\
&\quad +2r_{k,n}\|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n\|\|B_k\varDelta_n^{k-1}x_n - B_k p\|.
\end{aligned} \tag{3.16}$$

Combining (3.12) and (3.16), we have

$$\begin{aligned}
\|k_n - p\|^2 &\leq \|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\leq \|u_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\leq \|\varDelta_n^k x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\leq \|x_n - p\|^2 - \|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n\|^2 \\
&\quad +2r_{k,n}\|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n\|\|B_k\varDelta_n^{k-1}x_n - B_k p\| \\
&\quad +\gamma_n\|x_n - p\|^2 + c_n,
\end{aligned}$$

which implies

$$\begin{aligned}
\|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n\|^2 &\leq \|x_n - p\|^2 - \|k_n - p\|^2 + 2r_{k,n}\|\varDelta_n^{k-1}x_n \\
&\quad -\varDelta_n^k x_n\|\|B_k\varDelta_n^{k-1}x_n - B_k p\| + \gamma_n\|x_n - p\|^2 + c_n \\
&\leq \|x_n - k_n\|(\|x_n - p\| + \|k_n - p\|) \\
&\quad +2r_{k,n}\|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n\|\|B_k\varDelta_n^{k-1}x_n - B_k p\| \\
&\quad +\gamma_n\|x_n - p\|^2 + c_n.
\end{aligned}$$

From $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty} c_n = 0$, (3.11) and (3.15), we know that (3.13) holds. Hence we obtain

$$\begin{aligned} \|x_n - u_n\| &= \|\Delta_n^0 x_n - \Delta_n^M x_n\| \\ &\le \|\Delta_n^0 x_n - \Delta_n^1 x_n\| + \|\Delta_n^1 x_n - \Delta_n^2 x_n\| \\ &\quad + \cdots + \|\Delta_n^{M-1} x_n - \Delta_n^M x_n\| \\ &\to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{3.17}$$

Next we show that $\lim_{n\to\infty}\|A_i\Lambda_n^i u_n - A_i p\| = 0$, $i = 1,2,\ldots,N$. It follows from (2.1) that

$$\begin{aligned} \|\Lambda_n^i u_n - p\|^2 &= \|P_C(I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - P_C(I-\lambda_{i,n}A_i)p\|^2 \\ &\le \|(I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - (I-\lambda_{i,n}A_i)p\|^2 \\ &\le \|\Lambda_n^{i-1}u_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|A_i\Lambda_n^{i-1}u_n - A_i p\|^2 \\ &\le \|u_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|A_i\Lambda_n^{i-1}u_n - A_i p\|^2 \\ &\le \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|A_i\Lambda_n^{i-1}u_n - A_i p\|^2. \end{aligned} \tag{3.18}$$

Combining (3.12) and (3.18), we have

$$\begin{aligned} \|k_n - p\|^2 &\le \|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|\Lambda_n^i u_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|A_i\Lambda_n^{i-1}u_n - A_i p\|^2 \\ &\quad + \gamma_n\|x_n - p\|^2 + c_n, \end{aligned}$$

which implies

$$\begin{aligned} \lambda_{i,n}(2\eta_i - \lambda_{i,n})\|A_i\Lambda_n^{i-1}u_n - A_i p\|^2 &\le \|x_n - p\|^2 - \|k_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|x_n - k_n\|(\|x_n - p\| + \|k_n - p\|) \\ &\quad + \gamma_n\|x_n - p\|^2 + c_n. \end{aligned}$$

From $\{\lambda_{i,n}\}\subset[a_i,b_i]\subset(0,2\eta_i), i\in\{1,2,\ldots,N\}$, $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty}c_n = 0$ and (3.11), we obtain

$$\lim_{n\to\infty}\|A_i\Lambda_n^{i-1}u_n - A_i p\| = 0, \quad i = 1,2,\ldots,N. \tag{3.19}$$

By Proposition 2.1 (iii) and Lemma 2.3 (a), we obtain

$$\begin{aligned} \|\Lambda_n^i u_n - p\|^2 &= \|P_C(I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - P_C(I-\lambda_{i,n}A_i)p\|^2 \\ &\le \langle (I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - (I-\lambda_{i,n}A_i)p, \Lambda_n^i u_n - p\rangle \\ &= \frac{1}{2}(\|(I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - (I-\lambda_{i,n}A_i)p\|^2 + \|\Lambda_n^i u_n - p\|^2 \\ &\quad - \|(I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - (I-\lambda_{i,n}A_i)p - (\Lambda_n^i u_n - p)\|^2) \\ &\le \frac{1}{2}(\|\Lambda_n^{i-1}u_n - p\|^2 + \|\Lambda_n^i u_n - p\|^2 \\ &\quad - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}(A_i\Lambda_n^{i-1}u_n - A_i p)\|^2) \\ &\le \frac{1}{2}(\|u_n - p\|^2 + \|\Lambda_n^i u_n - p\|^2 \\ &\quad - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}(A_i\Lambda_n^{i-1}u_n - A_i p)\|^2) \end{aligned}$$

$$\begin{aligned}&\le \frac{1}{2}(\|x_n-p\|^2+\|\Lambda_n^i u_n-p\|^2\\&\quad -\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n-\lambda_{i,n}(A_i\Lambda_n^{i-1}u_n-A_ip)\|^2),\end{aligned}$$

which implies

$$\begin{aligned}\|\Lambda_n^i u_n-p\|^2 &\le \|x_n-p\|^2-\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n-\lambda_{i,n}(A_i\Lambda_n^{i-1}u_n-A_ip)\|^2\\&= \|x_n-p\|^2-\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\|^2-\lambda_{i,n}^2\|A_i\Lambda_n^{i-1}u_n-A_ip\|^2\\&\quad +2\lambda_{i,n}\langle \Lambda_n^{i-1}u_n-\Lambda_n^i u_n, A_i\Lambda_n^{i-1}u_n-A_ip\rangle\\&\le \|x_n-p\|^2-\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\|^2\\&\quad +2\lambda_{i,n}\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\|\|A_i\Lambda_n^{i-1}u_n-A_ip\|.\end{aligned}\tag{3.20}$$

Combining (3.12) and (3.20) we get

$$\begin{aligned}\|k_n-p\|^2 &\le \|z_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n\\&\le \|\Lambda_n^i u_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n\\&\le \|x_n-p\|^2-\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\|^2\\&\quad +2\lambda_{i,n}\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\|\|A_i\Lambda_n^{i-1}u_n-A_ip\|\\&\quad +\gamma_n\|x_n-p\|^2+c_n,\end{aligned}$$

which implies

$$\begin{aligned}\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\|^2 &\le \|x_n-p\|^2-\|k_n-p\|^2\\&\quad +2\lambda_{i,n}\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\|\|A_i\Lambda_n^{i-1}u_n-A_ip\|\\&\quad +\gamma_n\|x_n-p\|^2+c_n\\&\le \|x_n-k_n\|(\|x_n-p\|+\|k_n-p\|)\\&\quad +2\lambda_{i,n}\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\|\|A_i\Lambda_n^{i-1}u_n-A_ip\|\\&\quad +\gamma_n\|x_n-p\|^2+c_n.\end{aligned}$$

From (3.11), (3.19), $\lim_{n\to\infty}\gamma_n=0$ and $\lim_{n\to\infty}c_n=0$, we have

$$\lim_{n\to\infty}\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\|=0,\quad i=1,2,\dots,N.\tag{3.21}$$

From (3.21) we get

$$\begin{aligned}\|u_n-z_n\| &= \|\Lambda_n^0u_n-\Lambda_n^N u_n\|\\&\le \|\Lambda_n^0u_n-\Lambda_n^1u_n\|+\|\Lambda_n^1u_n-\Lambda_n^2u_n\|+\cdots+\|\Lambda_n^{N-1}u_n-\Lambda_n^N u_n\|\\&\to 0\quad \text{as } n\to\infty.\end{aligned}\tag{3.22}$$

By (3.17) and (3.22), we have

$$\begin{aligned}\|x_n-z_n\| &\le \|x_n-u_n\|+\|u_n-z_n\|\\&\to 0\quad \text{as } n\to\infty.\end{aligned}\tag{3.23}$$

From (3.8) and (3.23), we have

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \|z_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|+\|x_n-z_n\|\\&\to 0\quad \text{as } n\to\infty.\end{aligned}\tag{3.24}$$

By (3.11), (3.17) and (3.22), we get

$$\begin{aligned} \|k_n - z_n\| &\leq \|k_n - x_n\| + \|x_n - u_n\| + \|u_n - z_n\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.25}$$

We observe that

$$k_n - z_n = (1 - \delta_n)(S^n z_n - z_n).$$

From $\delta_n \leq d < 1$ and (3.25), we have

$$\lim_{n\to\infty} \|S^n z_n - z_n\| = 0. \tag{3.26}$$

We note that

$$\begin{aligned} \|S^n z_n - S^{n+1} z_n\| &\leq \|S^n z_n - z_n\| + \|z_n - z_{n+1}\| \\ &\quad + \|z_{n+1} - S^{n+1} z_{n+1}\| + \|S^{n+1} z_{n+1} - S^{n+1} z_n\|. \end{aligned}$$

From (3.24), (3.26) and Lemma 2.5, we obtain

$$\lim_{n\to\infty} \|S^n z_n - S^{n+1} z_n\| = 0. \tag{3.27}$$

On the other hand, we note that

$$\|z_n - S z_n\| \leq \|z_n - S^n z_n\| + \|S^n z_n - S^{n+1} z_n\| + \|S^{n+1} z_n - S z_n\|.$$

From (3.26), (3.27) and the uniform continuity of S, we have

$$\lim_{n\to\infty} \|z_n - S z_n\| = 0. \tag{3.28}$$

In addition, note that

$$\begin{aligned} \|z_n - W z_n\| &\leq \|z_n - k_n\| + \|k_n - W_n k_n\| + \|W_n k_n - W k_n\| + \|W k_n - W z_n\| \\ &\leq 2\|z_n - k_n\| + \|k_n - W_n k_n\| + \|W_n k_n - W k_n\|. \end{aligned}$$

So, from (3.10), (3.25) and [43, Remark 3.2] it follows that

$$\lim_{n\to\infty} \|z_n - W z_n\| = 0. \tag{3.29}$$

Step 4. We prove that $x_n \to v = P_\Omega x_0$ as $n \to \infty$.

Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ which converges weakly to some w. From (3.13) and (3.21)-(3.23), we have that $\Delta^k_{n_i} x_{n_i} \rightharpoonup w$, $\Lambda^m_{n_i} u_{n_i} \rightharpoonup w$ and $z_{n_i} \rightharpoonup w$, where $k \in \{1, 2, \ldots, M\}$ and $m \in \{1, 2, \ldots, N\}$. Since S is uniformly continuous, by (3.28) we get $\lim_{n\to\infty} \|z_n - S^m z_n\| = 0$ for any $m \geq 1$. Hence from Lemma 2.7, we obtain $w \in \text{Fix}(S)$. In the meantime, utilizing Lemma 2.16, we deduce from (3.29) and $z_{n_i} \rightharpoonup w$ that $w \in \text{Fix}(W) = \cap_{n=1}^\infty \text{Fix}(T_n)$. Next, we prove that $w \in \cap_{m=1}^N \text{VI}(C, A_m)$. Let

$$\widetilde{T}_m v = \begin{cases} A_m v + N_C v, & v \in C, \\ \emptyset, & v \notin C, \end{cases}$$

where $m \in \{1, 2, \ldots, N\}$. Let $(v, u) \in G(\widetilde{T}_m)$. Since $u - A_m v \in N_C v$ and $\Lambda^m_n u_n \in C$, we have

$$\langle v - \Lambda^m_n u_n, u - A_m v\rangle \geq 0.$$

On the other hand, from $\Lambda^m_n u_n = P_C(I - \lambda_{m,n} A_m)\Lambda^{m-1}_n u_n$ and $v \in C$, we have

$$\langle v - \Lambda^m_n u_n, \Lambda^m_n u_n - (\Lambda^{m-1}_n u_n - \lambda_{m,n} A_m \Lambda^{m-1}_n u_n)\rangle \geq 0,$$

and hence

$$\Big\langle v - \Lambda_n^m u_n, \frac{\Lambda_n^m u_n - \Lambda_n^{m-1} u_n}{\lambda_{m,n}} + A_m \Lambda_n^{m-1} u_n) \Big\rangle \geq 0.$$

Therefore we have

$$\begin{aligned}
\langle v - \Lambda_{n_i}^m u_{n_i}, u \rangle &\geq \langle v - \Lambda_{n_i}^m u_{n_i}, A_m v \rangle \\
&\geq \langle v - \Lambda_{n_i}^m u_{n_i}, A_m v \rangle \\
&\quad - \Big\langle v - \Lambda_{n_i}^m u_{n_i}, \frac{\Lambda_{n_i}^m u_{n_i} - (\Lambda_{n_i}^{m-1} u_{n_i}}{\lambda_{m,n_i}} + A_m \Lambda_{n_i}^{m-1} u_{n_i} \Big\rangle \\
&= \langle v - \Lambda_{n_i}^m u_{n_i}, A_m v - A_m \Lambda_{n_i}^m u_{n_i} \rangle \\
&\quad + \langle v - \Lambda_{n_i}^m u_{n_i}, A_m \Lambda_{n_i}^m u_{n_i} - A_m \Lambda_{n_i}^{m-1} u_{n_i} \rangle \\
&\quad - \Big\langle v - \Lambda_{n_i}^m u_{n_i}, \frac{\Lambda_{n_i}^m u_{n_i} - (\Lambda_{n_i}^{m-1} u_{n_i}}{\lambda_{m,n_i}} \Big\rangle \\
&\geq \langle v - \Lambda_{n_i}^m u_{n_i}, A_m \Lambda_{n_i}^m u_{n_i} - A_m \Lambda_{n_i}^{m-1} u_{n_i} \rangle \\
&\quad - \Big\langle v - \Lambda_{n_i}^m u_{n_i}, \frac{\Lambda_{n_i}^m u_{n_i} - (\Lambda_{n_i}^{m-1} u_{n_i}}{\lambda_{m,n_i}} \Big\rangle.
\end{aligned}$$

From (3.20) and since A_m is uniformly continuous, we obtain that $\lim_{n\to\infty} \|A_m \Lambda_n^m u_n - A_m \Lambda_n^{m-1} u_n\| = 0$. From $\Lambda_{n_i}^m u_{n_i} \rightharpoonup w$, $\{\lambda_{m,n}\} \subset [a_m, b_m] \subset (0, 2\eta_m)$, $\forall m \in \{1, 2, \ldots, N\}$ and (3.20), we have

$$\langle v - w, u \rangle \geq 0.$$

Since $\widetilde{T}_m$ is maximal monotone, we have $w \in \widetilde{T}_m^{-1} 0$ and hence $w \in \mathrm{VI}(C, A_m)$, $m = 1, 2, \ldots, N$, which implies $w \in \cap_{m=1}^N \mathrm{VI}(C, A_m)$. Next we prove that $w \in \cap_{k=1}^M \mathrm{GMEP}(\Theta_k, \varphi_k, B_k)$. Since $\Delta_n^k x_n = T_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n} B_k) \Delta_n^{k-1} x_n, n \geq 1, k \in \{1, 2, \ldots, M\}$, we have

$$\begin{aligned}
&\Theta_k(\Delta_n^k x_n, y) + \varphi_k(y) - \varphi_k(\Delta_n^k x_n) \\
&\qquad + \langle B_k \Delta_n^{k-1} x_n, y - \Delta_n^k x_n \rangle + \frac{1}{r_{k,n}} \langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1} x_n \rangle \geq 0.
\end{aligned}$$

By (A2), we have

$$\begin{aligned}
&\varphi_k(y) - \varphi_k(\Delta_n^k x_n) + \langle B_k \Delta_n^{k-1} x_n, y - \Delta_n^k x_n \rangle \\
&\qquad + \frac{1}{r_{k,n}} \langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1} x_n \rangle \geq \Theta_k(y, \Delta_n^k x_n).
\end{aligned}$$

Let $z_t = ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in C$. This implies that $z_t \in C$. Then, we have

$$\begin{aligned}
\langle z_t - \Delta_n^k x_n, B_k z_t \rangle &\geq \varphi_k(\Delta_n^k x_n) - \varphi_k(z_t) \\
&\quad + \langle z_t - \Delta_n^k x_n, B_k z_t \rangle - \langle z_t - \Delta_n^k x_n, B_k \Delta_n^{k-1} x_n \rangle \\
&\quad - \Big\langle z_t - \Delta_n^k x_n, \frac{\Delta_n^k x_n - \Delta_n^{k-1} x_n}{r_{k,n}} \Big\rangle + \Theta_k(z_t, \Delta_n^k x_n) \\
&= \varphi_k(\Delta_n^k x_n) - \varphi_k(z_t) + \langle z_t - \Delta_n^k x_n, B_k z_t - B_k \Delta_n^k x_n \rangle \qquad (3.30)
\end{aligned}$$

$$+\langle z_t - \varDelta_n^k x_n, B_k\varDelta_n^k x_n - B_k\varDelta_n^{k-1}x_n\rangle$$
$$-\Big\langle z_t - \varDelta_n^k x_n, \frac{\varDelta_n^k x_n - \varDelta_n^{k-1}x_n}{r_{k,n}}\Big\rangle$$
$$+\Theta_k(z_t, \varDelta_n^k x_n).$$

By (3.13), we have $\|B_k\varDelta_n^k x_n - B_k\varDelta_n^{k-1}x_n\| \to 0$ as $n \to \infty$. Furthermore, by the monotonicity of B_k, we obtain $\langle z_t - \varDelta_n^k x_n, B_k z_t - B_k\varDelta_n^k x_n\rangle \geq 0$. Then, by (A4) we obtain

$$\langle z_t - w, B_k z_t\rangle \geq \varphi_k(w) - \varphi_k(z_t) + \Theta_k(z_t, w). \tag{3.31}$$

Utilizing (A1), (A4) and (3.31), we obtain

$$\begin{array}{rcl} 0 &=& \Theta_k(z_t, z_t) + \varphi_k(z_t) - \varphi_k(z_t) \\ &\leq& t\Theta_k(z_t, y) + (1-t)\Theta_k(z_t, w) + t\varphi_k(y) + (1-t)\varphi_k(w) - \varphi_k(z_t) \\ &\leq& t[\Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)] + (1-t)\langle z_t - w, B_k z_t\rangle \\ &=& t[\Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)] + (1-t)t\langle y - w, B_k z_t\rangle, \end{array}$$

and hence

$$0 \leq \Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t) + (1-t)\langle y - w, B_k z_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq \Theta_k(w, y) + \varphi_k(y) - \varphi_k(w) + \langle y - w, B_k w\rangle.$$

This implies that $w \in \mathrm{GMEP}(\Theta_k, \varphi_k, B_k)$ and hence $w \in \cap_{k=1}^M \mathrm{GMEP}(\Theta_k, \varphi_k, B_k)$. Consequently, $w \in \Omega = \cap_{n=1}^\infty \mathrm{Fix}(T_n) \cap \cap_{k=1}^M \mathrm{GMEP}(\Theta_k, \varphi_k, B_k) \cap \cap_{m=1}^N \mathrm{VI}(C, A_m) \cap \mathrm{Fix}(S)$. This shows that $\omega_w(x_n) \subset \Omega$. From (3.7) and Lemma 2.13 we infer that $x_n \to v = P_\Omega x_0$ as $n \to \infty$.

Finally, assume additionally that $\gamma_n + c_n = o(\alpha_n)$ and $\|x_n - y_n\| = o(\alpha_n)$. Note that

$$\begin{array}{rcl} \mu\eta \geq \tau &\Leftrightarrow& \mu\eta \geq 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \\ &\Leftrightarrow& \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \geq 1 - \mu\eta \\ &\Leftrightarrow& 1 - 2\mu\eta + \mu^2\kappa^2 \geq 1 - 2\mu\eta + \mu^2\eta^2 \\ &\Leftrightarrow& \kappa^2 \geq \eta^2 \\ &\Leftrightarrow& \kappa \geq \eta. \end{array}$$

It is clear that

$$\langle(\mu F - \gamma f)x - (\mu F - \gamma f)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x, y \in H.$$

Hence by Lemma 2.18 we deduce from $0 \leq \gamma l < \tau \leq \mu\eta$ that $\mu F - \gamma f$ is $(\mu\eta - \gamma l)$-strongly monotone. In the meantime, it is easy to see that $\mu F - \gamma f$ is $(\mu\kappa + \gamma l)$-Lipschitzian with constant $\mu\kappa + \gamma l > 0$. Thus, there exists a unique solution p in Ω to the VIP

$$\langle(\mu F - \gamma f)p, u - p\rangle \geq 0, \quad \forall u \in \Omega.$$

Consequently, we deduce from (3.9) and $x_n \to v = P_\Omega x_0$ $(n \to \infty)$ that

$$\limsup_{n\to\infty}\langle(\gamma f - \mu F)p, y_n - p\rangle$$

$$
\begin{aligned}
(3.32) \qquad &= \limsup_{n\to\infty}(\langle(\gamma f-\mu F)p, x_n-p\rangle+\langle(\gamma f-\mu F)p, y_n-x_n\rangle) \\
&= \limsup_{n\to\infty}\langle(\gamma f-\mu F)p, x_n-p\rangle \\
&= \langle(\gamma f-\mu F)p, v-p\rangle \le 0.
\end{aligned}
$$

Furthermore, by Lemmas 2.2 and 2.19 we conclude from (3.1), (3.4) and (3.5) that

$$
\begin{aligned}
\|y_n-p\|^2 &= \|\alpha_n\gamma(f(x_n)-\mu Fp)+\beta_n(k_n-p)+((1-\beta_n)I-\alpha_n\mu F)W_nk_n \\
&\quad ((1-\beta_n)I-\alpha_n\mu F)W_np\|^2 \\
&= \|\alpha_n\gamma(f(x_n)-f(p))+\beta_n(k_n-p)+((1-\beta_n)I-\alpha_n\mu F)W_nk_n \\
&\quad -((1-\beta_n)I-\alpha_n\mu F)W_np+\alpha_n(\gamma f(p)-\mu Fp)\|^2 \\
&\le \|\alpha_n\gamma(f(x_n)-f(p))+\beta_n(k_n-p)+((1-\beta_n)I-\alpha_n\mu F)W_nk_n \\
&\quad -((1-\beta_n)I-\alpha_n\mu F)W_np\|^2+2\alpha_n\langle(\gamma f-\mu F)p, y_n-p\rangle \\
&\le \Big[\alpha_n\gamma\|f(x_n)-f(p)\|+\beta_n\|k_n-p\|+(1-\beta_n)\Big\|\Big(I-\frac{\alpha_n}{1-\beta_n}\mu F\Big)W_nk_n \\
&\quad -\Big(I-\frac{\alpha_n}{1-\beta_n}\mu F\Big)W_np\Big\|\Big]^2+2\alpha_n\langle(\gamma f-\mu F)p, y_n-p\rangle \\
&\le \Big[\alpha_n\gamma l\|x_n-p\|+\beta_n\|k_n-p\|+(1-\beta_n)\Big(1-\frac{\alpha_n\tau}{1-\beta_n}\Big)\|k_n-p\|\Big]^2 \\
&\quad +2\alpha_n\langle(\gamma f-\mu F)p, y_n-p\rangle \\
&= \Big[\alpha_n\tau\frac{\gamma l}{\tau}\|x_n-p\|+\beta_n\|k_n-p\|+(1-\beta_n-\alpha_n\tau)\|k_n-p\|\Big]^2 \\
&\quad +2\alpha_n\langle(\gamma f-\mu F)p, y_n-p\rangle \\
&= \Big[\alpha_n\tau\frac{\gamma l}{\tau}\|x_n-p\|+(1-\alpha_n\tau)\|k_n-p\|\Big]^2+2\alpha_n\langle(\gamma f-\mu F)p, y_n-p\rangle \\
&\le \alpha_n\tau\frac{(\gamma l)^2}{\tau^2}\|x_n-p\|^2+(1-\alpha_n\tau)\|k_n-p\|^2 \\
&\quad +2\alpha_n\langle(\gamma f-\mu F)p, y_n-p\rangle \\
&\le \alpha_n\tau\frac{(\gamma l)^2}{\tau^2}\|x_n-p\|^2+(1-\alpha_n\tau)((1+\gamma_n)\|z_n-p\|^2+c_n) \\
&\quad +2\alpha_n\langle(\gamma f-\mu F)p, y_n-p\rangle \\
&\le \alpha_n\frac{(\gamma l)^2}{\tau}\|x_n-p\|^2+(1-\alpha_n\tau)\|x_n-p\|^2 \\
&\quad +(1-\alpha_n\tau)(\gamma_n\|x_n-p\|^2+c_n) \\
&\quad +2\alpha_n\langle(\gamma f-\mu F)p, y_n-p\rangle \\
&= \Big(1-\alpha_n\frac{\tau^2-(\gamma l)^2}{\tau}\Big)\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n \\
&\quad +2\alpha_n\langle(\gamma f-\mu F)p, y_n-p\rangle,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\frac{\tau^2-(\gamma l)^2}{\tau}\|x_n-p\|^2 &\le \frac{\|x_n-p\|^2-\|y_n-p\|^2}{\alpha_n}+\frac{\gamma_n\|x_n-p\|^2+c_n}{\alpha_n} \\
&\quad +2\langle(\gamma f-\mu F)p, y_n-p\rangle
\end{aligned}
$$

$$\begin{aligned}&\leq \frac{\|x_n-y_n\|}{\alpha_n}(\|x_n-p\|+\|y_n-p\|)\\&\quad+\frac{\gamma_n+c_n}{\alpha_n}(\|x_n-p\|^2+1)\\&\quad+2\langle(\gamma f-\mu F)p,y_n-p\rangle.\end{aligned}$$

Since $\gamma_n+c_n=o(\alpha_n)$, $\|x_n-y_n\|=o(\alpha_n)$ and $x_n\to v=P_\Omega x_0$, we infer from (3.32) and $0\leq\gamma l<\tau$ that as $n\to\infty$

$$\frac{\tau^2-(\gamma l)^2}{\tau}\|v-p\|^2\leq 0.$$

That is, $p=v=P_\Omega x_0$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C\times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and $\varphi: C\to\mathbf{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let $B: H\to H$ and $A_i: C\to H$ be ζ-inverse strongly monotone and η_i-inverse-strongly monotone, respectively, for $i=1,2$. Let $S: C\to C$ be a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense for some $0\leq k<1$ with sequence $\{\gamma_n\}\subset[0,\infty)$ such that $\lim_{n\to\infty}\gamma_n=0$ and $\{c_n\}\subset[0,\infty)$ such that $\lim_{n\to\infty}c_n=0$. Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b\in(0,1)$. Let $F: H\to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa,\eta>0$. Let $f: H\to H$ be an l-Lipschitzian mapping with constant $l\geq 0$. Let $0<\mu<\frac{2\eta}{\kappa^2}$ and $0\leq\gamma l<\tau$, where $\tau=1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}$. Assume that $\Omega:=\cap_{n=1}^\infty\mathrm{Fix}(T_n)\cap\mathrm{GMEP}(\Theta,\varphi,B)\cap\mathrm{VI}(C,A_2)\cap\mathrm{VI}(C,A_1)\cap\mathrm{Fix}(S)$ is nonempty and bounded and that either (B1) or (B2) holds. Let W_n be the W-mapping defined by (2.2), and $\{\alpha_n\},\{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n+\beta_n\leq 1$ $(\forall n\geq 1)$, $\lim_{n\to\infty}\alpha_n=0$ and $k\leq\delta_n\leq d<1$. Pick any $x_0\in H$ and set $C_1=H$, $x_1=P_{C_1}x_0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases}\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\langle Bx_n,y-u_n\rangle\\ \qquad+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad\forall y\in C,\\ z_n=P_C(I-\lambda_{2,n}A_2)P_C(I-\lambda_{1,n}A_1)u_n,\\ k_n=\delta_nz_n+(1-\delta_n)S^nz_n,\\ y_n=\alpha_n\gamma f(x_n)+\beta_nk_n+[(1-\beta_n)I-\alpha_n\mu F]W_nk_n,\\ C_{n+1}=\{z\in C_n:\|y_n-z\|^2\leq\|x_n-z\|^2+\theta_n\},\\ x_{n+1}=P_{C_{n+1}}x_0,\quad\forall n\geq 0,\end{cases}$$

where $\theta_n=(\alpha_n+\gamma_n)\Gamma_n\varrho+c_n\varrho$, $\Gamma_n=\sup\{\|x_n-p\|^2+\|(\gamma f-\mu F)p\|^2: p\in\Omega\}<\infty$, and $\varrho=\frac{1}{1-\sup_{n\geq 1}\alpha_n}<\infty$. Assume that the following conditions are satisfied:

(i) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;
(ii) $\{\lambda_{i,n}\}\subset[a_i,b_i]\subset(0,2\eta_i)$ *for* $i=1,2$;
(iii) $\{r_n\}\subset[e,f]\subset(0,2\zeta)$.

Then the following statements hold:

(I) $\{x_n\}$ *converges strongly to* $v=P_\Omega x_0$;

(II) $\{x_n\}$ *converges strongly to* $v = P_\Omega x_0$, *which is a unique solution in* Ω *to the VIP*

$$\langle(\mu F - \gamma f)v, u - v\rangle \geq 0, \quad \forall u \in \Omega,$$

provided $\gamma_n + c_n = o(\alpha_n)$ *and* $\|x_n - y_n\| = o(\alpha_n)$.

Corollary 3.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* Θ *be a bifunction from* $C \times C$ *to* $\mathbf{R}$ *satisfying (A1)-(A4) and* $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* $B : H \to H$ *and* $A : C \to H$ *be* ζ*-inverse strongly monotone and* ξ*-inverse-strongly monotone, respectively. Let* $S : C \to C$ *be a uniformly continuous asymptotically* k*-strict pseudocontractive mapping in the intermediate sense for some* $0 \leq k < 1$ *with sequence* $\{\gamma_n\} \subset [0, \infty)$ *such that* $\lim_{n\to\infty} \gamma_n = 0$ *and* $\{c_n\} \subset [0, \infty)$ *such that* $\lim_{n\to\infty} c_n = 0$. *Let* $\{T_n\}_{n=1}^\infty$ *be a sequence of nonexpansive self-mappings on* C *and* $\{\lambda_n\}$ *be a sequence in* $(0, b]$ *for some* $b \in (0, 1)$. *Let* $F : H \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with positive constants* $\kappa, \eta > 0$. *Let* $f : H \to H$ *be an* l*-Lipschitzian mapping with constant* $l \geq 0$. *Let* $0 < \mu < \frac{2\eta}{\kappa^2}$ *and* $0 \leq \gamma l < \tau$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. *Assume that* $\Omega := \cap_{n=1}^\infty \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, B) \cap \mathrm{VI}(C, A) \cap \mathrm{Fix}(S)$ *is nonempty and bounded and that either (B1) or (B2) holds. Let* W_n *be the* W*-mapping defined by (2.2), and* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\delta_n\}$ *be sequences in* $(0, 1)$ *such that* $\alpha_n + \beta_n \leq 1$ $(\forall n \geq 1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $k \leq \delta_n \leq d < 1$. *Pick any* $x_0 \in H$ *and set* $C_1 = H$, $x_1 = P_{C_1} x_0$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle \\ \qquad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ k_n = \delta_n P_C(I - \rho_n A)u_n + (1 - \delta_n)S^n P_C(I - \rho_n A)u_n, \\ y_n = \alpha_n \gamma f(x_n) + \beta_n k_n + [(1 - \beta_n)I - \alpha_n \mu F]W_n k_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{3.33}$$

where $\theta_n = (\alpha_n + \gamma_n)\Gamma_n \varrho + c_n \varrho$, $\Gamma_n = \sup\{\|x_n - p\|^2 + \|(\gamma f - \mu F)p\|^2 : p \in \Omega\} < \infty$, *and* $\varrho = \frac{1}{1 - \sup_{n\geq 1} \alpha_n} < \infty$. *Assume that the following conditions are satisfied:*

(i) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(ii) $\{\rho_n\} \subset [a, b] \subset (0, 2\xi)$;
(iii) $\{r_n\} \subset [e, f] \subset (0, 2\zeta)$.

Then the following statements hold:

(I) $\{x_n\}$ *converges strongly to* $v = P_\Omega x_0$;
(II) $\{x_n\}$ *converges strongly to* $v = P_\Omega x_0$, *which is a unique solution in* Ω *to the VIP*

$$\langle(\mu F - \gamma f)v, u - v\rangle \geq 0, \quad \forall u \in \Omega,$$

provided $\gamma_n + c_n = o(\alpha_n)$ *and* $\|x_n - y_n\| = o(\alpha_n)$.

4. Weak convergence theorem

In this section, we will prove a weak convergence theorem for another iterative algorithm for finding a common element of the set of solutions of the set of solutions of finite generalized mixed equilibrium problems, the set of solutions of finite variational inequalities for inverse strong monotone mappings and the set of common fixed points of infinite nonexpansive mappings and asymptotically κ-strict pseudocontractive mapping $S : C \to C$ in the intermediate sense in a real Hilbert space. This iterative algorithm is based on the extragradient method, viscosity approximation method, Mann-type iterative method and hybrid steepest-descent method.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let M, N be two integers. Let Θ_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \dots, M\}$. Let $B_k : H \to H$ and $A_i : C \to H$ be μ_k-inverse strongly monotone and η_i-inverse-strongly monotone, respectively, where $k \in \{1, 2, \dots, M\}$, $i \in \{1, 2, \dots, N\}$. Let $S : C \to C$ be a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense for some $0 \le k < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$ and $\{c_n\} \subset [0, \infty)$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $f : H \to H$ be an l-Lipschitzian mapping with constant $l \ge 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \le \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \cap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, B_k) \cap \cap_{i=1}^{N} \mathrm{VI}(C, A_i) \cap \mathrm{Fix}(S)$ is nonempty and that either (B1) or (B2) holds. Let W_n be the W-mapping defined by (2.2), and $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n \le 1$ $(\forall n \ge 1)$ and $0 < k + \epsilon \le \delta_n \le d < 1$. Pick any $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
(4.1) \qquad \begin{cases} u_n = T_{r_{M,n}}^{(\Theta_M, \varphi_M)}(I - r_{M,n} B_M) T_{r_{M-1,n}}^{(\Theta_{M-1}, \varphi_{M-1})}(I - r_{M-1,n} B_{M-1}) \\ \qquad \cdots T_{r_{1,n}}^{(\Theta_1, \varphi_1)}(I - r_{1,n} B_1) x_n, \\ {}_n = P_C(I - \lambda_{N,n} A_N) P_C(I - \lambda_{N-1,n} A_{N-1}) \\ \qquad \cdots P_C(I - \lambda_{2,n} A_2) P_C(I - \lambda_{1,n} A_1) u_n, \\ k_n = \delta_n z_n + (1 - \delta_n) S^n z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n k_n + [(1 - \beta_n) I - \alpha_n \mu F] W_n k_n, \quad \forall n \ge 0, \end{cases}
$$

where $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, $i \in \{1, 2, \dots, N\}, k \in \{1, 2, \dots, M\}$. Assume that the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges weakly to $w = \lim_{n\to\infty} P_\Omega x_n$.

Proof. First, let us show that $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in \Omega$. Put

$$
\Delta_n^k = T_{r_{k,n}}^{(\Theta_k, \varphi_k)}(I - r_{k,n} B_k) T_{r_{k-1,n}}^{(\Theta_{k-1}, \varphi_{k-1})}(I - r_{k-1,n} B_{k-1}) \cdots T_{r_{1,n}}^{(\Theta_1, \varphi_1)}(I - r_{1,n} B_1) x_n,
$$

for all $k \in \{1, 2, \dots, M\}, n \ge 1$,

$$
\Lambda_n^i = P_C(I - \lambda_{i,n} A_i) P_C(I - \lambda_{i-1,n} A_{i-1}) \cdots P_C(I - \lambda_{1,n} A_1)
$$

for all $i \in \{1, 2, \dots, N\}, n \ge 1$, $\Delta_n^0 = \Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta_n^M x_n$ and $z_n = \Lambda_n^N u_n$. Take $p \in \Omega$ arbitrarily. Similarly to the proof of Theorem 3.1, we obtain that

$$\|u_n - p\| \leq \|x_n - p\|, \tag{4.2}$$

$$\|z_n - p\| \leq \|u_n - p\|, \tag{4.3}$$

$$\|\Delta_n^k x_n - p\|^2 \leq \|x_n - p\|^2 + r_{k,n}(r_{k,n} - 2\mu_k)\|B_k \Delta_n^{k-1} x_n - B_k p\|^2, \quad k \in \{1, 2, \ldots, M\}, \tag{4.4}$$

$$\begin{aligned}\|\Delta_n^k x_n - p\|^2 \leq{}& \|x_n - p\|^2 - \|\Delta_n^{k-1} x_n - \Delta_n^k x_n\|^2 \\ &+2r_{k,n}\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\|\|B_k \Delta_n^{k-1} x_n - B_k p\|, \quad k = 1, 2, \ldots, M,\end{aligned} \tag{4.5}$$

$$\|\Lambda_n^i u_n - p\|^2 \leq \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|A_i \Lambda_n^{i-1} u_n - A_i p\|^2, \quad i \in \{1, 2, \ldots, N\}, \tag{4.6}$$

$$\begin{aligned}\|\Lambda_n^i u_n - p\|^2 \leq{}& \|x_n - p\|^2 - \|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\|^2 \\ &+2\lambda_{i,n}\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\|\|A_i \Lambda_n^{i-1} u_n - A_i p\|, \quad i \in \{1, 2, \ldots, N\}.\end{aligned} \tag{4.7}$$

We observe that

$$\begin{aligned}\|k_n - p\|^2 &= \|\delta_n(z_n - p) + (1 - \delta_n)(S^n z_n - p)\|^2 \\ &= \delta_n\|z_n - p\|^2 + (1 - \delta_n)\|S^n z_n - p\|^2 - \delta_n(1 - \delta_n)\|z_n - S^n z_n\|^2 \\ &\leq \delta_n\|z_n - p\|^2 + (1 - \delta_n)[(1 + \gamma_n)\|z_n - p\|^2 + \kappa\|z_n - S^n z_n\|^2 + c_n] \\ &\quad -\delta_n(1 - \delta_n)\|z_n - S^n z_n\|^2 \\ &= [1 + \gamma_n(1 - \delta_n)]\|z_n - p\|^2 \\ &\quad +(1 - \delta_n)(\kappa - \delta_n)\|z_n - S^n z_n\|^2 + (1 - \delta_n)c_n \\ &\leq (1 + \gamma_n)\|z_n - p\|^2 + (1 - \delta_n)(\kappa - \delta_n)\|z_n - S^n z_n\|^2 + c_n \\ &\leq (1 + \gamma_n)\|z_n - p\|^2 + c_n.\end{aligned} \tag{4.8}$$

Taking into account that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\alpha_n + \beta_n \leq 1$, we get $\alpha_n/(1 - \beta_n) \leq 1$ for all $n \geq 1$. Then by Lemmas 2.2 and 2.19 we deduce from (4.2), (4.3) and (4.8) and $0 \leq \gamma l \leq \tau$ that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n\gamma(f(x_n) - \mu F p) + \beta_n(k_n - p) + ((1 - \beta_n)I - \alpha_n\mu F)W_n k_n \\ &\quad -((1 - \beta_n)I - \alpha_n\mu F)W_n p\|^2 \\ &= \|\alpha_n\gamma(f(x_n) - f(p)) + \beta_n(k_n - p) + ((1 - \beta_n)I - \alpha_n\mu F)W_n k_n \\ &\quad -((1 - \beta_n)I - \alpha_n\mu F)W_n p + \alpha_n(\gamma f(p) - \mu F p)\|^2 \\ &\leq \|\alpha_n\gamma(f(x_n) - f(p)) + \beta_n(k_n - p) + ((1 - \beta_n)I - \alpha_n\mu F)W_n k_n \\ &\quad -((1 - \beta_n)I - \alpha_n\mu F)W_n p\|^2 + 2\alpha_n\langle\gamma f(p) - \mu F p, x_{n+1} - p\rangle \\ &\leq \Big[\alpha_n\gamma\|f(x_n) - f(p)\| + \beta_n\|k_n - p\| \\ &\quad +(1 - \beta_n)\Big\|\Big(I - \frac{\alpha_n}{1 - \beta_n}\mu F\Big)W_n k_n - \Big(I - \frac{\alpha_n}{1 - \beta_n}\mu F\Big)W_n p\Big\|\Big]^2 \\ &\quad +2\alpha_n\|(\gamma f - \mu F)p\|\|x_{n+1} - p\|\end{aligned}$$

$$
\begin{aligned}
&\leq \Big[\alpha_n\gamma l\|x_n-p\|+\beta_n\|k_n-p\|+(1-\beta_n)\Big(1-\frac{\alpha_n\tau}{1-\beta_n}\Big)\Big\|k_n-p\|\Big]^2\\
&\quad +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|x_{n+1}-p\|^2)\\
&\leq [\alpha_n\tau\|x_n-p\|+\beta_n\|k_n-p\|+(1-\beta_n-\alpha_n\tau)\|k_n-p\|]^2\\
&\quad +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|x_{n+1}-p\|^2)\\
&= [\alpha_n\tau\|x_n-p\|+(1-\alpha_n\tau)\|k_n-p\|]^2\\
&\quad +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|x_{n+1}-p\|^2)\\
&\leq \alpha_n\tau\|x_n-p\|^2+(1-\alpha_n\tau)\|k_n-p\|^2\\
&\quad +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|x_{n+1}-p\|^2)\\
&\leq \alpha_n\tau\|x_n-p\|^2+(1-\alpha_n\tau)((1+\gamma_n)\|z_n-p\|^2+c_n)\\
&\quad +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|x_{n+1}-p\|^2)\\
&\leq \alpha_n\tau\|x_n-p\|^2+(1-\alpha_n\tau)((1+\gamma_n)\|x_n-p\|^2+c_n)\\
&\quad +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|x_{n+1}-p\|^2)\\
&= \|x_n-p\|^2+(1-\alpha_n\tau)\gamma_n\|x_n-p\|^2+(1-\alpha_n\tau)c_n\\
&\quad +\alpha_n(\|(\gamma f-\mu F)p\|^2+\|x_{n+1}-p\|^2)\\
&\leq (1+\gamma_n)\|x_n-p\|^2+c_n+\alpha_n(\|(\gamma f-\mu F)p\|^2+\|x_{n+1}-p\|^2),
\end{aligned}
$$

which hence yields

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \frac{1+\gamma_n}{1-\alpha_n}\|x_n-p\|^2+\frac{\alpha_n}{1-\alpha_n}\|(\gamma f-\mu F)p\|^2+\frac{1}{1-\alpha_n}c_n\\
&= \Big(1+\frac{\alpha_n+\gamma_n}{1-\alpha_n}\Big)\|x_n-p\|^2+\frac{\alpha_n}{1-\alpha_n}\|(\gamma f-\mu F)p\|^2+\frac{1}{1-\alpha_n}c_n \qquad (4.9)\\
&\leq [1+(\alpha_n+\gamma_n)\varrho]\|x_n-p\|^2+\alpha_n\varrho\|(\gamma f-\mu F)p\|^2+\varrho c_n,
\end{aligned}
$$

where $\varrho=\frac{1}{1-\sup_{n\geq 1}\alpha_n}<\infty$ (due to $\{\alpha_n\}\subset(0,1)$ and $\lim_{n\to\infty}\alpha_n=0$). From Lemma 2.10 and condition (i), we have that $\lim_{n\to\infty}\|x_n-p\|$ exists. Thus $\{x_n\}$ is bounded and so are the sequences $\{u_n\},\{z_n\}$ and $\{k_n\}$.

Also, utilizing Lemmas 2.2 and 2.3 (b) we obtain from (4.2), (4.3) and (4.8) that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(\gamma f(x_n)-\mu FW_nk_n)+\beta_n(k_n-p)+(1-\beta_n)(W_nk_n-p)\|^2\\
&\leq \|\beta_n(k_n-p)+(1-\beta_n)(W_nk_n-p)\|^2\\
&\quad +2\alpha_n\langle\gamma f(x_n)-\mu FW_nk_n,x_{n+1}-p\rangle\\
&= \beta_n\|k_n-p\|^2+(1-\beta_n)\|W_nk_n-p\|^2-\beta_n(1-\beta_n)\|k_n-W_nk_n\|^2\\
&\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|\\
&\leq \beta_n\|k_n-p\|^2+(1-\beta_n)\|k_n-p\|^2-\beta_n(1-\beta_n)\|k_n-W_nk_n\|^2\\
&\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\| \qquad (4.10)\\
&= \|k_n-p\|^2-\beta_n(1-\beta_n)\|k_n-W_nk_n\|^2\\
&\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|\\
&\leq (1+\gamma_n)\|z_n-p\|^2+c_n-\beta_n(1-\beta_n)\|k_n-W_nk_n\|^2\\
&\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|
\end{aligned}
$$

$$\leq (1+\gamma_n)\|x_n-p\|^2+c_n-\beta_n(1-\beta_n)\|k_n-W_nk_n\|^2 +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|,$$

which leads to

$$\begin{aligned}\beta_n(1-\beta_n)\|k_n-W_nk_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\gamma_n\|x_n-p\|^2+c_n\\ &\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\gamma_n=0$ and $\lim_{n\to\infty}c_n=0$, it follows from the existence of $\lim_{n\to\infty}\|x_n-p\|$ and condition (ii) that

$$\lim_{n\to\infty}\|k_n-W_nk_n\|=0. \tag{4.11}$$

Note that

$$x_{n+1}-k_n=\alpha_n(\gamma f(x_n)-\mu FW_nk_n)+(1-\beta_n)(W_nk_n-k_n),$$

which yields

$$\begin{aligned}\|x_{n+1}-k_n\| &\leq \alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|+(1-\beta_n)\|W_nk_n-k_n\|\\ &\leq \alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|+\|W_nk_n-k_n\|.\end{aligned}$$

So, from (4.11) and $\lim_{n\to\infty}\alpha_n=0$, we get

$$\lim_{n\to\infty}\|x_{n+1}-k_n\|=0. \tag{4.12}$$

In the meantime, we conclude from (4.2), (4.3), (4.8) and (4.10) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|k_n-p\|^2-\beta_n(1-\beta_n)\|k_n-W_nk_n\|^2\\ &\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|\\ &\leq \|k_n-p\|^2+2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|\\ &\leq (1+\gamma_n)\|z_n-p\|^2+(1-\delta_n)(\kappa-\delta_n)\|z_n-S^nz_n\|^2+c_n\\ &\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|\\ &\leq (1+\gamma_n)\|x_n-p\|^2+(1-\delta_n)(\kappa-\delta_n)\|z_n-S^nz_n\|^2+c_n\\ &\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|,\end{aligned}$$

which together with $0<k+\epsilon\leq\delta_n\leq d<1$, implies that

$$\begin{aligned}(1-d)\epsilon\|z_n-S^nz_n\|^2 &\leq (1-\delta_n)(\kappa-\delta_n)\|z_n-S^nz_n\|^2\\ &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\gamma_n\|x_n-p\|^2+c_n\\ &\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|.\end{aligned}$$

Consequently, from $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\gamma_n=0$, $\lim_{n\to\infty}c_n=0$ and the existence of $\lim_{n\to\infty}\|x_n-p\|$, we get

$$\lim_{n\to\infty}\|z_n-S^nz_n\|=0. \tag{4.13}$$

Since $k_n-z_n=(1-\delta_n)(S^nz_n-z_n)$, from (4.13) we have

$$\lim_{n\to\infty}\|k_n-z_n\|=0. \tag{4.14}$$

Combining (4.4), (4.8) and (4.10), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|k_n-p\|^2+2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|\\ &\leq \|z_n-p\|^2+\gamma_n\|z_n-p\|^2+c_n\end{aligned}$$

$$
\begin{aligned}
& +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\| \\
\leq\ & \|u_n-p\|^2+\gamma_n\|z_n-p\|^2+c_n \\
& +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\| \\
\leq\ & \|\Delta_n^k x_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n \\
& +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\| \\
\leq\ & \|x_n-p\|^2+r_{k,n}(r_{k,n}-2\mu_k)\|B_k\Delta_n^{k-1}x_n-B_kp\|^2 \\
& +\gamma_n\|x_n-p\|^2+c_n+2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|,
\end{aligned}
$$

which implies

$$
\begin{aligned}
r_{k,n}(2\mu_k-r_{k,n})\|B_k\Delta_n^{k-1}x_n-B_kp\|^2 \leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
& +\gamma_n\|x_n-p\|^2+c_n \\
& +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|.
\end{aligned}
$$

From $\{r_{k,n}\}\subset[e_k,f_k]\subset(0,2\mu_k),\ k\in\{1,2,\ldots,M\},\ \lim_{n\to\infty}\alpha_n=0,\ \lim_{n\to\infty}\gamma_n=0,\ \lim_{n\to\infty}c_n=0$ and the existence of $\lim_{n\to\infty}\|x_n-p\|$, we get

$$
\lim_{n\to\infty}\|B_k\Delta_n^{k-1}x_n-B_kp\|=0. \tag{4.15}
$$

Combining (4.5), (4.8) and (4.10), we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq\ & \|\Delta_n^k x_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n \\
& +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\| \\
\leq\ & \|x_n-p\|^2-\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\|^2 \\
& +2r_{k,n}\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\|\|B_k\Delta_n^{k-1}x_n-B_kp\| \\
& +\gamma_n\|x_n-p\|^2+c_n+2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\|^2 \leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
& +2r_{k,n}\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\|\|B_k\Delta_n^{k-1}x_n-B_kp\| \\
& +\gamma_n\|x_n-p\|^2+c_n+2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|.
\end{aligned}
$$

From (4.15), $\lim_{n\to\infty}\alpha_n=0,\ \lim_{n\to\infty}\gamma_n=0,\ \lim_{n\to\infty}c_n=0$ and the existence of $\lim_{n\to\infty}\|x_n-p\|$, we obtain

$$
\lim_{n\to\infty}\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\|=0,\quad k\in\{1,2,\ldots,M\}. \tag{4.16}
$$

From (4.16), we have

$$
\begin{aligned}
\|x_n-u_n\| =\ & \|\Delta_n^0 x_n-\Delta_n^M x_n\| \\
\leq\ & \|\Delta_n^0 x_n-\Delta_n^1 x_n\|+\|\Delta_n^1 x_n-\Delta_n^2 x_n\| \\
& +\cdots+\|\Delta_n^{M-1}x_n-\Delta_n^M x_n\| \\
& \to 0\quad \text{as } n\to\infty.
\end{aligned} \tag{4.17}
$$

Combining (4.6), (4.8) and (4.10), we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq\ & \|k_n-p\|^2+2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\| \\
\leq\ & \|z_n-p\|^2+\gamma_n\|z_n-p\|^2+c_n
\end{aligned}
$$

$$\begin{aligned}
&\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\| \\
&\leq \|\Lambda_n^i u_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n \\
&\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\| \\
&\leq \|x_n-p\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|A_i\Lambda_n^{i-1}u_n-A_ip\|^2 \\
&\quad +\gamma_n\|x_n-p\|^2+c_n+2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|,
\end{aligned}$$

where $i\in\{1,2,\dots,N\}$, which implies

$$\begin{aligned}
\lambda_{i,n}(2\eta_i-\lambda_{i,n})\|A_i\Lambda_n^{i-1}u_n-A_ip\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +\gamma_n\|x_n-p\|^2+c_n \\
&\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|.
\end{aligned}$$

From $\{\lambda_{i,n}\}\subset[a_i,b_i]\subset(0,2\eta_i), i\in\{1,2,\dots,N\}$, $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\gamma_n=0$, $\lim_{n\to\infty}c_n=0$ and the existence of $\lim_{n\to\infty}\|x_n-p\|$, we obtain

$$\lim_{n\to\infty}\|A_i\Lambda_n^{i-1}u_n-A_ip\|=0,\quad i\in\{1,2,\dots,N\}. \tag{4.18}$$

Combining (4.7), (4.8) and (4.10), we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|k_n-p\|^2+2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\| \\
&\leq \|z_n-p\|^2+\gamma_n\|z_n-p\|^2+c_n \\
&\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\| \\
&\leq \|\Lambda_n^i u_n-p\|^2+\gamma_n\|x_n-p\|^2+c_n \\
&\quad +2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\| \\
&\leq \|x_n-p\|^2-\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\|^2 \\
&\quad +2\lambda_{i,n}\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\|\|A_i\Lambda_n^{i-1}u_n-A_ip\| \\
&\quad +\gamma_n\|x_n-p\|^2+c_n+2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|,
\end{aligned}$$

which implies

$$\begin{aligned}
\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +2\lambda_{i,n}\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\|\|A_i\Lambda_n^{i-1}u_n-A_ip\| \\
&\quad +\gamma_n\|x_n-p\|^2+c_n+2\alpha_n\|\gamma f(x_n)-\mu FW_nk_n\|\|x_{n+1}-p\|.
\end{aligned}$$

From (4.18), $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\gamma_n=0$, $\lim_{n\to\infty}c_n=0$ and the existence of $\lim_{n\to\infty}\|x_n-p\|$, we obtain

$$\lim_{n\to\infty}|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\|=0,\quad i\in\{1,2,\dots,N\}. \tag{4.19}$$

By (4.19), we have

$$\begin{aligned}
\|u_n-z_n\| &= \|\Lambda_n^0u_n-\Lambda_n^Nu_n\| \\
&\leq \|\Lambda_n^0u_n-\Lambda_n^1u_n\|+\|\Lambda_n^1u_n-\Lambda_n^2u_n\| \\
&\quad +\cdots+\|\Lambda_n^{N-1}u_n-\Lambda_n^Nu_n\| \\
&\to 0\quad \text{as } n\to\infty.
\end{aligned} \tag{4.20}$$

From (4.17) and (4.20), we have

$$\|x_n-z_n\| \leq \|x_n-u_n\|+\|u_n-z_n\|$$

$$\rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{4.21}$$

By (4.14) and (4.21), we obtain

$$\begin{aligned} \|k_n - x_n\| &\leq \|k_n - z_n\| + \|z_n - x_n\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \tag{4.22}$$

which together with (4.12) and (4.22), implies that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - k_n\| + \|k_n - x_n\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \tag{4.23}$$

On the other hand, we observe that

$$\|z_{n+1} - z_n\| \leq \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\|.$$

By (4.21) and (4.23), we have

$$\lim_{n\rightarrow\infty} \|z_{n+1} - z_n\| = 0. \tag{4.24}$$

We note that

$$\begin{aligned} \|z_n - Sz_n\| \leq\ & \|z_n - z_{n+1}\| + \|z_{n+1} - S^{n+1}z_{n+1}\| \\ & + \|S^{n+1}z_{n+1} - S^{n+1}z_n\| + \|S^{n+1}z_n - Sz_n\|. \end{aligned}$$

From (4.13), (4.24), Lemma 2.5 and the uniform continuity of S, we obtain

$$\lim_{n\rightarrow\infty} \|z_n - Sz_n\| = 0. \tag{4.25}$$

In addition, note that

$$\|k_n - Wk_n\| \leq \|k_n - W_nk_n\| + \|W_nk_n - Wk_n\|$$

So, from (4.11) and Remark 2.3 it follows that

$$\lim_{n\rightarrow\infty} \|k_n - Wk_n\| = 0. \tag{4.26}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to w. From (4.21) and (4.22), we have that $z_{n_i} \rightharpoonup w$ and $k_{n_i} \rightharpoonup w$. From (4.25) and the uniform continuity of S, we have $\lim_{n\rightarrow\infty} \|z_n - S^m z_n\| = 0$ for any $m \geq 1$. So, from Lemma 2.7, we have $w \in \text{Fix}(S)$. In the meantime, by (4.26) and Lemma 2.16, we get $w \in \text{Fix}(W) = \cap_{n=1}^{\infty}\text{Fix}(T_n)$. Utilizing the similar arguments to those in the proof of Theorem 3.1, we can derive $w \in \cap_{k=1}^{M}\text{GMEP}(\Theta_k, \varphi_k, B_k) \cap \cap_{i=1}^{N}\text{VI}(C, A_i)$. Consequently, $w \in \Omega$. This shows that $\omega_w(x_n) \subset \Omega$.

Next let us show that $\omega_w(x_n)$ is a single-point set. As a matter of fact, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w'$. Then we get $w' \in \Omega$. If $w \neq w'$, from the Opial condition, we have

$$\begin{aligned} \lim_{n\rightarrow\infty} \|x_n - w\| &= \lim_{i\rightarrow\infty} \|x_{n_i} - w\| < \lim_{i\rightarrow\infty} \|x_{n_i} - w'\| \\ &= \lim_{n\rightarrow\infty} \|x_n - w'\| = \lim_{j\rightarrow\infty} \|x_{n_j} - w'\| \\ &< \lim_{j\rightarrow\infty} \|x_{n_j} - w\| = \lim_{n\rightarrow\infty} \|x_n - w\|. \end{aligned}$$

This attains a contradiction. So we have $w = w'$. Put $v_n = P_\Omega x_n$. Since $w \in \Omega$, we have $\langle x_n - v_n, v_n - w\rangle \geq 0$. By Lemma 2.12, we have that $\{v_n\}$ converges strongly to some $w_0 \in \Omega$. Since $\{x_n\}$ converges weakly to w, we have

$$\langle w - w_0, w_0 - w\rangle \geq 0.$$

Therefore we obtain $w = w_0 = \lim_{n\to\infty} P_\Omega x_n$. This completes the proof. □

Corollary 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $B : H \to H$ and $A_i : C \to H$ be ζ-inverse strongly monotone and η_i-inverse-strongly monotone, respectively, for $i = 1, 2$. Let $S : C \to C$ be a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense for some $0 \leq k < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$ and $\{c_n\} \subset [0, \infty)$. Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $f : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := \cap_{n=1}^\infty \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, B) \cap \mathrm{VI}(C, A_2) \cap \mathrm{VI}(C, A_1) \cap \mathrm{Fix}(S)$ is nonempty and that either (B1) or (B2) holds. Let W_n be the W-mapping defined by (2.2), and $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n \leq 1$ $(\forall n \geq 1)$ and $0 < k + \epsilon \leq \delta_n \leq d < 1$. Pick any $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$(4.27)\qquad \begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle \\ \qquad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1)u_n, \\ k_n = \delta_n z_n + (1 - \delta_n)S^n z_n, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n k_n + [(1 - \beta_n)I - \alpha_n\mu F]W_n k_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\{r_n\} \subset [e, f] \subset (0, 2\zeta)$ for $i = 1, 2$. Assume that the following conditions are satisfied:

(i) $\sum_{n=1}^\infty \alpha_n < \infty$, $\sum_{n=1}^\infty \gamma_n < \infty$ *and* $\sum_{n=1}^\infty c_n < \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges weakly to $w = \lim_{n\to\infty} P_\Omega x_n$.

Corollary 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $B : H \to H$ and $A : C \to H$ be ζ-inverse strongly monotone and ξ-inverse-strongly monotone, respectively. Let $S : C \to C$ be a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense for some $0 \leq k < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$ and $\{c_n\} \subset [0, \infty)$. Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $f : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := \cap_{n=1}^\infty \mathrm{Fix}(T_n) \cap$*

GMEP$(\Theta, \varphi, B) \cap$VI$(C, A) \cap$Fix$(S)$ *is nonempty and that either (B1) or (B2) holds. Let* W_n *be the* W*-mapping defined by (2.2), and* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\delta_n\}$ *be sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n \leq 1$ $(\forall n \geq 1)$ *and* $0 < k + \epsilon \leq \delta_n \leq d < 1$*. Pick any* $x_1 \in H$ *and let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle \\ \qquad + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ k_n = \delta_n P_C(I - \rho_n A) u_n + (1 - \delta_n) S^n P_C (I - \rho_n A) u_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n k_n + [(1 - \beta_n) I - \alpha_n \mu F] W_n k_n, \quad \forall n \geq 0, \end{cases} \tag{4.28}$$

where $\{\rho_n\} \subset [a, b] \subset (0, 2\xi)$*,* $\{r_n\} \subset [e, f] \subset (0, 2\zeta)$*. Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$*;*
(ii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$.

Then $\{x_n\}$ *converges weakly to* $w = \lim_{n \to \infty} P_\Omega x_n$*.*

References

[1] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 197–228.

[2] R. E. Bruck, T. Kuczumow and S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property*, Colloq. Math. **65** (1993), 169–179.

[3] G. Cai and S. Q. Bu, *Strong and weak convergence theorems for general mixed equilibrium problems and variational inequality problems and fixed point problems in Hilbert spaces*, J. Comput. Appl. Math. **247** (2013), 34–52.

[4] L. C. Ceng, Q. H. Ansari and S. Schaible, *Hybrid extragradient-like methods for generalized mixed equilibrium problems, system of generalized equilibrium problems and optimization problems*, J. Glob. Optim. **53** (2012), 69–96.

[5] L. C. Ceng, Q. H. Ansari, M. M. Wong and J. C. Yao, *Mann type hybrid extragradient method for variational inequalities, variational inclusions and fixed point problems*, Fixed Point Theory **13** (2012), 403–422.

[6] L. C. Ceng, S. M. Guu and J. C. Yao, *Hybrid iterative method for finding common solutions of generalized mixed equilibrium and fixed point problems*, Fixed Point Theory Appl. 2012, 2012:92, 19pp.

[7] L. C. Ceng, N. Hadjisavvas, N. C. Wong, *Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems*, J. Glob. Optim. **46** (2010), 635–646.

[8] L. C. Ceng, H. Y. Hu and M. M. Wong, *Strong and weak convergence theorems for generalized mixed equilibrium problem with perturbation and fixed point problem of infinitely many nonexpansive mappings*, Taiwanese J. Math. **15** (2011), 1341–1367.

[9] L. C. Ceng and A. Petrusel, *Relaxed extragradient-like method for general system of generalized mixed equilibria and fixed point problem*, Taiwanese J. Math. **16** (2012), 445–478.

[10] L. C. Ceng, A. Petrusel and J. C. Yao, *Iterative approaches to solving equilibrium problems and fixed point problems of infinitely many nonexpansive mappings*, J. Optim. Theory Appl. **143** (2009), 37–58.

[11] L. C. Zeng and J. C. Yao, *Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems*, Taiwanese J. Math. **10** (2006), 1293–1303.

[12] L. C. Ceng and J. C. Yao, *An extragradient-like approximation method for variational inequality problems and fixed point problems*, Appl. Math. Comput. **190** (2007), 205–215.

[13] L. C. Ceng and J. C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math. **214** (2008), 186–201.

[14] L. C. Ceng and J. C. Yao, *A relaxed extragradient-like method for a generalized mixed equilibrium problem, a general system of generalized equilibria and a fixed point problem*, Nonlinear Anal. **72** (2010), 1922–1937.

[15] L. C. Ceng and J. C. Yao, *Strong convergence theorems for variational inequalities and fixed point problems of asymptotically strict pseudocontractive mappings in the intermediate sense*, Acta Appl. Math. **115** (2011), 167–191.

[16] V. Colao, G. Marino and H. K. Xu, *An iterative method for finding common solutions of equilibrium and fixed point problems*, J. Math. Anal. Appl. **344** (2008), 340–352.

[17] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Volume I and Volume II, Springer-Verlag, New York, 2003.

[18] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174.

[19] K. Goebel and W. A. Kirk, *Topics on Metric Fixed-Point Theory*, Cambridge University Press, Cambridge, 1990.

[20] S. Huang, *Hybrid extragradient methods for asymptotically strict pseudo-contractions in the intermediate sense and variational inequality problems*, Optimization **60** (2011), 739–754.

[21] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.

[22] J. Gornicki, *Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces*, Comment. Math. Univ. Carolin. **30** (1989), 249–252.

[23] T. H. Kim and H. K. Xu, *Convergence of the modified Mann's iteration method for asymptotically strict pseudocontractions*, Nonlinear Anal. **68** (2008), 2828–2836.

[24] G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Matecon. **12** (1976), 747–756.

[25] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[26] G. Marino and H. K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl. **329** (2007), 336–346.

[27] C. Martinez-Yanes and H. K. Xu, *Strong convergence of the CQ method for fixed point iteration process*, Nonlinear Anal. **64** (2006), 2400–2411.

[28] N. Nadezhkina and W. Takahashi, *Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **128** (2006), 191–201.

[29] N. Nadezhkina and W. Takahashi, *Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings*, SIAM J. Optim. **16** (2006), 1230–1241.

[30] J. T. Oden, *Quantitative Methods on Nonlinear Mechanics*, Prentice-Hall, Englewood Cliffs, NJ 1986.

[31] J. G. O'Hara, P. Pillay and H. K. Xu, *Iterative approaches to convex feasibility problems in Banach spaces*, Nonlinear Anal. **64** (2006), 2022–2042.

[32] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[33] M. O. Osilike, S. C. Aniagbosor and B. G. Akuchu, *Fixed points of asymptotically demicontractive mappings in arbitrary Banach space*, Panamer. Math. J. **12** (2002), 77–88.

[34] J. W. Peng and J. C. Yao, *A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems*, Taiwanese J. Math. **12** (2008), 1401–1432.

[35] R. T. Rockafellar, *Monotone operators and the proximal point algorithms*, SIAM J. Control Optim. **14** (1976), 877–898.

[36] D. R. Sahu, H. K. Xu, J. C. Yao,*Asymptotically strict pseudocontractive mappings in the intermediate sense*, Nonlinear Anal. **70** (2009), 3502–3511.

[37] S. Takahashi and W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space*, Nonlinear Anal. **69** (2008), 1025–1033.

[38] W. Takahashi, *Nonlinear Functional Analysis, –Fixed Point Thoery and its Applications–*, Yokohama Publishers, Yokohama, Japan, 2000.

[39] K. K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), 301–308.

[40] H. K. Xu, *Existence and convergence for fixed points for mappings of asymptotically nonexpansive type*, Nonlinear Anal. **16** (1991), 1139–1146.

[41] H. K. Xu and T. H. Kim, *Convergence of hybrid steepest-descent methods for variational inequalities*, J. Optim. Theory. Appl. **119** (2003), 185–201.

[42] I. Yamada, *The hybrid steepest-descent method for the variational inequality problems over the intersection of the fixed-point sets of nonexpansive mappings*, in: Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, D. Batnariu, Y. Censor, and S. Reich (eds.), Amsterdam, North-Holland, Holland, 2001, pp. 473–504.

[43] Y. Yao, Y.C. Liou and J. C. Yao, *Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings*, Fixed Point Theory Appl. 2007 (2007) Article ID 64363, 12pp.

[44] Y. Yao, H. Zhou, Y. C. Liou, *Weak and strong convergence theorems for an asymptotically k-strict pseudocontraction and a mixed equilibrium problem*, J. Korean Math. Soc. **46** (2009), 561–576.

[45] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, Springer-Verlag, New York, 1985.

Manuscript received February 18, 2014
revised November 25, 2015

Lu-Chuan Ceng
Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
E-mail address: `zenglc@hotmail.com`

Chi-Ming Chen
Department of Applied Mathematics, National Hsinchu University of Education, Hsinchu 30033, Taiwan
E-mail address: `ming@mail.nhcue.edu.tw`

Ching-Feng Wen
Center for Fundamental Science and Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung 807, Taiwan
E-mail address: `cfwen@kmu.edu.tw`