Journal of Nonlinear and Convex Analysis
Volume 16, Number 12, 2015, 2401–2417

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

LEVITIN-POLYAK WELL-POSEDNESS FOR PARAMETRIC GENERALIZED QUASIVARIATIONAL INEQUALITY PROBLEM OF THE MINTY TYPE

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

Abstract. In this paper, we introduce the notions of Levitin-Polyak (LP) well-posedness and Levitin-Polyak well-posedness in the generalized sense, for a parametric generalized quasivariational inequality problem of the Minty type. Metric characterizations of well-posedness and generalized LP well-posedness, in terms of the approximate solution sets are presented. Numerous examples are provided to explain that all the assumptions we impose are very relaxed and cannot be dropped.

1. Introduction

Let K be a nonempty, closed and convex set of $\mathbb{R}^n$. For the given mapping $T : \mathbb{R}^n \to \mathbb{R}$, the classical Minty variational inequality problem [28] is to find $x \in K$, such that

$$\langle T(y), x - y\rangle \leq 0, \ \ \forall y \in K.$$

The relevance of Minty problem to applications was studied in [5, 8]. It was observed that Minty variational inequality is a sufficient optimality condition for the minimization problem

$$\underset{x\in K}{\text{minimize}} f(x)$$

under the assumption that there exists a convex and differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, such that $T(x) = \nabla f(x), \ \forall x \in \mathbb{R}^n$. John [18] considered the Minty problem with a single valued mapping and proved that the Minty solutions can be considered as a subset of the stable equilibria within the set of all equilibria of a dynamical system associated with the map.

It is well known that the well-posedness is very important for both optimization theory and numerical methods of optimization problems, which guarantees that, for approximating solution sequences, there is a subsequence which converges to a solution. The study of well-posedness originates from Tikhonov [31], which means the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Levitin-Polyak [22] introduced a new notion of well-posedness that strengthened Tykhonov's concept as it required the convergence to

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2010 *Mathematics Subject Classification.* 47H05, 47H10, 47J25.

Key words and phrases. Levitin-Polyak Well-posedness, parametric generalized quasivariational inequality problem, metric characterizations.

The authors would like to thank The Thailand Research Fund and Naresuan university for financial support.

the optimal solution of each sequence belonging to a larger set of minimizing sequences. Subsequently, some authors studied the Levitin-Polyak well-posedness for convex optimization problems with functional constraints (Konsulova and Revalski [19]), general constrained nonconvex optimization problems (Huang and Yang [15]), general constrained vector optimization problems (Huang and Yang [16]) and generalized variational inequality problems with functional constraints (Huang and Yang [17]).

The notion of well-posedness for variational inequality problems was introduced by Lucchetti and Patrone [27] based on the fact that an optimization problem of minimizing a function can be formulated as a variational inequality problem involving the derivative of the objective. After that several researchers [10, 11, 12, 24, 23, 26] have explored the forms of well-posedness for various forms of variational inequality problems. Recently, Hu and Fang [11] studied LP well-posedness of a classical variational inequality problems. Huang [14] studied various types of LP well-posedness for scalar and vector optimization problems with functional constraints with applications to the convergence analysis of augmented Lagrangian methods and penalty methods for constrained scalar or vector optimization problems. Lalitha and Bhatia [21] performed and extended parametric quasivariational inequality of the Stampacchia type, the study of well-posedness and generalized well-posedness to an optimization problem with quasivariational inequality constraints. Fang and Hu [10] considered notions of well-posedness for both Stampacchia and Minty variational inequalities in terms of bifunction which were further extended for parametric quasivariational inequalities by Hu et al. [12]. Recently, Lalitha and Bhatia [20] proposed the notions of LP well-posedness and LP well-posedness in the generalized sense, for a parametric quasivariational inequality problem of the Minty type. Further, they studied the metric characterizations of LP well-posedness and generalized LP well-posedness, in terms of the approximate solution sets. In [25], Luc and Tan introduced and studied the existence results for a general variational inclusion problem with constraints which can deduce to the variational inclusion of Minty type (VIM). Further, a parametric generalized quasivariational inequality of the Minty type (for short GMVI) can be viewed as a special case for VIM. However, to the best of our knowledge, there is no a result concerning the LP well-posedness for GMVI. It is natural to raise and give an answer to the following conjecture :

Conjecture : Can one give some criteria and characterizations of the LP well-posedness for GMVI ?

Inspired and motivated by researches going on this direction, the aim of this paper is to give positive answers to the above question. We first give the notions of Levitin-Polyak (LP) well-posedness and Levitin-Polyak well-posedness in the generalized sense, for a parametric generalized quasivariational inequality problem of the Minty type. We establish some metric characterizations of well-posedness and generalized LP well-posedness, in terms of the approximate solution sets. The paper organization is described below.

In Section 2, the concept of LP well-posedness for a parametric generalized quasivariational inequality problem of the Minty type and present metric characterizations for LP well-posedness in terms of the approximate solution sets is introduced.

Under suitable conditions, including a compactness condition, it is shown that LP well-posedness of the problem is analogous to the existence and uniqueness of its solution. In Section 3, the notion of generalized LP well-posedness for parametric generalized quasivariational inequalities possessing more than one solution is introduced. We propose various characterizations for generalized LP well-posedness and sufficient conditions for generalized LP well-posedness in this section.

The following well known notions for the set-valued maps are required throughout the paper. For a set-valued map $F : X \rightrightarrows Y$ the domain of F, denoted by dom F, is given as

$$\text{dom}F = \{x \in X : F(x) \neq \emptyset\}.$$

Definition 1.1. The set-valued map F is said to be

(1) upper semicontinuous (usc) at $x \in \text{dom } F$ if for any open set U satisfying $F(x) \subset U$, there exists a $\delta > 0$ such that $F(y) \subset U$, for every $y \in B(x, \delta)$;
(2) lower semicontinuous (lsc) at $x \in \text{dom } F$ if for any open set U satisfying $F(x) \cap U \neq \emptyset$, there exists a $\delta > 0$ such that $F(y) \cap U \neq \emptyset$, for every $y \in B(x, \delta)$;
(3) closed at $x \in \text{dom } F$ if for each sequence $\{x_n\} \subseteq X$ converging to x and $\{y_n\}$ in Y converging to y such that $y_n \in F(x_n)$, we have $y \in F(x)$.

If $S \subseteq X$, then F is said to be usc (lsc, closed respectively) on the set S if F is usc (lsc, closed respectively) at every $x \in \text{dom } F \cap S$.

Remark 1.2. An equivalent formulation of Definition 1.1(ii) is as follows: F is said to be lsc at $x \in \text{dom } F$ if for each sequence $\{x_n\} \subseteq \text{dom } F$ converging to x and for any $y \in F(x)$, there exists a sequence $\{y_n\}$ in $F(x_n)$ converging to y.

Let A, B be two subsets of a metric space X. The Hausdorff distance between A and B is defined as follows

$$H(A, B) = \max\{H^*(A, B), H^*(B, A)\},$$

where $H^*(A, B) = \sup_{a \in A} d(a, B)$, and $d(x, A) = \inf_{y \in A} d(x, y)$.

2. LP well-posedness for the parametric generalized quasivariational inequality problem of the Minty type

Let X be a nonempty closed subset of $\mathbb{R}^n$ and A be nonempty closed convex subset of $\mathbb{R}^m$. Let $K_1, K_2 : X \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ and $T : X \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ be set-valued maps with K_1, K_2 being closed valued maps. We assume throughout that domK_1 = domK_2 = dom$T$ = $X \times \mathbb{R}^m$.

We consider the following *the generalized quasivariational inequality problem of the Minty type*, corresponding to a parameter $\bar{x} \in X$.

$$(\text{GMVI}(\bar{x})) \quad \begin{cases} \text{Find } \bar{u} \in K_1(\bar{x}, \bar{u}) \cap A \text{ such that} \\ \langle t, \bar{u} - v \rangle \leq 0, \quad \forall v \in K_2(\bar{x}, \bar{u}), \ \forall t \in T(\bar{x}, v). \end{cases}$$

In [25], Luc and Tan studied the existence of solutions to the variational inclusion of the Minty type, defined in terms of set-valued maps with nonempty values $\phi : B \times A \times A \rightrightarrows Z, T' : A \times A \rightrightarrows B$, where X, Y, Z are Hausdorff topological vector spaces and $A \subset X, B \subset Y$ are nonempty sets. Let $S_1, S_2 : A \rightrightarrows A$ be set-valued maps with nonempty values. They considered the following variational inclusion of Minty type;

$$\text{(VIM)} \begin{cases} \text{Find } \bar{u} \in S_1(\bar{u}) \cap A \text{ such that} \\ \phi(t, v, \bar{u}) \subseteq \phi(t, \bar{u}, \bar{u}) + C, \quad \forall v \in S_2(\bar{u}),\ \forall t \in T'(v, v), \end{cases}$$

where $\emptyset \neq C \subset Z$ is a cone. If $Z = \mathbb{R}$, $C = \mathbb{R}_+$, $K_i(\bar{x}, \bar{u}) = S_i(\bar{u})$ for all $i \in \{1, 2\}, T(\bar{x}, v) = T'(v, v)$ and $\phi(t, v, \bar{u}) = \{\langle t, v - \bar{u}\rangle\}$ then (GMVI) is of the form considered by Luc and Tan [25] in the same setting of spaces.

When $K_1 = K_2 := K : X \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, the (GMVI) collapses to *the quasivariational inequality problem of the Minty type (MVI) corresponding to a parameter* $\bar{x} \in X$:

$$(\text{MVI}(\bar{x})) \begin{cases} \text{Find } \bar{u} \in K(\bar{x}, \bar{u}) \cap A \text{ such that} \\ \langle t, \bar{u} - v\rangle \leq 0, \quad \forall v \in K(\bar{x}, \bar{u}),\ \forall t \in T(\bar{x}, v). \end{cases}$$

In [20], Lalitha and Bhatia introduced the notions of Levitin-Polyak (LP) well-posedness and Levitin-Polyak well-posedness in the generalized sense, for (MVI). They obtained some sufficient conditions for a family of such problems to be LP well-posed at the reference point. In [12], Hu et al. considered a parametric Minty variational inequality problem, defined in terms of a function $h : X \times A \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$, where A is a nonempty subset of a Banach space Y, X is a parametric Banach space and $K : X \times A \rightrightarrows A$ is a set-valued map. The problem is to find $\bar{u} \in K(\bar{x}, \bar{u}) \cap A$ such that

$$h(\bar{x}, v, \bar{u} - v) \leq 0, \quad \forall v \in K(\bar{x}, \bar{u}).$$

If the map T is single valued and $h(\bar{x}, v, \bar{u} - v) = \langle t, \bar{u} - v\rangle$, then $(\text{MVI}(\bar{x}))$ is of the form considered by Hu et al. [12] in the same setting of spaces.

Let $GM(\bar{x})$ denote the solution set of $(\text{GMVI}(\bar{x}))$, that is,

$$GM(\bar{x}) := \{u \in K_1(\bar{x}, \bar{u}) \cap A : \langle t, u - v\rangle \leq 0, \quad \forall v \in K_2(\bar{x}, u),\ \forall t \in T(\bar{x}, v)\}.$$

For $\delta, \varepsilon \geq 0$, define the set of *approximation solutions* for the problem $(\text{GMVI}(\bar{x}))$ as

$$GM(\bar{x}, \delta, \varepsilon) := \bigcup_{x \in B(\bar{x}, \delta) \cap X} \left\{ \begin{array}{l} u \in \mathbb{R}^m : d(u, K_1(x, u) \cap A) \leq \varepsilon \text{ and } \langle t, u - v\rangle \leq \varepsilon, \\ \forall v \in K_2(x, u), \forall t \in T(x, v), \end{array} \right\}$$

where $B(\bar{x}, \delta)$ denotes the closed ball centered at $\bar{x}$ with radius δ. Observe that

$$GM(\bar{x}, 0, 0) = GM(\bar{x})$$

and

$$GM(\bar{x}) \subseteq GM(\bar{x}, \delta, \varepsilon), \quad \forall \delta, \varepsilon > 0.$$

Next, we present the sufficient conditions ensuring the closedness of the approximate solution set.

Proposition 2.1. *If the following conditions hold:*

(i) *K_1 is closed and K_2 is lsc on $X \times \mathbb{R}^m$;*
(ii) *T is lsc on $X \times \mathbb{R}^m$;*
(iii) *A is a compact subset of $\mathbb{R}^m$;*

then $GM(\bar{x}, \delta, \varepsilon)$ is closed, for all δ and $\varepsilon > 0$.

Proof. Suppose that there exist $\delta, \varepsilon > 0$ such that $GM(\bar{x}, \delta, \varepsilon)$ is not closed. Then there exists a sequence $u_n \in GM(\bar{x}, \delta, \varepsilon)$ with $u_n \to u'$ such that

$$u' \notin GM(\bar{x}, \delta, \varepsilon).$$

Since $u_n \in GM(\bar{x}, \delta, \varepsilon)$, there exists a sequence $x_n \subseteq B(\bar{x}, \delta) \cap X$ such that

$$d(u_n, K_1(x_n, u_n) \cap A) \le \varepsilon, \tag{2.1}$$

and

$$\langle t, u_n - v \rangle \le \varepsilon, \quad \forall v \in K_2(x_n, u_n), \ \forall t \in T(x_n, v). \tag{2.2}$$

Since $B(\bar{x}, \delta)$ is compact, we can assume that

$$x_n \to x' \in B(\bar{x}, \delta),$$

which further implies that $x' \in X$, as X is closed. Since $K_1(x_n, u_n) \cap A$ is a closed set in $\mathbb{R}^m$ and from (2.1), we can choose $z_n \in K_1(x_n, u_n) \cap A$ such that

$$\|u_n - z_n\| \le \varepsilon, \quad \forall n \in \mathbb{N}.$$

Since A is compact, without loss of generality, we can assume that

$$z_n \to z' \in A.$$

Since K_1 is closed at (x', u'), it follows that $z' \in K_1(x', u')$ and hence

$$d(u', K_1(x', u') \cap A) \le \|u' - z'\| = \lim_{n \to \infty} \|u_n - z_n\| \le \varepsilon. \tag{2.3}$$

Next, since K_2 is lsc at (x', u') and $(x_n, u_n) \to (x', u')$, it follows that for any $v \in K_2(x', u')$ there exists a sequence $v_n \in K_2(x_n, u_n)$, such that

$$v_n \to v, \text{ as } n \to \infty.$$

By assumption (ii), we have T is lsc at (x', v) and $(x_n, v_n) \to (x', v)$, then it follows that for any $t \in T(x', v)$, there exists a sequence $t_n \in T(x_n, v_n)$ such that

$$t_n \to t.$$

Taking limit as $n \to \infty$ in (2.2), we have

$$\langle t, u' - v \rangle \le \varepsilon, \text{ for all } t \in T(x', v). \tag{2.4}$$

By (2.3) and (2.4), we conclude that $u' \in GM(\bar{x}, \delta, \varepsilon)$, which leads to a contradiction, therefore $GM(\bar{x}, \delta, \varepsilon)$ is closed. $\square$

In general, $GM(\bar{x}) \subseteq GM(\bar{x}, \delta, \varepsilon), \ \forall \delta, \varepsilon > 0$ and hence

$$GM(\bar{x}) \subseteq \bigcap_{\delta, \varepsilon > 0} GM(\bar{x}, \delta, \varepsilon).$$

Next, we provide the sufficient conditions for the two sets to coincide.

Proposition 2.2. *If the following conditions hold:*

(i) *K_1 is closed and K_2 is lsc on $X \times \mathbb{R}^m$;*
(ii) *T is lsc on $X \times \mathbb{R}^m$;*
(iii) *A is a closed subset of $\mathbb{R}^m$,*

then

$$\bigcap_{\delta,\varepsilon>0} GM(\bar{x},\delta,\varepsilon) = GM(\bar{x}).$$

Proof. Let $\bar{u} \in \bigcap_{\delta,\varepsilon>0} GM(\bar{x},\delta,\varepsilon)$. Hence, there exist two sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ in $(0,\infty)$ such that $\delta_n \to 0, \varepsilon_n \to 0$ and

$$\bar{u} \in GM(\bar{x},\delta_n,\varepsilon_n), \text{ for all } n \in \mathbb{N}.$$

Hence, for each $n \in \mathbb{N}$, it follows that there exists a sequence $x_n \subseteq B(\bar{x},\delta_n) \cap X$ such that

$$d(\bar{u}, K_1(x_n,\bar{u}) \cap A) \leq \varepsilon_n, \tag{2.5}$$

and

$$\langle t, \bar{u} - v\rangle \leq \varepsilon_n, \quad \forall v \in K_2(x_n,\bar{u}), \ \forall t \in T(x_n,v). \tag{2.6}$$

Since $K_1(x_n,\bar{u}) \cap A$ is a closed set, it follows from (2.5) that we can choose $u_n \in K_1(x_n,u_n) \cap A$ such that

$$\|\bar{u} - u_n\| \leq \varepsilon_n, \text{ for all } n \in \mathbb{N}, \tag{2.7}$$

Hence, one has

$$u_n \to \bar{u}, \text{ as } n \to \infty.$$

Since K_2 is lsc at $(\bar{x},\bar{u})$, and $(x_n,u_n) \to (\bar{x},\bar{u})$, it follows that, for any $v \in K_2(\bar{x},\bar{u})$, there exists a sequence $v_n \in K_2(x_n,u_n)$ such that

$$v_n \to v, \text{ as } n \to \infty.$$

Also, since T is lsc at $(\bar{x},v)$ and $(x_n,v_n) \to (\bar{x},v)$, it is clear that, for any $t \in T(\bar{x},v)$, there exists a sequence $t_n \in T(x_n,v_n)$ such that

$$t_n \to t, \text{ as } n \to \infty.$$

On taking $v = v_n$ and $t = t_n$ in (2.6) and taking limit as $n \to \infty$, we have $\langle t, \bar{u}-v\rangle \leq 0$ and hence, $\bar{u} \in GM(\bar{x})$. The proof is completed. □

Definition 2.3. Let $\{x_n\}$ be a sequence in X such that $x_n \to \bar{x}$. A sequence $\{u_n\}$ is said to be an *LP approximating sequence* for (GMVI($\bar{x}$)) with respect to $\{x_n\}$, if there exists a positive sequence $\{\varepsilon_n\}$ in $\mathbb{R}$ with $\varepsilon_n \to 0$ such that, for each $n \in \mathbb{N}$,

(i) $d(u_n, K_1(x_n,u_n) \cap A) \leq \varepsilon_n$;
(ii) $\langle t, u_n - v\rangle \leq \varepsilon_n, \ \forall v \in K_2(x_n,u_n)$ and $\forall t \in T(x_n,v)$.

Definition 2.4. The problem (GMVI($\bar{x}$)) is said to be *LP well-posed* if

(i) there exists a unique solution $\bar{u}$ of (GMVI($\bar{x}$));
(ii) for any sequence $\{x_n\}$ converging to $\bar{x}$, every LP approximating sequence $\{u_n\}$ with respect to $\{x_n\}$ converges to $\bar{u}$.

We now present a metric characterization for LP well-posedness in terms of the behavior of the approximate solution set. We recall that the diameter of a nonempty set A in $\mathbb{R}^m$, is defined as

$$\text{diam } A := \sup_{a,b\in A} \|a - b\|.$$

Theorem 2.1. *Suppose that all conditions of Proposition 2.2 are satisfied. Then (GMVI($\bar{x}$)) is LP well-posed if and only if*

$$GM(\bar{x},\delta,\varepsilon)\neq\emptyset,\ \forall\delta,\varepsilon>0\ \ \textit{and}\ \ \operatorname{diam} GM(\bar{x},\delta,\varepsilon)\to 0\ \ \textit{as}\ \ (\delta,\varepsilon)\to(0,0). \tag{2.8}$$

Proof. Suppose that the problem (GMVI($\bar{x}$)) is LP well-posed. Then it has a unique solution $\bar{u}\in GM(\bar{x})$ and hence

$$GM(\bar{x},\delta,\varepsilon)\neq\emptyset,\ \forall\delta,\varepsilon>0\ \text{ as } GM(\bar{x})\subseteq GM(\bar{x},\delta,\varepsilon).$$

Assume, on the contrary, that $\operatorname{diam} GM(\bar{x},\delta,\varepsilon)\nrightarrow 0$ as $(\delta,\varepsilon)\to(0,0)$. Hence, there exist $r>0$, a positive integer m, sequences $\delta_n>0,\varepsilon_n>0$ with $(\delta_n,\varepsilon_n)\to(0,0)$ as $n\to\infty$, and $u_n,u_n'\in GM(\bar{x},\delta_n,\varepsilon_n)$ such that

$$\|u_n-u_n'\|>r,\quad\forall n\geq m. \tag{2.9}$$

For each $n\in\mathbb{N}$, as $u_n,u_n'\in GM(\bar{x},\delta_n,\varepsilon_n)$, there exist $\{x_n\}$ and $\{x_n'\}$ which belong to $B(\bar{x},\delta_n)\cap X$ such that

$$d(u_n,K_1(x_n,u_n)\cap A)\leq\varepsilon_n,\quad d(u_n',K_1(x_n',u_n')\cap A)\leq\varepsilon_n,$$

$$\langle t,u_n-v\rangle\leq\varepsilon_n,\quad\forall v\in K_2(x_n,u_n),\ \forall t\in T(x_n,v)$$

and

$$\langle t,u_n'-v\rangle\leq\varepsilon_n,\quad\forall v\in K_2(x_n',u_n'),\ \forall t\in T(x_n',v).$$

Since $x_n\to\bar{x}$ and $x_n'\to\bar{x}$, it follows that $\{u_n\}$ and $\{u_n'\}$ are LP approximating sequences for (GMVI($\bar{x}$)). Since (GMVI($\bar{x}$)) is LP well-posed, both the sequences converge to the unique solution $\bar{u}$, which gives a contradiction with (2.9). Therefore, we obtain $\operatorname{diam} GM(\bar{x},\delta,\varepsilon)\to 0$ as $(\delta,\varepsilon)\to(0,0)$.

Conversely, let $\{x_n\}$ be a sequence in X converging to $\bar{x}$ and $\{u_n\}$ be an LP approximating sequence with respect to $\{x_n\}$. Hence, there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n\to 0$ as $n\to\infty$ such that

$$d(u_n,K_1(x_n,u_n)\cap A)\leq\varepsilon_n$$

and

$$\langle t,u_n-v\rangle\leq\varepsilon_n,\quad\forall v\in K_2(x_n,u_n),\ \forall t\in T(x_n,v). \tag{2.10}$$

For each $n\in\mathbb{N}$, putting $\delta_n=\|x_n-\bar{x}\|$, we have that $\delta_n\to 0$ as $n\to\infty$. Furthermore, we get that

$$u_n\in GM(\bar{x},\delta_n,\varepsilon_n),\ \text{ for all } n\in\mathbb{N}.$$

Since $\operatorname{diam} GM(\bar{x},\delta_n,\varepsilon_n)\to 0$ as $(\delta_n,\varepsilon_n)\to(0,0)$, it follows that $\{u_n\}$ is a Cauchy sequence in $\mathbb{R}^m$. Since $\mathbb{R}^m$ is complete, we have $\{u_n\}$ converges to a point $\bar{u}\in\mathbb{R}^m$. Since $d(u_n,K_1(x_n,u_n)\cap A)\leq\varepsilon_n$ for each positive integer n, we can choose $u_n'\in K_1(x_n,u_n)\cap A$ so that

$$\|u_n-u_n'\|\leq\varepsilon_n,$$

which further implies that $u_n'\to\bar{u}$. Since A is closed and K_1 is a closed map on $X\times\mathbb{R}^m$, it follows that $\bar{u}\in K_1(\bar{x},\bar{u})\cap A$. Next, we will show that $\bar{u}\in GM(\bar{x})$. Suppose on the contrary $\bar{u}\notin GM(\bar{x})$, that is, there exist $\bar{v}\in K_2(\bar{x},\bar{u})$ and $\bar{t}\in T(\bar{x},\bar{v})$ such that

$$\langle\bar{t},\bar{u}-\bar{v}\rangle>0. \tag{2.11}$$

Since K_2 is lsc at $(\bar{x}, \bar{u})$ and $(x_n, u_n) \to (\bar{x}, \bar{u})$, there exists a sequence $v_n \in K_2(x_n, u_n)$ such that

$$v_n \to \bar{v}, \text{ as } n \to \infty.$$

Again, since T is lsc at $(\bar{x}, \bar{v})$ and $(x_n, v_n) \to (\bar{x}, \bar{v})$, there exists a sequence $t_n \in T(x_n, v_n)$ such that

$$t_n \to \bar{t}, \text{ as } n \to \infty.$$

Considering $v = v_n$ and $t = t_n$ in (2.10) and taking $n \to \infty$, we have

$$\langle \bar{t}, \bar{u} - \bar{v} \rangle \leq 0,$$

which leads to a contradiction (2.11). The uniqueness of $\bar{u}$ follows from (2.8). □

Corollary 2.5. *If the conditions of Theorem 2.1 hold then (GMVI($\bar{x}$)) is LP well-posed if and only if $GM(\bar{x}) \neq \emptyset$ and*

$$\mathit{diam}\; GM(\bar{x}, \delta, \varepsilon) \to 0 \quad \mathit{as} \quad (\delta, \varepsilon) \to (0, 0).$$

We now give an example as an application of the metric characterization of LP well-posedness.

Example 2.6. Let $X = [-1, 1]$ and $A = \mathbb{R}$. Define set-valued maps $K_1, K_2 : X \times \mathbb{R} \rightrightarrows \mathbb{R}$ and $T : X \times \mathbb{R} \rightrightarrows \mathbb{R}$ as follows

$$K_1(x, u) = \begin{cases} \{0\}, & \text{if } u \geq 0; \\ [0, |x|] & \text{if } u < 0, \end{cases}$$

$$K_2(x, u) = \begin{cases} \{1\}, & \text{if } u \geq 0; \\ [0, 1], & \text{if } u < 0 \end{cases}$$

and

$$T(x, u) = \begin{cases} \left\{\frac{1}{4}\right\}, & \text{if } u \geq 0; \\ [0, 1], & \text{if } u < 0. \end{cases}$$

For $\bar{x} = 0$, the map K_1 is closed on $\{\bar{x}\} \times \mathbb{R}$ and T is lsc on $\{\bar{x}\} \times \mathbb{R}$. By definition of K_2, we have K_2 is lsc but not closed. Indeed if we choose $u_n = -\frac{1}{n} \to 0$ and sequence y_n in $K_2(\bar{x}, u_n) = [0, 1]$ which does not converges to 1, then limit point of y_n is not belong to $\{1\} = K_2(\bar{x}, 0)$. Therefore K_2 is not closed on $\{\bar{x}\} \times \mathbb{R}$. Next, it can be observed that $GM(\bar{x}) = \{0\}$. For $\varepsilon > 0$ and $x \in B(\bar{x}, \delta) \cap X$, for any $\delta > 0$

$$\{u \in \mathbb{R} : d(u, K_1(x, u) \cap A) \leq \varepsilon\} = [-\varepsilon, \varepsilon].$$

For every $u \in [-\varepsilon, \varepsilon]$ and $x \in B(\bar{x}, \delta) \cap X$ and any $\delta > 0$, it can be seen that

$$\langle t, u - v \rangle \leq \varepsilon, \quad \forall v \in K_2(x, u), \; \forall t \in T(x, v).$$

Hence it follows that $GM(\bar{x}, \delta, \varepsilon) = [-\varepsilon, \varepsilon]$ and $\mathrm{diam} GM(\bar{x}, \delta, \varepsilon) = 2\varepsilon \to 0$ as $\varepsilon \to 0$. By Theorem 2.1, we canclude that the problem (GMVI($\bar{x}$)) is LP well-posed.

Next, we show that the assumptions of Theorem 2.1 cannot be dispensed as indicated in the following examples.

Example 2.7 (the lower semicontinuity of K_2)**.** Let $X = [-1, 1]$ and $A = \mathbb{R}$. Define set-valued maps $K_1, K_2 : X \times \mathbb{R} \rightrightarrows \mathbb{R}$ and $T : X \times \mathbb{R} \rightrightarrows \mathbb{R}$ as follows

$$K_1(x, u) = [0, 1],\ K_2(x, u) = \begin{cases} \{1\}, & \text{if } u \geq 0; \\ f\{0\}, & \text{if } u < 0 \end{cases}$$

and

$$T(x, u) = \begin{cases} \left\{\frac{1}{2}\right\}, & \text{if } u \geq 0; \\ [0, 1], & \text{if } u < 0. \end{cases}$$

For $\bar{x} \in X$, we see that K_1 is closed on $\bar{x} \times \mathbb{R}$ and T is lsc on $\bar{x} \times \mathbb{R}$ but K_2 is not lsc because we can choose $x_n = \frac{1}{n}, u_n = \frac{-1}{n} \to 0$ and for any $y \in K_2(x, 0) = \{1\}$, we can not find sequence in $K_2(x_n, u_n) = \{0\}$ which converges to y. Next, it can be observed that $GM(\bar{x}) = [0, 1]$. For $\varepsilon > 0$ and $x \in B(\bar{x}, \delta) \cap X$, for any $\delta > 0$

$$\{u \in \mathbb{R} : d(u, K_1(x, u) \cap A) \leq \varepsilon\} = [-\varepsilon, \varepsilon + 1].$$

For every $u \in [-\varepsilon, \varepsilon + 1]$ and $x \in B(\bar{x}, \delta) \cap X$ and any $\delta > 0$, it can be seen that

$$\langle t, u - v\rangle \leq \varepsilon, \quad \forall v \in K_2(x, u),\ \forall t \in T(x, v).$$

Hence it follows that $GM(\bar{x}, \delta, \varepsilon) = [-\varepsilon, \varepsilon + 1]$ and $\text{diam}GM(\bar{x}, \delta, \varepsilon) = 1 + 2\varepsilon \to 1$ as $\varepsilon \to 0$. Then we conclude that the problem (GMVI($\bar{x}$)) is not LP well-posed.

Example 2.8 (the closedness of K_1)**.** Let $X = [-1, 1]$ and $A = \mathbb{R}$. Define set-valued maps $K_1, K_2 : X \times \mathbb{R} \rightrightarrows \mathbb{R}$ and $T : X \times \mathbb{R} \rightrightarrows \mathbb{R}$ as follows

$$K_1(x, u) = \begin{cases} [\frac{1}{2}, 1], & \text{if } u \geq 0; \\ [0, 1], & \text{if } u < 0, \end{cases}$$

$$K_2(x, u) = \{1\},\ T(x, u) = \{\frac{1}{2}\}.$$

For $\bar{x} \in X$, we see that K_2 is lsc on $\{\bar{x}\}\times\mathbb{R}$ and T is lsc on $\{\bar{x}\}\times\mathbb{R}$ but K_1 is not closed on $\{\bar{x}\} \times \mathbb{R}$. Indeed we choose $u_n = \frac{-1}{n} \to 0 = u$ and $y_n = 0 \in K_1(\bar{x}, u_n) = [0, 1]$ which converges to 0 but $0 \notin K_1(x, u) = [\frac{1}{2}, 1]$. Next, it can be observed that $GM(\bar{x}) = [0, 1]$. For $\varepsilon > 0$ and $x \in B(\bar{x}, \delta) \cap X$, for any $\delta > 0$

$$\{u \in \mathbb{R} : d(u, K_1(x, u) \cap A) \leq \varepsilon\} = [-\varepsilon, \varepsilon + 1].$$

For every $u \in [-\varepsilon, \varepsilon + 1]$ and $x \in B(\bar{x}, \delta) \cap X$ and any $\delta > 0$, it can be seen that

$$\langle t, u - v\rangle \leq \varepsilon, \quad \forall v \in K_2(x, u),\ \forall t \in T(x, v).$$

Hence it follows that $GM(\bar{x}, \delta, \varepsilon) = [-\varepsilon, \varepsilon + 1]$ and $\text{diam}GM(\bar{x}, \delta, \varepsilon) = 1 + 2\varepsilon \to 1$ as $\varepsilon \to 0$. Then we conclude that the problem (GMVI($\bar{x}$)) is not LP well-posed.

Example 2.9 (the lower semicontinuity of T)**.** Let $X = [-1, 1]$ and $A = \mathbb{R}$. Define set-valued maps $K_1, K_2 : X \times \mathbb{R} \rightrightarrows \mathbb{R}$ and $T : X \times \mathbb{R} \rightrightarrows \mathbb{R}$ as follows

$$K_1(x, u) = [0, 1],\ K_2(x, u) = \begin{cases} \{1\}, & \text{if } u \geq 0; \\ [0, 1], & \text{if } u < 0 \end{cases}$$

and

$$T(x, u) = \begin{cases} [0, 1], & \text{if } u \geq 0; \\ \left\{\frac{1}{2}\right\}, & \text{if } u < 0. \end{cases}$$

For $\bar{x} \in X$, we see that K_1 is closed on $\{\bar{x}\} \times \mathbb{R}$ and K_2 is lsc on $\{\bar{x}\} \times \mathbb{R}$ but T is not lsc on $\{\bar{x}\} \times \mathbb{R}$. Indeed we choose sequence $x_n = \frac{1}{n} \to 0$ and $u_n = \frac{-1}{n} \to 0$ and choose $y = 1 \in T(0,0) = [0,1]$, we can't find some sequence in $T(x_n, u_n) = \{\frac{1}{2}\}$ which converges to y. Next, it can be observed that $GM(\bar{x}) = [0,1]$. For $\varepsilon > 0$ and $x \in B(\bar{x}, \delta) \cap X$, for any $\delta > 0$

$$\{u \in \mathbb{R} : d(u, K_1(x,u) \cap A) \leq \varepsilon\} = [-\varepsilon, \varepsilon + 1].$$

For every $u \in [-\varepsilon, \varepsilon + 1]$ and $x \in B(\bar{x}, \delta) \cap X$ and any $\delta > 0$, it can be seen that

$$\langle t, u - v \rangle \leq \varepsilon, \quad \forall v \in K_2(x,u), \ \forall t \in T(x,v).$$

Hence it follows that $GM(\bar{x}, \delta, \varepsilon) = [-\varepsilon, \varepsilon + 1]$ and $\operatorname{diam} GM(\bar{x}, \delta, \varepsilon) = 1 + 2\varepsilon \not\to 0$ as $\varepsilon \to 0$. Then we conclude that the problem (GMVI($\bar{x}$)) is not LP well-posed.

We give the relation between the LP well-posedness of the problem (GMVI($\bar{x}$)) with the existence and uniqueness of its solution as follows.

Theorem 2.2. *If the conditions Proposition 2.2 hold and A is compact, then (GMVI($\bar{x}$)) is LP well-posed if and only if it has a unique solution.*

Proof. It is obvioused that if (GMVI($\bar{x}$)) is LP well-posed, then it has a unique solution. Thus, we next prove the converse, assume that (GMVI($\bar{x}$)) has a unique solution u'. Let $\{x_n\}$ be a sequence in X which converges to $\bar{x}$. Let $\{u_n\}$ be an LP approximating sequence with respect to $\{x_n\}$. Then there exists a sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$d(u_n, K_1(x_n, u_n) \cap A) \leq \varepsilon_n \tag{2.12}$$

and

$$\langle t, u_n - v \rangle \leq \varepsilon_n, \quad \forall v \in K_2(x_n, u_n), \ \forall t \in T(x_n, v). \tag{2.13}$$

Using (2.12) and the closedness of $K_1(x_n, u_n) \cap A$, for each positive integer n, we can choose $u'_n \in K_1(x_n, u_n) \cap A$ so that

$$\|u_n - u'_n\| \leq \varepsilon_n. \tag{2.14}$$

Since A is a compact set, the sequence $\{u'_n\}$ has a subsequence $\{u'_{n_k}\}$ which converges to a point $\bar{u} \in A$. Using (2.14), we conclude that the corresponding subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converges to $\bar{u}$. We use the assumption in K_1, it follows that

$$\bar{u} \in K_1(\bar{x}, \bar{u}).$$

By similar argument as the proof in Theorem 2.1, we can show that

$$\bar{u} \in GM(\bar{x}).$$

Consequently, $\bar{u}$ coincides with u'. Again, by the uniqueness of the solution, it is obvious that every possible subsequence converges to the unique solution u' and hence the sequence $\{u_n\}$ converges to u'. Hence the LP well-posedness of (GMVI($\bar{x}$)) is satisfied. □

We are going to use the notions of measures of noncompactness in a normed space $Y := \mathbb{R}^m$ for interpreting the equivalence between LP well-posedness and the measure of the set of approximation solutions under suitable conditions.

Definition 2.10. Let M be a nonempty subset of a metric space Y.

(i) The *Kuratowski measure* of M is

$$\mu(M) = \inf\Big\{\varepsilon > 0 | M \subseteq \bigcup_{k=1}^{n} M_k \text{ and } \operatorname{diam} M_k \leq \varepsilon, k = 1, \ldots, n, \ \exists n \in \mathbb{N}\Big\}.$$

(ii) The *Hausdorff measure* of M is

$$\eta(M) = \inf\Big\{\varepsilon > 0 | M \subseteq \bigcup_{k=1}^{n} B(x_k, \varepsilon), x_k \in X, \text{ for some } n \in \mathbb{N}\Big\}.$$

Daneš [6] obtained the following inequalities:

$$\eta(M) \leq \mu(M) \leq 2\eta(M). \tag{2.15}$$

The measures μ and η share many common properties and we will use γ in the sequel to denote either one of them. γ is a regular measure (see [2, 30]), i.e., it enjoys the following properties

(1) $\gamma(M) = +\infty$ if and only if the set M is unbounded;
(2) $\gamma(M) = \gamma(\mathrm{cl}M)$;
(3) from $\gamma(M) = 0$ it follows that M is totally bounded set;
(4) if X is a complete space and if $\{A_n\}$ is a sequence of closed subsets of X such that $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \gamma(A_n) = 0$, then $K := \bigcap_{n\in\mathbb{N}} A_n$ is a nonempty compact set and $\lim_{n\to+\infty} H(A_n, K) = 0$, where H is the Hausdorff metric;
(5) from $M \subseteq N$ it follows that $\gamma(M) \leq \gamma(N)$.

Theorem 2.3. (i) *If (GMVI($\bar{x}$)) is LP well-posed, then $\gamma(GM(\bar{x}, \delta, \varepsilon)) \to 0^+$ as $(\delta, \varepsilon) \to (0^+, 0^+)$.*

(ii) *Conversely, if X is complete and the following conditions hold*
(a) *K_1 is closed and K_2 is lsc on $X \times \mathbb{R}^m$;*
(b) *T is lsc on $X \times \mathbb{R}^m$;*
(c) *A is a closed subset of $\mathbb{R}^m$,*

then GMVI($\bar{x}$) is LP well-posed, provided that $\gamma(GM(\bar{x}, \delta, \varepsilon)) \to 0^+$ as $(\delta, \varepsilon) \to (0^+, 0^+)$.

Proof. Let γ be the Hausdorff measure η (for the Kuratowski measure case the argument is similar).

(i) Assume that (GMVI($\bar{x}$)) is LP well-posed. For any δ, ε in $(0, \infty)$, we have $GM(\bar{x}) \subseteq GM(\bar{x}, \delta, \varepsilon)$, and hence

$$H(GM(\bar{x}, \delta, \varepsilon), GM(\bar{x})) = H^*(GM(\bar{x}, \delta, \varepsilon), GM(\bar{x})).$$

Let $\{u_n\}$ be arbitrary sequence in $GM(\bar{x})$. Then, of course, $\{u_n\}$ is an LP approximating sequence for (GMVI($\bar{x}$)), there is a subsequence convergent to some point of $GM(\bar{x})$. Therefore, $GM(\bar{x})$ is compact. Consequently, for any $\varepsilon > 0$, there exist $z_1, z_2, \cdots, z_n$, for some $n \in \mathbb{N}$, such that

$$GM(\bar{x}) \subseteq \cup_{k=1}^{n} B(z_k, \varepsilon),$$

which gives that

$$GM(\bar{x},\delta,\varepsilon) \subseteq \bigcup_{k=1}^{n} B(z_k, \varepsilon + H(GM(\bar{x},\delta,\varepsilon), GM(\bar{x}))).$$

Then,

$$\gamma(GM(\bar{x},\delta,\varepsilon)) \leq H(GM(\bar{x},\delta,\varepsilon), GM(\bar{x})) + \gamma(GM(\bar{x})).$$

Since $GM(\bar{x})$ is compact, we have $\gamma(GM(\bar{x})) = 0$. Thus, we obtain that

$$\gamma(GM(\bar{x},\delta,\varepsilon)) \leq H(GM(\bar{x},\delta,\varepsilon), GM(\bar{x})).$$

Next, we claim that $H(GM(\bar{x},\delta,\varepsilon), GM(\bar{x})) \to 0^+$ as $(\delta,\varepsilon) \to (0^+,0^+)$. Suppose that there are $\rho > 0, \{(\delta_n,\varepsilon_n)\} \to (0^+,0^+)$ and $\{u'_n\} \subseteq GM(\bar{x},\delta_n,\varepsilon_n)$ such that, for all $n \in \mathbb{N}$,

$$d(u'_n, GM(\bar{x})) \geq \rho.$$

Since $\{u'_n\}$ is an LP approximating sequence for (GMVI($\bar{x}$)), there is a subsequence $\{u'_{n_k}\}$ converging to a point of $GM(\bar{x})$ which leads to a contradiction. Hence, we conclude that $\gamma(GM(\bar{x},\delta,\varepsilon)) \to 0^+$ as $(\delta,\varepsilon) \to (0^+,0^+)$.

(ii) Suppose that $\gamma(GM(\bar{x},\delta,\varepsilon)) \to 0^+$ as $(\delta,\varepsilon) \to (0^+,0^+)$. First, we will show that $GM(\bar{x},\delta,\varepsilon)$ is closed for all positive δ and ε. Let the sequence $\{u_n\}$ in $GM(\bar{x},\delta,\varepsilon)$ be such that $u_n \to u$, as $n \to \infty$. Then, for each $n \in \mathbb{N}$, there is $\{x_n\} \subseteq B(\bar{x},\delta)$ such that

(2.16)
$$d(u_n, K_1(x_n,u_n) \cap A) \leq \varepsilon \text{ and } \langle t, u_n - v\rangle \leq \varepsilon, \ \ \forall v \in K_2(x_n,u_n), \forall t \in T(x_n,v).$$

Since $B(\bar{x},\delta)$ is compact, we can assume that $x_n \to x$ for some $x \in B(\bar{x},\delta)$. By similar argument as the proof in Theorem 2.1, we have $u \in K_1(x,u) \cap A$. Next, we claim that, for all $v \in K_2(x,u)$ and $t \in T(x,v)$,

$$\langle t, u - v\rangle \leq \varepsilon. \tag{2.17}$$

Suppose that there is $v \in K_2(x,u)$ and $t \in T(x,v)$ such that

$$\langle t, u - v\rangle > \varepsilon. \tag{2.18}$$

Since K_2 is lsc at (x,u), there is $v_n \in K_2(x_n,u_n)$ such that $v_n \to v$. By using (b), there is $t_n \in T(x_n,v_n)$ such that $t_n \to t$. By the continuity of $\langle \cdot,\cdot\rangle$, we have

$$\langle t_n, u_n - v_n\rangle \to \langle t, u - v\rangle.$$

Put $\varepsilon' = \langle t, u - v\rangle - \varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$|\langle t, u - v\rangle - \langle t_n, u_n - v_n\rangle| < \varepsilon' \text{ for all } n \geq n_0$$

This implies that

$$\varepsilon = \langle t, u - v\rangle - \varepsilon' < \langle t_n, u_n - v_n\rangle \text{ for all } n \geq n_0,$$

which leads to a contradiction with (2.16). Therefore, (2.17) is proved. Furthermore, since $x \in B(\bar{x},\delta)$, we can conclude that $u \in GM(\bar{x},\delta,\varepsilon)$. Hence $GM(\bar{x},\delta,\varepsilon)$ is closed. By Proposition 2.2, we get that

$$\bigcap_{\delta,\varepsilon>0} GM(\bar{x},\delta,\varepsilon) = GM(\bar{x}).$$

Since $\gamma(GM(\bar{x},\delta,\varepsilon)) \to 0^+$ as $(\delta,\varepsilon) \to (0^+,0^+)$, the regular measure properties of γ imply that $GM(\bar{x})$ is compact and

$$H(GM(\bar{x},\delta,\varepsilon), GM(\bar{x})) \to 0^+ \text{ as } (\delta,\varepsilon) \to (0^+,0^+).$$

Finally, let $\{u_n\}$ be an approximating sequence for (GMVI($\bar{x}$)) with respect to a sequence $\{x_n\}$ converging to $\bar{x}$. Hence, t here is $\varepsilon_n \to 0^+$ such that for all $v \in K_2(x_n,u_n), t \in T(x_n,v)$ and for all $n \in \mathbb{N}$,

$$\langle t, u_n - v\rangle \leq \varepsilon_n,$$

which gives that $u_n \in GM(\bar{x},\delta_n,\varepsilon_n)$ with $\delta_n := d(\bar{x},x_n)$. We see that

$$d(u_n, GM(\bar{x})) \leq H(GM(\bar{x},\delta_n,\varepsilon_n), GM(\bar{x})) \to 0^+.$$

Hence, there is $\bar{u}_n \in GM(\bar{x})$ such that $d(u_n,\bar{u}_n) \to 0$ as $n \to \infty$. By the compactness of $GM(\bar{x})$, there is a subsequence $\{\bar{u}_{n_k}\}$ of $\{\bar{u}_n\}$ converging to a point $\bar{u} \in GM(\bar{x})$. Therefore, the corresponding subsequence $\{u_{n_k}\}$ of $\{u_n\}$ tends to $\bar{u}$. Hence GMVI($\bar{x}$) is LP well-posed. The proof is completed. □

3. LP well-posedness in the parametric generalized sense for the generalized quasivariational inequality problem of the Minty type

In many practical situations, the problem (GMVI($\bar{x}$)) may not always possess a unique solution. Hence, in this section, we introduce a generalization of LP well-posedness defined as follows.

Definition 3.1. *The problem GMVI($\bar{x}$) is said to be* LP well-posed in the generalized sense *if*

(i) *the solution set $GM(\bar{x})$ is nonempty;*
(ii) *for any sequence $\{x_n\}$ converging to $\bar{x}$, every LP approximating sequence $\{u_n\}$ with respect to $\{x_n\}$ has a subsequence converging to some point of $GM(\bar{x})$.*

Proposition 3.2. *If GMVI($\bar{x}$) is LP well-posed in the generalized sense, then its solution set $GM(\bar{x})$ is a nonempty compact set.*

Proof. Let $\{u_n\}$ be any sequence in $GM(\bar{x})$. Then, of course, it is an LP approximating sequence with respect to sequences $x_n := \bar{x}$ and $\varepsilon_n := \frac{1}{n}$, for every $n \in \mathbb{N}$. The generalized LP well-posedness of GMVI($\bar{x}$) ensures the existence of a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging to a point of in $GM(\bar{x})$. Therefore, we conclude that $GM(\bar{x})$ is a nonempty compact set. The proof is completed. □

Next, we present a metric characterization for the generalized LP well-posedness of (GMVI($\bar{x}$)) in terms of the upper semicontinuity of the approximate solution set.

Theorem 3.1. *GMVI($\bar{x}$) is LP well-posed in the generalized sense if and only if $GM(\bar{x})$ is a nonempty, compact set and $GM(\bar{x},\cdot,\cdot)$ is usc at $(\delta,\varepsilon) := (0,0)$.*

Proof. Assume that GMVI($\bar{x}$) is LP well-posed in the generalized sense it follows that

$$GM(\bar{x}) \neq \emptyset$$

and using Proposition 3.2, we have $GM(\bar{x})$ is compact. Next, suppose on the contrary that $GM(\bar{x},\cdot,\cdot)$ is not usc at $(\delta,\varepsilon) := (0,0)$. Then there exist open set U

containing $GM(\bar{x},0,0) = GM(\bar{x})$ and sequences $\delta_n > 0$ with $\delta_n \to 0$ and $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$GM(\bar{x},\delta_n,\varepsilon_n) \nsubseteq U.$$

Thus, there exists an LP approximating sequence, such that none of its subsequences converges to a point of $GM(\bar{x})$, which is a contradiction. Therefore, $GM(\bar{x},\cdot,\cdot)$ is usc at $(\delta,\varepsilon) := (0,0)$.

Conversely, let $\{u_n\}$ be an LP approximating sequence with respect to $\{x_n\}$ converging $\bar{x}$ as $n \to \infty$. For each $n \in \mathbb{N}$, let a sequence $\delta_n := \|x_n - \bar{x}\|$, we have

$$\delta_n \to 0 \quad \text{and} \quad u_n \in GM(\bar{x},\delta_n,\varepsilon_n).$$

As $GM(\bar{x},\delta,\varepsilon)$ is usc at $(\delta,\varepsilon) = (0,0)$ and $GM(\bar{x}) \neq \emptyset$, it follows that for every $\alpha > 0$,

$$GM(\bar{x},\delta_n,\varepsilon_n) \subseteq GM(\bar{x}) + B(0,\alpha) \quad \text{for } n \text{ sufficiently large.}$$

Thus $u_n \in GM(\bar{x})+B(0,\alpha)$, for n sufficiently large and hence there exists a sequence $\bar{u}_n \in GM(\bar{x})$ such that

$$\|u_n - \bar{u}_n\| \leq \alpha. \tag{3.1}$$

Since $GM(\bar{x})$ is compact, there exists a subsequence $\{\bar{u}_{n_k}\}$ of $\{\bar{u}_n\}$ converging to $\bar{u} \in GM(\bar{x})$. From (3.1), we conclude that the corresponding subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converges to $\bar{u} \in GM(\bar{x})$. Hence, GMVI($\bar{x}$) is LP well-posed in the generalized sense . The proof is completed. □

The following example illustrates that the compactness assumption for the solution set can't be relaxed in the previous theorem.

Example 3.3. Let $X = \mathbb{R}$ and $A = [-1,\infty)$. Define set-valued maps $K_1, K_2, T : X \times \mathbb{R} \rightrightarrows \mathbb{R}$ as follows

$$K_1(x,u) = \begin{cases} [u, 1+|x|], & \text{if } u \leq 1; \\ [1, 2u-1], & \text{if } u > 1, \end{cases}$$

$$K_2(x,u) = \begin{cases} \{1\}, & \text{if } u \leq 1; \\ [0,1], & \text{if } u > 1 \end{cases}$$

and

$$T(x,u) = \begin{cases} [0,1], & \text{if } u < 1; \\ \{0\}, & \text{if } u \geq 1. \end{cases}$$

For $\bar{x} = 0$, it can be observed that $GM(\bar{x}) = [-1,\infty)$,

$$GM(\bar{x},\delta,\varepsilon) = [-1-\varepsilon,\infty)$$

and $GM(\bar{x},\delta,\varepsilon)$ is usc at $(\delta,\varepsilon) = (0,0)$. The sequence $\{x_n\}$ where $x_n = \frac{1}{n}$ converges to $\bar{x}$, the sequence $\{u_n\}$ where $u_n = n$ satisfies

$$d(u_n, K_1(x_n,u_n) \cap A) \leq \varepsilon_n$$

and

$$\langle t, u_n - v\rangle \leq \varepsilon_n, \ \ \forall v \in K_2(x_n,u_n), \ \forall t \in T(x_n,v)$$

for every sequence ε_n with $\varepsilon_n \to 0$. Thus $\{u_n\}$ is an LP approximating sequence for (GMVI($\bar{x}$)) but posseses no convergent subsequence, consequently (GMVI($\bar{x}$)) is not generalized LP well-posed.

The following result illustrates the fact that generalized LP well-posedness of the parametric quasivariational inequality ensures the stability, in terms of the upper semicontinuity of the solution set.

Theorem 3.2. *If (GMVI($\bar{x}$)) is LP well-posed in the generalized sense, then GM is usc at $\bar{x}$.*

Proof. Suppose that M is not usc at $\bar{x}$. Then there is an open set U containing $GM(\bar{x})$ such that for every sequence $x_n \to \bar{x}$, there exists $u_n \in GM(x_n)$ such that $u_n \notin U$, for every n. Since $x_n \to \bar{x}, \{u_n\}$ is an LP approximating sequence for (GMVI($\bar{x}$)) and none of its subsequences converge to a point of $GM(\bar{x})$, hence we have a contradiction to the fact that (GMVI($\bar{x}$)) is LP well-posed in the generalized sense. □

It is easy to get that if $GM(\bar{x}, \delta, \varepsilon)$ is usc at $(\delta, \varepsilon) = (0, 0)$, then

$$H^*(GM(\bar{x}, \delta, \varepsilon), GM(\bar{x})) \to 0 \text{ as } (\delta, \varepsilon) \to (0, 0).$$

The converse implication holds if $GM(\bar{x})$ is a nonempty compact set. For more details, refer to [13, 29]. Hence (GMVI($\bar{x}$)) is LP well-posed in the generalized sense if and only if $GM(\bar{x})$ is nonempty, compact and

$$H^*(GM(\bar{x}, \delta, \varepsilon), GM(\bar{x})) \to 0 \text{ as } (\delta, \varepsilon) \to (0, 0).$$

The proof of the following theorem is similar to that of Theorem 2.2

Theorem 3.3. *If the conditions (i)-(iii) of Proposition 2.2 hold then GMVI($\bar{x}$) is LP well-posed in the generalized sense if and only if the solution set $GM(\bar{x}) \neq \emptyset$.*

Finally, in this section we provide sufficient conditions which ensure that the generalized LP well-posedness in terms of upper semicontinuity of a particular type of approximate solution set.

Theorem 3.4. *If the conditions (i) and (ii) of Proposition 2.2 hold and if for each $x \in X$ there exists $\varepsilon > 0$ such that $GM(x, \varepsilon, \varepsilon)$ is nonempty and bounded, then (GMVI($\bar{x}$)) is LP well-posed in the generalized sense.*

Proof. Let $\{x_n\}$ be a sequence in X with $x_n \to \bar{x}$ and $\{u_n\}$ be an LP approximating sequence, with respect to $\{x_n\}$. Then there exists a sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that

$$d(u_n, K_1(x_n, u_n) \cap A) \leq \varepsilon_n$$

and

$$\langle t, u_n - v\rangle \leq \varepsilon_n, \quad \forall v \in K_2(x_n, u_n), \ \forall t \in T(x_n, v). \tag{3.2}$$

Setting $\delta_n = \|x_n - \bar{x}\|$, we have

$$\delta_n \to 0 \text{ and } u_n \in GM(\bar{x}, \delta_n, \varepsilon_n).$$

Let $\varepsilon > 0$ be the number such that $GM(x, \varepsilon, \varepsilon)$ is nonempty and bounded. Then there is a positive integer m such that

$$u_n \in GM(\bar{x}, \varepsilon, \varepsilon) \text{ for all } n \geq m.$$

This implies that $\{u_n\}$ is bounded and hence, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$u_{n_k} \to \bar{u} \text{ as } k \to \infty.$$

On taking the subsequence $\{u_{n_k}\}$ in (3.2), we can show that $\bar{u} \in GM(\bar{x})$ by Theorem 2.1. □

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Manuscript received August 11, 2014
revised November 2, 2015

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