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CONVERGENCE ANALYSIS OF THE JACOBI PSEUDO-SPECTRAL METHOD FOR SECOND ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

XIAO-YONG ZHANG

Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday.

Abstract. The Jacobi pseudo-spectral method for the second order Volterra integro-differential equations of the second kind is proposed in this paper. We provide a rigorous error analysis for the proposed method, which indicates that numerical errors (in the $L^2_{\omega_{\alpha,\beta}}$-norm and the L^∞-norm) will decay exponentially provided that the source function is sufficiently smooth. Numerical examples are given to illustrate the theoretical results.

1. Introduction

In practical applications one frequently encounters the second order Volterra integro-differential equations of the second kind of the form

$$y''(x) = a(x)y(x) + b(x)y'(x) + c(x) + \int_0^x K(x,s)y(s)ds, \ 0 < x \le T \tag{1.1}$$

with the given initial condition $y(0) = y_0, y'(0) = y'_0$. Where the unknown function $y(x)$ is defined in $0 < x \le T < \infty$. $a(x), b(x), c(x)$ are three given functions and $K(x,s)$ is a given kernel.

Equations of this type arise in the mathematical model of physical and biological phenomena. Due to the wide application of these equations, they must be solved successfully with efficient numerical methods. For these problems, many numerical approaches can be applied directly, such as collocation methods, which have been provided ([1,11]), Sine-collocation method see, e.g., [18] and references therein. But the spectral collocation methods are similar to the finite-difference approach. It makes use of values of interpolation points to present coefficients of expanded form of the numerical solution, and as a result its computing scheme is complex and the corresponding error analysis is tedious as it does not fit in a unified framework. So to find a simple and efficient method is very meaningful for solving the VIDES.

In this paper, we propose a kind of novel algorithm for second order Volterra integro-differential equations, which is called the pseudo-spectral method, and it differs from the spectral-collocation method and has several advantages. Firstly, Although both the pseudo-spectral method and the spectral-collocation algorithm

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possess the spectral accuracy, in the pseudo-spectral method, we put the approximation scheme under the general inner product type framework and take advantage of the property of orthogonal polynomials sufficiently, the results are that the computing schemes of the pseudo-spectral method are more simple and the relevant convergence theories, as will be seen from Sections 2,4, are cleaner and more reasonable than those obtained in [9,10,16]. Secondly, compared with the finite-difference method,etc., the pseudo-spectral method possesses high accuracy.

The paper is organized as follows. In Section 2, we introduce the Jacobi pseudo-spectral approaches for the second order Volterra integro-differential equations (1.1). Some preliminaries and useful lemmas are provided in Section 3. In Section 4, the convergence analysis is given. We prove the error estimates in the $L^2_{\omega_{\alpha,\beta}}$-norm and L^∞-norm. The numerical experiments are carried out in Section 5, which will be used to verify the theoretical results obtained in Section 4. The final section contains conclusions.

2. Jacobi pseudo-spectral Galerkin method

In this section, we formulate the Jacobi pseudo-spectral schemes for problem (1.1). For this purpose, Let $\omega_{\alpha,\beta} = (1-t)^\alpha(1+t)^\beta$ be a weight function in the usual sense, for $\alpha, \beta > -1. J_k^{\alpha,\beta}(t),\ k = 0, 1, \ldots,$ denote the Jacobi polynomials. The set of Jacobi polynomials $\{J_k^{\alpha,\beta}\}_{k=0}^\infty$ forms a complete $L^2_{\omega_{\alpha,\beta}}(-1,1)$-orthogonal system. Before using pseudo-spectral methods, we need to restate problem (1.1). The usual way (see [6]) to deal with the original problem is: writing $z(x) = y^{'}(x), z_1(x) = y^{''}(x)$, (1.1) is equivalent to a linear Volterra integral equations of the second kind with respect to y, z, z_1.

$$
\left\{
\begin{aligned}
& y(x) = y_0 + \int_0^x z(s)ds, \\
& z(x) = y_0^{'} + \int_0^x z_1(s)ds, \\
& z_1(x) = y_0 a(x) + y_0^{'} b(x) + c(x) \\
& \qquad\qquad + \int_0^x (K(x,s)y(s) + a(x)z(s) + b(x)z_1(s))ds.
\end{aligned}
\right.
\tag{2.1}
$$

For the sake of applying the theory of orthogonal polynomials conveniently, by the linear transformation

$$
x = \frac{T(1+t)}{2}, \quad s = \frac{T(1+\tau)}{2}
$$

letting

$$
\begin{aligned}
u(t) &= y\Big(\frac{T(1+t)}{2}\Big),\ v(t) = z\Big(\frac{T(1+t)}{2}\Big),\ w(t) = z_1\Big(\frac{T(1+t)}{2}\Big), \\
g(t) &= y_0 a\Big(\frac{T(1+t)}{2}\Big) + y_0^{'} b\Big(\frac{T(1+t)}{2}\Big) + c\Big(\frac{T(1+t)}{2}\Big), \\
\widetilde{K}(t,\tau) &= K\Big(\frac{T(1+t)}{2}, \frac{T(1+\tau)}{2}\Big),
\end{aligned}
$$

$$\widetilde{a}(t) = a\Big(\frac{T(1+t)}{2}\Big),\ \widetilde{b}(t) = b\Big(\frac{T(1+t)}{2}\Big),\ \Lambda = [-1,1],$$

$$(2.2)\qquad \begin{cases} u(t) = y_0 + \dfrac{T}{2}\displaystyle\int_{-1}^{t} v(\tau)d\tau,\ v(t) = y_0^{'} + \dfrac{T}{2}\displaystyle\int_{-1}^{t} w(\tau)d\tau, \\ w(t) = g(t) + \dfrac{T}{2}\displaystyle\int_{-1}^{t} (\widetilde{K}(t,\tau)u(\tau) + \widetilde{a}(t)v(\tau) + \widetilde{b}(t)w(\tau))d\tau. \end{cases}$$

The weak form of (2.2) is to find $u, v, w \in L^2_{\omega_{\alpha,\beta}}(\Lambda)\times L^2_{\omega_{\alpha,\beta}}(\Lambda)\times L^2_{\omega_{\alpha,\beta}}(\Lambda)$, such that

$$(2.3)\quad \begin{cases} (u,\phi)_{\omega_{\alpha,\beta}} = \Big(y_0 + \dfrac{T}{2}\displaystyle\int_{-1}^{t} v(\tau)d\tau, \phi\Big)_{\omega_{\alpha,\beta}}, \\ (v,\varphi)_{\omega_{\alpha,\beta}} = \Big(y_0^{'} + \dfrac{T}{2}\displaystyle\int_{-1}^{t} w(\tau)d\tau, \varphi\Big)_{\omega_{\alpha,\beta}}, \\ (w,\psi)_{\omega_{\alpha,\beta}} = \Big(g(t) + \dfrac{T}{2}\displaystyle\int_{-1}^{t} (\widetilde{K}(t,\tau)u(\tau) + \widetilde{a}(t)v(\tau) + \widetilde{b}(t)w(\tau))d\tau, \psi\Big)_{\omega_{\alpha,\beta}} \end{cases}$$
$$\forall \phi,\varphi,\psi \in L^2_{\omega_{\alpha,\beta}}(\Lambda)\times L^2_{\omega_{\alpha,\beta}}(\Lambda)\times L^2_{\omega_{\alpha,\beta}}(\Lambda).$$

where $(\cdot,\cdot)_{\omega_{\alpha,\beta}}$ denotes the usual inner product in the $L^2_{\omega_{\alpha,\beta}}$-space.

Now, let N be any positive integer and $\mathcal{P}_N(\Lambda)$ be the set of all algebraic polynomials of degree at most N. Obviously, the Jacobi polynomials $J_0^{\alpha,\beta}(t)$, $J_1^{\alpha,\beta}(t), \dots, J_N^{\alpha,\beta}(t)$ are the basis functions of $\mathcal{P}_N(\Lambda)$.

Next, we denote the collocation points by $\{t_i\}_{i=0}^N$, which is the set of (N+1) Jacobi Gauss point. We also define the Jacobi interpolating polynomial $I_N^{\alpha,\beta}v \in \mathcal{P}_N(\Lambda)$, satisfying

$$I_N^{\alpha,\beta}v(t_i) = v(t_i),\ 0 \le i \le N.$$

It can be written as an expression of the form

$$(2.4)\qquad I_N^{\alpha,\beta}v(t) = \sum_{i=0}^{N} v(t_i)F_i(t),$$

where $F_i(t)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{t_i\}_{i=0}^N$.

Now we describe the Jacobi pseudo-spectral method. For this purpose, set $\tau = \tau(t,\theta) = \frac{t-1}{2} + \frac{t+1}{2}\theta,\ \theta \in [-1,1]$. We define that

$$(2.5)\qquad \mathcal{M}u(t) = \frac{T}{2}\int_{-1}^{t} u(\tau)d\tau = \frac{T}{2}\int_{-1}^{1}\Big(\frac{t+1}{2}\Big)u(\tau(t,\theta))d\theta,$$

$$(2.6)\qquad \overline{\mathcal{M}}u(t) = \frac{T}{2}\int_{-1}^{t} \widetilde{a}(t)u(\tau)d\tau = \frac{T}{2}\int_{-1}^{1}\Big(\frac{t+1}{2}\Big)\widetilde{a}(t)u(\tau(t,\theta))d\theta,$$

$$(2.7)\qquad \widehat{\mathcal{M}}u(t) = \frac{T}{2}\int_{-1}^{t} \widetilde{b}(t)u(\tau)d\tau = \frac{T}{2}\int_{-1}^{1}\Big(\frac{t+1}{2}\Big)\widetilde{b}(t)u(\tau(t,\theta))d\theta,$$

and

$$(\widetilde{\mathcal{M}}u)(t) = \frac{T}{2}\int_{-1}^{t}\widetilde{K}(t,\tau)u(\tau)d\tau$$
$$= \frac{T}{2}\int_{-1}^{1}\Big(\frac{t+1}{2}\Big)\widetilde{K}(t,\tau(t,\theta))u(\tau(t,\theta))d\theta. \tag{2.8}$$

Using (N +1)-point Gauss-Jacobi quadrature formula with weight $\omega_{\alpha,\beta}$ to approximate (2.5)-(2.8) yields

$$\mathcal{M}u(t) \approx \mathcal{M}_N u(t) := \frac{T}{2}\sum_{j=0}^{N}\Big(\frac{t+1}{2}\Big)u(\tau(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)\omega_j, \tag{2.9}$$

$$\overline{\mathcal{M}}u(t) \approx \overline{\mathcal{M}}_N u(t) := \frac{T}{2}\sum_{j=0}^{N}\Big(\frac{t+1}{2}\Big)\widetilde{a}(t)u(\tau(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)\omega_j, \tag{2.10}$$

$$\widehat{\mathcal{M}}u(t) \approx \widehat{\mathcal{M}}_N u(t) := \frac{T}{2}\sum_{j=0}^{N}\Big(\frac{t+1}{2}\Big)\widetilde{b}(t)u(\tau(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)\omega_j, \tag{2.11}$$

and

$$\widetilde{\mathcal{M}}u(t) \approx \widetilde{\mathcal{M}}_N u(t) := \frac{T}{2}\sum_{j=0}^{N}\Big(\frac{t+1}{2}\Big)\widetilde{K}(t,\tau(t,\theta_j))u(\tau(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)\omega_j, \tag{2.12}$$

where $\{\theta_j\}_{j=0}^{N}$are the (N + 1)-degree Jacobi-Gauss points associated with $\omega_{\alpha,\beta}$, and $\{\omega_j\}_{j=0}^{N}$ are the corresponding Jacobi weights. On the other hand, instead of the continuous inner product, the discrete inner product will be implemented by the following equality,

$$(u,v)_N = \sum_{j=0}^{N} u(\theta_j)v(\theta_j)\omega_j.$$

As a result,

$$(\varphi,\psi)_{\omega_{\alpha,\beta}} = (\varphi,\psi)_N, \text{ if } \varphi\psi \in \mathcal{P}_{2N+1}(\Lambda). \tag{2.13}$$

By the definition of $I_N^{\alpha,\beta}$, we have

$$(u,v)_N = (I_N^{\alpha,\beta}u,v)_N. \tag{2.14}$$

The Jacobi pseudo-spectral method is to find

$$u_N(t) = \sum_{j=0}^{N}\widetilde{u}_j J_j^{\alpha,\beta}(t),\ v_N(t) = \sum_{j=0}^{N}\widetilde{v}_j J_j^{\alpha,\beta}(t),\ w_N(t) = \sum_{j=0}^{N}\widetilde{w}_j J_j^{\alpha,\beta}(t) \in \mathcal{P}_N(\Lambda)$$

such that

$$\begin{cases}(u_N,\phi)_N = (y_0 + \mathcal{M}_N v_N,\phi)_N\\ (v_N,\varphi)_N = (y_0^{'} + \mathcal{M}_N w_N,\varphi)_N\\ (w_N,\psi)_N = (g(t) + \widetilde{\mathcal{M}}_N u_N + \overline{\mathcal{M}}_N v_N + \widehat{\mathcal{M}}_N w_N,\psi)_N,\end{cases} \tag{2.15}$$
$$\forall\phi,\varphi,\psi \in \mathcal{P}_N(\Lambda).$$

where $\{\widetilde{u}_j\}_{j=0}^N, \{\widetilde{v}_j\}_{j=0}^N$and $\{\widetilde{w}_j\}_{j=0}^N$ are determined by

$$\left\{\begin{aligned} &\sum_{j=0}^{N}(J_j^{-\mu,-\mu}, J_i^{\alpha,\beta})_N\widetilde{u}_j - \sum_{j=0}^{N}(\mathcal{M}_N J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N\widetilde{v}_j = (y_0, J_i^{\alpha,\beta})_N, \\ &\sum_{j=0}^{N}(J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N\widetilde{v}_j - \sum_{j=0}^{N}(\mathcal{M}_N J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N\widetilde{w}_j = (y_0^{'}, J_i^{\alpha,\beta})_N \\ &\qquad - \sum_{j=0}^{N}(\widetilde{\mathcal{M}}_N J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N\widetilde{u}_j + \sum_{j=0}^{N}(\overline{\mathcal{M}}_N J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N\widetilde{v}_j \\ &\qquad + \sum_{j=0}^{N}((J_j^{\alpha,\beta} - \widehat{\mathcal{M}}_N J_j^{\alpha,\beta}), J_i^{\alpha,\beta})_N\widetilde{w}_j = (g(t), J_i^{\alpha,\beta})_N. \end{aligned}\right. \tag{2.16}$$

Denoting $\widetilde{X} = [\widetilde{u}_0, \widetilde{u}_1, \ldots, \widetilde{u}_N, \widetilde{v}_0, \widetilde{v}_1, \ldots, \widetilde{v}_N, \widetilde{w}_0, \widetilde{w}_1, \ldots, \widetilde{w}_N]^\top$, (2.16) yields a equation of the matrix form

$$A\widetilde{X} = g_N, \tag{2.17}$$

where

$$g_N(i) = \begin{cases} (y_0, J_i^{\alpha,\beta})_N, & 0 \le i \le N, \\ (y_0^{'}, J_{i-N-1}^{\alpha,\beta})_N, & N+1 \le i \le 2N+1, \\ (g(t), J_{i-2N-2}^{\alpha,\beta})_N, & 2N+2 \le i \le 3N+2. \end{cases}$$

$$A(i,j) = \begin{cases} (J_j^{\alpha,\beta}, J_i^{\alpha,\beta})_N, & 0 \le i \le N, 0 \le j \le N, \\ 0, & N+1 \le i \le 2N+1, \\ & 0 \le j \le N, \\ -(\widetilde{\mathcal{M}}_N J_j^{\alpha,\beta}, J_{i-2N-2}^{\alpha,\beta})_N, & 2N+2 \le i \le 3N+2, \\ & 0 \le j \le N, \\ -(\mathcal{M}_N J_{j-N-1}^{\alpha,\beta}, J_i^{\alpha,\beta})_N, & 0 \le i \le N, \\ & N+1 \le j \le 2N+1, \\ (J_{j-N-1}^{\alpha,\beta}, J_{i-N-1}^{\alpha,\beta})_N, & N+1 \le i \le 2N+1, \\ & N+1 \le j \le 2N+1, \\ (\overline{\mathcal{M}}_N J_{j-N-1}^{\alpha,\beta}, J_{i-2N-2}^{\alpha,\beta})_N, & 2N+2 \le i \le 3N+2, \\ & N+1 \le j \le 2N+1, \\ 0, & 0 \le i \le N, \\ & 2N+2 \le j \le 3N+2, \\ -(\mathcal{M}_N J_{j-2N-2}^{\alpha,\beta}, J_{i-N-1}^{\alpha,\beta})_N, & N+1 \le i \le 2N+1, \\ & 2N+2 \le j \le 3N+2, \\ ((J_{j-2N-2}^{\alpha,\beta} - \widehat{\mathcal{M}}_N J_{j-2N-2}^{\alpha,\beta}), J_{i-2N-2}^{\alpha,\beta})_N, & 2N+2 \le i \le 3N+2, \\ & 2N+2 \le j \le 3N+2. \end{cases}$$

3. Some useful lemmas

We first introduce some Hilbert spaces. For simplicity, denote $\partial_t v(t) = (\partial/\partial_t)v(t)$, etc. For a nonnegative integer m, define

$$H^m_{\omega_{\alpha,\beta}}(-1,1) := \{v : \partial_t^k v(t) \in L^2_{\omega_{\alpha,\beta}}(-1,1),\ 0 \le k \le m\},$$

with the semi-norm and the norm as

$$|v|_{L^2_{\omega_{\alpha,\beta}}} = \|\partial_t^m v(t)\|_{L^2_{\omega_{\alpha,\beta}}},\ \|v\|_m = \left(\sum_{k=0}^{m} \|\partial_t^k v(t)\|^2_{L^2_{\omega_{\alpha,\beta}}}\right)^{\frac{1}{2}},$$

respectively. It is convenient sometime to introduce the semi-norms

$$|v|_{H^{m,N}_{\omega_{\alpha,\beta}}(\Lambda)} = \left(\sum_{k=\min(m,N+1)}^{m} |\partial_t^k v(t)|_{L^2_{\omega_{\alpha,\beta}}(\Lambda)}\right)^{\frac{1}{2}}.$$

For bounding some approximation error of Jacobi polynomials, we need the following nonuniformly-weighted Sobolev spaces:

$$H^m_{\omega_{\alpha,\beta},*}(-1,1) := \{v : \partial_t^k v(t) \in L^2_{\omega_{\alpha+k,\beta+k}}(-1,1),\ 0 \le k \le m\},$$

equipped with the inner product and the norm as

$$(u,v)_{m,*} = \sum_{k=0}^{m} (\partial_t^k u, \partial_t^k v)_{\omega_{\alpha+k,\beta+k}},\ \|v\|_{m,*} = \sqrt{(v,v)_{m,*}}.$$

Next, we define the orthogonal projection $P_N : L^2(\Lambda) \longrightarrow \mathcal{P}_N(\Lambda)$ as

$$(u - P_N u, v) = 0, \quad \forall v \in \mathcal{P}_N(\Lambda).$$

P_N possesses the following approximation properties (5.4.11), (5.4.12) and (5.4.24) on pp. 283-287 in Ref. ([8]):

$$\|u - P_N u\|_{L^2(\Lambda)} \le cN^{-m}\|u\|_{H^m(\Lambda)} \tag{3.1}$$

and

$$\|u - P_N u\|_{L^\infty} \le cN^{\frac{3}{4}-m}\|u\|_{m,\infty} \tag{3.2}$$

We have the following optimal error estimate for the interpolation polynomials based on the Jacobi Gauss points (cf. [10]).

Lemma 3.1. *For any function v satisfying $v \in H^m_{\omega_{\alpha,\beta},*}(-1,1)$, we have*

$$\|v - I_N^{\alpha,\beta} v\|_{L^2_{\omega_{\alpha,\beta}}(\Lambda)} \le cN^{-m}\|\partial_t^m v\|_{L^2_{\omega_{\alpha+m,\beta+m}}}, \tag{3.3}$$

for the Jacobi Gauss points and Jacobi Gauss-Radau points.

Lemma 3.2. *If $v \in H^m_{\omega_{\alpha,\beta},*}(-1,1)$, for some $m \ge 1$ and $\phi \in \mathcal{P}_N(\Lambda)$, then for the Jacobi Gauss and Jacobi Gauss-Radau integration we have (cf. [10])*

$$\begin{aligned} |(v,\phi)_{\omega_{\alpha,\beta}} - (v,\phi)_N| &\le \|v - I_N^{\alpha,\beta} v\|_{L^2_{\omega_{\alpha,\beta}}} \|\phi\|_{L^2_{\omega_{\alpha,\beta}}} \\ &\le cN^{-m}\|\partial_t^m v\|_{L^2_{\omega_{\alpha+m,\beta+m}}} \|\phi\|_{L^2_{\omega_{\alpha,\beta}}}. \end{aligned} \tag{3.4}$$

We have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials; (cf. [10]).

Lemma 3.3. *Let $\{F_j(t)\}_{j=0}^N$ be the N-th Lagrange interpolation polynomials associated with the Gauss, or Gauss-Radau, or Gauss-Lobatto points of the Jacobi polynomials. Then*

$$(3.5)\qquad \|I_N^{\alpha,\beta}\|_{L^\infty} := \max_{t\in[-1,1]} \sum_{j=0}^{N} |F_j(t)| = \begin{cases} c\log N & -1<\alpha,\beta\le -\frac{1}{2}, \\ cN^{\gamma+\frac{1}{2}}, & \gamma = max(\alpha,\beta),\ otherwise. \end{cases}$$

We now introduce some notation. For $r \ge 0$ and $\kappa \in [0,1], C^{r,\kappa}([-1,1])$ will denote the space of functions whose r-th derivatives are Hölder continuous with exponent κ, endowed with the usual norm $||\cdot||_{r,\kappa}$. When $\kappa = 0, C^{r,0}([-1,1])$ denotes the space of functions with r continuous derivatives on [0, T], also denoted by $C^r([-1,1])$, and with norm $||\cdot||_r$.

We will make use of a result of Ragozin ([12], [13]), which states that,for each nonnegative integer r and $\kappa\in[0,1]$, there exists a constant $C_{r,\kappa}>0$ such that for any function $v \in C^{r,\kappa}([-1,1])$, there exists a polynomial function $\tau_N v \in \mathcal{P}_N$ such that

$$(3.6)\qquad \|v-\tau_N v\|_{L^\infty} \le C_{r,\kappa} N^{-(r+\kappa)} \|v\|_{r,\kappa}.$$

where $||\cdot||_\infty$ is the norm of the space $L^\infty([-1,1])$, and when the function $v \in C([-1,1])$. Actually,τ_N is a linear operator from $C^{r,\kappa}([-1,1])$ to $\mathcal{P}_N$.

We will need the fact that $\widetilde{\mathcal{M}}$,which be defined by (2.8), is compact as an operator from $C([0,T])$ to $C^{r,\kappa}([-1,1])$ for any $0<\kappa<1$. (see [9].)

Lemma 3.4. *Let $0<\kappa<1$. then, for any function $v\in C([-1,1])$,there exists a positive constant C such that*

$$\frac{|\widetilde{\mathcal{M}}v(t')-\widetilde{\mathcal{M}}v(t'')|}{|t'-t''|^\kappa} \le c \max_{-1\le t\le 1}|v(t)|.$$

Proof. We only need to prove that $\widetilde{\mathcal{M}}$ is Hölder continuous. For any$t',t''\in[-1,1]$ and $t'\neq t''$,

$$\frac{|\widetilde{\mathcal{M}}v(t')-\widetilde{\mathcal{M}}v(t'')|}{|t'-t''|^\kappa} = \frac{|\frac{T}{2}(\int_{t'}^{t''} Kv(\theta)d\theta - \int_0^{t'} \partial_t K(t'-t'')v(\theta)d\theta)|}{|t'-t''|^\kappa} \le c \max_{-1\le t\le 1}|v(t)|.$$

This implies that

$$(3.7)\qquad \|\widetilde{\mathcal{M}}v\|_{0,\kappa} \le C\|v\|_{L^\infty},\ 0<\kappa<1.$$

Clearly, $\mathcal{M},\overline{\mathcal{M}}$ and $\widehat{\mathcal{M}}$ also satisfy (3.7). □

To prove the error estimate, we will apply the standard Gronwall Lemma. We call such a function $v=v(t)$ locally integrable on the interval $[-1,1]$ if for each $t\in[-1,1]$, its Lebesgue integral $\int_0^t v(s)ds$ is finite.

Lemma 3.5. *Suppose that $v(t), w_*(t)$ are nonnegative and*

$$v(t) \le w_*(t) + c\int_0^t v(s)ds,\ t\in[0,T].$$

Then

$$v(t) \le w_*(t) + \widetilde{c}\int_0^t w_*(s)ds,\ \ t \in [-1,1].$$

In our analysis, we will need the following estimate for the Lagrange interpolation associated with the Jacobi Gaussian collocation points.

Lemma 3.6. *For every bounded function v, there exists a constant C independent of v such that*

$$\|I_N^{\alpha,\beta}v(t)\|_{L^\infty} = \left\|\sum_{j=0}^{N}|v(t_j)F_j(t)|\right\|_{L^\infty} \le \begin{cases} c\log N\|v\|_{L^\infty} & -1<\alpha,\beta\le -\dfrac{1}{2}, \\ cN^{\gamma+\frac{1}{2}}\|v\|_{L^\infty}, & \gamma = max(\alpha,\beta), otherwise \end{cases}$$

where $F_j(t)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{t_j\}_{j=0}^N$.

Proof. It is obvious that

$$\begin{aligned} \|I_N^{\alpha,\beta}v(t)\|_{L^\infty} &= \left\|\sum_{j=0}^{N}|v(t_j)F_j(t)|\right\|_{L^\infty} \\ &\le \max_{t\in[-1,1]}\sum_{j=0}^{N}|v(t_j)||F_j(t)| \\ &\le \left(\max_{t\in[-1,1]}\sum_{j=0}^{N}|F_j(t)|\right)\|v\|_{L^\infty}. \end{aligned} \tag{3.8}$$

By Lemma 3.3, we obtain the desired result. □

Lemma 3.7. *For every bounded function v, there exists a constant C independent of v such that*

$$\|I_N^{\alpha,\beta}v(t)\|_{L^2_{\omega_{\alpha,\beta}}} \le c\|v\|_{L^\infty},$$

where $F_j(t)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{t_j\}_{j=0}^N$.

Proof. It is obvious that

$$\|I_N^{\alpha,\beta}v(t)\|^2_{L^2_{\omega_{\alpha,\beta}}} = \int_{-1}^{1}(I_N^{\alpha,\beta}v)^2\omega^{\alpha,\beta}dt = \sum_{j=0}^{N}v^2(t_j)\omega_j \le \|v\|^2_{L^\infty}\sum_{j=0}^{N}\omega_j = \gamma_0\|v\|^2_{L^\infty}.$$

where $\gamma_0 = (J_0^{\alpha,\beta}, J_0^{\alpha,\beta})_{\omega_{\alpha,\beta}}$. As a consequence,

$$\sup_N\|I_N^{\alpha,\beta}v(t)\|_{L^2_{\omega_{\alpha,\beta}}} \le C\|v\|_{L^\infty},$$

with $C = \sqrt{\gamma_0}$. □

4. Convergence for Jacobi pseudo-spectral method

As $I_N^{\alpha,\beta}$ is the interpolation operator which is based on the (N + 1)-degree Jacobi-Gauss points with weight $\omega_{\alpha,\beta}$, in terms of (2.13), (2.14) and (2.15), the pseudo-spectral solution u_N, v_N, w_N satisfies

$$(4.1)\quad \begin{cases} (u_N,\phi)_{\omega_{\alpha,\beta}} - (I_N^{\alpha,\beta}\mathcal{M}_N v_N,\phi)_{\omega_{\alpha,\beta}} = (I_N^{\alpha,\beta}y_0,\phi)_{\omega_{\alpha,\beta}}, \\ (v_N,\varphi)_{\omega_{\alpha,\beta}} - (I_N^{\alpha,\beta}\mathcal{M}_N w_N,\varphi)_{\omega_{\alpha,\beta}} = (I_N^{\alpha,\beta}y_0',\varphi)_{\omega_{\alpha,\beta}}, \\ (w_N - I_N^{\alpha,\beta}(\widetilde{\mathcal{M}}_N u_N + \overline{\mathcal{M}}_N v_N + \widehat{\mathcal{M}}_N w_N),\psi)_{\omega_{\alpha,\beta}} = (I_N^{\alpha,\beta}g(t),\psi)_{\omega_{\alpha,\beta}}, \\ \hfill \forall\phi,\varphi,\psi\in\mathcal{P}_N(\Lambda). \end{cases}$$

where

$$\mathcal{M}_N v_N = \mathcal{M}v_N - (\mathcal{M}v_N - \mathcal{M}_N v_N) = \mathcal{M}v_N - Q(t),$$

with

$$\begin{aligned} Q(t) &= \mathcal{M}v_N - \mathcal{M}_N v_N \\ (4.2)\qquad &= \int_{-1}^{1}\Big(\frac{t+1}{2}\Big)v_N(\tau(t,\theta))d\theta - \sum_{j=0}^{N}\Big(\frac{t+1}{2}\Big)v_N(\tau(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)\omega_j \\ &= \Big(\Big(\frac{t+1}{2}\Big)\omega_{-\alpha,-\beta}, v_N(\tau(t,\cdot))\Big)_{\omega_{\alpha,\beta}} - \Big(\Big(\frac{t+1}{2}\Big)\omega_{-\alpha,-\beta}, v_N(\tau(t,\cdot))\Big)_N, \end{aligned}$$

$$Q_1(t) = \Big(\Big(\frac{t+1}{2}\Big)\omega_{-\alpha,-\beta}, w_N(\tau(t,\cdot))\Big)_{\omega_{\alpha,\beta}} - \Big(\Big(\frac{t+1}{2}\Big)\omega_{-\alpha,-\beta}, w_N(\tau(t,\cdot))\Big)_N,$$

in which $(\cdot,\cdot)_{\omega_{\alpha,\beta}}$ represents the continuous inner product with respect to θ, and $(\cdot,\cdot)_N$ is the corresponding discrete inner product defined by the Gauss-Jacobi quadrature formula. Similar to (4.2),we have that

$$\overline{\mathcal{M}}_N v_N = \overline{\mathcal{M}}v_N - (\overline{\mathcal{M}}v_N - \overline{\mathcal{M}}_N v_N) = \overline{\mathcal{M}}v_N - \overline{Q}(t),$$

with

$$\begin{aligned} \overline{Q}(t) &= \overline{\mathcal{M}}v_N - \overline{\mathcal{M}}_N v_N \\ &= \int_{-1}^{1}\Big(\frac{t+1}{2}\Big)\widetilde{a}(t)v_N(\tau(t,\theta))d\theta \\ (4.3)\qquad & - \sum_{j=0}^{N}\Big(\frac{t+1}{2}\Big)\widetilde{a}(t)v_N(\tau(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)\omega_j \\ &= \Big(\Big(\frac{t+1}{2}\Big)\widetilde{a}(t)\omega_{-\alpha,-\beta}, v_N(\tau(t,\cdot))\Big)_{\omega_{\alpha,\beta}} - \Big(\Big(\frac{t+1}{2}\Big)\widetilde{a}(t)\omega_{-\alpha,-\beta}, v_N(\tau(t,\cdot))\Big)_N, \end{aligned}$$

$$\widehat{\mathcal{M}}_N w_N = \widehat{\mathcal{M}}w_N - (\widehat{\mathcal{M}}w_N - \widehat{\mathcal{M}}_N w_N) = \widehat{\mathcal{M}}w_N - \widehat{Q}(t),$$

with

$$\begin{aligned} \widehat{Q}(t) &= \widehat{\mathcal{M}}w_N - \widehat{\mathcal{M}}_N w_N \\ &= \int_{-1}^{1}\Big(\frac{t+1}{2}\Big)\widetilde{b}(t)w_N(\tau(t,\theta))d\theta \end{aligned}$$

$$
\begin{aligned}
(4.4) \qquad & -\sum_{j=0}^{N}\Big(\frac{t+1}{2}\Big)\widetilde{b}(t)w_N(\tau(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)\omega_j \\
= \;& \Big(\Big(\frac{t+1}{2}\Big)\widetilde{b}(t)\omega_{-\alpha,-\beta}, w_N(\tau(t,\cdot))\Big)_{\omega_{\alpha,\beta}} - \Big(\Big(\frac{t+1}{2}\Big)\widetilde{b}(t)\omega_{-\alpha,-\beta}, w_N(\tau(t,\cdot))\Big)_N,
\end{aligned}
$$

and

$$
\widetilde{\mathcal{M}}_N u_N = \widetilde{\mathcal{M}}u_N - (\widetilde{\mathcal{M}}u_N - \widetilde{\mathcal{M}}_N u_N) = \widetilde{\mathcal{M}}u_N - \widetilde{Q}(t),
$$

with

$$
\begin{aligned}
\widetilde{Q}(t) &= \widetilde{\mathcal{M}}u_N - \widetilde{\mathcal{M}}_N u_N \\
&= \int_{-1}^{1}\Big(\frac{t+1}{2}\Big)\widetilde{K}(t,\tau(t,\theta))u_N(\tau(t,\theta))d\theta \\
&\quad -\sum_{j=0}^{N}\Big(\frac{t+1}{2}\Big)\widetilde{K}(t,\tau(t,\theta_j))\omega_{-\alpha,-\beta}(\theta_j)u_N(\tau(t,\theta_j))\omega_j \qquad (4.5)\\
&= \Big\{\Big(\Big(\frac{t+1}{2}\Big)\widetilde{K}(t,\tau(t,\cdot))\omega_{-\alpha,-\beta}(\cdot), u_N(\tau(t,\cdot))\Big)_{\omega_{\alpha,\beta}} \\
&\quad -\Big(\Big(\frac{t+1}{2}\Big)\widetilde{K}(t,\tau(t,\cdot))\omega_{-\alpha,-\beta}(\cdot), u_N(\tau(t,\cdot))\Big)_N\Big\},
\end{aligned}
$$

The combination of (4.1)-(4.5) yields

$$
\left\{
\begin{array}{l}
(u_N + I_N^{\alpha,\beta}Q(t) - I_N^{\alpha,\beta}\mathcal{M}v_N, \phi)_{\omega_{\alpha,\beta}} = (I_N^{\alpha,\beta}y_0, \phi)_{\omega_{\alpha,\beta}}, \\
(v_N + I_N^{\alpha,\beta}Q_1(t) - I_N^{\alpha,\beta}\mathcal{M}w_N, \varphi)_{\omega_{\alpha,\beta}} = (I_N^{\alpha,\beta}y_0', \varphi)_{\omega_{\alpha,\beta}}, \\
(w_N + I_N^{\alpha,\beta}\widetilde{Q}(t) - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}u_N + I_N^{\alpha,\beta}\overline{Q}(t) - I_N^{\alpha,\beta}\overline{\mathcal{M}}v_N + I_N^{\alpha,\beta}\widehat{Q}(t) \\
-I_N^{\alpha,\beta}\widehat{\mathcal{M}}w_N, \psi)_{\omega_{\alpha,\beta}} = (I_N^{\alpha,\beta}g(t), \psi)_{\omega_{\alpha,\beta}}.
\end{array}
\right.
$$

which gives rise to

$$
(4.6) \qquad
\begin{cases}
u_N + I_N^{\alpha,\beta}Q(t) - I_N^{\alpha,\beta}\mathcal{M}v_N = I_N^{\alpha,\beta}y_0, \\
v_N + I_N^{\alpha,\beta}Q_1(t) - I_N^{\alpha,\beta}\mathcal{M}w_N = I_N^{\alpha,\beta}y_0', \\
w_N + I_N^{\alpha,\beta}\widetilde{Q}(t) - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}u_N \\
+I_N^{\alpha,\beta}\overline{Q}(t) - I_N^{\alpha,\beta}\overline{\mathcal{M}}v_N + I_N^{\alpha,\beta}\widehat{Q}(t) - I_N^{\alpha,\beta}\widehat{\mathcal{M}}w_N = I_N^{\alpha,\beta}g(t).
\end{cases}
$$

By the discussion above, (2.15), (4.1) and (4.6) are equivalent.

We first consider an auxiliary problem. We want to find $\widehat{u}_N, \widehat{v}_N, \widehat{w}_N \in \mathcal{P}_N(\Lambda)$ such that

$$
(4.7) \qquad
\begin{cases}
(\widehat{u}_N, \phi)_N - (\mathcal{M}\widehat{v}_N, \phi)_N = (y_0, \phi)_N, \\
(\widehat{v}_N, \varphi)_N - (\mathcal{M}\widehat{w}_N, \varphi)_N = (y_0', \varphi)_N, \ \forall \phi, \varphi, \psi \in \mathcal{P}_N(\Lambda) \\
(\widehat{w}_N - \widetilde{\mathcal{M}}\widehat{u}_N - \overline{\mathcal{M}}\widehat{v}_N - \widehat{\mathcal{M}}\widehat{w}_N, \psi)_N = (g(t), \psi)_N,
\end{cases}
$$

where $\mathcal{M}, \overline{\mathcal{M}}, \widetilde{\mathcal{M}}$ and $\widehat{\mathcal{M}}$ are integral operators defined in Sect 2, and $(\cdot,\cdot)_N$ is still the discrete inner product based on the (N +1)-degree Jacobi-Gauss points. In terms of the definition of $I_N^{\alpha,\beta}$, (4.7) can be written as

$$
(4.8) \qquad
\begin{cases}
\widehat{u}_N - I_N^{\alpha,\beta}\mathcal{M}\widehat{v}_N = y_0, \ \widehat{v}_N - I_N^{\alpha,\beta}\mathcal{M}\widehat{w}_N = y_0', \\
\widehat{w}_N - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\widehat{u}_N - I_N^{\alpha,\beta}\overline{\mathcal{M}}\widehat{v}_N - I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{w}_N = I_N^{\alpha,\beta}g(t).
\end{cases}
$$

When $y_0 = y_0' = g = 0$, (4.8) can be written as

$$\begin{cases} \widehat{u}_N - I_N^{\alpha,\beta}\mathcal{M}\widehat{v}_N = 0, \ \widehat{v}_N - I_N^{\alpha,\beta}\mathcal{M}\widehat{w}_N = 0, \\ \widehat{w}_N - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\widehat{u}_N - I_N^{\alpha,\beta}\overline{\mathcal{M}}\widehat{v}_N - I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{w}_N = 0. \end{cases}$$

In terms of the fact that

$$\begin{cases} \widehat{u}_N - I_N^{\alpha,\beta}\mathcal{M}\widehat{v}_N = \widehat{u}_N - \mathcal{M}\widehat{v}_N + (\mathcal{M}\widehat{v}_N - I_N^{\alpha,\beta}\mathcal{M}\widehat{v}_N), \\ \widehat{w}_N - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\widehat{u}_N - I_N^{\alpha,\beta}\overline{\mathcal{M}}\widehat{v}_N - I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{w}_N \\ \qquad = \begin{cases} \widehat{w}_N - \widetilde{\mathcal{M}}\widehat{u}_N + (\widetilde{\mathcal{M}}\widehat{u}_N - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\widehat{u}_N) \\ -\overline{\mathcal{M}}\widehat{v}_N + (\overline{\mathcal{M}}\widehat{v}_N - I_N^{\alpha,\beta}\overline{\mathcal{M}}\widehat{v}_N) \\ -\widehat{\mathcal{M}}\widehat{w}_N + (\widehat{\mathcal{M}}\widehat{w}_N - I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{w}_N). \end{cases} \end{cases}$$

Suppose that $\max\{|\widetilde{K}(t,\tau)|, |\widetilde{a}(t)|, |\widetilde{b}(t)|, 1\} \leq L$. It is clear that from (2.5)-(2.8)

$$\begin{cases} \widehat{u}_N = \dfrac{T}{2}\displaystyle\int_{-1}^{t} \widehat{v}_N(\tau)d\tau + I_N^{\alpha,\beta}\mathcal{M}\widehat{v}_N - \mathcal{M}\widehat{v}_N, \\ \widehat{v}_N = \dfrac{T}{2}\displaystyle\int_{-1}^{t} \widehat{w}_N(\tau)d\tau + I_N^{\alpha,\beta}\mathcal{M}\widehat{w}_N - \mathcal{M}\widehat{w}_N, \\ \widehat{w}_N = \dfrac{T}{2}\displaystyle\int_{-1}^{t} (\widetilde{K}(t,\tau)\widehat{u}_N(\tau) + \widetilde{a}(t)\widehat{v}_N(\tau) + \widetilde{b}(t)\widehat{w}_N(\tau))d\tau + I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\widehat{u}_N \\ \qquad -\widetilde{\mathcal{M}}\widehat{u}_N + I_N^{\alpha,\beta}\overline{\mathcal{M}}\widehat{v}_N - \overline{\mathcal{M}}\widehat{v}_N + I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{w}_N - \widehat{\mathcal{M}}\widehat{w}_N. \end{cases}$$

which yields

$$\begin{aligned} (|\widehat{u}_N| + |\widehat{v}_N| + |\widehat{w}_N|) &\leq c\int_{-1}^{t} (|\widehat{u}_N(\tau)| + |\widehat{v}_N| + |\widehat{w}_N(\tau)|)d\tau \\ &\quad + |I_1| + |I_2| + |I_3| + |I_4| + |I_5|. \end{aligned}$$

where $I_1 = I_N^{\alpha,\beta}\mathcal{M}\widehat{v}_N - \mathcal{M}\widehat{v}_N, I_2 = I_N^{\alpha,\beta}\mathcal{M}\widehat{w}_N - \mathcal{M}\widehat{w}_N, I_3 = I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\widehat{u}_N - \widetilde{\mathcal{M}}\widehat{u}_N, I_4 = I_N^{\alpha,\beta}\overline{\mathcal{M}}\widehat{v}_N - \overline{\mathcal{M}}\widehat{v}_N, I_5 = I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{w}_N - \widehat{\mathcal{M}}\widehat{w}_N$. Using Lemma 3.5 leads to

$$\begin{aligned} (|\widehat{u}_N| + |\widehat{v}_N| + |\widehat{w}_N|) &\leq c\int_{-1}^{t} (|I_1| + |I_2| + |I_3| + |I_4| + |I_5|)ds \\ &\quad + |I_1| + |I_2| + |I_3| + |I_4| + |I_5| \\ &\leq c(\|I_1\|_{L^\infty} + \|I_2\|_{L^\infty} + \|I_3\|_{L^\infty} + \|I_4\|_{L^\infty} + \|I_5\|_{L^\infty}). \end{aligned}$$

This gives,

$$\begin{aligned} &\| |\widehat{u}_N| + |\widehat{v}_N| + |\widehat{w}_N| \|_{L^\infty} \\ &\quad \leq c(\|I_1\|_{L^\infty} + \|I_2\|_{L^\infty} + \|I_3\|_{L^\infty} + \|I_4\|_{L^\infty} + \|I_5\|_{L^\infty}). \end{aligned} \tag{4.9}$$

We now estimate $\|I_1\|_{L^\infty}, \|I_2\|_{L^\infty}, \|I_3\|_{L^\infty}, \|I_4\|_{L^\infty}$ and $\|I_5\|_{L^\infty}$. By virtue of (3.6), (3.7) and Lemma 3.3, we obtain that

$$\begin{aligned} \|I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\widehat{u}_N - \widetilde{\mathcal{M}}\widehat{u}_N\|_{L^\infty} &= \|(I - I_N^{\alpha,\beta})\widetilde{\mathcal{M}}\widehat{u}_N\|_{L^\infty} \\ &= \|(I - I_N^{\alpha,\beta})(\widetilde{\mathcal{M}}\widehat{u}_N - \tau_N\widetilde{\mathcal{M}}\widehat{u}_N)\|_{L^\infty} \\ &\leq (1 + \|I_N^{\alpha,\beta}\|_{L^\infty})\|\widetilde{\mathcal{M}}\widehat{u}_N - \tau_N\widetilde{\mathcal{M}}\widehat{u}_N\|_{L^\infty} \end{aligned}$$

$$\leq \begin{cases} c\log N\|\widetilde{\mathcal{M}}\widehat{u}_N - \tau_N\widetilde{\mathcal{M}}\widehat{u}_N\|_{L^\infty} \leq c\log N N^{-\kappa}\|\widehat{u}_N\|_{L^\infty}, \\ \ -1<\alpha,\beta\leq -\frac{1}{2} \\ cN^{\frac{1}{2}+\gamma}\|\widetilde{\mathcal{M}}\widehat{u}_N - \tau_N\widetilde{\mathcal{M}}\widehat{u}_N\|_{L^\infty} \leq cN^{\frac{1}{2}-\kappa+\gamma}\|\widehat{u}_N\|_{L^\infty}, \\ \ \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases}$$

Similarly,

$$\|I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{w}_N - \widehat{\mathcal{M}}\widehat{w}_N\|_{L^\infty} \leq \begin{cases} c\log N N^{-\kappa}\|\widehat{w}_N\|_{L^\infty},\ -1<\alpha,\beta\leq -\frac{1}{2} \\ cN^{\frac{1}{2}-\kappa+\gamma}\|\widehat{w}_N\|_{L^\infty},\ \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases}$$

$$\|I_N^{\alpha,\beta}\overline{\mathcal{M}}\widehat{v}_N - \overline{\mathcal{M}}\widehat{v}_N\|_{L^\infty} \leq \begin{cases} c\log N N^{-\kappa}\|\widehat{v}_N\|_{L^\infty},\ -1<\alpha,\beta\leq -\frac{1}{2} \\ cN^{\frac{1}{2}-\kappa+\gamma}\|\widehat{v}_N\|_{L^\infty},\ \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases}$$

and

$$\|I_N^{\alpha,\beta}\mathcal{M}\widehat{w}_N - \mathcal{M}\widehat{w}_N\|_{L^\infty} \leq \begin{cases} c\log N N^{-\kappa}\|\widehat{w}_N\|_{L^\infty},\ -1<\alpha,\beta\leq -\frac{1}{2} \\ cN^{\frac{1}{2}-\kappa+\gamma}\|\widehat{w}_N\|_{L^\infty},\ \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases}$$

These, together with (4.9), give

$$\||\widehat{u}_N|+|\widehat{v}_N|+|\widehat{w}_N|\|_{L^\infty} \leq \begin{cases} c\log N N^{-\kappa}\||\widehat{u}_N|+|\widehat{v}_N|+|\widehat{w}_N|\|_{L^\infty},\ -1<\alpha,\beta\leq -\frac{1}{2} \\ cN^{\frac{1}{2}-\kappa+\gamma}\||\widehat{u}_N|+|\widehat{v}_N|+|\widehat{w}_N|\|_{L^\infty},\ \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases}$$

which implies, taking $\kappa\in(0,1)$ such that $\kappa>\frac{1}{2}+\gamma$, when N is large enough, $\widehat{u}_N=\widehat{v}_N=\widehat{w}_N=0$. Hence, $\widehat{u}_N,\widehat{v}_N$ and $\widehat{w}_N$ are existent and unique as $\mathcal{P}_N(\Lambda)$ is finite-dimensional.

Lemma 4.1. *Suppose that $u\in H^m_{\omega_{m+\alpha,m+\beta}}(\Lambda)$ and max$\{|\widetilde{K}(t,\tau)|,|\widetilde{a}(t)|,|\widetilde{b}(t)|,1\}\leq L$, then we have*

$$\||u-\widehat{u}_N|+|v-\widehat{v}_N|+|w-\widehat{w}_N|\|_{L^\infty} \leq \begin{cases} c\log N N^{\frac{3}{4}-m}\|u\|_{m+2,\infty},\ -1<\alpha,\beta\leq -\frac{1}{2} \\ cN^{\frac{5}{4}-m+\gamma}\|u\|_{m+2,\infty},\ \gamma=max(\alpha,\beta), \textit{otherwise}. \end{cases} \tag{4.10}$$

and

$$\||u-\widehat{u}_N|+|v-\widehat{v}_N|+|w-\widehat{w}_N|\|_{L^2_{\omega_{\alpha,\beta}}(\Lambda)} \leq \begin{cases} cN^{-m}\Big(\sum_{k=0}^{2}\|\partial_t^{m+k}u\|_{\omega_{m+\alpha,m+\beta}}\Big) \\ \quad +c\log N N^{\frac{3}{4}-m}\|u\|_{m+2,\infty},\ -1<\alpha,\beta\leq -\frac{1}{2}, \\ cN^{-m}\Big(\sum_{k=0}^{2}\|\partial_t^{m+k}u\|_{\omega_{m+\alpha,m+\beta}}\Big) \\ \quad +cN^{\frac{5}{4}-m+\gamma}\|u\|_{m+2,\infty},\ \gamma=max(\alpha,\beta), \textit{otherwise}. \end{cases} \tag{4.11}$$

Proof. Subtracting (4.8) from (2.2) yields

$$\begin{cases} u(t) - \widehat{u}_N + I_N^{\alpha,\beta}\mathcal{M}\widehat{v}_N - \mathcal{M}v(t) = 0, \\ v(t) - \widehat{v}_N + I_N^{\alpha,\beta}\mathcal{M}\widehat{w}_N - \mathcal{M}w(t) = 0, \\ w(t) - \widehat{w}_N + I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\widehat{u}_N - \widetilde{\mathcal{M}}u + I_N^{\alpha,\beta}\overline{\mathcal{M}}\widehat{v}_N - \overline{\mathcal{M}}v(t) \\ +I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{w}_N - \widehat{\mathcal{M}}w(t) \\ = g(t) - I_N^{\alpha,\beta}g(t). \end{cases} \tag{4.12}$$

Set $\epsilon = u(t) - \widehat{u}_N, \overline{\epsilon} = v(t) - \widehat{v}_N, \widehat{\epsilon} = w(t) - \widehat{w}_N$. Direct computation shows that

$$\begin{aligned} \mathcal{M}v(t) - I_N^{\alpha,\beta}\mathcal{M}\widehat{v}_N &= \mathcal{M}v - I_N^{\alpha,\beta}\mathcal{M}v + I_N^{\alpha,\beta}\mathcal{M}(v - \widehat{v}_N) \\ &= \mathcal{M}v - I_N^{\alpha,\beta}\mathcal{M}v + \mathcal{M}(v - \widehat{v}_N) \\ &\quad -[\mathcal{M}(v - \widehat{v}_N) - I_N^{\alpha,\beta}\mathcal{M}(v - \widehat{v}_N)] \\ &= u - y_0 - I_N^{\alpha,\beta}(u - y_0) + \mathcal{M}(v - \widehat{v}_N) \\ &\quad -[\mathcal{M}(v - \widehat{v}_N) - I_N^{\alpha,\beta}\mathcal{M}(v - \widehat{v}_N)] \\ &= u - I_N^{\alpha,\beta}u + \mathcal{M}\overline{\epsilon} - [\mathcal{M}\overline{\epsilon} - I_N^{\alpha,\beta}\mathcal{M}\overline{\epsilon}]. \end{aligned} \tag{4.13}$$

Similarly

$$\begin{aligned} &\widetilde{\mathcal{M}}u - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\widehat{u}_N + \overline{\mathcal{M}}v - I_N^{\alpha,\beta}\overline{\mathcal{M}}\widehat{v}_N + \widehat{\mathcal{M}}w(t) - I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{w}_N \\ &\quad = w - I_N^{\alpha,\beta}w - g(t) + I_N^{\alpha,\beta}g(t) + \widetilde{\mathcal{M}}\epsilon - [\widetilde{\mathcal{M}}\epsilon - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\epsilon] \\ &\qquad + \overline{\mathcal{M}}\overline{\epsilon} - [\overline{\mathcal{M}}\overline{\epsilon} - I_N^{\alpha,\beta}\overline{\mathcal{M}}\overline{\epsilon}] + \widehat{\mathcal{M}}\widehat{\epsilon} - [\widehat{\mathcal{M}}\widehat{\epsilon} - I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{\epsilon}]. \end{aligned} \tag{4.14}$$

The insertion of (4.13), (4.14) into (4.12) yields

$$\begin{cases} \epsilon = u - I_N^{\alpha,\beta}u + \mathcal{M}\overline{\epsilon} - [\mathcal{M}\overline{\epsilon} - I_N^{\alpha,\beta}\mathcal{M}\overline{\epsilon}], \\ \overline{\epsilon} = v - I_N^{\alpha,\beta}v + \mathcal{M}\widehat{\epsilon} - [\mathcal{M}\widehat{\epsilon} - I_N^{\alpha,\beta}\mathcal{M}\widehat{\epsilon}], \\ \widehat{\epsilon} = w - I_N^{\alpha,\beta}w + \widetilde{\mathcal{M}}\epsilon - [\widetilde{\mathcal{M}}\epsilon - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\epsilon] + \overline{\mathcal{M}}\overline{\epsilon} - [\overline{\mathcal{M}}\overline{\epsilon} - I_N^{\alpha,\beta}\overline{\mathcal{M}}\overline{\epsilon}] \\ \quad +\widehat{\mathcal{M}}\widehat{\epsilon} - [\widehat{\mathcal{M}}\widehat{\epsilon} - I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{\epsilon}], \end{cases}$$

which implies that

$$\begin{aligned} |\epsilon| + |\overline{\epsilon}| + |\widehat{\epsilon}| &\le |J_1| + |J_2| + |J_3| + |J_4| + |J_5| + |J_6| + |J_7| + |J_8| \\ &\quad +c\int_{-1}^{t}(|\epsilon(\tau)| + |\overline{\epsilon}(s)| + |\widehat{\epsilon}(\tau)|)d\tau, \end{aligned} \tag{4.15}$$

where $J_1 = u - I_N^{\alpha,\beta}u,\ J_2 = v - I_N^{\alpha,\beta}v,\ J_3 = w - I_N^{\alpha,\beta}w,\ J_4 = \widetilde{\mathcal{M}}\epsilon - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\epsilon,\ J_5 = \overline{\mathcal{M}}\overline{\epsilon} - I_N^{\alpha,\beta}\overline{\mathcal{M}}\overline{\epsilon},\ J_6 = \widehat{\mathcal{M}}\widehat{\epsilon} - I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{\epsilon},\ J_7 = \mathcal{M}\widehat{\epsilon} - I_N^{\alpha,\beta}\mathcal{M}\widehat{\epsilon},\ J_8 = \mathcal{M}\overline{\epsilon} - I_N^{\alpha,\beta}\mathcal{M}\overline{\epsilon}$. Using Lemma 3.5 gives

$$\begin{aligned} |\epsilon| + |\overline{\epsilon}| + |\widehat{\epsilon}| &\le |J_1| + |J_2| + |J_3| + |J_4| + |J_5| + |J_6| + |J_7| + |J_8| \\ &\quad +c\int_{-1}^{t}(|J_1| + |J_2| + |J_3| + |J_4| + |J_5| + |J_6| + |J_7| + |J_8|)d\tau. \end{aligned} \tag{4.16}$$

Similar to (4.9), we have that

$$\begin{aligned} \||\epsilon|+|\overline{\epsilon}|+|\widehat{\epsilon}|\|_{L^\infty} &\le c(\|J_1\|_{L^\infty}+\|J_2\|_{L^\infty}+\|J_3\|_{L^\infty}+\|J_4\|_{L^\infty} \\ &\quad +\|J_5\|_{L^\infty}+\|J_6\|_{L^\infty}+\|J_7\|_{L^\infty}+\|J_8\|_{L^\infty}). \end{aligned} \tag{4.17}$$

By using (3.2), Lemma 3.3, we obtain that

$$\begin{aligned} \|u-I_N^{\alpha,\beta}u\|_{L^\infty} &= \|(I-I_N^{\alpha,\beta})(u-P_Nu)\|_{L^\infty} \\ &\le c(1+\|I_N^{\alpha,\beta}\|_\infty)\|u-P_Nu\|_{L^\infty} \\ &\le \begin{cases} c\log N N^{\frac{3}{4}-m}\|u\|_{m,\infty}, & -1<\alpha,\beta\le -\frac{1}{2} \\ cN^{\frac{5}{4}-m+\gamma}\|u\|_{m,\infty}, & \gamma=\max(\alpha,\beta),\text{otherwise.} \end{cases} \end{aligned} \tag{4.18}$$

$$\begin{aligned} &\|v-I_N^{\alpha,\beta}v\|_{L^\infty} \\ &\le \begin{cases} c\log N N^{\frac{3}{4}-m}\|v\|_{m,\infty}\le c\log N N^{\frac{3}{4}-m}\|u'\|_{m,\infty}, & -1<\alpha,\beta\le -\frac{1}{2} \\ cN^{\frac{5}{4}-m+\gamma}\|v\|_{m,\infty}\le cN^{\frac{5}{4}-m+\gamma}\|u'\|_{m,\infty}, & \gamma=\max(\alpha,\beta),\text{otherwise.} \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\|w-I_N^{\alpha,\beta}w\|_{L^\infty} \\ &\le \begin{cases} c\log N N^{\frac{3}{4}-m}\|w\|_{m,\infty}\le c\log N N^{\frac{3}{4}-m}\|u''\|_{m,\infty}, & -1<\alpha,\beta\le -\frac{1}{2} \\ cN^{\frac{5}{4}-m+\gamma}\|w\|_{m,\infty}\le cN^{\frac{5}{4}-m+\gamma}\|u''\|_{m,\infty}, & \gamma=\max(\alpha,\beta),\text{otherwise.} \end{cases} \end{aligned} \tag{4.19}$$

We now estimate J_4. It is clear that $\epsilon\in C[-1,1]$. Consequently, using (3.6), (3.7) and Lemma 3.3 it follows that

$$\begin{aligned} \|J_4\|_{L^\infty} &= \|(I-I_N^{\alpha,\beta})(\widetilde{M}\epsilon-\tau_N\widetilde{M}\epsilon)\|_{L^\infty} \\ &\le (1+\|I_N^{\alpha,\beta}\|_{L^\infty})\|\widetilde{M}\epsilon-\tau_N\widetilde{M}\epsilon\|_{L^\infty} \\ &\le c(1+\|I_N^{\alpha,\beta}\|_{L^\infty})N^{-\kappa}\|\widetilde{M}\epsilon\|_{0,\kappa} \\ &\le \begin{cases} c\log N N^{-\kappa}\||\epsilon|+|\overline{\epsilon}|+|\widehat{\epsilon}|\|_{L^\infty}, & -1<\alpha,\beta\le -\frac{1}{2} \\ cN^{\frac{1}{2}-\kappa+\gamma}\||\epsilon|+|\overline{\epsilon}|+|\widehat{\epsilon}|\|_{L^\infty}, & \gamma=\max(\alpha,\beta),\text{otherwise.} \end{cases} \end{aligned} \tag{4.20}$$

where $\kappa\in(0,1)$ and $\tau_N\mathcal{M}\epsilon\in\mathcal{P}_N(\Lambda)$. (4.20) also holds for $\|J_5\|_{L^\infty},\|J_6\|_{L^\infty},\|J_7\|_{L^\infty}$ and $\|J_8\|_{L^\infty}$. Taking $\kappa\in(0,1)$ such that $\kappa>\frac{1}{2}+\gamma$, the estimate (4.10) follows from (4.17)-(4.20), provider that N is large enough.

Next we prove (4.11). Using the standard Gronwall inequality, we have from (4.15) that

$$\begin{aligned} \||\epsilon|+|\overline{\epsilon}|+|\widehat{\epsilon}|\|^2_{L^2_{\omega_{\alpha,\beta}}} &\le c(\|J_1\|^2_{L^2_{\omega_{\alpha,\beta}}}+\|J_2\|^2_{L^2_{\omega_{\alpha,\beta}}}+\|J_3\|^2_{L^2_{\omega_{\alpha,\beta}}} \\ &\quad +\|J_4\|^2_{L^2_{\omega_{\alpha,\beta}}}+\|J_5\|^2_{L^2_{\omega_{\alpha,\beta}}}+\|J_6\|^2_{L^2_{\omega_{\alpha,\beta}}} \\ &\quad +\|J_7\|^2_{L^2_{\omega_{\alpha,\beta}}}+\|J_8\|^2_{L^2_{\omega_{\alpha,\beta}}}). \end{aligned} \tag{4.21}$$

We obtain that from (3.6), (3.7) and Lemma 3.7

$$\begin{aligned} \|J_4\|_{L^2_{\omega_{\alpha,\beta}}} &= \|(I-I_N^{\alpha,\beta})\widetilde{\mathcal{M}}\epsilon\|_{L^2_{\omega_{\alpha,\beta}}} \\ &= \|(I-I_N^{\alpha,\beta})(\widetilde{\mathcal{M}}\epsilon-\tau_N\widetilde{\mathcal{M}}\epsilon)\|_{L^2_{\omega_{\alpha,\beta}}} \end{aligned}$$

$$
\begin{aligned}
&\leq \; c\|\widetilde{\mathcal{M}}\epsilon - \tau_N \widetilde{\mathcal{M}}\epsilon\|_{L^\infty} \\
&\leq \; cN^{-\kappa}\|\epsilon\|_{L^\infty} \leq cN^{-\kappa}\||\epsilon| + |\overline{\epsilon}| + |\widehat{\epsilon}|\|_{L^\infty}.
\end{aligned}
$$

It also holds for $\|J_5\|_{L^2_{\omega_{\alpha,\beta}}}, \|J_6\|_{L^2_{\omega_{\alpha,\beta}}}, \|J_7\|_{L^2_{\omega_{\alpha,\beta}}}, \|J_8\|_{L^2_{\omega_{\alpha,\beta}}}$. These result, together with the estimates (3.3), (4.10) and (4.21), yields (4.11). □

Now subtracting (4.6) from (4.8) leads to

$$
\begin{cases}
\widehat{u}_N - u_N - I_N^{\alpha,\beta}Q(t) + I_N^{\alpha,\beta}\mathcal{M}v_N - I_N^{\alpha,\beta}\mathcal{M}\widehat{v}_N = 0, \\
\widehat{v}_N - v_N - I_N^{\alpha,\beta}Q_1(t) + I_N^{\alpha,\beta}\mathcal{M}w_N - I_N^{\alpha,\beta}\mathcal{M}\widehat{w}_N = 0, \\
\widehat{w}_N - w_N - I_N^{\alpha,\beta}\widetilde{Q}(t) - I_N^{\alpha,\beta}\overline{Q}(t) - I_N^{\alpha,\beta}\widehat{Q}(t) + I_N^{\alpha,\beta}\widetilde{\mathcal{M}}u_N - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}\widehat{u}_N \\
+I_N^{\alpha,\beta}\overline{\mathcal{M}}v_N - I_N^{\alpha,\beta}\overline{\mathcal{M}}\widehat{v}_N + I_N^{\alpha,\beta}\widehat{\mathcal{M}}w_N - I_N^{\alpha,\beta}\widehat{\mathcal{M}}\widehat{w}_N = 0,
\end{cases}
$$

which can be simplified as, by setting $E = \widehat{u}_N - u_N, E_1 = \widehat{v}_N - v_N, E_2 = \widehat{w}_N - w_N$

$$
\begin{cases}
E - I_N^{\alpha,\beta}Q(t) - I_N^{\alpha,\beta}\mathcal{M}E_1 = 0, \\
E_1 - I_N^{\alpha,\beta}Q_1(t) - I_N^{\alpha,\beta}\mathcal{M}E_2 = 0, \\
E_2 - I_N^{\alpha,\beta}\widetilde{Q}(t) - I_N^{\alpha,\beta}\overline{Q}(t) - I_N^{\alpha,\beta}\widehat{Q}(t) - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}E \\
-I_N^{\alpha,\beta}\overline{\mathcal{M}}E_1 - I_N^{\alpha,\beta}\widehat{\mathcal{M}}E_2 = 0.
\end{cases}
\tag{4.22}
$$

Let $e_N = u - u_N$, $\overline{e}_N = v - v_N$ and $\widehat{e}_N = w - w_N$ be the error corresponding to Jacobi pseudo-spectral solution u_N, v_N, w_N of (2.15). Now we are prepared to get our global convergence result for problem (2.2).

Theorem 4.2. *Suppose that* $max\{|\widetilde{K}(t,\tau)|, |\widetilde{a}(t)|, |\widetilde{b}(t)|, 1\} \leq L$ *and the solution of (2.2) is sufficiently smooth. For the Jacobi pseudo spectral solution defined in (2.15), we have the following estimates*

(1) L^∞ *norm of* $|e_N| + |\overline{e}_N| + |\widehat{e}_N|$ *satisfies,*

$$
\begin{aligned}
&\|(|e_N| + |\overline{e}_N| + |\widehat{e}_N|)\|_{L^\infty} \\
&\leq \begin{cases}
c\log N N^{\frac{3}{4}-m}\|u\|_{m+2,\infty} + c\log N N^{-m}\|u\|_{2,\infty}, \; -1 < \alpha, \beta \leq -\frac{1}{2} \\
cN^{\frac{5}{4}-m+\gamma}\|u\|_{m+2,\infty} + cN^{\frac{1}{2}-m+\gamma}\|u\|_{2,\infty}, \; \gamma = max(\alpha,\beta), \textit{otherwise.}
\end{cases}
\end{aligned}
\tag{4.23}
$$

(2) *The Jacobi spectral error* $|e_N| + |\overline{e}_N| + |\widehat{e}_N|$ *satisfies,*

$$
\begin{aligned}
&\||e_N| + |\overline{e}_N| + |\widehat{e}_N|\|_{L^2_{\omega_{\alpha,\beta}}} \\
&\leq \begin{cases}
c\log N N^{\frac{3}{4}-m}\|u\|_{m+2,\infty} + c\log N N^{-m}\|u\|_{2,\infty} \\
+N^{-m}\displaystyle\sum_{k=0}^{2}\|\partial_t^{m+k}u\|_{\omega_{m+\alpha,m+\beta}}, \; -1 < \alpha, \beta \leq -\frac{1}{2}, \\
+cN^{\frac{5}{4}-m+\gamma}\|u\|_{m+2,\infty} + cN^{\frac{1}{2}-m+\gamma}\|u\|_{2,\infty} \\
+cN^{-m}\displaystyle\sum_{k=0}^{2}\|\partial_t^{m+k}u\|_{\omega_{m+\alpha,m+\beta}}, \; \gamma = max(\alpha,\beta), \textit{otherwise.}
\end{cases}
\end{aligned}
\tag{4.24}
$$

Proof. We first prove the existence and uniqueness of the Jacobi pseudo-spectral solution u_N. As the dimension of $\mathcal{P}_N(\Lambda)$ is finite and (2.15) and (4.6) are equivalent, we only need to prove that the solution of (4.6) is $u_N = v_N = w_N = 0$ when $g = y_0 = y_0^{'} = 0$. For this purpose, we consider equations

$$\left\{ \begin{array}{l} u_N + I_N^{\alpha,\beta} Q(t) - I_N^{\alpha,\beta}\mathcal{M}v_N = 0,\ v_N + I_N^{\alpha,\beta}Q_1(t) - I_N^{\alpha,\beta}\mathcal{M}w_N = 0, \\ w_N + I_N^{\alpha,\beta}\widetilde{Q}(t) + I_N^{\alpha,\beta}\widehat{Q}(t) + I_N^{\alpha,\beta}\overline{Q}(t) - I_N^{\alpha,\beta}\widetilde{\mathcal{M}}u_N \\ \qquad\qquad\qquad\qquad -I_N^{\alpha,\beta}\overline{\mathcal{M}}v_N - I_N^{\alpha,\beta}\widehat{\mathcal{M}}w_N = 0. \end{array} \right. \tag{4.25}$$

Obviously (4.25) can be written as

$$\left\{ \begin{array}{l} u_N - \mathcal{M}v_N = I_N^{\alpha,\beta}\mathcal{M}v_N - I_N^{\alpha,\beta}Q(t) - \mathcal{M}v_N = R_1 + R_2, \\ v_N - \mathcal{M}w_N = I_N^{\alpha,\beta}\mathcal{M}w_N - I_N^{\alpha,\beta}Q_1(t) - \mathcal{M}w_N = R_3 + R_4, \\ w_N - \widetilde{\mathcal{M}}u_N - \overline{\mathcal{M}}v_N - \widehat{\mathcal{M}}w_N = I_N^{\alpha,\beta}\widetilde{\mathcal{M}}u_N + I_N^{\alpha,\beta}\overline{\mathcal{M}}v_N + I_N^{\alpha,\beta}\widehat{\mathcal{M}}w_N - I_N^{\alpha,\beta}\widetilde{Q}(t) \\ \qquad -I_N^{\alpha,\beta}\overline{Q}(t) - I_N^{\alpha,\beta}\widehat{Q}(t) - \widetilde{\mathcal{M}}u_N - \overline{\mathcal{M}}v_N - \widehat{\mathcal{M}}w_N \\ \quad = R_5 + R_6 + R_7 + R_8 + R_9 + R_{10}. \end{array} \right.$$

namely,

$$\left\{ \begin{array}{l} u_N = \dfrac{T}{2}\displaystyle\int_{-1}^{t} v_N(\tau)d\tau + R_1 + R_2,\ v_N = \dfrac{T}{2}\displaystyle\int_{-1}^{t} w_N(\tau)d\tau + R_3 + R_4, \\ w_N = \dfrac{T}{2}\displaystyle\int_{-1}^{t} (\widetilde{K}(t,s)u_N(\tau) + \widetilde{a}(t)v_N(\tau) + \widetilde{b}(t)w_N(\tau))d\tau \\ \qquad +R_5 + R_6 + R_7 + R_8 + R_9 + R_{10}. \end{array} \right. \tag{4.26}$$

with $R_1 = I_N^{\alpha,\beta}\mathcal{M}v_N - \mathcal{M}v_N,\ R_2 = -I_N^{\alpha,\beta}Q(t), R_3 = I_N^{\alpha,\beta}\mathcal{M}w_N - \mathcal{M}w_N,\ R_4 = -I_N^{\alpha,\beta}Q_1(t),\ R_5 = I_N^{\alpha,\beta}\widetilde{\mathcal{M}}u_N - \widetilde{\mathcal{M}}u_N, R_6 = I_N^{\alpha,\beta}\overline{\mathcal{M}}v_N - \overline{\mathcal{M}}v_N,\ R_7 = I_N^{\alpha,\beta}\widehat{\mathcal{M}}w_N - \widehat{\mathcal{M}}w_N, R_8 = -I_N^{\alpha,\beta}\widetilde{Q}(t), R_9 = -I_N^{\alpha,\beta}\overline{Q}(t), R_{10} = -I_N^{\alpha,\beta}\widehat{Q}(t)$. Using (4.26) gives

$$\begin{aligned} (|u_N| + |v_N| + |w_N|) \quad &\le \quad c\int_{-1}^{t} (|u_N(\tau)| + |v_N(\tau)| + |w_N(\tau)|)d\tau \\ &\quad +|R_1| + |R_2| + |R_3| + |R_4| + |R_5| + |R_6| \\ &\quad +|R_7| + |R_8| + |R_9| + |R_{10}|. \end{aligned} \tag{4.27}$$

Using Lemma 3.5 yields

$$\begin{aligned} \|(|u_N| + |v_N| + |w_N|)\|_{L^\infty} \quad &\le \quad c(\|R_1\|_{L^\infty} + \|R_2\|_{L^\infty} + \|R_3\|_{L^\infty} + \|R_4\|_{L^\infty} \\ &\quad +\|R_5\|_{L^\infty} + \|R_6\|_{L^\infty} + \|R_7\|_{L^\infty} \\ &\quad +\|R_8\|_{L^\infty} + \|R_9\|_{L^\infty} + \|R_{10}\|_{L^\infty}). \end{aligned} \tag{4.28}$$

On the other hand, according to Lemma 3.3,

$$\begin{aligned} \|R_8\|_{L^\infty}^2 \quad &= \quad \|I_N^{\alpha,\beta}\widetilde{Q}(t)\|_{L^\infty}^2 \\ &\le \left\{ \begin{array}{l} c(\log N)^2\|\widetilde{Q}(t)\|_{L^\infty}^2,\ -1 < \alpha,\beta \le -\frac{1}{2} \\ cN^{1+2\gamma}\|\widetilde{Q}(t)\|_{L^\infty}^2,\ \gamma = \max(\alpha,\beta), \text{otherwise}. \end{array} \right. \end{aligned} \tag{4.29}$$

By the expression of $\widetilde{Q}(t)$ in (4.5), we have from Lemma 3.2

$$\begin{aligned} |\widetilde{Q}(t)| \quad &\le \quad cN^{-m}\Big\|\partial_\theta^m\Big(\Big(\frac{t+1}{2}\Big)\widetilde{K}(t,\tau(t,\theta)\Big)\omega_{-\alpha,-\beta}(\theta)\Big\|_{L^2_{\omega_{m+\alpha,m+\beta}}}\|u_N\|_{L^2_{\omega_{\alpha,\beta}}} \\ &\le \quad cN^{-m}\|u_N\|_{L^2_{\omega_{\alpha,\beta}}}, \end{aligned}$$

which, together with (4.29), gives

$$\|R_8\|_{L^\infty} \le \begin{cases} c\log N N^{-m}\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty}, & -1<\alpha,\beta\le -\frac{1}{2}, \\ cN^{\frac{1}{2}-m+\gamma}\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty}, & \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases} \tag{4.30}$$

Similarly, (4.30) holds for $\|R_2\|_{L^\infty}, \|R_4\|_{L^\infty}, \|R_9\|_{L^\infty}$ and $\|R_{10}\|_{L^\infty}$.

The combination of (4.20) and (4.30) yields

$$\begin{aligned} &\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty} \\ &\le \begin{cases} c(\log N N^{-m}+\log N N^{-\kappa})\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty}, \\ \quad -1<\alpha,\beta\le -\frac{1}{2}, \\ c(N^{\frac{1}{2}-m+\gamma}+N^{\frac{1}{2}-\kappa+\gamma})\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty}, \\ \quad \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases} \end{aligned} \tag{4.31}$$

Based on (4.31) and taking $\kappa > \frac{1}{2}+\gamma$, when N is large enough,$u_N=v_N=w_N=0$. As a result, the existence and uniqueness of Jacobi pseudo-spectral solutions u_N, v_N, w_N are proved. □

Now we turn to the L^∞ error estimate. Actually (4.22) can be transformed into

$$\begin{cases} E = \frac{T}{2}\int_{-1}^{t} E_1(\tau)d\tau + I_N^{\alpha,\beta}\mathcal{M}E_1 - \mathcal{M}E_1 + I_N^{\alpha,\beta}Q(t), \\ E_1 = \frac{T}{2}\int_{-1}^{t} E_2(\tau)d\tau + I_N^{\alpha,\beta}\mathcal{M}E_2 - \mathcal{M}E_2 + I_N^{\alpha,\beta}Q_1(t), \\ E_2 = \frac{T}{2}\int_{-1}^{t} (\widetilde{K}(t,\tau)E(\tau)+\widetilde{a}(t)E_1(\tau)+\widetilde{b}(t)E_2(\tau))d\tau \\ \qquad -\widetilde{\mathcal{M}}E - \overline{\mathcal{M}}E_1 - \widehat{\mathcal{M}}E_2 + I_N^{\alpha,\beta}\widetilde{\mathcal{M}}E + I_N^{\alpha,\beta}\overline{\mathcal{M}}E_1 + I_N^{\alpha,\beta}\widehat{\mathcal{M}}E_2 \\ \qquad +I_N^{\alpha,\beta}\widetilde{Q}(t) + I_N^{\alpha,\beta}\overline{Q}(t) + I_N^{\alpha,\beta}\widehat{Q}(t). \end{cases} \tag{4.32}$$

which yields

$$\begin{aligned} |E|+|E_1|+|E_2| \le\ & c\int_{-1}^{t}(|E(\tau)|+|E_1(\tau)|+|E_2(\tau)|)d\tau \\ & +|R_2|+|R_4|+|R_8|+|R_9|+|R_{10}|+|R_{11}| \\ & +|R_{12}|+|R_{13}|+|R_{14}|+|R_{15}|. \end{aligned} \tag{4.33}$$

with $R_{11}=I_N^{\alpha,\beta}\mathcal{M}E_1-\mathcal{M}E_1,\ R_{12}=I_N^{\alpha,\beta}\mathcal{M}E_2-\mathcal{M}E_2, R_{13}=I_N^{\alpha,\beta}\widetilde{\mathcal{M}}E-\widetilde{\mathcal{M}}E, R_{14}=I_N^{\alpha,\beta}\overline{\mathcal{M}}E_1-\overline{\mathcal{M}}E_1, R_{15}=I_N^{\alpha,\beta}\widehat{\mathcal{M}}E_2-\widehat{\mathcal{M}}E_2$.

Similar to (4.9), it follows from (3.9), (4.33) and Lemma 3.5 that

$$\begin{aligned} \||E|+|E_1|+|E_2|\|_{L^\infty} \le\ & c(\|R_2\|_{L^\infty}+\|R_4\|_{L^\infty}+\|R_8\|_{L^\infty}+\|R_9\|_{L^\infty} \\ & +\|R_{10}\|_{L^\infty}+\|R_{11}\|_{L^\infty}+\|R_{12}\|_{L^\infty}+\|R_{13}\|_{L^\infty} \\ & +\|R_{14}\|_{L^\infty}+\|R_{15}\|_{L^\infty}). \end{aligned} \tag{4.34}$$

Similar to the estimate of (4.20), we obtain

$$\|R_{13}\|_{L^\infty} \le \begin{cases} c\log N N^{-\kappa}\|E\|_{L^\infty}, & -1<\alpha,\beta\le -\frac{1}{2}, \\ cN^{\frac{1}{2}-\kappa+\gamma}\|E\|_{L^\infty}, & \gamma=\max(\alpha,\beta), \text{otherwise}. \end{cases} \tag{4.35}$$

It also holds for R_{11}, R_{12}, R_{14} and R_{15}. In terms of (4.30), (4.34) and (4.35), when N is large enough, we obtain

$$\begin{aligned}
(4.36)\quad & \||E|+|E_1|+|E_2|\|_{L^\infty} \\
& \le \begin{cases} c\log N N^{-m}\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty} \\ \le c\log N N^{-m}(\|(|u|+|v|+|w|)\|_{L^\infty} \\ +\|(|u-u_N|+|v-v_N|+|w-w_N|)\|_{L^\infty}),\ -1<\alpha,\beta\le -\frac{1}{2}, \\ cN^{\frac{1}{2}-m+\gamma}\|(|u_N|+|v_N|+|w_N|)\|_{L^\infty} \\ \le cN^{\frac{1}{2}-m+\gamma}(\|(|u|+|v|+|w|)\|_{L^\infty} \\ +\|(|u-u_N|+|v-v_N|+|w-w_N|)\|_{L^\infty}),\ \gamma=\max(\alpha,\beta),\text{otherwise}. \end{cases}
\end{aligned}$$

By the triangular inequality,

$$\begin{aligned}
\||u-u_N|+|v-v_N|+|w-w_N|\|_{L^\infty} \quad &\le \quad \||u-\widehat{u}_N|+|v-\widehat{v}_N|+|w-\widehat{w}_N|\|_{L^\infty} \\
(4.37)\qquad\qquad &\qquad +\||E|+|E_1|+|E_2|\|_{L^\infty}
\end{aligned}$$

as well as (4.36), (4.37) and Lemma 4.1, we can obtain the estimated (4.23) provided N is sufficiently large.

Next we prove (4.24). Using the standard Gronwall inequality, one obtains that from (4.33).

$$\begin{aligned}
\||E|+|E_1|+|E_2|\|^2_{L^2_{\omega_{\alpha,\beta}}} \quad &\le \quad c\Big(\|R_2\|^2_{L^2_{\omega_{\alpha,\beta}}}+\|R_4\|^2_{L^2_{\omega_{\alpha,\beta}}}+\sum_{i=0}^{7}\|R_{8+i}\|^2_{L^2_{\omega_{\alpha,\beta}}}\Big) \\
(4.38)\qquad\qquad &\le \quad c\Big(\|R_2\|^2_{L^\infty}+\|R_4\|^2_{L^\infty}+\sum_{i=0}^{7}\|R_{8+i}\|^2_{L^\infty}\Big)
\end{aligned}$$

The combination of (4.30), (4.35) and (4.36) yields

$$\begin{aligned}
(4.39)\qquad & \||E|+|E_1|+|E_2|\|_{L^2_{\omega_{\alpha,\beta}}} \\
& \le \begin{cases} c\log N N^{-m}(\|(|u|+|v|+|w|)\|_{L^\infty} \\ +\||e_N|+|\overline{e}_N|+|\widehat{e}_N|)\|_{L^\infty}),\ -1<\alpha,\beta\le -\frac{1}{2}, \\ cN^{\frac{1}{2}-m+\gamma}(\|(|u|+|v|+|w|)\|_{L^\infty} \\ +\||e_N|+|\overline{e}_N|+|\widehat{e}_N|\|_{L^\infty}),\ \gamma=\max(\alpha,\beta),\text{otherwise}. \end{cases}
\end{aligned}$$

By the triangular inequality again,

$$\begin{aligned}
\||e_N|+|\overline{e}_N|+|\widehat{e}_N|\|_{L^2_{\omega_{\alpha,\beta}}} \quad &\le \quad \||u-\widehat{u}_N|+|v-\widehat{v}_N|+|w-\widehat{w}_N|\|_{L^2_{\omega_{\alpha,\beta}}} \\
(4.40)\qquad\qquad &\qquad +\||E|+|E_1|+|E_2|\|_{L^2_{\omega_{\alpha,\beta}}}.
\end{aligned}$$

In terms of (4.23), (4.39), (4.40) and Lemma 4.1, we obtain the desired result.

5. Numerical results

We give two numerical example to confirm our analysis.

Example 5.1. Consider the second order Volterra integro-differential equation

$$\begin{aligned} u''(t) &= tu(t) + (1-t)u' + (2+t)e^{2t} + \frac{2t-t^2-1}{(2-t)^2}e^{(2-t)t} \\ &\quad - \frac{3-t}{(2-t)^2}e^{t-2} + \int_{-1}^{t} \tau e^{-t\tau} u(\tau)d\tau. \end{aligned}$$

The exact solution is $u(t) = e^{2t}$. Fig. 1 shows the errors $u - u_N$ and $u' - u'_N$ of approximate solution in L^∞ and Fig. 2 shows the errors weighted $L^2_{\omega_{\alpha,\beta}}$ norms obtained by using the pseudo-spectral methods described above. It is observed that the desired exponential rate of convergence is obtained.

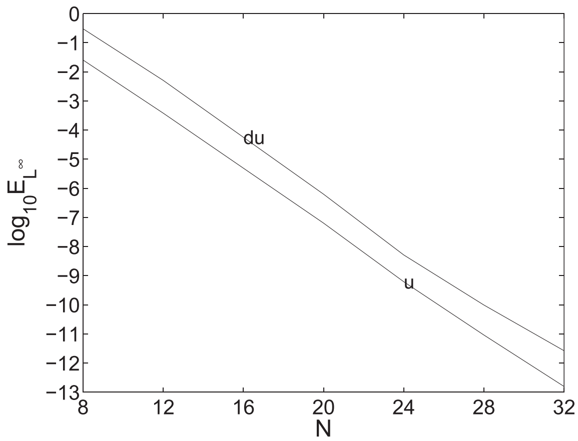

FIGURE 1. L^∞ error of Example 5.1

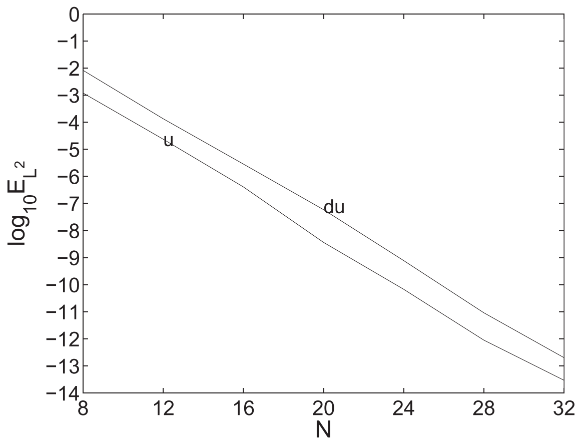

FIGURE 2. $L^2_{\omega_{\alpha,\beta}}$ error of Example 5.1

Example 5.2. Consider the second order Volterra equation integro-differential equation

$$u''(t) = e^t u(t) + 2tu'(t) + t\sin(0.5t) - \left(\frac{1}{4} + e^t\right)\cos(0.5t) + \frac{1-t^2}{2} + \int_{-1}^{t} \frac{\tau}{\cos(0.5\tau)} u(\tau)d\tau.$$

The corresponding exact solution is given by $u(t) = \cos(0.5t)$.

Fig.3 and Fig.4 plot the errors $u - u_N$ and $u' - u'_N$ for $3 \le N \le 18$ in L^∞ and $L^2_{\omega_{\alpha,\beta}}$ norms. Once again the desired spectral accuracy is obtained.

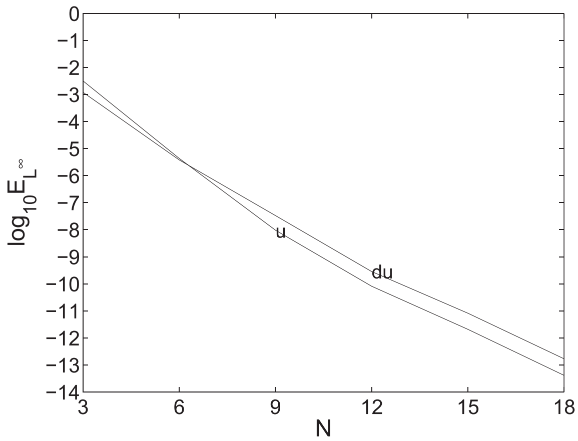

FIGURE 3. L^∞ error of Example 5.2

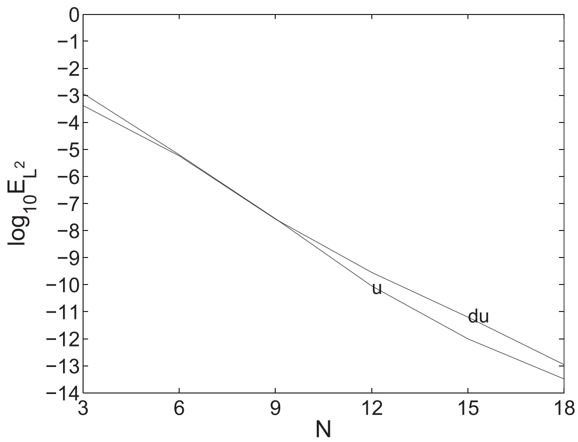

FIGURE 4. $L^2_{\omega_{\alpha,\beta}}$ error of Example 5.2

6. Concluding remarks

This paper proposes a numerical method for second order Volterra integro-differential equations based on a Jacobi pseudospectral approach. To facilitate the use of the method, we first restate the original second order Volterra integro-differential equation as three simple integral equations of the second kind. The most important contribution of this work is that we are able to demonstrate rigorously that the errors of approximations decay exponentially in $L^{\infty}$and$L^2_{\omega_{\alpha,\beta}}$, which is a desired feature for a spectral method.

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Xiao-yong Zhang
Department of Mathematics, Shanghai Maritime University, Haigang Avenue, 1550, Shanghai, 201306, China
E-mail address: zxyo20@163.com